Finite-size scaling tests for spectra in SU(3) lattice gauge theory coupled to 12 fundamental flavor fermions

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Abstract

I carry out a finite-size scaling study of the correlation length in SU(3) lattice gauge theory coupled to 12 fundamental flavor fermions, using recent data published by Fodor, Holland, Kuti, Nógradi and Schroeder [1]. I make the assumption that the system is conformal in the zero-mass, infinite volume limit, that scaling is violated by both nonzero fermion mass and by finite volume, and that the scaling function in each channel is determined self-consistently by the data. From several different observables I extract a common exponent for the scaling of the correlation length ξ with the fermion mass m_q, $\xi \sim m_q^{-1/y_m}$ with $y_m \sim 1.35$. Shortcomings of the analysis are discussed.

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A recent paper by Fodor, et al [1] presents an analysis of spectroscopy for $SU(3)$ gauge theory coupled to $N_f = 12$ flavors of fermions. This theory is a potential candidate for beyond - Standard Model physics. However, it is a subject of some recent controversy. Appelquist et al [2, 3], performing a calculation of a running coupling constant, concluded that in the limit of vanishing fermion mass, it had an infrared-attractive fixed point (IRFP). In that limit it exhibits conformal behavior at long distances. From the first analysis of spectroscopy for this model [4] the dependence of bound state masses on the fermion mass has been observed to be quite different from what is seen in ordinary (low N_f) QCD. The authors of Ref. [1] collected spectroscopic data at one value of the gauge coupling, four simulation volumes, and eight fermion masses, for a total of twelve volume - mass combinations. They analyzed their data under the competing assumptions that the system was confining and chirally broken, or conformal, and concluded that their data favored the confining and chirally broken scenario. Other references relevant to this controversy include [5–13].

The "conformal scenario" for a system like the one under discussion assumes that the long distance behavior of the theory is described by one relevant coupling, the fermion mass m_q. In infinite volume, tuning the mass to zero causes the correlation length to diverge algebraically,

$$\xi \sim m_q^{-\frac{1}{y_m}}. \tag{1}$$

The quantity y_m is the leading relevant exponent for the system, in statistical physics language. This exponent is related to the anomalous dimension γ_m of the mass operator $\bar{\psi}\psi$, and determines the running of the mass parameter according to

$$\mu\frac{dm(\mu)}{d\mu} = -\gamma_m(g^2)m(\mu), \tag{2}$$

$y_m = 1 + \gamma_m(g_*)$. All other couplings, including the gauge coupling (more properly, the distance of the gauge coupling from its fixed point value) are irrelevant.

However, no simulation is ever done in infinite volume. The system size L is also a relevant parameter since the correlation length only diverges in the $1/L \to 0$ limit. When the correlation length measured in a system of size L (call it ξ_L) becomes comparable to L, ξ_L saturates at L even as m_q vanishes. However, if the only large length scales in the problem are ξ and L, then overall factors of length can only involve ξ and L. For the correlation length itself, this argument says that

$$\xi_L = LF(\xi/L) \tag{3}$$

where $F(x)$ is some unknown function of ξ/L. A somewhat more useful version of this relation invokes Eq. 1, to say

$$\xi_L = Lf(L^{y_m}m_q). \tag{4}$$

Then one can plot ξ_L/L vs $L^{y_m}m_q$ for many L's, and vary y_m. Under this variation, data from different L's will march across the x axis at different rates. The exponent can be determined by tuning y_m to collapse the data onto a single curve.

The authors of Ref. [1] confronted a subset of their data with Eq. 1 and all their data with Eq. 3. The reason for this note and my analysis is that they chose a particular functional form for $F(x)$ in their fit. However, in general, the actual form of the scaling function is unknown, Limiting behavior is known: for example, at small x, $F(x) = x$ and at large x, $F(x)$ goes to a constant. This means that finding $F(x)$ or $f(x)$ is itself part of the fit. In addition, there is no reason for different observables to have the same $F(x)$. The finite box size can – and the data show that it does – affect them differently.

Not having a-priori knowledge of the scaling function means that it is difficult to assign a goodness-of-fit parameter, like a chi-squared, to a determination of y_m. All one can do is to compare the y_m's from different data sets, and ask if they are consistent. It also means that the resulting value of y_m will have a large uncertainty. However, I think it is still a potentially informative task, to ask, whether the data of Ref. [1] is consistent with the finite size scaling hypothesis, while letting the data itself determine the scaling function. That is the subject of this note.

