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String or brane-like solutions in four-dimensional Einstein gravity in the presence of cosmological constant

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We investigate string or brane-like solutions for four-dimensional vacuum Einstein equations in the presence of cosmological constant. For the case of negative cosmological constant, the BTZ black string is the only warped stringlike solution. The general solutions for non-warped branelike configurations are found and they are characterized by the ADM mass density and two tensions. Interestingly, the sum of these tensions is equal to the minus of the mass density. Other than the well known black string and AdS soliton spacetimes, all the static solutions possess naked singularities. The time-dependent solutions can be regarded as the AdS extension of the well-known Kasner solutions. The speciality of those static regular solutions and the implication of singular solutions are also discussed in the context of cylindrical matter collapse. For the case of positive cosmological constant, the Kasner-de Sitter spacetime appears as time-dependent solutions and all static solutions are found to be naked singular.

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I. INTRODUCTION

One may define hypercylindrical spacetimes as spacetime configurations having translational symmetries along some spatial directions. Black p -brane solutions have such configurations and have been used in various studies in string theory. In the context of the extra-dimensional models, a black hole observed in our four-dimensional universe could be a part of the full dimensional black string or brane configuration rather than a part of the full dimensional black hole confined around our universe. Gregory and Laflamme found that black string or p -brane backgrounds are generically unstable on the small perturbations in the string or brane directions. However, some black p -brane configurations such as BTZ black string show absence of the Gregory-Laflamme instability. The origin and the evolution of such instability have not been fully understood as yet. Therefore, it is very important to know the full solution space for string or brane configurations.

In general relativity, hypercylindrical static vacuum solutions in five spacetime dimensions were found repeatedly by several authors [1–3]. Initially, these solutions were investigated in the context of the Kaluza-Klein dimensional reduction, including the interpretation of integral constants as mass and scalar charge. The correct interpretation of the integral constants as ADM mass density and gravitational tension and the study on the geometrical properties were given recently in Refs. [3, 4]. The extension to spacetime dimensions higher than five and the inclusion of dilatonic scalar and antisymmetric form fields are considered in Refs. [5, 6] and in Refs. [7], respectively. (See also references therein.) Stationary stringlike solutions having a momentum flow along the string direction were found by Chodos and Detweiler [2], and their geometrical properties were analyzed by Kim and Lee [8] in detail. Gravitational energy conditions on ADM mass and momentum densities and tension have also been analyzed recently [9], with presenting classification of solutions under boost transformations.

In this paper we search for the same type of solutions in the presence of cosmological constant. The inclusion of cosmological constant may change the behaviors of spacetime solutions drastically. For instance, the spacetime

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becomes asymptotically non-flat. The Gregory-Laflamme instability [10] generically occurring for black string or brane solutions disappears for a certain class of such solutions with a negative cosmological constant [11]. Furthermore, in the presence of a positive cosmological constant, it was shown that the cosmological anisotropy will be exponentially washed away if the matter fields satisfy the dominant energy condition except for special cases in Bianchi IX [12]. Note also that our universe has a non-vanishing positive cosmological constant. Therefore, it is very interesting to see how string or branelike solutions behave differently in the presence of cosmological constant.

At the present paper, we consider the case of four-dimensional spacetime only. In the absence of cosmological constant, black string solutions cannot exist in four dimensions [13], but start to appear from five dimensions. In the presence of cosmological constant, however, black string or brane solutions such as BTZ black string and AdS C-metrics in Ref. [14] exist even in four dimensions. Therefore, instead of considering general higher dimensional spacetime solutions, we search for a wider class of string or branelike configurations in four dimensions for simplicity. In addition, we also examine whether or not a wider class of the Kasner solutions with cosmological constant [15] exist.

In section II, we consider a class of string-like solutions that are warped along a spatial direction. It is shown that the BTZ black string is indeed the unique nontrivial string-like solution. In section III, general string-like solutions without warping factor are considered. Static and time-dependent solutions are presented for the cases of negative and positive cosmological constants. In sections IV and V, the causal structures and geometrical properties of those solutions are summarized. Conclusions are followed in section VI and physical implications are also discussed.

