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Hamiltonian structure of Hořava gravity

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The Hamiltonian formulation of Hořava gravity is derived. In a closed universe the Hamiltonian is a sum of generators of gauge symmetries, the foliation-preserving diffeomorphisms, and vanishes on shell. The scalar constraint is second class, except for a global, first-class part that generates time reparameterizations. A reduced phase space formulation is given in which the local part of the scalar constraint is solved formally for the lapse as a function of the 3-metric and its conjugate momentum. In the infrared limit the scalar constraint is linear in the square root of the lapse. For asymptotically flat boundary conditions the Hamiltonian is a sum of bulk constraints plus a boundary term that gives the total energy. This energy expression is identical to the one for Einstein-aether theory which, for static spherically symmetric solutions, is the usual ADM energy of general relativity with a rescaled Newton constant.

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I. INTRODUCTION

Hořava [1] has proposed a theory of gravity that is closely related to general relativity (GR), but is power-counting renormalizable and possesses a preferred space-like foliation of spacetime that breaks the spacetime diffeomorphism symmetry down to time-dependent 3-dimensional diffeomorphism symmetry and a reparameterization of global time. Several variants of this theory have been considered (see Ref. [2] for a review). In this paper we study the “consistent extension” of Blas et. al. [3]. This theory is just the “non-projectable” version of Hořava’s original proposal with the inclusion of terms involving $(\ln N)_{,i}$, the spatial gradient of the log of the lapse function N , which are compatible with the symmetry but were not explicitly mentioned in Ref. [1]. This variant of the theory is free of known pathologies (instabilities, over-constrained evolution, or strong coupling at low energies) which afflict some other variants. We will refer to this theory simply as Hořava gravity.

In this paper, we consider the Hamiltonian formulation of Hořava gravity. While this was already considered in Ref. [4] (see also comments in Ref. [3]), there are several reasons to revisit the analysis. In Ref. [4], it was argued that, unlike other theories with time-reparameterization symmetry, Hořava gravity has a non-vanishing Hamiltonian. A vanishing Hamiltonian is one aspect of the so-called “problem of time” in quantum gravity: quantization leads to a theory in which there is no unitary evolution with respect to an external time [5]. Here we find that although the Hamiltonian density is a sum of second-class constraints, the total Hamiltonian is a sum of first-class constraints, one of which was overlooked pre-

viously. Once this constraint is recognized, we see that the Hamiltonian for a closed space is indeed a sum of constraints, and therefore a global version of the problem of time persists.

We also consider the Hamiltonian formulation with asymptotically flat boundary conditions. In Ref. [6] it was shown that in the IR limit Hořava gravity is closely related to Einstein-aether theory with a hypersurface-orthogonal aether field, in the sense that every hypersurface-orthogonal aether solution is also a Hořava solution.¹ We show that the energy of asymptotically flat solutions of Hořava gravity can be expressed as a surface integral at spatial infinity that agrees with the expression for energy in Einstein-aether theory.

II. HOŘAVA GRAVITY

In a covariant formulation [8], the dynamical objects of Hořava gravity are a 4-metric and a preferred foliation. In the Hamiltonian formulation one specifies an additional foliation, which need not coincide with the preferred foliation. However it will be useful to use the preferred foliation in the Hamiltonian formulation, since then the Lagrangian depends only on first time derivatives of the fields. We will therefore present the theory in coordinates (t, x^i) adapted to the preferred foliation. This is the

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¹ In fact there can be a global topological obstruction to this equivalence. If Σ has a non-trivial first homotopy group then it is possible to have an aether field that is locally hypersurface orthogonal but is not normal to any global foliation. A simple example is provided by the “tilted aether” Bianchi type I cosmologies discussed in Ref. [7], if a homogeneous aether field on a homogeneous spacetime with topology $R^3 \times S^1$ is tilted along the S^1 direction.