This methodology was used in Ref. [14] to measure $y_m(g^2)$ in the $SU(3)$ - $N_f = 2$ sextet system. It produced relatively noisy exponents. The technique of using correlation functions in the Schrödinger functional is much more accurate, but the finite-size scaling exponents agreed reasonably well with these better measurements [15].

The technique has already been described in Ref. [14], so we will just proceed to results. I have analyzed the mass spectra of the following states from the data sets of Ref. [1]: the pseudoscalar (would-be Goldstone boson in a chirally broken theory), vector and axial vector mesons, baryon, and pseudoscalar decay constant, f_π. I will define the correlation length ξ_L to be just the inverse mass, (or $1/f_\pi$) in a lattice whose spatial length is L.

To begin the analysis, we have to see if the data shows the appropriate qualitative behavior: does ξ_L seem to flatten out at small m_q, at an L - dependent value? Fig. 1 shows that (with one exception) the trend is as expected, and furthermore larger ξ_L correlates with

larger L. Data at smaller masses on larger volumes would be desirable (was that not always so?) to push to the m_q independent regime, but in addition, smaller volume data would do as well. The size of the finite volume effect is, not surprisingly, different for different observables.

The exception is f_π, panel (b), which has a tiny L dependence with an unexpected order, ξ_L falls with L, and no plateau yet observed, as f_π always monotonically decreases with m_q. The axial vector matrix element $\langle 0|A_0|\pi\rangle \sim m_\pi f_\pi$ does approach a plateau because m_π does.

By eye, before the different L data separate, the correlation lengths seems to show the power law behavior of Eq. 1. Nevertheless, the figure illustrates the danger of a simple fit to a power law: if the system is conformal in the zero mass limit, the smallest fermion mass data is presumably closest to conformality, and yet it is the most contaminated by finite volume effects. While the data does not show this, presumably the largest masses could be far enough away from the critical region that they might be outside it, following some different scaling law. One is then forced to consider cutting the data from both the high and low mass ends, to produce a fit to Eq. 1, not a desirable procedure.

Next we perform a scan of ξ_L/L vs $L^{y_m}m_q$, varying y_m, using the cleanest data set, the pseudoscalar. This is shown in Fig. 2, for $y_m = 1.0, 1.2, 1.4, 1.6$. Collapse to a single scaling curve seems to be occurring. Notice that ξ_L/L is not zero; the volume is not infinite.

Finally, I attempt to determine a best fit value of y_m. The method is that of Bhattacharjee and Seno [16]. The idea is to use each data set (each different L values, for the same channel) to estimate the scaling curve and to find the y_m which pulls the other L sets onto it. This is done inclusively; all data sets take a turn at being the fiducial. The quantity to minimize is

$$P(y_m) = \frac{1}{N_{over}} \sum_p \sum_{j \neq p} \sum_{i,over} \left(\frac{\xi_L(m_{i,j})}{L_j} - f_p(L_j^{y_m} m_{i,j}) \right)^2. \tag{5}$$

Data set p is used to estimate the scaling function $f(x)$. This is done by interpolation, either by polynomials or rational functions, using the recorded values of ξ_L/L. The label "over" indicates that the sum only includes data from set j whose x values, $L_j^{y_m}m_{i,j}$, overlap the range of x's of set p. The overall factor of $1/N_{over}$ counts the total number of points and guards against recording a zero value of P if there are no overlap. P is minimized by the optimal y_m. This is folded into a jackknife.

