Gravity waves from quantum stress tensor fluctuations in inflation

Chun-Hsien Wu, Jen-Tsung Hsiang, L. H. Ford, and Kin-Wang Ng

Phys. Rev. D **84**, 103515 — Published 11 November 2011

DOI: 10.1103/PhysRevD.84.103515

Gravity Waves from Quantum Stress Tensor Fluctuations in Inflation

Chun-Hsien Wu,[1,*] Jen-Tsung Hsiang,[2,†] L. H. Ford,[3,‡] and Kin-Wang Ng[4,§]

[1]*Department of Physics, Soochow University, 70 Linhsi Road, Shihlin, Taipei 111, Taiwan*

[2]*Department of Physics, National Dong Hwa University, Hualien 97401, Taiwan*

[3]*Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, MA 02155, USA*

[4]*Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan*

Abstract

We consider the effects of the quantum stress tensor fluctuations of a conformal field in generating gravity waves in inflationary models. We find a non-scale invariant, non-Gaussian contribution which depends upon the total expansion factor between an initial time and the end of inflation. This spectrum of gravity wave perturbations is an illustration of a negative power spectrum, which is possible in quantum field theory. We discuss possible choices for the initial conditions. If the initial time is taken to be sufficiently early, the fluctuating gravity waves are potentially observable both in the CMB radiation and in gravity wave detectors, and could offer a probe of transplanckian physics. The fact that they have not yet been observed might be used to constrain the duration and energy scale of inflation. However, this conclusion is contingent upon including the contribution of modes which were transplanckian at the beginning of inflation.

PACS numbers: 98.80.Cq, 04.30.-w, 04.62.+v, 05.40.-a

*Electronic address: `chunwu@scu.edu.tw`

†Electronic address: `cosmology@gmail.com`

‡Electronic address: `ford@cosmos.phy.tufts.edu`

§Electronic address: `nkw@phys.sinica.edu.tw`

I. INTRODUCTION

Inflationary models predict a nearly scale invariant spectrum of both scalar and tensor perturbations, both of which arise from the quantum fluctuations of nearly free fields and are Gaussian in character. The scalar perturbations arise from the the quantum fluctuations of an inflaton field [1–5], and have apparently been observed in the temperature fluctuations of the cosmic microwave background [6]. The tensor perturbations arise from the fluctuations of quantized linear perturbations of de Sitter spacetime [7–9], but have not yet been observed. In both cases, the Gaussian nature of the fluctuations and the approximate scale invariance arise from the properties of free quantum fields.

Coupling of the inflaton or graviton fields to other fields can modify these conclusions. For example, the coupling of graviton modes to the expectation value of the quantum stress tensor of a conformal field was recently treated in Ref. [10]. It was shown that graviton modes can acquire a one-loop correction which increases their amplitude in a way which depends upon the duration of inflation and upon the wavenumber of the mode. This effect will tend to lead to a blue tilt to the spectrum of tensor perturbations, but will not change their Gaussian character at the one-loop level.

However, an additional source of perturbations is quantum fluctuations of the stress tensor. The effects of stress tensor fluctuations in generating density perturbations have recently been studied in Refs. [11, 12], where a non-Gaussian, non-scale invariant contribution was found. Furthermore, this contribution can also depend upon the duration of inflation and potentially be used to place limits on this duration. The effect studied in Refs. [11, 12] arises from the quantum fluctuations of the comoving energy density of a conformal field in its vacuum state. The resulting density perturbations are a non-Gaussian, non-scale invariant component to be added to the effect of inflaton field fluctuations [1–5].

Fluctuations of other components of the stress tensor are capable of creating tensor perturbations. The purpose of the present paper is to address the creation of gravity wave fluctuations by stress tensor fluctuations of a conformal field in its vacuum state. These can be called passive fluctuations of gravity, as opposed to the active fluctuations discussed in Refs. [7–9]. The radiation of gravity waves by stress tensor fluctuations of matter fields in thermal states in flat spacetime was discussed in Ref. [13]. Matter fields in the vacuum state in flat spacetime cannot radiate due to energy conservation, but in a time-dependent spacetime, such radiation is possible.

Unless otherwise noted, units in which $G = c = \hbar = 1$ will be used, where G is Newton's constant.

II. GRAVITATIONAL RADIATION IN AN EXPANDING UNIVERSE

Here we review the formalism needed to compute gravitational radiation by a time-dependent source. We consider a spatially flat Robertson-Walker universe, for which the metric may be written as

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) = a^2(\eta)\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right). \tag{1}$$

Here t is the comoving time, and η the conformal time. Let $\gamma_{\mu\nu}$ be this background metric, and $h_{\mu\nu}$ be a linear perturbation,

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}\,. \tag{2}$$

Here we are concerned with tensor perturbations, and impose the transverse, tracefree gauge in which

$$h^{\mu\nu}{}_{;\nu} = 0, \quad h^{\mu}_{\mu} = h = 0, \quad \text{and} \quad h^{\mu\nu}u_\nu = 0\,, \tag{3}$$

where $u^\nu = \delta^\nu_t$ is the four velocity of the comoving observers, and the semicolon denotes the covariant derivative on the background spacetime. These conditions remove all of the gauge freedom, and leave the two degrees of freedom associated with the polarizations of a gravity wave.

Lifshitz [14] showed that, in the absence of a source, the mixed components h^ν_μ satisfy the *scalar* wave equation,

$$\Box h_\mu{}^\nu = 0\,, \tag{4}$$

where

$$\Box = \frac{1}{\sqrt{-\gamma}}\partial_\mu\left(\sqrt{-\gamma}\,\gamma^{\mu\nu}\,\partial_\nu\right) \tag{5}$$

is the scalar wave operator for the metric of Eq. (1). A consequence of this result is that gravitons in the spatially flat Robertson-Walker spacetime behave as a pair of massless, minimally coupled quantum scalar fields [15].