original formulation of Hořava[1].² In these coordinates Hořava gravity has the symmetry of foliation-preserving diffeomorphisms

$$t \rightarrow t'(t), \quad x^i \rightarrow x'^i(x^i, t). \quad (1)$$

The metric can be expressed in Arnowitt-Deser-Misner (ADM) form in terms of the lapse N , shift vector N^i and 3-dimensional metric g_{ij} ,

$$ds^2 = N^2 dt^2 - g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad (2)$$

where letters $i, j, \dots = 1, \dots, 3$ denote spatial indices. The extrinsic curvature K_{ij} of the constant t surfaces and its trace K are given by

$$K_{ij} = \frac{1}{2N} [\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i], \quad K = g^{ij} K_{ij} \quad (3)$$

where the dot denotes partial derivative with respect to t , and ∇_i is the 3-dimensional covariant derivative compatible with the metric g_{ij} . The acceleration of the congruence normal to the constant t surfaces has the spatial projection

$$a_i \equiv \nabla_i \ln N. \quad (4)$$

In terms of these variables the Lagrangian density of Hořava gravity is

$$\mathcal{L} = \frac{1}{16\pi G_H} \sqrt{g} N (K_{ij} K^{ij} - \lambda K^2 - V(g_{ij}, a_i)), \quad (5)$$

where V is a potential that depends on the 3-metric and the acceleration. The potential contains all terms that are spatial scalars of dimension up to six, where the spatial coordinates x^i are taken to have dimension -1 . In the infrared (IR) limit, the potential is dominated by the lowest dimension terms

$$V(g_{ij}, a_i) = -\xi R - \alpha a_i a^i + \dots, \quad (6)$$

where R is the Ricci scalar of the 3-metric g_{ij} , and \dots refers to terms containing more than two spatial derivatives. The term $\nabla \cdot a$ is also of second order in derivatives, but its contribution to the Lagrangian differs from that of a^2 by a total derivative since $N \nabla \cdot a = \nabla^2 N - N a^2$. The dimensionless free parameters of the IR limit of the theory are α , ξ and λ . When $\alpha = 0$ and $\lambda = \xi = 1$, the theory reduces to general relativity. From here on we adopt units where $16\pi G_H = 1$. In this paper we do not consider matter couplings.

III. HAMILTONIAN AND CONSTRAINTS

We now consider the Hamiltonian formulation of the theory, following Ref. [4]. First, since the time derivatives of N and N^i do not appear in the action, the corresponding conjugate momenta p_N and p_i vanish, i.e. we have primary constraints

$$p_N = 0, \quad p_i = 0. \quad (7)$$

The momentum conjugate to g_{ij} is

$$p^{ij} \equiv \sqrt{g}(K^{ij} - \lambda K g^{ij}), \quad (8)$$

and this relationship can be inverted yielding

$$K^{ij} = \frac{1}{\sqrt{g}} \left(p^{ij} + \frac{\lambda}{1 - 3\lambda} p g^{ij} \right), \quad (9)$$

where $p \equiv p^{ij} g_{ij}$.

The Hamiltonian density \mathcal{H} has the usual form $p^{ij} \dot{g}_{ij} - \mathcal{L}$ plus the primary constraints times Lagrange multipliers. Up to total derivatives it takes the form

$$\mathcal{H} = N \mathcal{H}_t + N^i \mathcal{H}_i + v^i p_i + v p_N \quad (10)$$

$$\mathcal{H}_t = \frac{1}{\sqrt{g}} \left(p^{ij} p_{ij} + \frac{\lambda}{1 - 3\lambda} p^2 + gV \right) \quad (11)$$

$$\mathcal{H}_i = -2g_{ik} \nabla_j p^{jk} \quad (12)$$

For now we will assume that Σ has no boundary, so that total derivative terms play no role, but they will be considered in section V when asymptotically flat boundary conditions are imposed. The Hamiltonian is then given by

$$H = \int_{\Sigma} d^3y \mathcal{H}(y) \equiv \int \mathcal{H}. \quad (13)$$

Here and below we adopt a notation in which \int means $\int_{\Sigma} d^3y$ and the dependence on spatial coordinates of integration is suppressed.

The primary constraints (7) must be preserved in time, in other words they must have vanishing Poisson bracket with the Hamiltonian. Among these constraints there is a particular combination $\int N p_N$ that generates constant rescalings of N . Since a_i is invariant under such rescalings, only explicit N and p_N dependence need be considered. Preservation of this constraint in time thus requires that

$$\left\{ \int N p_N, H \right\} = \int (N \mathcal{H}_t + v p_N) = 0, \quad (14)$$

which shows that $\int N \mathcal{H}_t = 0$ when the constraints are preserved in time. It follows that the Hamiltonian is a sum of constraints. Moreover, the requirement that all constraints are preserved in time implies that the Hamiltonian is first class, i.e. it has zero Poisson brackets with all constraints.

² It can happen that such coordinates do not cover the complete manifold as, for example, in black hole formation by collapse [9].