Again, the scaling function is not identical for different particle correlators, and so I choose not to combine the data further. All data sets tested so far produce similar values for
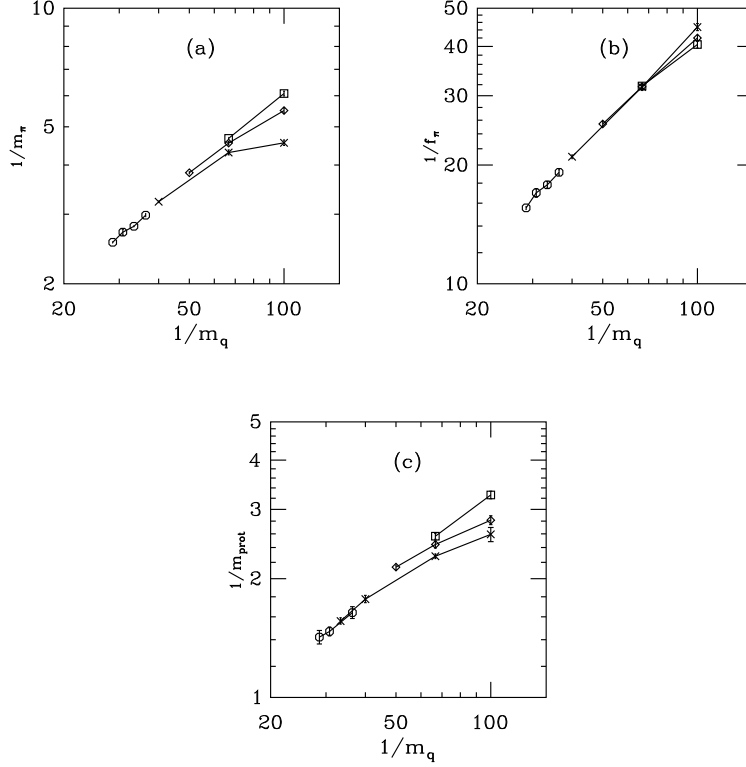
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FIG. 1: Correlation length versus inverse quark mass. Panels use the inverse pion mass (a), f_π (b), and proton mass (c). Plotting symbols are for different simulation volumes, squares, $L = 48$; diamonds, $L = 40$; crosses, $L = 32$; octagons, $L = 24$.

y_m, about 1.35. Results are given in Table I, with errors from a single-elimination jackknife. Bhattacharjee and Seno advocate taking an error from an approximation to the second derivative of P,

$$\Delta y_m = \eta y_m \left(2 \ln \frac{P(y_m(1+\eta))}{P(y_m)} \right)^{-1/2} \tag{6}$$

This gives slightly smaller uncertainty estimates, about 0.1 at $\eta = 1$. The scaling curves for this y_m are shown in Fig. 3.

The authors of Ref. [1] confronted their data with Eq. 1 and concluded that the hypothesis was disfavored. They repeated their analysis using Eq. 3, making a specific choice for $F(x)$, and reached a similar conclusion. The choice of a particular functional form for $F(L/\xi)$ is, I have already remarked, unjustified, and the authors' analysis may simply show that their functional form for $F(x)$ is not the one which actually describes the data.

My analysis of the data of Ref. [1] with the assumption that the infinite volume theory
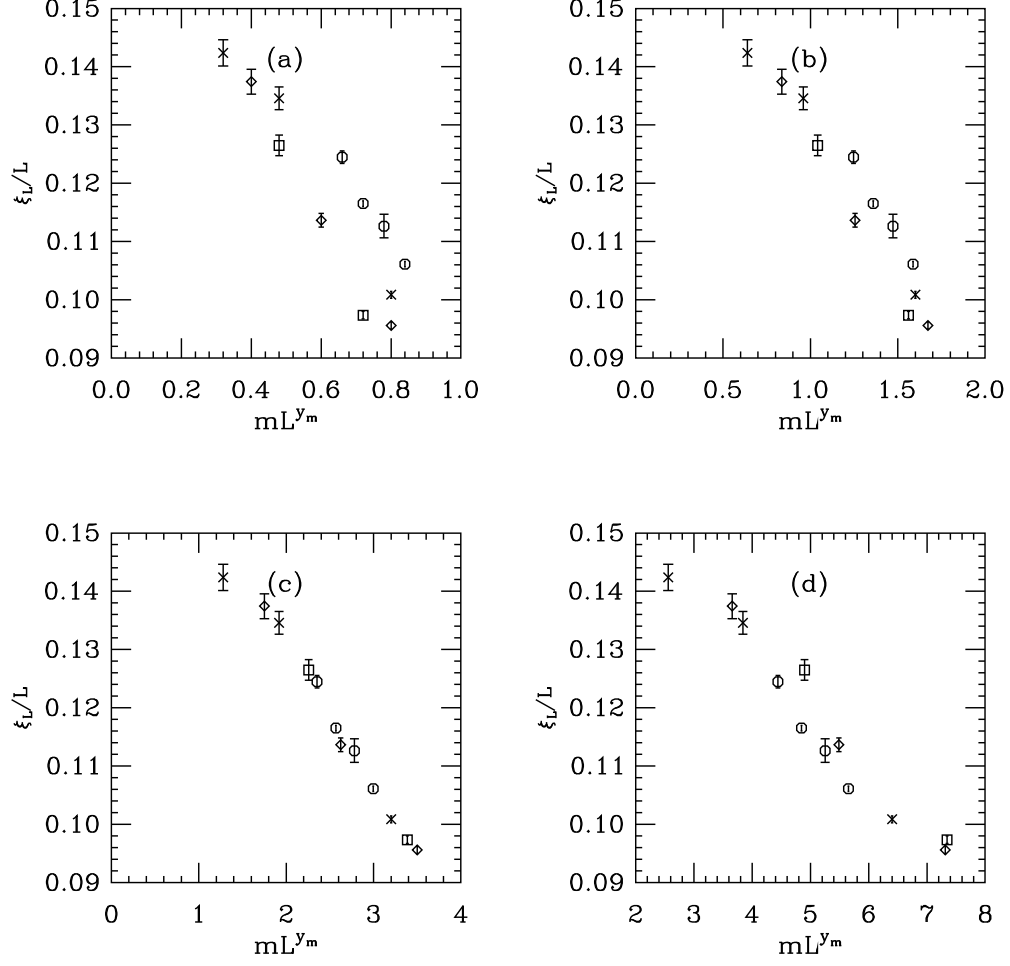
FIG. 2: Plots of ξ_L/L vs $m_q L^{y_m}$ with $\xi_L = 1/m_\pi$ for four choices of y_m: (a) $y_m = 1.0$, (b) $y_m = 1.2$, (c) $y_m = 1.4$ (d) $y_m = 1.6$. Plotting symbols are for different spatial sizes, squares, $L = 48$; diamonds, $L = 40$; crosses, $L = 32$; octagons, $L = 24$.