In the presence of a source, the metric perturbation satisfies an inhomogeneous equation

$$\Box h_\mu^{\ \nu} = -16\pi\, S_\mu^{\ \nu} \,, \tag{6}$$

where $S_\mu^{\ \nu}(x)$ is the transverse, tracefree part of the stress tensor of the source. It satisfies the conditions in Eq. (3), and is most conveniently defined in momentum space. The solutions of Eq. (4) in the spatially flat Robertson-Walker spacetime may be taken to be plane waves of the form

$$h_\mu^{\ \nu}(x) = e_\mu^{\ \nu}\, f_k(\eta)\, e^{i\mathbf{k}\cdot\mathbf{x}} \,, \tag{7}$$

where $f_k(\eta)$ is a solution of

$$\frac{d}{d\eta}\left(a^2\,\frac{df}{d\eta}\right) + k^2 a^2\, f = 0 \,, \tag{8}$$

and $e_\mu^\nu = e_\mu^\nu(\mathbf{k},\lambda)$ is a polarization tensor which satisfies

$$e_\mu^{\ \mu} = e_\mu^{\ \nu}\, u_\nu = e_\mu^{\ \nu}\, k_\nu = 0 \,. \tag{9}$$

If we take vector $\mathbf{k}$ to be in the z-direction, then the independent linear polarization tensors can be taken to have the nonzero components

$$e_x^{\ x} = -e_y^{\ y} = \frac{1}{\sqrt{2}} \,, \tag{10}$$

for the + polarization, or

$$e_x^{\ y} = e_y^{\ x} = \frac{1}{\sqrt{2}} \,, \tag{11}$$

for the $\times$ polarization.

Denote the spatial Fourier transform of any field $A(\eta,\mathbf{x})$ by

$$\hat{A}(\eta,\mathbf{k}) \equiv \frac{1}{(2\pi)^3}\int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}} A(\eta,\mathbf{x}) \,. \tag{12}$$

In momentum space, the transverse, tracefree part of a stress tensor is defined by the projection

$$\hat{S}_\mu^{\ \nu}(\eta,\mathbf{k}) = \sum_\lambda e_\mu^{\ \beta}(\mathbf{k},\lambda)\; e_\alpha^{\ \nu}(\mathbf{k},\lambda)\, \hat{T}^\alpha_{\ \beta}(\eta,\mathbf{k}) \,. \tag{13}$$

Thus given a stress tensor $T^\alpha_{\ \beta}(\eta,\mathbf{x})$ in coordinate space, we first take a Fourier transform to find $\hat{T}^\alpha_{\ \beta}(\eta,\mathbf{k})$, then find $\hat{S}_\mu^{\ \nu}(\eta,\mathbf{k})$ using Eq. (13), and finally take an inverse Fourier transform to find $S_\mu^{\ \nu}(\eta,\mathbf{x})$.

Let $G_R(x,x')$ be the retarded Green's function for the scalar wave operator, which satisfies

$$\Box G_R(x,x') = -\frac{\delta(x-x')}{\sqrt{-\gamma}} \,, \tag{14}$$

and $G_R(x,x') = 0$ if $t < t'$. Here $\Box$ is understood to act at the point x. The gravity wave radiated by the source S^ν_μ can be written as an integral over the past lightcone of the point x as

$$h_\mu{}^\nu(x) = 16\pi \int d^4x' \sqrt{-\gamma(x')}\, G_R(x,x')\, S_\mu{}^\nu(x')\,. \tag{15}$$

The source here could represent either classical matter or quantum fields. In the latter case, the average effect of a quantum stress tensor can be described by the semiclassical theory, in which the renormalized expectation value $\langle T_{\mu\nu}\rangle$ is used as a source in the Einstein equation.

The effects of a conformal quantum field upon graviton modes in de Sitter spacetime has recently been treated in the context of the semiclassical theory [10]. It was found that there is a correction to the graviton modes which grows with increasing duration of inflation, analogous to the effects found in Refs. [11, 12] and to be discussed in this paper. However, the effect found in Ref. [10] comes only from the the expectation value of the stress tensor, not from stress tensor fluctuations. If $h_\mu{}^\nu$ is a classical solution of Eq. (4), then there is a correction term $h'_\mu{}^\nu$. In the case that the conformal field is the electromagnetic field, the fractional correction is

$$\Gamma = \left|\frac{h'_\mu{}^\nu}{h_\mu{}^\nu}\right| = \frac{1}{5\pi}\ell_p^2\, H\, S\, k\,, \tag{16}$$

where ℓ_p is the Planck length, H is the Hubble parameter during inflation, and S is the expansion factor during inflation. This effect grows with increasing S and k, but its total magnitude is limited by the requirement that $\Gamma \lesssim 1$ for the one-loop approximation to hold.

III. A FLUCTUATING SOURCE

Now we consider the case where the source $S_\mu{}^\nu(x)$ is undergoing fluctuations, leading to a fluctuating tensor perturbation, $h_\mu{}^\nu(x)$. The correlation function for the perturbation is

$$K_\mu{}^\nu{}_\rho{}^\sigma(x,x') = \langle h_\mu{}^\nu(x)\; h_\rho{}^\sigma(x')\rangle - \langle h_\mu{}^\nu(x)\rangle\langle h_\rho{}^\sigma(x')\rangle\,, \tag{17}$$

and that for the source is

$$C_\mu{}^\nu{}_\rho{}^\sigma(x,x') = \langle S_\mu{}^\nu(x)\; S_\rho{}^\sigma(x')\rangle - \langle S_\mu{}^\nu(x)\rangle\langle S_\rho{}^\sigma(x')\rangle\,. \tag{18}$$

Their relation follows from Eq. (15):

$$K_\mu{}^\nu{}_\rho{}^\sigma(x,x') = (16\pi)^2 \int d^4x_1 \sqrt{-\gamma(x_1)}\, d^4x_2 \sqrt{-\gamma(x_2)}\, G_R(x,x_1)\, G_R(x',x_2)\, C_\mu{}^\nu{}_\rho{}^\sigma(x_1,x_2)\,. \tag{19}$$

The spatial Fourier transform of this equation may be expressed as

$$\hat{K}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}(\eta,\eta',k) = 64(2\pi)^8 \int d\eta_1 d\eta_2\, a^4(\eta_1)\, a^4(\eta_2)\, \hat{G}(\eta,\eta_1,k)\, \hat{G}(\eta',\eta_2,k)\, \hat{C}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}(\eta_1,\eta_2,k)\,, \tag{20}$$

where $\hat{C}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}(\eta_1,\eta_2,k)$ and $\hat{G}(\eta,\eta',k)$ are the Fourier transforms of $C_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}(x_1,x_2)$ and of the retarded Green's function $G_R(x,x')$, respectively.