The fact that the Hamiltonian is a sum of first-class constraints is a feature of systems that have time-reparameterization symmetry. What distinguishes Hořava gravity from GR is that arbitrary deformations of the constant-time surfaces cannot be regarded as pure gauge. In this regard, Hořava gravity is like a partial gauge-fixing of GR in which the global time-reparameterization freedom is left unfixed.

Preservation of the primary constraints in time leads to secondary constraints

$$C = \delta H / \delta N = 0, \quad C_i = \delta H / \delta N^i = 0, \quad (15)$$

where

$$C = \mathcal{H}_t - \frac{1}{N} \nabla_i V^i, \quad (16)$$

$$C_i = \mathcal{H}_i, \quad (17)$$

and we have defined the vector density

$$V^i(x) = \frac{\delta}{\delta a_i(x)} \int \sqrt{g} N V. \quad (18)$$

The vector constraint C_i is the same as in GR, but the scalar constraint C is modified. Note that the constraint C is invariant under a constant rescaling of N . The source of this rescaling freedom is the reparameterization symmetry (1), under which the lapse transforms as $N \rightarrow N/f'(t)$.

In the IR limit,

$$V^i = -2\alpha\sqrt{g}\nabla^i N, \quad (19)$$

and the constraint $C = 0$ becomes

$$\frac{1}{g} \left(p^{ij} p_{ij} + \frac{\lambda}{1-3\lambda} p^2 \right) - \xi R - \alpha \frac{(\nabla N)^2}{N^2} + 2\alpha \frac{\nabla^2 N}{N} = 0. \quad (20)$$

There are different strategies for solving the scalar constraint. For example, one could solve for the conformal factor of the metric while keeping the other degrees of freedom fixed as is often done in GR. In Hořava gravity there is the scalar degree of freedom N , and it is natural to view C as an equation for N [3, 4]. Note, however, that Eq. (20) can determine N at most up to a constant rescaling.

The constraint equation in the IR limit can be linearized by the change of variables³ $N = n^2$, resulting in the equation

$$Ln = 0 \quad (21)$$

where L is the linear differential operator

$$L \equiv -4\alpha\nabla^2 - \frac{1}{g} \left(p^{ij} p_{ij} + \frac{\lambda}{1-3\lambda} p^2 \right) + \xi R. \quad (22)$$

Such an equation admits a solution if and only if the spectrum of the Schrödinger-like operator L contains zero. Moreover, if the foliation by constant t surfaces is to be smooth the lapse must be positive everywhere, which requires that $n(x)$ is positive for all x . A solution with positive n exists if and only if zero is the least eigenvalue of L : this is the familiar statement that the Schrödinger equation admits a unique eigenstate with no nodes, and this state is a ground state [10]. Thus Eq. (21) contains a condition on the metric g_{ij} and its conjugate momentum p^{ij} . If this condition is met the constraint has a unique (up to rescaling) positive solution for n , and hence the lapse is determined up to a constant scaling.

A. Propagation of constraints

The constraint equations $C = 0$ and $C_i = 0$ must hold at each time. One way to satisfy this requirement is to solve these two equations as independent constraints at each time. For example, one could imagine solving the constraint C for N as a function of the metric and its conjugate momentum, up to a time-dependent pre-factor. If this could be done, the constraint would hold for all times and there would be no need to add an independent condition to ensure its preservation with time. The time evolution of N is governed by the Lagrange multiplier v , which is entirely free to begin with, so can always be chosen so as to produce the required time evolution of N . (The situation would be quite different however if V had no N -dependence. Then the constraint would impose a relation between g_{ij} and p^{ij} alone that is not generally consistent with their evolution equations, except in the GR case).

Rather than solving the constraint at all times, one can instead solve it at one time, and then choose the Lagrange multiplier such that the constraint is preserved. We now proceed to analyze the preservation of the constraints in this way. This approach implements the general formalism for constrained Hamiltonian systems discussed in Ref. [11], and allows one to identify the first-class constraints without any guesswork.

1. Propagation of diffeomorphism constraints

Let us consider first the diffeomorphism constraint C_i . Since it generates spatial diffeomorphisms, and the Hamiltonian is a spatial scalar (a number, not a field), one might think that the Poisson bracket $\{C_i, H\}$ would vanish, implying that C_i is constant and therefore the constraint $C_i = 0$ is preserved in time. There is a catch in this reasoning however, since C_i only generates diffeomorphisms of g_{ij} and p^{ij} , whereas the Hamiltonian also depends on N and N^i , so in fact $\{C_i, H\} \neq 0$. The impact of the N^i dependence is transparent, since N^i only enters H linearly in the combination $N^i C_i$, and the Pois-

³ This was pointed out to us by M. Grillakis.

son bracket algebra of diffeomorphism generators closes,

$$\{C_i(x), C_j(y)\} = C_i(y)\delta_{,j}(x, y) + C_j(x)\delta_{,i}(x, y). \quad (23)$$

This N^i dependence therefore produces a contribution to the time derivative of $C_i(x)$ that is proportional to the constraint itself, which vanishes when the constraint is satisfied.