is conformal while the conformality is broken by both the quark mass and the finite volume produces a consistent picture, that the leading relevant exponent at their simulation parameters is $y_m = 1.35$ or $\gamma_m = 0.35$, with unfortunately unimpressively large uncertainty. Of course, this is an analysis for which the data set was not designed. It could be improved by more mass values at all chosen volumes.

The exponent is similar to the value reported by Ref. [1] and by Ref. [17], which also analyzed the data of Ref. [1]. It is similar to, but a bit smaller than, the exponent reported by Ref. [4] from a pure power law fit to their pseudoscalar and vector channel mass data from one volume but over a range of bare couplings, $y_m \sim 1.64(5) - 1.52(2)$. The small
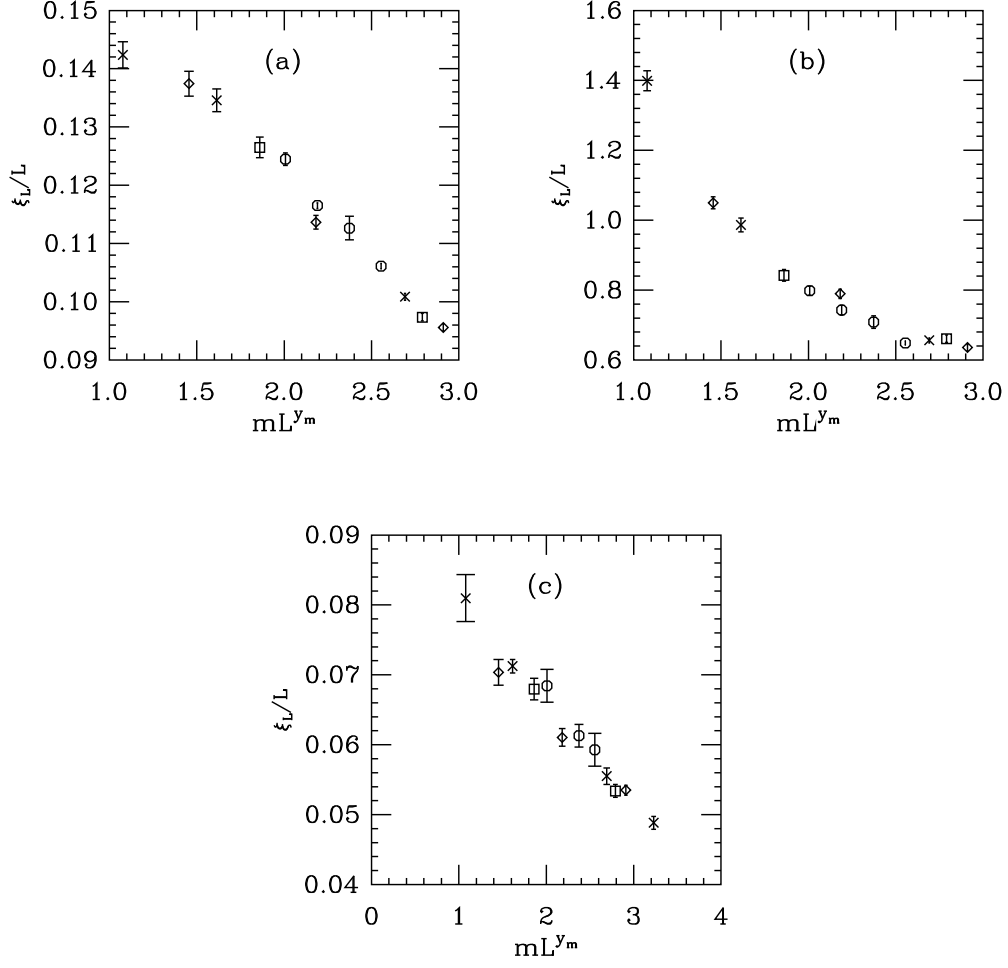
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FIG. 3: Plots of ξ_L/L vs $m_q L^{y_m}$ for $y_m = 1.35$: (a) pseudoscalar (would-be Goldstone) (b) f_π, (c) proton. Plotting symbols are for different spatial sizes, squares, $L = 48$; diamonds, $L = 40$; crosses, $L = 32$; octagons, $L = 24$.