If $\mathbf{k}$ is in the z-direction, then the nonzero components of $\hat{C}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}(\eta_1,\eta_2,k)$ for the + polarization are

$$\hat{C}_+ = \hat{C}_x{}^x{}_x{}^x = \hat{C}_y{}^y{}_y{}^y = -\hat{C}_x{}^x{}_y{}^y = -\hat{C}_y{}^y{}_x{}^x\,. \tag{21}$$

Similarly, the nonzero components for the $\times$ polarization are

$$\hat{C}_\times = \hat{C}_x{}^y{}_x{}^y = \hat{C}_y{}^x{}_y{}^x = \hat{C}_y{}^x{}_x{}^y = \hat{C}_x{}^y{}_y{}^x\,. \tag{22}$$

In fact, the stress tensor correlation functions for both polarizations are equal in our case, so we may drop the polarization label and write

$$\hat{C}(\eta_1,\eta_2,k) = \hat{C}_+(\eta_1,\eta_2,k) = \hat{C}_\times(\eta_1,\eta_2,k)\,. \tag{23}$$

Furthermore, the correlation function $\hat{C}(\eta_1,\eta_2,k)$ for the conformal field in Robertson-Walker spacetime may be related to the corresponding correlation function for the conformal field in flat spacetime by a conformal transformation. First consider a classical stress tensor $T_{\mu}{}^{\nu}$ in Robertson-Walker spacetime which is conformally related to $\mathcal{T}_{\mu}{}^{\nu}$ in Minkowski spacetime. The spatial components of these tensors are related by $T_i{}^j = a^{-4}\mathcal{T}_i{}^j$. The same conformal transformation applies to the quantum stress tensor correlation function. Although the conformal anomaly in the expectation value of a quantum stress tensor operator breaks the conformal symmetry, the conformal anomaly for free fields is a c-number which cancels in the correlation function. Consequently, we can write

$$\hat{C}(\eta_1,\eta_2,k) = a^{-4}(\eta_1)\, a^{-4}(\eta_2)\, \hat{C}_M(\eta_1-\eta_2,k)\,, \tag{24}$$

where $\hat{C}_M(\eta_1-\eta_2,k)$ is the Fourier transform of the Minkowski spacetime correlation function for a fixed component of $\mathcal{T}_i{}^j$. As is shown in Appendix A , it may be expressed as

$$\hat{C}_M(\eta_1-\eta_2,k) = -\frac{k^5}{512\pi^5} \int_0^1 du\, (1-u^2)^2\, \cos[ku(\eta_1-\eta_2)]\,. \tag{25}$$

This result applies for either polarization.

IV. THE POWER SPECTRUM IN INFLATIONARY COSMOLOGY

The well-known Wiener-Khinchin [16, 17] theorem states that the Fourier transform of a correlation function is a power spectrum. A corollary of this theorem is that the power spectrum can normally be written as the expectation value of a squared quantity, and hence must be positive. However, the latter result can fail in quantum field theory, and negative power spectra are possible. This has recently been discussed in Ref. [18]. Let $C(t-t', \mathbf{x}-\mathbf{x}')$ be a flat spacetime correlation function. We define the associated power spectrum by a spatial Fourier transform at $t = t'$:

$$P(k) = \frac{1}{(2\pi)^3} \int d^3u \, \mathrm{e}^{i\,\mathbf{k}\cdot\mathbf{u}} \, C(0, \mathbf{u}) \,. \tag{26}$$

Given a power spectrum, we can find the correlation function in space at equal times as an inverse Fourier transform:

$$C(0, \mathbf{u}) = \int d^3k \, \mathrm{e}^{-i\,\mathbf{k}\cdot\mathbf{u}} \, P(k) \,. \tag{27}$$

(Here we will use C to denote either a generic or a stress tensor correlation function, and K to denote metric correlation functions.)

The power spectrum for the gravity wave fluctuations is just a spatial component of $\hat{K}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}(\eta, \eta', k)$ in the limit that $\eta' = \eta$, and is the same for both polarizations. Thus we may combine Eqs. (20) and (24) to write the power spectrum at conformal time $\eta = \eta_r$ as

$$P(k) = 64(2\pi)^8 \int^{\eta_r} d\eta_1 d\eta_2 \, \hat{G}(\eta_r, \eta_1, k) \, \hat{G}(\eta_r, \eta_2, k) \, \hat{C}_M(\eta_1 - \eta_2, k) \,. \tag{28}$$

The possible treatments of the lower limits of integration will be discussed below. We should note that the quantity which is usually called the power spectrum in cosmology is not $P(k)$, but rather

$$\mathcal{P}(k) = 4\pi k^3 \, P(k) \,. \tag{29}$$

It is $\mathcal{P}(k)$ which is approximately independent of k for the active gravity wave fluctuations. The probability distribution for quantum stress tensor fluctuations is a skewed, hence non-Gaussian distribution with non-zero odd moments, although the explicit form has only been found in two-dimensional spacetime [19]. Consequently, the gravity wave fluctuations produced by stress tensor fluctuations will also be non-Gaussian.

The Green's function $\hat{G}(\eta, \eta', k)$ satisfies

$$\left(\partial_\eta^2 + 2\frac{a'}{a}\,\partial_\eta + k^2\right) \, \hat{G}(\eta, \eta', k) = \frac{\delta(\eta - \eta')}{(2\pi)^3 \, a^2(\eta')} \,, \tag{30}$$

as may be verified by taking a spatial Fourier transform of Eq. (14). Here $a' = da/d\eta$. Now we wish to specialize to the case of de Sitter spacetime, for which the scale factor is

$$a(\eta) = -\frac{1}{H\,\eta} \tag{31}$$

with $\eta < 0$. We may set the scale factor to be unity at the end of inflation, $\eta = \eta_r$, in which case $\eta_r = -1/H$. Now Eq. (30) becomes

$$\left(\partial_\eta^2 + 2Ha\partial_\eta + k^2\right)\,\hat{G}(\eta,\eta',k) = \frac{\delta(\eta-\eta')}{(2\pi)^3\,a^2(\eta')}\,. \tag{32}$$

Comparison of this result with Eq. (71) of Ref. [11] reveals that $\hat{G}(\eta,\eta',k)$ differs from the Green's function defined in the latter reference by a factor of $1/[(2\pi)^3\,a^2(\eta')]$. Consequently, we may use the result of Ref. [11] to write

$$\hat{G}(\eta,\eta',k) = \frac{H^2}{(2\pi k)^3}\,\left\{-k(\eta-\eta')\,\cos[k(\eta-\eta')] + (1+k^2\,\eta\eta')\,\sin[k(\eta-\eta')]\right\}\,. \tag{33}$$

Next we turn to a discussion of some the possible initial conditions which can be imposed on solutions of Eq. (28).