The impact of the N dependence is more subtle. However, a simple way to see that it does not spoil the conservation of the diffeomorphism constraint is to modify the diffeomorphism constraint to include the term $N_{,i} p_N$ that generates diffeomorphisms of N (and p_N), which vanishes when the primary constraints (7) are satisfied⁴. Similarly one could add the term that generates diffeomorphisms of N^i . The resulting extended diffeomorphism constraint \tilde{C}_i is defined such that

$$\begin{aligned} \int \xi^i \tilde{C}_i &= \int p_N \mathcal{L}_\xi N + p_i \mathcal{L}_\xi N^i + p^{ij} \mathcal{L}_\xi g_{ij} \\ &= \int \xi^i \left[(\nabla_i N) p_N + \mathcal{L}_{\tilde{N}} p_i - 2g_{ik} \nabla_j p^{jk} \right] \end{aligned} \quad (24)$$

where \mathcal{L}_ξ is the Lie derivative. This extended constraint generates diffeomorphisms of *all* the variables, so it actually does have vanishing bracket with the Hamiltonian. Since the secondary constraints (15) already imply that the primary constraints (7) are preserved in time, the time independence of \tilde{C}_i also implies the preservation of the secondary constraint $C_i = 0$.

2. Propagation of scalar constraint

Next we turn to the issue of propagation in time of the scalar secondary constraint $C = 0$. First we recall how it works in GR, then we consider the case of non-projectable Hořava gravity without the a_i dependence in V , and finally we include the effects of this dependence. In GR, the potential $V = -\xi R$ is independent of N , so that $C = \mathcal{H}_t$. Moreover, these constraints are first class, i.e. their Poisson brackets with each other are combinations of the constraints themselves. This is obvious for the Poisson brackets with C_i , since C_i just generates diffeomorphisms. The only other bracket is [12]

$$\{C(x), C(y)\} = \left(g^{ij}(x) C_i(x) + g^{ij}(y) C_i(y) \right) \delta_{,j}(x, y), \quad (25)$$

which closes on the C_i constraint (for any value of ξ). Thus the time derivative of C is a combination of the constraints, hence the constraints are preserved in time.

In Hořava gravity, let us consider first the case where $V(g_{ij})$ is independent of N . Then, when $\lambda \neq 1$ and/or

$V \neq -\xi R$, the bracket of two C 's does not close on a constraint, so the constraints are not first class. Preservation of C then imposes a further tertiary constraint that depends on g_{ij} , p^{ij} , and N . This case was analyzed in Ref. [13] (see also Ref. [14]), where it was shown that for generic g_{ij} and p^{ij} , the only solution for N is $N = 0$. This is unacceptable since the kinetic term of the action (5) is proportional to $1/N$ and the Hamiltonian generates no evolution, so we will assume $N \neq 0$ everywhere. When $V = -\xi R$, the tertiary constraint then holds if and only if π/\sqrt{g} is constant, and the theory is equivalent to GR in constant mean curvature (CMC) gauge (provided such a gauge is accessible) [15]. The case where V is independent of N has also been considered in the presence of R^2 -type terms both in the linearized[16] and nonlinear[17] settings, where it was found that the structure of the constraint algebra as well as the number of propagating degrees of freedom depend on which R^2 terms are included. In what follows we will consider the generic case where V depends on N and $\alpha \neq 0$.

In the generic case, N and its spatial derivative occur in the a_i dependence of $V(g_{ij}, a_i)$, so that the constraint C does not commute with p_N , i.e. p_N and C are second class. A possible way to proceed is to simply solve the constraint $C(x) = 0$ for one of the dynamical variables, such as N as suggested in Ref. [4]. The subtlety in doing this reduction is that N can only be so determined up to an arbitrary time-dependent, spatially constant multiple. This is related to the fact that among the constraints $p_N(x)$ and $C(x)$ are two linear combinations that are first class. These first-class combinations should not be set to zero strongly, since the symplectic form pulled back to such a subspace would be degenerate, and therefore the Poisson brackets would be ill-defined [11]. Thus to carry out the reduction we identify the full set of first-class constraints.