II. WARPED SOLUTIONS

The action of the Einstein gravity with cosmological constant in four dimensions is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda). \quad (1)$$

Here we are interested in searching for string-like solutions as general as possible. Let us first summarize some results on the string-like solutions for the case of five-dimensional Einstein gravity without cosmological constant (*i.e.*, $\Lambda = 0$). Consider hypercylindrical uniform static vacuum solutions in five dimensions whose metric forms are given by

$$ds^2 = -F(\rho)dt^2 + G(\rho) (d\rho^2 + \rho^2 d\Omega_2^2) + d\eta^2 \quad (2)$$

in the isentropic coordinate system. We find that the only solution is the Schwarzschild black string where

$$F(\rho) = \left(\frac{1 - K/\rho}{1 + K/\rho} \right)^2, \quad G(\rho) = \left(1 + \frac{K}{\rho} \right)^4. \quad (3)$$

This solution appears to be characterized by a single parameter, namely, the ADM mass density $M (= 2K)$. However, we expect that there should be another physical quantity (*i.e.*, the ADM tension) associated with the translational symmetry along the η direction in this metric. Actually, the above solution turns out to be a special case where the ADM tension is given by $\tau = M/2 = K$. More general string-like solutions having arbitrary values of tension were indeed obtained by allowing the metric component $g_{\eta\eta}$ arbitrary, *e.g.*, $g_{\eta\eta} = 1 \rightarrow Z(\rho)$, and the solutions are [3, 4]

$$F = \left| \frac{1 - \frac{K}{\rho}}{1 + \frac{K}{\rho}} \right|^{\frac{2(2-a)}{\sqrt{3(a^2-a+1)}}} = Z^{(2-a)/(2a-1)}, \quad G = \left(1 + \frac{K}{\rho} \right)^4 \left| \frac{1 + \frac{K}{\rho}}{1 - \frac{K}{\rho}} \right|^{\frac{2(a+1)}{\sqrt{3(a^2-a+1)}}-2}. \quad (4)$$

These solutions are characterized by two parameters (*e.g.*, mass density and tension) and are still asymptotically flat at $\eta = \text{constant}$ surfaces. However, most of them have naked singularity at $\rho = K$ surface and the only regular solutions occur when the tension to mass ratio ($\tau/M \equiv a$) is $a = 1/2, 2$, which correspond to the Schwarzschild black string and the Kaluza-Klein bubble, respectively.

Now let us consider what happens if a cosmological constant is present. For simplicity we consider four dimensions. In four-dimensional Einstein gravity with a negative cosmological constant ($\Lambda < 0$), the well-known black string metric is the BTZ solution given by

$$ds^2 = \frac{-3/\Lambda}{\sin^2 \eta} \left[-(-M + r^2) dt^2 + \frac{dr^2}{-M + r^2} + r^2 d\theta^2 + d\eta^2 \right]. \quad (5)$$

Note that this solution is asymptotically AdS_4 at $r \rightarrow \infty$. Note also that the string direction is not uniform, but warped. In order to find a more general class of string-like static vacuum solutions we consider the following metric ansatz, similarly to the case of the Schwarzschild black string above,

$$ds^2 = H^{-2}(\eta, \rho) [-F(\rho)dt^2 + G(\rho) (d\rho^2 + \rho^2 d\theta^2) + Z(\rho)d\eta^2]. \quad (6)$$

Note here that the ρ -dependence is allowed on the warping factor in addition to the arbitrary ρ -dependence in the $g_{\eta\eta}$ -component.

The (ρ, η) -component of the field equations gives

$$\frac{\partial}{\partial \eta} \left[2 \frac{\partial H(\eta, \rho)}{\partial \rho} Z(\rho) - H(\eta, \rho) \frac{\partial Z(\rho)}{\partial \rho} \right] = 0. \quad (7)$$

Note first that this equation is easily solved if the warping factor is a function of ρ only, *i.e.*, $H(\eta, \rho) = H(\rho)$. For such case that the geometry is not warped in the η -direction, we can set $H = 1$ by redefining the functions F , G and Z , and it will be considered separately in the next section.