A. Sudden Switching

Here we impose the initial condition that the metric fluctuations vanish at $\eta = \eta_0$. The power spectrum of tensor fluctuations at the end of inflation, $\eta = \eta_r = -1/H$, is then given by

$$P(k) = P_s(k) = 64(2\pi)^8 \int_{\eta_0}^{-1/H} d\eta_1 \int_{\eta_0}^{-1/H} d\eta_2\,\hat{G}(\eta,\eta_1,k)\,\hat{G}(\eta,\eta_2,k)\;\hat{C}_M(\eta_1-\eta_2,k)\,. \tag{34}$$

The integrals in Eq. (34) may be evaluated, using for example the algebraic computer program *Mathematica*. In the limit that $k|\eta_0| \gg 1$, the result is approximately

$$P_s(k) \approx -\frac{H^4\,\eta_0^2}{3\pi^3\,k}(1+k^2\,H^{-2})\,. \tag{35}$$

There are several remarkable features of this result: its negative sign, its blue tilt, and the fact that it grows with increasing $|\eta_0|$. The possibility of negative power spectra was discussed in Ref. [18], where it was shown that such spectra arise naturally in quantum field theory for the fluctuations of quadratic operators, such as quantum stress tensors. Indeed, the power spectrum associated with the fluctuations of the transverse, tracefree part of the electromagnetic stress tensor is given by the $\eta_1 = \eta_2$ limit of Eq. (25),

$$\hat{C}_M(0,k) = -\frac{k^5}{960\pi^5}\,, \tag{36}$$

which is negative. Negative power spectra are always associated with coordinate space correlation functions which are singular in the coincidence limit. This is the case for stress tensor correlation functions. They are also associated with the opposite correlation versus anti-correlation behavior as compared with a positive power spectrum with the same functional form. This means that $C(r)$ changes sign of the sign of $P(k)$ changes, so events that were correlated become anticorrelated and vice versa. The spectrum is also not scale invariant, and tilted toward the blue end of the spectrum because $|\mathcal{P}_s(k)|$ grows with increasing k.

Another feature of Eq. (35) is that the power spectrum for the gravity waves grows as η_0^2, which means that it is proportional to the square of the scale factor change between the initial time and the end of inflation. This is analogous to the results found in Refs. [11, 12] for the power spectrum of density fluctuations produced by quantum stress tensor fluctuations. In both cases, the growth of fluctuations can potentially be used to place upper limits on the duration of inflation, as will be discussed in Sec. V. The net expansion factor during inflation is

$$S = H\,|\eta_0|\,, \tag{37}$$

so we may write Eq. (35) as

$$P_s(k) = -\frac{H^2\,S^2}{3\pi^3\,k}(1 + k^2\,H^{-2})\,. \tag{38}$$

The coordinate space correlation function associated with this power spectrum is given by Eq. (27) to be

$$K_s(r) = -\frac{4H^2\,S^2}{3\pi^2\,r^2}\left(1 - \frac{2}{H^2\,r^2}\right)\,. \tag{39}$$

This gives the correlation of points at spatial separation r at equal times. Note that it may be either positive or negative.

Although the power spectrum and the associated correlation function grow with increasing S or energy scale k, the perturbative approach used here require that $|K(r)| \ll 1$, which places a limit on the magnitude of the effect.

It is informative to compare the results of this subsection with the flat spacetime limit. If we set $a = 1$ in Eq. (30) and solve for the flat space Green's function, the result is

$$\hat{G}_M(\eta - \eta', k) = \frac{1}{(2\pi)^3\, k}\, \sin[k(\eta - \eta')]\,. \tag{40}$$

If we use this Green's function in Eq. (28), the resulting power spectrum becomes

$$P_M(k) = -\frac{5k}{6\pi^3}\,, \tag{41}$$

where a rapidly oscillating term which depends upon the integration interval has been dropped. The associated coordinate space metric correlation function at equal times is

$$K_M(r) = \frac{20\ell_p^4}{3\pi^2\, r^4}\,. \tag{42}$$

This function simply describes Planck-scale fluctuations, which are presumably unobservable. The main point is that the fluctuations do not accumulate in flat spacetime due to anticorrelations. In a curved spacetime, such a de Sitter space, this is no longer the case, and the anticorrelated fluctuations need not cancel. In the calculations, the crucial difference is between the flat spce Green's function, Eq. (40), and that in de Sitter space, Eq. (33).

B. Exponential Switching

In the previous subsection, the interaction between the quantum stress tensor and the gravitational field was taken to be switched on suddenly at $\eta = \eta_0$. One might be concerned that either the sign of $P(k)$, or its growth with increasing $|\eta_0|$ are artifacts of this sudden switching. Here we investigate a model in which the interaction is switched on gradually. We replace the step function $\theta(\eta - \eta_0)$ by an exponential function, $e^{p\eta}$, with $p > 0$. This function vanishes as $\eta \to -\infty$, and in the limit of small p, is close to unity by the end of inflation. The effect of this switching function is effectively to switch on the interaction on a conformal time scale of order $|\eta_0|$, where $\eta_0 = -1/p$. Equation (34) is replaced by

$$P_e(k) = 64(2\pi)^8 \int_{-\infty}^{-1/H} d\eta_1\, s_e(\eta_1) \int_{-\infty}^{-1/H} d\eta_2\, s_e(\eta_2)\, \hat{G}(\eta, \eta_1, k)\, \hat{G}(\eta, \eta_2, k)\, \hat{C}_{flat}(\eta_1 - \eta_2, k)\,, \tag{43}$$

where the switching function is

$$s_e(\eta) = e^{p\eta}\,. \tag{44}$$

In the limit of small p, Eq. (43) leads to

$$P_e(k) \approx -\frac{H^4(1+k^2/H^2)}{8\pi^2\, k^2\, p} + O(\ln p) = -\frac{H^3(1+k^2/H^2)\, S}{8\pi^2\, k^2}\,, \tag{45}$$

where S, given by Eq. (37), is the expansion between $\eta = \eta_0 = -1/p$ and the end of inflation.