B. First-class constraints and Hamiltonian

For convenience in the analysis, we write the Lagrange multiplier v in the form

$$v = Nw + N^i \nabla_i N, \quad (26)$$

where the function w is initially arbitrary. In terms of w the Hamiltonian density (10) takes the form

$$\mathcal{H} = N\mathcal{H}_t + N^i \tilde{C}_i + v^i p_i + Nwp_N \quad (27)$$

up to a total derivative. The condition that C be preserved in time is then

$$\{C(x), H\} = \int N \left(\{C(x), \mathcal{H}_t\} + w \{C(x), p_N\} \right) = 0 \quad (28)$$

where we have dropped the term proportional to $\{\tilde{C}^i, C\}$, which vanishes when $C = 0$. By solving for w , we ensure that the constraint $C = 0$ is preserved in time, assuming

⁴ Such a modified constraint was considered in Ref. [4] but not explicitly identified as a generator of diffeomorphisms.

it has been solved at some initial time. Thus rather than solving the non-linear equation $C = 0$ for N separately at each instant of time, one can instead solve a linear equation for w . The presence of the free function w can solve the over-constraining problem, provided the bracket (28) can be set to zero by solving for w .

The Lagrange multiplier w is not completely determined by preservation of C , instead the equation (28) determines w up to a solution of the homogeneous equation

$$\int Nw\{C(x), p_N\} = 0. \quad (29)$$

Each such solution is a gauge freedom in the evolution and corresponds to a primary first-class constraint, $\int wNp_N$. In this case $w(x) = \kappa$ is such a solution, where κ is constant. The existence of this solution follows from the fact that the constraint

$$\int Np_N \quad (30)$$

generates constant rescalings of N , and C is invariant under such a rescaling.

Eq. (28) is a linear partial differential equation for w of up to sixth order. In the IR limit it reduces to a second-order equation,

$$\partial_i(N\sqrt{g}g^{ij}\partial_j w) = -\frac{N}{2\alpha}\left\{C, \int N\mathcal{H}_t\right\}. \quad (31)$$

This equation is elliptic, and in fact the left hand side can be written as $\sqrt{g}\tilde{\nabla}^2 w$ where $\tilde{\nabla}$ is the covariant derivative compatible with the metric $\tilde{g}_{ij} = N^2 g_{ij}$. It follows that the solution exists for w provided the integral of the right hand side is zero

$$\left\{\int NC, \int N\mathcal{H}_t\right\} = 0 \quad (32)$$

which follows immediately from the definition of C (16). This solution is unique up to the addition of the homogeneous solution $w(x) = \kappa$.

Beyond the IR limit, the equation determining w is of higher order and might admit non-constant homogeneous solutions. However any theory admitting additional solutions to Eq. (29) will have additional first-class constraints, which may be regarded as generators of gauge transformations [11]. For a generic choice of parameters, we expect there will be no additional gauge symmetry, so that the general solution of (28) has the form

$$w = \bar{w}[g_{ij}, p^{ij}, N] + \kappa, \quad (33)$$

where κ is constant and \bar{w} is a particular solution of Eq. (28).

The solution for w can be substituted back into the Hamiltonian to obtain a Hamiltonian that preserves the constraints in time,

$$H = \int N(\mathcal{H}_t + \bar{w}p_N) + \kappa \int Np_N + \int (N^i \tilde{C}_i + v^i p_i). \quad (34)$$

By varying the Hamiltonian by a constant rescaling of N , we can see that the first two terms in Eq. (34) are the Poisson brackets of the Hamiltonian with the first-class constraint $\int Np_N$,

$$\left\{H, \int Np_N\right\} = \int N(\mathcal{H}_t + \bar{w}p_N). \quad (35)$$

The Hamiltonian H is therefore a sum of first-class constraints generating the two types of foliation-preserving diffeomorphisms: global time reparameterizations, and spatial diffeomorphisms.

Observables in Hořava gravity must be gauge-invariant functions; that is they must commute with all first-class constraints. Since the Hamiltonian is a sum of first-class constraints, observables must also have zero Poisson bracket with the Hamiltonian and hence be conserved in time. For example, the volume of the spatial slice Σ is diffeomorphism-invariant, but is not time-independent since it fails to commute with the constraint $\int NC$:

$$\left\{\int NC, \int \sqrt{g}\right\} = \frac{1}{1-3\lambda} \int Np. \quad (36)$$

Hence the volume is not an observable in Hořava gravity, contrary to what was claimed in Ref. [4]. This reflects the fact that it is meaningless to label an observable by the “ t ” coordinate in a theory that has t -reparameterization symmetry.

