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Remarks on the renormalization of primordial cosmological perturbations

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We briefly review the need to perform renormalization of inflationary perturbations to properly work out the physical power spectra. We also summarize the basis of (momentum-space) renormalization in curved spacetime and address several misconceptions found in recent literature on this subject.

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I. Introduction. The inflationary universe [1] opens an exciting window to observationally test fundamental aspects of the theory of quantized fields in curved space-time [2–5]. In a curved background, the vacuum energy of quantum fields enters into the gravitational field equations. As is well known, infinities arise in the computation of vacuum energy, and other expectation values quadratic in fields, due to the ultraviolet (UV) behavior of the field theory. Therefore, the potential gravitational effects of the quantum vacuum must be handled with care. Methods have been developed to define regularization and renormalization procedures to physically account for the effects of vacuum energy in free and interacting quantum field theory in curved spacetime.

In the cosmological scenario, it was shown in the early eighties that vacuum fluctuations can induce a primordial spectrum of density perturbations during an inflationary expansion. Remarkably, the calculated spectrum satisfactorily accounts for the origin of the cosmic inhomogeneities that we observe in the present universe. Let φ represent a generic field describing (scalar or tensor) perturbations during inflation. The quantum fluctuations of φ can be quantified by the mean square fluctuation in the vacuum state (see, for instance, [6])

$$\langle \varphi^2(\vec{x}, t) \rangle = \int d^3k |\varphi_k(t)|^2 \equiv \int_0^\infty \frac{dk}{k} \Delta_\varphi^2(k, t) . \quad (1)$$

where $\varphi_k(t)$ is defined by the expansion in Fourier modes of a free field operator

$$\varphi(\vec{x}, t) = \int d^3k (A_{\vec{k}} \varphi_k(t) + A_{-\vec{k}}^\dagger \varphi_k^*(t)) e^{i\vec{k}\vec{x}} , \quad (2)$$

where $A_{\vec{k}}$ and $A_{\vec{k}}^\dagger$ are creation and annihilation operators, such that $A_{\vec{k}}|0\rangle = 0$. In the last equality of Eq. (1) we have defined the power spectrum $\Delta_\varphi^2(k, t)$, which is the quantity conventionally used in cosmology to quantify the variance (1). For a single k , the power spectrum $\Delta_\varphi^2(k, t)$ is well defined. However, the formal variance $\langle \varphi^2(\vec{x}, t) \rangle$ diverges in the UV. One could argue that this

UV divergence concerns ultrashort wavelength modes, and it is not going to affect the predictions regarding the finite range of wavelengths that we actually observe in cosmic inhomogeneities. But in view of the fact that the perturbations are a consequence of quantum field theory in curved spacetime, it was proposed in [7–10] that the physical power spectrum should be defined in terms of the renormalized mean square

$$\langle \varphi^2(\vec{x}, t) \rangle_{ren} = \int_0^\infty \frac{dk}{k} \tilde{\Delta}_\varphi^2(k, t) . \quad (3)$$

They found that, when methods of renormalization in curved space-time are applied, the power spectrum at wavelengths of observational interest may be affected in ways that can be tested by observations in the not-too-distant future.

It has been argued recently that the renormalization procedure used in [7–10] is not well-defined at the time scales at which cosmic inhomogeneities are created during inflation, particularly at the time t_k that each mode crosses the Hubble sphere [11, 12]. In the present work, we briefly summarize the basis of renormalization (including its effects on the momentum-space power spectrum) in an expanding universe and address the criticisms raised in [11, 12]. We believe that the arguments and conclusions proposed in [11, 12] conflict with some of the basic principles of renormalization in curved space-time. The last paragraphs in sections II and III summarize the arguments provided here in relation to the criticism of [11, 12].

II. Renormalization in curved space-time. The renormalization of expectation values like those corresponding to the mean square fluctuation and the stress tensor of a quantized field in a curved space-time is more involved than in Minkowski space, even for the simplest case of a free field. This is rooted in the fact that in a general curved space-time there are insufficient isometries to uniquely determine a global vacuum state. Moreover, the presence of space-time curvature yields new types of divergences not present in Minkowski space. This is true

for free as well as interacting fields.