γ_m measured here resembles results from other nearly-conformal theories observed to date [15, 18–20].

Notice, finally, that this is far from being a complete story. For the fit itself, one could be concerned with, and include, non-scaling contributions. (See Ref. [17] which does this. The authors specified their $F(x)$ rather than letting the data do so.)

More importantly, the y_m which comes out is very likely not to be an actual scaling exponent. A few moment's reflection shows why: Because the gauge coupling runs so slowly, simulations done over a small range of volumes cannot flow to a fixed point (if it exists) unless they begin very close to it.

TABLE I: Exponent y_m from various hadronic channels from my analysis of the data of [1] at its one bare gauge coupling. Errors are from a single-elimination jackknife.

channel	y_m
pseudoscalar	1.35(23)
nucleon	1.43(26)
f_π	1.23(31)
vector	1.33(22)
axial vector	1.32(12)

That this is expected, is easy to see from the one-loop beta function result, where under a scale change s the inverse coupling is shifted by

$$\frac{1}{g^2(s)} - \frac{1}{g^2(1)} \sim \frac{b_1}{8\pi^2} \log s + \ldots \tag{7}$$

In three-flavor $SU(3)$, $b_1 = 9$ and in 12 flavor $SU(3)$, $b_1 = 3$, so that the equivalent scale changes in the two theories, for an equal change in coupling, are $s_{12} = s_3^3$. In ordinary QCD, the coupling runs from weak at a distance of 0.1 fm to strong at a distance 1.0 fm, or over $s = 10$. It is hard to get a large aspect ratio s from a set of numerical simulations at a single set of bare parameters. Therefore, whether or not a running coupling in one of these many-fermion theories actually has an IR fixed point, it runs so slowly that for all practical purposes its running can be neglected. Then the zero mass limit is effectively conformal. That is all that is needed to motivate a scaling analysis as is done here.

(The lowest order beta function argument is merely suggestive of problems, but if the theory has an IRFP, than the situation on the weak coupling side calls for even slower running than the one loop formula.)

Taking this argument further, it says that a measurement of y_m at one set of bare coupling values does not answer the question of whether the theory actually has an IRFP. Simulations which have made predictions for an exponent map out g^2 in some prescription and $\gamma_m(g^2)$, both from simulations at many values of the bare parameters. They then separately determine the critical coupling (in some scheme) g_*^2 and read off $\gamma_m(g_*^2)$. (Compare Refs. [15, 18–20].) Most likely, finite size scaling studies of spectroscopy are just not the method of choice for discovering whether the theory has an IRFP, and if it does, accurately

determining either g_*^2 or $\gamma_m(g_*^2)$.

Acknowledgments

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