IV. REDUCED PHASE SPACE

We have expressed Hořava gravity as a Hamiltonian system with second-class constraints. In the presence of second-class constraints it is possible to reduce the phase space by solving the constraints for one or more of the dynamical variables as a function of the others. The Hamiltonian and symplectic form are then restricted to the reduced phase space, and the restriction of the symplectic form defines a non-degenerate Poisson bracket on the reduced phase space known as the “Dirac bracket” [11, 18].

In Refs. [3, 4], it was proposed that the constraint $C = 0$ (15) be solved for N . There are two related issues with this strategy. First, N can be determined only up to a time-dependent rescaling. Second, among the constraints are two first-class constraints; setting these to zero strongly would result in a degenerate symplectic form and therefore an undefined Dirac bracket.

To reduce the system, we need to impose as many linear combinations of the constraints as possible without setting the Hamiltonian to zero. In order to have a set of constraints that determines N completely, we choose the gauge in which the average lapse is one. Once this new constraint is introduced the only first-class constraint that remains (apart from the diffeomorphism constraints) is the Hamiltonian. We therefore impose the

following constraints strongly:

$$\int N \sqrt{g} = \int \sqrt{g}, \quad p_N = 0, \quad C = C_0 \sqrt{g} \quad (37)$$

and eliminate N from the phase space. Here C_0 is a function of time whose presence is necessary to ensure that the Hamiltonian is not set to zero strongly. C_0 can be expressed in terms of N by integrating the identity $C = C_0 \sqrt{g}$, or alternatively, since N is completely determined by the constraints, C_0 can be written in terms of the metric variables g_{ij}, p^{ij} .

The reduced phase space has coordinates g_{ij}, p^{ij} and its dynamics are expressed in terms of the Hamiltonian and the Dirac brackets [11]. Although generically the Dirac brackets are different from the Poisson brackets, in the case where the second-class constraints can be written in the form $N = f[g_{ij}, p^{ij}]$, $p_N = 0$, the Dirac and Poisson brackets between g_{ij} and p^{ij} coincide. Since the constraints (37) do not restrict the values of g_{ij} and p^{ij} , they are of this special form and the Dirac bracket $\{\cdot, \cdot\}^*$ on the reduced phase space is same as the canonical Poisson bracket,

$$\{g_{ij}(x), p^{kl}(y)\}^* = \{g_{ij}(x), p^{kl}(y)\} = \delta_i^{(k} \delta_j^{l)} \delta(x, y). \quad (38)$$

The Hamiltonian on the reduced phase space is found by substituting the second-class constraints (37) into the Hamiltonian (34):

$$H = \mathcal{V} C_0 + \int \left(N^i \tilde{C}_i + v^i p_i \right) \quad (39)$$

where \mathcal{V} is the volume of the spatial slice Σ . Note that the terms involving \bar{w} are absent, since their only role was to preserve the constraint $C = 0$, which always holds on the reduced phase space. The equations of motion are then

$$\dot{g}_{ij} = \mathcal{V} \frac{\delta C_0}{\delta p^{ij}} + \mathcal{L}_{\tilde{N}} g_{ij} \quad (40)$$

$$\dot{p}^{ij} = -\mathcal{V} \frac{\delta C_0}{\delta g_{ij}} + \mathcal{L}_{\tilde{N}} p^{ij} \quad (41)$$

where we have used the fact that $C_0 = 0$ on shell.

In the IR limit, $-C_0$ can be expressed as the smallest eigenvalue of a linear operator,

$$Ln = -C_0 n, \quad (42)$$

where L is given by Eq. (22), and $n = \sqrt{N}$. The time evolution then depends on first-order changes of the eigenvalue C_0 with respect to g , and p . These can be found by first-order perturbation theory, e.g.