It is instructive to briefly consider, in a general curved space-time, an interacting scalar field φ with a $\lambda\varphi^4$ interaction term in its Lagrangian. In flat space-time, one usually renormalizes such an interacting theory by going to momentum-space and adding counter-terms to the Lagrangian that will cancel the regularized UV divergences in the momentum space integrals corresponding to Feynman diagrams. The latter method is not directly applicable in a general curved space-time because of the absence of a global and generally-covariant momentum-space expansion. However, this problem was overcome for a general smooth curved space-time in [13] by making use of a Riemann normal coordinate system (RNC) with its origin at a given space-time point x' . Such a coordinate system, based on the system of geodesics leaving the point x' , exists in any smooth space-time in a normal neighborhood of x' (i.e., one in which these outgoing geodesics do not intersect). In a RNC, the metric infinitesimally close to x' is Minkowskian and has vanishing first derivatives with respect to the space and time coordinates of the RNC. In addition, the metric has a well defined expansion in powers of the RNC coordinates y , which are defined for each space-time point x in the normal neighborhood of x' by means of the tangent vector at x' of the unique geodesic that connects x' to x and the invariant length of that geodesic.

In [13], Bunch and Parker (BP) defined a local momentum-space “Fourier” transform of the Feynman propagator $G(x, x')$ based on an RNC with its origin at x' . Working in a *general* curved space-time, they used this local momentum-space method to evaluate the Feynman diagrams necessary to renormalize the *interacting* scalar field theory to second order in the interaction coupling constant λ appearing in the $\lambda\varphi^4$ self-interaction. Dimensional regularization was used to replace the UV infinities that are present in 4-dimensions by covariant well-defined expressions. For the interacting theory, they introduced into the Lagrangian the minimal set of generally covariant counter-terms necessary to absorb the regularized UV “infinities” into the values of the constant coefficients of these counter-terms. This process leaves one with a Lagrangian having terms of the same form as the original counter-terms, but with “renormalized” constant coefficients that are assumed to be finite, well-defined, and in principle measurable.

The terms that involve the curvature at the space-time point x' were shown to drop out of the final result. This step is non-trivial and necessary for the renormalized interacting theory to be covariant in a general curved space-time. This calculation showed that interacting $\lambda\varphi^4$ theory is renormalizable in a general curved space-time to second order in λ . The reason for using the *minimal* set of counter-terms is to alter the original Lagrangian as little as possible, thus avoiding the arbitrary introduction of interactions that are not necessary for renormalization.

They also calculated the leading terms of their local momentum-space expansion of the propagator that

remain when $\lambda = 0$, (i.e., for the non-interacting, free field). For example, for a minimally coupled *free* field, φ , the local momentum-space Fourier transform of $|g(x)|^{-1/4}G(x, x')|g(x')|^{-1/4} \equiv \bar{G}(x, x')$ has an asymptotic expansion for large k^2 that is given by [13]

$$\bar{G}(k; x') \sim \frac{1}{-k^2 + m^2} + \frac{R(x')}{6(-k^2 + m^2)^2} + \cdots, \quad (4)$$

where k is the 4-momentum, m is the mass of the field, and the ellipsis (\cdots) includes terms that go as the third and fourth powers of $1/(-k^2 + m^2)$. The coefficients of those terms are formed from contractions of Riemann tensors and their derivatives, evaluated at point x' , and are given explicitly in Eq. (2.21) of [13], with k^2 in their results replaced by $-k^2$ because we are using the opposite metric signature, namely $(+, -, -, -)$, in the present paper. By carrying out the inverse Fourier transformation back to curved space-time, they were able to recast this series into the form of the proper-time or heat kernel expansion of $\bar{G}(x, x')$ [14].

As is well known, the expectation values $\langle T_{\mu\nu}(x') \rangle$ of the energy-momentum tensor of the free scalar field in physical states can be found from the exact expression for the Green function $G(x, x')$ by applying to $G(x, x')$ the appropriate second-order differential operator that produces the expression for the operator $T_{\mu\nu}(x')$ in the coincidence limit as $x \rightarrow x'$ and taking the expectation value in the physical state of interest. From the asymptotic series for $\bar{G}(k; x')$ discussed above, it is clear that the terms in that series up to those that go as $1/(-k^2 + m^2)^3$, will produce divergent contributions to $\langle T_{\mu\nu}(x') \rangle$ when the Riemann normal coordinate y is taken to zero and the dimension n is taken to the value 4 in the n -dimensional Fourier transform. It can be seen that, when the two spatial derivatives in $T_{\mu\nu}$ are taken into account, the first term in (4) gives a contribution to $\langle T_{\mu\nu}(x') \rangle$ that has a quartic UV divergence that does not depend on the curvature at x' , the second term has a quadratic UV divergence that depends on the Ricci scalar curvature $R(x')$, and the subsequent terms have logarithmic UV divergences that depend in a more complicated way on contractions and spatial derivatives of the Riemann tensor, evaluated at x' .