Again the power spectrum is negative, blue tilted, and grows with increasing S although now linearly. In this case, the equal time spatial correlation function is an inverse Fourier transform of $P_e(k)$ given by

$$K_e(r) = -\frac{H^3\, S}{4r}\,. \tag{46}$$

Here we have dropped a delta-function term proportional to $\delta(\mathbf{x})$, which will not contribute to measurements made at distinct spatial locations. Note that because $a(\eta) = 1/(H|\eta|)$, if the switching time $\Delta\eta$ is of order $|\eta_0|$, then the scale factor approximately doubles during the switch-on. For example, $a(\eta/2) = 2a(\eta)$. In terms of comoving time t, where $a(t) = e^{Ht}$, this corresponding to a time interval of $\Delta t \approx 1/H$, or one horizon crossing time. Thus the switch-on time in this model is of order of the horizon crossing time.

C. Adjustable Width Switching

The switching function $s_e(\eta)$ used in the previous subsection contains only one parameter, p, which regulates both the effective duration of inflation and the period over which the switching occurs. It is instructive to consider a more general function with two parameters:

$$s_{aw}(\eta) = \frac{1}{1+e^{(\eta_0-\eta)/\alpha}}\,. \tag{47}$$

This function, analogous to the Fermi-Dirac distribution function, changes from zero to unity when $\eta \approx \eta_0$ over a time scale of $\Delta\eta \approx \alpha$. The resulting power spectrum, $P_{aw}(k)$, is given by Eq. (43), with $s_e(\eta)$ replaced by $s_{aw}(\eta)$, and may be expressed as

$$P_{aw}(k) = -\frac{H^4}{2\pi^3\, k^3} \int_0^1 du\, (1-u^2)^2\, (I_C^2 + I_S^2)\,. \tag{48}$$

Here

$$I_C = \int_{x_r}^{\infty} dx\, g(x, x_r)\, s(x)\, \cos(ux)\,, \tag{49}$$

and

$$I_S = \int_{x_r}^{\infty} dx\, g(x, x_r)\, s(x)\, \sin(ux)\,, \tag{50}$$

with

$$g(x, x_r) = (x - x_r)\cos(x - x_r) - (1 + x x_r)\sin(x - x_r)\,. \tag{51}$$

We use the notation, $x = -k\eta$, $x_r = -k\eta_r$, and $s(x) = s_{aw}(\eta)$. The dominant contributions to the integrals in I_C and I_S come from values of x of order $x_0 = -k\eta_0 \gg x_r$, so we may write

$$g(x, x_r) \approx x(\cos x - x_r \sin x)\,. \tag{52}$$

The resulting integrals may be evaluated using Eqs. (B8) and (B9), which are derived in Appendix B. The final result, when $\alpha \gtrsim 1/k$, is

$$P_{aw}(k) \approx -\frac{H^4\,\eta_0^2}{2\pi\,k^2\,\alpha}(1 + k^2\,H^{-2})\,. \tag{53}$$

Apart from numerical factors, this result contains both Eqs. (35) and (45) as special cases. If $\alpha \approx 1/k$, then we return to the sudden switching case of Eq. (35). On the other hand, if $\alpha \approx |\eta_0|$, we find Eq. (45), up to numerical constants. The fact that the constants do not match exactly may be due to the approximation used in deriving Eqs. (B8) and (B9) ($q \ll 1$) not being very good near $u = 0$. In summary, if $\Delta\eta \lesssim 1/k$, we obtain the sudden result, Eq. (38), proportional to S^2, and if $\Delta\eta \approx |\eta_0|$, we obtain Eq. (45), proportional to S. Intermediate switching times lead to Eq. (53).

Note that in this section, we have been discussing the effects of different rates at which the coupling between the conformal field and gravity is switched on. This issue is distinct from the choice of the initial quantum state for either the conformal field or the gravitons. If the conformal field is not in its vacuum state, then its particle content should rapidly redshift. Soon, it will be indistinguishable from the vacuum, and we may regard our analysis as beginning at that time. Variation of the state of the gravitons essentially adds an additional term to the power spectrum of the tensor perturbations. Here we are concerned with the tensor perturbations generated by the quantum stress tensor fluctuations, and do not explicitly treat other sources of tensor perturbations.

V. IMPLICATIONS OF THE POWER SPECTRUM

A. Initial Conditions and the Transplanckian Issue

Although the results in the previous section depend somewhat on the rate at which the coupling between the quantum stress tensor fluctuations and the gravitational field is switched on, in all cases the power spectrum grows as a power of S, the expansion from the initial time to the end of inflation. Thus we need an interpretation which suggests a reasonable value for this time, $|\eta_0|$. One possibility is to take this time to be the onset of inflation. This imposes the initial condition that the gravity wave perturbations vanish at the beginning of inflation. In this case, S becomes the total expansion factor during inflation. A possible objection to this interpretation is that it can lead to contributions from transplanckian modes. This raises the question of whether our perturbative treatment can be trusted, as relations such as Eq. (15) are lowest order approximations in the dimensionless coupling constant $(\ell_p k)^2$. The transplanckian issue has been extensively discussed in the contexts of the Hawking effect and of cosmology. Hawking's original derivation [20] of black hole radiance relies upon modes which begin far above the Planck energy. The fact that the Hawking effect gives a beautiful unification of gravity, thermodynamics, and quantum theory can be considered to be a powerful argument to take transplanckian modes seriously. It is true that it is possible to derive the Hawking effect without transplanckian modes [21, 22], but only at the price of introducing modified dispersion relations which break local Lorentz symmetry and hence postulate new physics. There has been an extensive discussion of the possible role of transplanckian modes in inflationary cosmology. (See Ref. [12] for a lengthy list of references.) The effect discussed in this paper has the potential to serve as an observational probe of transplanckian physics.