$$\mathcal{V} \frac{\delta C_0}{\delta g_{ij}} = -\frac{\delta}{\delta g_{ij}} \int \sqrt{g} n L n \quad (43)$$

and similarly for p . This formula can then be rewritten

in terms of N :

$$\mathcal{V} \frac{\delta C_0}{\delta p^{ij}} = \frac{2N}{\sqrt{g}} \left(p_{ij} + \frac{\lambda}{1-3\lambda} p g_{ij} \right), \quad (44)$$

$$\begin{aligned} \mathcal{V} \frac{\delta C_0}{\delta g_{ij}} = & \sqrt{g} N (\alpha a_k a^k + \xi R) g^{ij} \\ & + \frac{N}{\sqrt{g}} \left(p^{kl} p_{kl} + \frac{\lambda}{1-3\lambda} p^2 \right) g^{ij} \\ & + \alpha \sqrt{g} N a^i a^j \\ & + \frac{2N}{\sqrt{g}} \left(p^{ik} p_k^j + \frac{\lambda}{1-3\lambda} p p^{ij} \right) \\ & + \xi \sqrt{g} (\nabla^i \nabla^j - R^{ij} - g^{ij} \nabla^2) N. \end{aligned} \quad (45)$$

These equations are equivalent to those obtained by varying the Lagrangian (5).

V. ASYMPTOTICALLY FLAT CASE

In the preceding discussion, it was assumed that the spatial manifold Σ is compact with no boundary. When there is a boundary (or in the case of asymptotic flatness, an asymptotic region with prescribed falloff conditions), variation of the Hamiltonian will result in a total derivative that can be written as a boundary term. To have a well-defined variational principle, additional terms must be added to the Hamiltonian to make this boundary term vanish. The appropriate boundary terms in the asymptotically flat case are determined by the falloff conditions on the fields,

$$g_{ij} = \delta_{ij} + O(r^{-1}), \quad K_{ij} = O(r^{-2}), \quad R = O(r^{-3}) \quad (46)$$

where r is the radial coordinate of a coordinate system in which the metric asymptotically approaches the Euclidean metric δ_{ij} . In order for the Hamiltonian to define an asymptotic time translation, we must have $N^i \rightarrow 0$ as $r \rightarrow \infty$, and we choose a falloff on N such that

$$N = 1 + O(r^{-1}), \quad \nabla_i N = O(r^{-2}) \quad (47)$$

as $r \rightarrow \infty$. The contributions to the boundary term H_∂ in the Hamiltonian then come only from the potential V . For any term in V that is fourth order or higher in derivatives, the corresponding boundary term is of at least third order in derivatives and hence vanishes subject to (46). The remaining terms in V , which are of second order in derivatives, are R and a^2 . The a^2 term does not contribute a boundary term because $a_i \delta N = O(r^{-3})$, and the variation of R with respect to g_{ij} can be cancelled by adding to the Hamiltonian the boundary term

$$H_\partial = \xi \oint (\partial_i g_{jk} - \partial_j g_{ik}) \delta^{ik} n^j. \quad (48)$$

Here $\oint = \oint_{\partial\Sigma} d^2\sigma \sqrt{h}$ where σ are coordinates on the sphere at infinity $\partial\Sigma$, h is the induced metric on $\partial\Sigma$, and

n^i is the outward unit normal. This term is nothing but the usual boundary term for general relativity, the ADM mass, with a factor of ξ coming from the action (5). We have therefore obtained a Hamiltonian that leads to a consistent variational principle in the asymptotically flat setting, given by

$$H = \int \mathcal{H} + H_{\partial}, \quad (49)$$

where the first term is given by (10-13).

The primary and secondary constraints (7) and (15) remain the same in the asymptotically flat setting. The difference comes when solving for the undetermined part of the Lagrange multiplier w to ensure the propagation of the constraint C (29). In the asymptotically flat setting there is no global first-class constraint $\int N p_N$, when the global time-reparameterization symmetry is broken by the asymptotic value of N . Formally, the first-class constraint is absent because in order for the flow generated by $\int w N p_N$ to preserve the boundary condition (47), we must have $w(x) \rightarrow 0$ as $x \rightarrow \infty$. Therefore the homogeneous solution $w = 1$ to Eq. (29) is not admissible and the constraints p_N and C are purely second-class.

The Hamiltonian can be expressed as a sum of bulk constraints (13) and a boundary term by multiplying the definition of C (16) by N and integrating,

$$\int N \mathcal{H}_t = \int N C + \oint n_i V^i. \quad (50)$$

Terms in V with four or more derivatives contribute terms in V^i with three or more derivatives, and these vanish subject to the falloff conditions (46) so that we can replace V^i with its IR limit, Eq. (19). Combining these results with (27) we find that the Hamiltonian is

$$H = \int \left(N C + \overline{w} N p_N + N^i \tilde{C}_i + v^i p_i \right) + \mathcal{E}, \quad (51)$$

which is a sum of local constraints, plus a boundary term \mathcal{E} , the total energy. The total energy \mathcal{E} is the on-shell value of the Hamiltonian, which is the sum of two boundary terms

$$\mathcal{E} = \xi \oint (\partial_i g_{jk} - \partial_j g_{ik}) \delta^{ik} n^j - 2\alpha \oint n^i \partial_i N. \quad (52)$$