It is also clear that these UV divergent contributions to $\langle T_{\mu\nu}(x') \rangle$ have the same state-independent expression in terms of the curvature tensor and its derivatives at x' . The leading (quartic) divergence reduces to the vacuum energy in flat space-time and is customarily subtracted from the physical or renormalized value of $\langle T_{\mu\nu}(x') \rangle$ in a general curved space-time. The other state-independent divergences coming from the leading terms in the asymptotic series in (4), are also subtracted. Alternatively, one can rewrite the full Lagrangian (including the gravitational terms) to include counter-terms having the same form as the expressions involving the Riemann tensor and its derivatives in these UV divergences. Then one can absorb the regularized expressions for these UV infinities into the constants in front of similar terms added to the

original Lagrangian (similar to the procedure described above for the interacting field). Thus, the renormalization of the free field requires covariant terms involving the curvature tensor to be added to the Lagrangian with coupling constants that are in principle measurable.

Whether one thinks simply of subtracting these state-independent UV-infinite terms from the formal expression for $T_{\mu\nu}$ or thinks of those terms as renormalizing the coupling constants of terms present in the full Lagrangian, the result is the same, namely, a quantity that has no UV infinities when its expectation value is evaluated for any physical state in 4-dimensions. Its expectation value is what we will refer to as the renormalized expectation value of the energy-momentum tensor, denoted by $\langle T_{\mu\nu}(x') \rangle_{\text{ren}}$.

Now let us consider the renormalization of the variance $\langle \varphi^2(\vec{x}, t) \rangle$ within the local momentum space method we are considering. The expression for the renormalized variance $\langle \varphi^2(\vec{x}', t') \rangle_{\text{ren}}$ in any given physical state is obtained by subtracting the terms in the asymptotic series that would give UV infinities from the solution for $\bar{G}(k; x')$ in the normal neighborhood of x' . The variance involves no derivatives of $\varphi(x)$, so only the first two terms in Eq.(4) are subtracted from the local Fourier transform of $\bar{G}(k; x')$. Then the inverse of the local momentum space Fourier transform can be performed, giving a well-defined operator that we will denote by $\bar{G}(x; x')_{\text{ren}}$. The coincidence limit ($x \rightarrow x'$) of the expectation value of this quantity, then gives the renormalized value of the variance at x' in any given physical state. Thus, we obtain the result,

$$\langle \varphi^2(x') \rangle_{\text{ren}} = \lim_{x \rightarrow x'} \bar{G}(x; x')_{\text{ren}} \quad (5)$$

As explained previously, we are subtracting only the minimum number of terms needed to give an expression having no UV divergences. Even though we found it necessary to subtract four terms of the asymptotic series to obtain $\langle T_{\mu\nu}(x') \rangle_{\text{ren}}$, since the renormalization of the variance involves only the first two counter-terms, there is no reason to subtract additional terms in the series. In the gravitational part of the Lagrangian of the renormalized theory the additional covariant terms needed to renormalize the energy-momentum tensor will still appear, but it would be uneconomical and would lead to unnecessary complications to subtract the corresponding momentum space terms from $\bar{G}(k; x')$ in calculating the renormalized variance because those terms are not necessary to regularize UV divergences in the variance. One does not wish to introduce physical effects into the theory that are not required by the actual renormalization process. Hence, one uses the principle of minimal subtraction.

In the above discussion of renormalization in a general curved space-time, the asymptotic expansion of $\bar{G}(k; x')$ in (4) was used to identify the set of UV divergent terms in the local momentum space expansion in the normal neighborhood of x' . The asymptotic expansion was *not* used to *approximate* a solution of the differential equation for $\bar{G}(k; x')$ that would be the local momentum space

Fourier transform of some particular global Green function $G(x, x')$ for values of x in the normal neighborhood of x' . As noted above, one subtracts the minimum number of terms in the asymptotic expansion (4) that are required to renormalize the UV divergences of the variance, namely, the first two terms in that expansion. It has been argued [11, 12] that subtracting these two terms is not reliable for values of k sufficiently small that the second term in the expansion (4) is not small compared to the first term. As we see it, such an argument seems based on the incorrect view that the large-momentum series should approximate the exact solution for the Green function $\bar{G}(k; x')$ within the normal neighborhood of x' . Because in doing renormalization one generally uses an asymptotic series, which does not necessarily converge, to identify the terms that must be subtracted, minimal subtraction to achieve a UV finite result is a well-defined and reasonable procedure. Neither the behavior of the asymptotic series for finite momentum, nor the size of the normal neighborhood, is relevant to defining the UV renormalization subtraction terms.