There is an alternative possibility [12], which is to take the initial time at which the perturbation vanishes to depend upon the mode, and to be the time at which a given mode redshifts below the Planck scale in the comoving frame. This avoids the transplanckian issue, but at the price of introducing a non-local and frame dependent prescription, which is analogous to introducing non-Lorentz invariant modified dispersion relations. In the remainder of this paper, we will explore the consequences of adopting the former prescription whereby S is the total expansion factor during inflation. However, all of our conclusions depend upon this assumption.

The dependence of the gravity wave spectrum upon a positive power of S might seems to contradict a theorem due to Weinberg [23], which was generalized by Chaicherdsukal [24]. This theorem states that radiative corrections during inflation should not grow faster than a logarithm of the scale factor. However, as was discussed in more detail in Ref. [12], density perturbations which are proportional to a power of S are really due to high frequency modes at the initial time, and are hence always large rather than growing. This comment also applies to the effects found in Ref. [10] and in the present paper.

The fact that the effect which we calculate come from high frequency modes does not mean that it can be removed by a renormalization. Our key coordinate space result, Eq. (19), involves an integral of the full stress tensor correlation function, with no renormalizations. Our view is that the finiteness of this integral implies that no renormalization is needed.

B. Numerical Estimates

We may use the coordinate space correlation functions, $K_s(r)$ and $K_e(r)$, to estimate the physical effects of the gravity wave fluctuations on various scales. However, these functions describe the primordial fluctuations at the end of inflation. After the end of inflation, modes which are outside the horizon remain approximately constant until they re-enter the horizon. (For a more detailed discussion, see, for example, Ref. [25].) After that point, they redshift with their amplitude proportional to $1/a$. Let a_{Hc} be the value of the scale factor at which a mode associated with coordinate length r re-enters the horizon, and a_{now} be the present value of the scale factor. The present value of the correlation function is then

$$K_{now}(r) = K(r) \left(\frac{a_{Hc}}{a_{now}}\right)^2 . \tag{54}$$

For the sudden switch model, this becomes

$$K_{now-S}(r) = -\frac{4H^2\, S^2\, \ell_p^4}{3\pi^2\, r^2} \left(1 - \frac{2}{H^2 r^2}\right) \left(\frac{a_{Hc}}{a_{now}}\right)^2 , \tag{55}$$

and for the exponential switch model it is

$$K_{now-E}(r) = -\frac{H^3\, S\, \ell_p^4}{4r} \left(\frac{a_{Hc}}{a_{now}}\right)^2 . \tag{56}$$

Here the factors of the Planck length ℓ_p are written explicitly.

Let E_R be the reheating energy at the end of inflation. This energy has since been redshifted to that of the cosmic microwave background. We set $a = 1$ at the end of inflation, so that

$$a_{now} \approx \frac{E_R}{2.5 \times 10^{-4} eV} \,. \tag{57}$$

The proper length scale today associated with coordinate distance r is

$$\ell = a_{now}\, r \,. \tag{58}$$

We assume that reheating is efficient, so the vacuum energy at the end of inflation is of order E_R^4, and

$$H^2 = \frac{8\pi}{3}\, \ell_p^2\, E_R^4 \,. \tag{59}$$

We also assume that the scale of interest was outside the horizon at the end of inflation, so that $Hr > 1$. We may combine all of these results to write

$$|K_{now-S}| = 10^{45} \left(\frac{\ell_p}{\ell}\right)^2 \left(\frac{E_R}{10^{16} GeV}\right)^6 \left(\frac{a_{Hc}}{a_{now}}\right)^2 S^2 \,, \tag{60}$$

and

$$|K_{now-E}| = 10^{11} \left(\frac{\ell_p}{\ell}\right) \left(\frac{E_R}{10^{16} GeV}\right)^7 \left(\frac{a_{Hc}}{a_{now}}\right)^2 S \,. \tag{61}$$

Let us first consider the case of perturbations of the order of the present horizon size, $\ell \approx 10^{61}\, \ell_p$. In this case, $a_{Hc} \approx a_{now}$. Data from the WMAP satellite [6] constrain these perturbations to satisfy $h \lesssim 10^{-5}$, so that $|K_{now}| \lesssim 10^{-10}$. Consequently, the sudden switch model leads to

$$S \lesssim 10^{34} \left(\frac{10^{16} GeV}{E_R}\right)^3 \,, \tag{62}$$

and the exponential switch model to

$$S \lesssim 10^{40} \left(\frac{10^{16} GeV}{E_R}\right)^7 \,. \tag{63}$$

These constraints on the total expansion during inflation are compatible with adequate inflation to solve the horizon and flatness problems, $S \gtrsim 10^{23}$. Because $K < 0$, quantum stress tensor fluctuations during inflation will tend to produce anti-correlated gravity wave fluctuations. Note that in this example, $|K_{now}| \approx K(r) \lesssim 10^{-10}$, so the criterion for the validity of the perturbative calculation, $|K(r)| \ll 1$, is satisfied.

Now we wish to consider perturbations which are well within the present horizon. For this purpose, we need an approximate model for the current matter content of the Universe.

Although the dominant component today is the dark energy, this is likely to be a recent phenomenon. If the dark energy is due to a cosmological constant term, it does not redshift and hence does not grow as we go backwards in time. Here we assume that the Universe was radiation dominated, $a \propto t^{1/2}$, for $t \lesssim t_{eq}$ and subsequently matter dominated, $a \propto t^{2/3}$. Furthermore, we assume

$$\frac{a_{eq}}{a_{now}} \approx 10^{-4}\,, \tag{64}$$

so that $t_{eq}/t_{now} \approx 10^{-6}$. A perturbation with proper length ℓ enters the horizon at $t = t_{Hc} = \ell$. If we assume that $\ell < t_{eq}$, then we may write

$$\left(\frac{a_{Hc}}{a_{now}}\right)^2 \approx 10^{-63}\,\frac{\ell}{\ell_p}\,. \tag{65}$$

If we insert this relation into Eq. (60), the result is

$$|K_{now-S}| = 10^{-58}\,\left(\frac{100\,km}{\ell}\right)\,\left(\frac{E_R}{10^{16}GeV}\right)^6\,S^2\,. \tag{66}$$

In the case of the exponential switch model, the factors of ℓ cancel,

$$|K_{now-E}| = 10^{-52}\,\left(\frac{E_R}{10^{16}GeV}\right)^7\,S\,, \tag{67}$$

leading to a scale independent correlation function on scales $\ell \lesssim 10^{23}\,cm$.