We emphasize that this is the total energy for the full Hořava gravity, not just in the IR limit.⁵

In the limit of large r , the metric can be treated as a perturbation of flat spacetime. In Newtonian gauge,

$$N = 1 + \psi, \quad g_{ij} = (1 - 2\phi) \delta_{ij}, \quad (53)$$

the energy takes the form

$$\mathcal{E} = \oint n^i \partial_i (4\xi \phi - 2\alpha \psi). \quad (54)$$

Note that the ADM energy in GR depends only on ϕ , whereas in Hořava gravity there is also dependence on ψ .

In the static spherically symmetric solutions, $\phi = \psi$ and the asymptotic form of the solution to $O(1/r)$ is [3]

$$N = 1 - \frac{r_0}{2r}, \quad g_{ij} = \left(1 + \frac{r_0}{r}\right) \delta_{ij}, \quad (55)$$

where r_0 is a parameter with dimensions of length. Returning to units in which $16\pi G_H \neq 1$, the energy of this solution is

$$\mathcal{E} = \frac{r_0}{2G} \left(\xi - \frac{1}{2}\alpha \right). \quad (56)$$

The gravitational constant that appears in the Newtonian limit is not G_H but $G_N \equiv G_H / (\xi - \frac{1}{2}\alpha)$ [3, 6]. Thus, if we identify $r_0 = 2G_N M$ in the weak field limit then $\mathcal{E} = M$.

Horava gravity in the IR limit can be thought of as the hypersurface orthogonal restriction of Einstein-aether theory, where the restriction is at the level of the Lagrangian [6, 8, 20]. Because of this close relation, it is not surprising that the total energy expression (54) is identical to the expression for total energy in Einstein-aether theory found in Ref. [21], when $\nabla_i \ln N$ is identified with the acceleration of the aether, and when the parameters of Hořava gravity are identified with the parameters of Einstein-aether theory as in Ref. [6].

The reduction of the phase space with asymptotically flat boundary conditions proceeds in much the same way as in the compact case. The difference is that the constraints C are all second class, so the equation $C = 0$ can be solved strongly as a differential equation for N with the boundary condition $N \rightarrow 1$ at spatial infinity. The reduced phase space has coordinates g_{ij}, p^{ij} with the canonical Poisson brackets (38). The Hamiltonian on the reduced phase space is

$$H = \int N C + \int \left(N^i \tilde{C}_i + v^i p_i \right) + \mathcal{E}, \quad (57)$$

where N is treated as a functional of g_{ij} and p^{ij} defined by the constraint.

VI. CONCLUSION

We have derived the Hamiltonian formulation of Hořava gravity for both closed and asymptotically flat spatial manifolds, extending the analysis initiated in Ref. [4]. In contrast with GR, the scalar constraints are not generators of surface deformations. These constraints are of second class, except for a single linear combination that is first class, which generates transformations between leaves of the preferred foliation. The Hamiltonian

⁵ An expression for the total energy in Horava gravity was recently found in Ref. [19] using the Noether charge formalism.

is a sum of constraints that generate 3-dimensional diffeomorphisms and global reparameterizations of time.

We have also considered a phase space reduction of Hořava gravity in which the second-class constraints are solved formally for the lapse. This is complicated by the fact that the constraints have a global first-class part that cannot be set to zero strongly. The second-class constraints take the form of a non-linear differential equation for N , but in the IR limit they are linear in \sqrt{N} . In the IR limit, the Hamiltonian is expressible in terms of the least eigenvalue of a certain differential operator, and is constrained to vanish.

In the asymptotically flat setting, the time-reparameterization symmetry is fixed by the constant value of the lapse at infinity, so the associated global constraint is absent. Instead the Hamiltonian acquires additional boundary terms that contribute to the total energy. The energy depends on both Newtonian potentials ϕ and ψ , not just on ϕ as in GR, in exact agreement with the result from Einstein-aether theory. For the static spherically symmetric solutions, $\phi = \psi$ and the energy takes the same form as GR, with a rescaled Newton constant. We do not know whether there are solutions in which $\phi \neq \psi$, for which the energy would not be just a rescaling of the GR energy.

Note added: Shortly after version 1 of this paper appeared on the arXiv, J. Bellorín and A. Restuccia posted Ref. [22], which treats many of the same issues addressed here.

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