III. Adiabatic regularization in an RW universe. A spatially flat RW universe, for which $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$, is often used as the background space-time in discussions of inflation. In such a universe, the 3-dimensional hypersurfaces of constant t are homogeneous and the relevant quantized perturbation field can be expanded as in Eq.(2). The method of *adiabatic regularization* in the RW universe [2, 15] starts with the formal expression for $\langle T_{\mu\nu}(x') \rangle$, which is a function of t' in the RW coordinate system. As described above in connection with the local momentum space method, one can make a large momentum asymptotic expansion of the 3-dimensional Fourier transform of this quantity and identify the leading terms in that expansion that would give UV divergences in the integration over \vec{k} . These terms are the same for any physical state. Now, instead of using dimensional regularization or some other regularization method to find the covariant form of the counter-terms that would be added to the Lagrangian in the course of renormalization, in adiabatic regularization one goes directly to the step of subtracting the relevant leading terms in this momentum space asymptotic expansion from the formal expression for the spatial 3-dimensional Fourier transform of $\langle T_{\mu\nu}(x') \rangle$. This step insures that the integrand of this 3-dimensional Fourier transform will not give any UV divergences when integrated over 3-momentum. The renormalized expression for $\langle T_{\mu\nu}(x') \rangle_{\text{ren}}$ is defined by this momentum integral.

It is worth remarking that just as explained earlier in connection with the $\lambda\phi^4$ interaction, the minimal number of terms in the asymptotic series are subtracted in the process of adiabatic subtraction and renormalization, so as to change the form of the original unrenormalized Lagrangian as little as possible in constructing the renormalized Lagrangian (including the gravitational part of the Lagrangian). Note that after making the subtractions in the above method of regularization the integrand

of the momentum integral is already regularized (i.e., the momentum space integrand no longer has any UV infinities when it is considered as a whole). One of the nice points of this process of adiabatic regularization is that it directly displays the spectral properties of the renormalized physical quantity, such as $\langle T_{\mu\nu}(x') \rangle_{\text{ren}}$.

Regarding the variance $\langle \varphi^2(x') \rangle$, one proceeds in parallel to the method outlined above for the energy-momentum tensor. The relevant leading terms in the momentum space *asymptotic* expansion have to be subtracted from the formal expression for the spatial 3-dimensional Fourier transform of $\langle \varphi^2(x') \rangle$. Minimal subtraction requires keeping only up to second adiabatic order subtraction terms. The renormalized expression for $\langle \varphi^2(x') \rangle_{\text{ren}}$ is then defined by the momentum integral, regardless of the expansion rate of the universe (assuming the expansion is sufficiently smooth and that infrared divergences that may occur for zero mass are dealt with properly). Therefore, we disagree with the argument [11, 12] that adiabatic renormalization fails when the expansion rate H is larger than the physical momentum scale $k/a(t)$.

IV. Renormalization in the inflationary universe. In the physical situation in which the field $\varphi(\vec{x}, t)$ represents scalar or (a polarization mode of) tensorial metric perturbation during inflation, $\varphi(\vec{x}, t)$ must be treated as a massless field. The term V'' appearing in the wave equation of scalar perturbations, where V is the potential of a slow-roll inflationary model, should be considered as a second-order adiabatic term [9, 10]. In the massless limit, the subtraction terms defined by the BP renormalization [13] and those obtained with the adiabatic renormalization coincide, thus defining a unique expression for the (renormalized) power spectrum [10], namely, $\tilde{\Delta}_\varphi^2(k, t) = 4\pi k^3(|\varphi_k(t)|^2 - C_k(t))$. The subtraction terms $C_k(t)$ can be obtained easily from the two first terms in the expansion (4) (or equivalently by the adiabatic expansion of the modes).[16]

For scalar perturbations in single-field, slow-roll inflation one finds (see [10] for details)

$$C_k^{(\delta\phi)}(t) = \frac{1}{2(2\pi)^3 a^3} \left[\frac{a}{k} + \frac{a^3}{2k^3} H^2 (2 - 3\eta + 5\epsilon) \right], \quad (6)$$

where $\delta\phi$ is the inflaton perturbation (in the spatially flat gauge), a is the expansion factor, $H \equiv \dot{a}/a$, and ϵ and η are the usual slow-roll parameters; while for tensorial perturbations one gets

$$C_k^{(h)}(t) = \frac{16\pi G}{2(2\pi)^3 a^3} \left[\frac{a}{k} + \frac{a^3}{2k^3} H^2 (2 - \epsilon) \right], \quad (7)$$

where h stands for one of the two independent fluctuating modes. Taking into account the overall factor of a^{-3} , the first terms in square brackets in (6) and (7) decay rapidly with time during inflation. However, the second terms in square brackets change very slowly during slow-roll inflation and, while maintaining the nearly scale-invariant behavior, produce a non-negligible effect

in the power spectrum even at late times during inflation, as explicitly worked out in [10].

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