If the magnitude of these fluctuations is sufficiently large, they should produce background noise in gravitational wave detectors, which has not been observed. LIGO has placed limits [26] of $h \lesssim 10^{-24}$ on scales of the order of $10^2\,km$, corresponding to $|K_{now}| < 10^{-48}$, and leading to the constraints

$$S < 10^{23}\,\left(\frac{10^{10}GeV}{E_R}\right)^3\,. \tag{68}$$

for the sudden switch model, and

$$S < 10^{25}\,\left(\frac{10^{13}GeV}{E_R}\right)^7\,, \tag{69}$$

for the exponential switch model. However, these results are compatible with adequate inflation to solve the horizon and flatness problems only if

$$E_R \lesssim 10^{10}\,GeV \tag{70}$$

for the sudden switch model, and

$$E_R \lesssim 10^{13}\,GeV \tag{71}$$

for the exponential switch model. In this example, $|K(r)| \approx 10^{23} K_{now} \lesssim 10^{-25}$, so again the requirement that $|K|$ be small is fulfilled.

VI. SUMMARY AND DISCUSSION

We have seen that quantum stress tensor fluctuations are capable of creating gravity waves during inflation. The resulting spectrum has several properties, including negative power and an amplitude which grows with increasing duration of the inflationary period. Negative power spectra, although forbidden by the Wiener-Khinchin theorem [16, 17], can arise in quantum field theory [18], especially in quantum stress tensor fluctuations. A negative power spectrum can be viewed as interchanging correlations and anti-correlations, as compared to a positive power spectrum of the same functional form.

We find that the amplitude of the gravity wave spectrum is proportional to a positive power of S, the change in scale factor during inflation. A similar dependence was also found in Refs. [11, 12], for the effects of stress tensor fluctuations on density perturbation and in Ref. [10] for the correction to gravity wave modes from expectation value of the stress tensor of a conformal field. The gravity wave power depends somewhat upon the details of the initial conditions, being S^2 if one integrates the equations directly from a state of zero fluctuations, and being S if the interaction between the fluctuating matter stress tensor is supposed to be switched on over a finite interval of the order of the horizon size in comoving time. In all cases, the primordial power spectrum of gravity wave fluctuations is negative and greater in magnitude at shorter wavelengths. This non-scale invariant spectrum of fluctuations will be highly non-Gaussian, due to the non-Gaussian character of quantum stress tensor fluctuations.

Our conclusions are contingent upon the assumptions which we have made, especially concerning the transplanckian modes. We have chosen to include all modes in the conformal field theory, including those which are above the Planck scale at the beginning of inflation. Because our results depend crucially on this assumption, one can regard the predicted power spectra as probes of transplanckian physics.

The gravity wave fluctuations are potentially observable. Longer wavelengths could alter the polarization of the CMB, and be detected in the same way as the active fluctuations. Shorter wavelengths could potentially be detected by Earth or space based gravity wave detectors. The fact that they have not yet been detected might be used to infer constraints on the duration and energy scale of inflation.

Acknowledgments

We would like to thank Shun-Pei Miao, Richard Woodard, and the participants of the 14th and 15th Peyresq workshops for valuable discussions. This work is partially supported by the National Science Council, Taiwan, ROC under the Grants NSC 98-2112-M-001-009-MY3 and NSC99-2112-M-031-002-MY3, and by the U.S. National Science Foundation under Grant PHY-0855360.

Appendix A: Flat Space Stress Tensor Correlation Functions

In this appendix, we derive the explicit expressions for the flat space correlation functions utilized in Sect. III, especially Eq. (25). All of the expressions in this appendix refer to flat spacetime, so here we drop the subscript "M". We may use the results of Ref. [27], where the electromagnetic field stress tensor correlation function was shown to be

$$\begin{aligned} C^{\mu\nu\sigma\lambda}(x,x') \;=\;& 4\,(\partial_\mu\partial_\nu D)\,(\partial_\sigma\partial_\lambda D)\;+2\,g_{\mu\nu}\,(\partial_\sigma\partial_\alpha D)\,(\partial_\lambda\partial^\alpha D)\;+2\,g_{\sigma\lambda}\,(\partial_\mu\partial_\alpha D)\,(\partial_\nu\partial^\alpha D)\\ -&\;2\,g_{\mu\sigma}\,(\partial_\nu\partial_\alpha D)\,(\partial_\lambda\partial^\alpha D)\;-2\,g_{\nu\sigma}\,(\partial_\mu\partial_\alpha D)\,(\partial_\lambda\partial^\alpha D)\\ -&\;2\,g_{\nu\lambda}\,(\partial_\mu\partial_\alpha D)\,(\partial_\sigma\partial^\alpha D)\;-2\,g_{\mu\lambda}\,(\partial_\nu\partial_\alpha D)\,(\partial_\sigma\partial^\alpha D)\\ +&\;(g_{\mu\sigma}g_{\nu\lambda}+g_{\nu\sigma}g_{\mu\lambda}-g_{\mu\nu}g_{\sigma\lambda})\,(\partial_\rho\partial_\alpha D)\,(\partial^\rho\partial^\alpha D)\,. \end{aligned} \tag{A1}$$

Here

$$D = D(x-x') = \frac{1}{4\pi^2(x-x')^2} \tag{A2}$$

is the Hadamard (symmetric two-point) function for the massless scalar field. For our purposes, it is sufficient to compute a single component, such as C^{xyxy}. The result is

$$C^{xyxy}(\tau,r) = \frac{3}{\pi^2[(t-t')^2-r^2]^4}\,, \tag{A3}$$

where $r = |\mathbf{x}-\mathbf{x}'|$ and $\tau = t-t'$. The spatial Fourier transform of this expression is

$$\hat{C}^{xyxy}(\tau,k) = -\frac{1}{512\pi^5}\left(\frac{d^4}{d\tau^4}+2k^2\,\frac{d^2}{d\tau^2}+k^4\right)\left(\frac{\sin k\tau}{\tau}\right)\,, \tag{A4}$$

or equivalently,

$$\hat{C}^{xyxy}(\tau,k) = -\frac{k^5}{512\pi^5}\int_0^1 du\,(1-u^2)^2\,\cos(ku\tau)\,, \tag{A5}$$

which is Eq. (25). This expression may be verified by checking that

$$C^{xyxy}(\tau,r) = \int d^3k\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\,\hat{C}^{xyxy}(\tau,k)\,. \tag{A6}$$

Appendix B: Evaluation of Sampling Function Integrals

In this appendix, we will evaluate some of the integrals needed in Sect. IV C, which involve the function s_{aw}, defined in Eq. (47). We begin with expression 3.411.23 in Ref. [28], which states that

$$\int_{-\infty}^{\infty} \frac{x\, e^{\mu x}}{1+e^x}\, dx = -\pi^2 \csc(\pi\mu)\, \cot(\pi\mu)\,, \tag{B1}$$

for $0 < Re(\mu) < 1$. This implies that

$$\int_{-\infty}^{\infty} \frac{e^{\mu x}}{1+e^x}\, dx = \pi \csc(\pi\mu)\,. \tag{B2}$$

This may be verified by taking a derivative of Eq. (B2) with respect to μ, and by noting that when $\mu = 1/2$, this relation becomes

$$\int_{-\infty}^{\infty} \frac{e^{x/2}}{1+e^x}\, dx = 2\int_{0}^{\infty} \frac{1}{1+y^2}\, dy = \pi\,. \tag{B3}$$

This confirms that there is no additional constant in Eq. (B2) . Next we may take the limit in which $\mu \to iq$ to write

$$\int_{-\infty}^{\infty} \frac{e^{iqx}}{1+e^x}\, dx = -\frac{2\pi i}{e^{\pi q} - e^{-\pi q}}\,. \tag{B4}$$

However, we need integrals over a semi-infinite range of the form

$$\int_{x_r}^{\infty} \frac{e^{iqx}}{1+e^{(x-x_0)/b}}\, dx = b\, e^{iqx_0} \left[\int_{-\infty}^{\infty} \frac{e^{iqbz}}{1+e^z}\, dz - \int_{-\infty}^{\frac{x_r-x_0}{b}} \frac{e^{iqbz}}{1+e^z}\, dz\right]. \tag{B5}$$

The second integral on the right-hand side of the above equation may be approximated by setting the denominator of the integrand to unity:

$$\int_{-\infty}^{\frac{x_r-x_0}{b}} \frac{e^{iqbz}}{1+e^z}\, dz \approx \frac{i}{q\,b}\, e^{iq(x_r-x_0)} + O(e^{-x_0/b})\,. \tag{B6}$$

Thus,

$$\int_{x_r}^{\infty} \frac{e^{iqx}}{1+e^{(x-x_0)/b}}\, dx \approx -\frac{2\pi i\, e^{iqx_0}}{e^{\pi q} - e^{-\pi q}} + \frac{i}{q}\, e^{iqx_r}\,. \tag{B7}$$

If we take a derivative with respect to q, then the real and imaginary parts of the resulting expression become, for $q\,b \gg 1$,

$$\int_{x_r}^{\infty} \frac{x\, \sin qx}{1+e^{(x-x_0)/b}}\, dx \approx -2\pi x_0\, b\, \cos(qx_0)\, e^{-qb} \tag{B8}$$

and

$$\int_{x_r}^{\infty} \frac{x\, \cos qx}{1+e^{(x-x_0)/b}}\, dx \approx 2\pi x_0\, b\, \sin(qx_0)\, e^{-qb}\,. \tag{B9}$$

[1] V. Mukhanov and G. Chibisov, JETP Lett. **33**, 532 (1981).

[2] A.H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).

[3] S.W. Hawking, Phys. Lett. B **115**, 295 (1982).

[4] A.A. Starobinsky, Phys. Lett. B **117**, 175 (1982).

[5] J.M. Bardeen, P.J. Steinhardt, and M.S. Turner, Phys. Rev. D **28**, 679 (1983).

[6] E. Komatsu *et al.*, Astrophys. J. Suppl. **192**, 18 (2011), arXiv:1001.4538.

[7] A.A. Starobinsky, JETP Lett. **30**, 682 (1979).

[8] L.F. Abbott and M.B. Wise, Nucl. Phys. B **244**, 541 (1984).

[9] B. Allen, Phys. Rev. D **37**, 2078 (1988).

[10] J.-T. Hsiang, L.H. Ford, D.-S. Lee, and H.-L. Yu, Phys. Rev. D, **83**, 084027 (2011), arXiv:1012.1582.

[11] C.-H. Wu, K.-W. Ng, and L.H. Ford, Phys. Rev. D **75**, 103502 (2007), arXiv:gr-qc/0608002.

[12] L.H. Ford, S.-P. Miao, K.-W. Ng, R.P. Woodard, and C.-H. Wu, Phys. Rev. D **82**, 043501 (2010), arXiv:1005.4530.

[13] S. del Campo and L.H. Ford, Phys. Rev. D **38**, 3657 (1988).

[14] E. M. Lifshitz, Zh. Eksp. Teor. Phys. **16** 587 (1946).

[15] L.H. Ford and L. Parker, Phys. Rev. D **16**, 1601 (1977).

[16] N. Wiener, Acta. Math, Stockholm **55**, 117 (1930).

[17] A. Khinchin, Math. Ann. **109**, 604 (1934).

[18] J.-T. Hsiang, C.-H. Wu, and L. H. Ford, Phys. Lett. A **375**, 2296 (2011), arXiv:1012.3226.

[19] C.J. Fewster, L.H. Ford, and T.A. Roman, Phys. Rev. D **81**, 121901(R) (2010), arXiv:1004.0179.

[20] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[21] W.G. Unruh, Phys. Rev. D **51**, 2827 (1995), gr-qc/9409008.

[22] S. Corley and T. Jacobson, Phys. Rev. D **54**, 1568 (1996), hep-th/9601073.

[23] S. Weinberg, Phys. Rev. D **72**, 043514 (2005); **74**, 023508 (2006).

[24] K. Chaicherdsakul, Phys. Rev. D **75**, 063522 (2007).

[25] K.-W. Ng, Int. J. Mod. Phys. A **11**, 3175 (1996).

[26] P.J. Sutton, J. Phys. Conf. Ser. **110**, 062024 (2008).

[27] L.H. Ford and C.-H. Wu, Int. J. Theor. Phys. **42**, 15 (2003), gr-qc/0102063.

[28] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products, 6th Ed.* (Academic Press, New York, 2000).