Now, if the geometry is warped in the η direction, *i.e.*, $\partial_\eta H \neq 0$, Eq. (7) gives

$$H(\eta, \rho) = \sqrt{Z(\rho)} [h(\eta) + g(\rho)]. \quad (8)$$

Thus, we can set

$$Z(\rho) = 1, \quad H(\eta, \rho) = h(\eta) + g(\rho) \quad (9)$$

by redefining the functions F and G in Eq. (6). We first consider the case that the warping factor is independent of the ρ -coordinate, *e.g.*, $g(\rho) = 0$. Then, solving the rest of the field equations gives

$$F = C_1 \tan^2 \sqrt{M} \ln(\rho/\rho_0), \quad G = \frac{-M/\Lambda_3}{\rho^2 \cos^2 \sqrt{M} \ln(\rho/\rho_0)}, \quad H = C_2 e^{\sqrt{\Lambda_3} \eta} + C_3 e^{-\sqrt{\Lambda_3} \eta}. \quad (10)$$

Here the integral constants are C_1, C_2, C_3, ρ_0, M and Λ_3 with $C_2 C_3 = \Lambda/(12\Lambda_3)$, and they must be chosen suitably to make the metric functions real.

i) $\Lambda < 0$; In this case the spacetime is asymptotically AdS_4 . The warping function in Eq. (10) can be reexpressed as

$$H = \begin{cases} \sqrt{\Lambda/3\Lambda_3} \sin \sqrt{-\Lambda_3}(\eta - \eta_0) & \text{for } \Lambda_3 < 0, \\ \sqrt{-\Lambda/3\Lambda_3} \sinh \sqrt{\Lambda_3}(\eta - \eta_0) & \text{for } \Lambda_3 > 0, \\ \sqrt{-\Lambda/3}(\eta - \eta_0) & \text{for } \Lambda_3 = 0, \end{cases} \quad (11)$$

The integral constants are restricted accordingly to the sign of Λ_3 . That is, for $\Lambda_3 < 0$, $M, C_1 > 0$ and $C_2 = C_3^*$ with $|C_2| = \sqrt{\Lambda/(12\Lambda_3)}$, and the metric is written as

$$ds^2 = \frac{3\Lambda_3/\Lambda}{\sin^2 \sqrt{-\Lambda_3} \eta} \left[-\tan^2 \sqrt{M} \ln \rho dt^2 + \frac{M/(-\Lambda_3)}{\rho^2 \cos^2 \sqrt{M} \ln \rho} (d\rho^2 + \rho^2 d\theta^2) + d\eta^2 \right]. \quad (12)$$

Here C_1 is absorbed by rescaling the t -coordinate, and $\eta - \eta_0 \rightarrow \eta$ and $\rho/\rho_0 \rightarrow \rho$ are used. For $\Lambda_3 > 0$, $M, C_1 < 0$, and the metric is

$$ds^2 = \frac{3\Lambda_3/(-\Lambda)}{\sinh^2 \sqrt{\Lambda_3} \eta} \left[-\tanh^2 \sqrt{M} \ln \rho dt^2 + \frac{\bar{M}/\Lambda_3}{\rho^2 \cosh^2 \sqrt{M} \ln \rho} (d\rho^2 + \rho^2 d\theta^2) + d\eta^2 \right]. \quad (13)$$

Here $\bar{M} = -M > 0$. When $\Lambda_3 = 0$, we take $M \rightarrow 0$ and $C_1 \rightarrow \infty$ with $-M/\Lambda_3 \equiv G_0 (> 0)$ and $C_1 M \equiv F_0 (> 0)$ fixed. Hence one finds

$$ds^2 = \frac{-3/\Lambda}{\eta^2} \left[-\ln^2 \rho dt^2 + \frac{1}{\rho^2} (d\rho^2 + \rho^2 d\theta^2) + d\eta^2 \right]. \quad (14)$$

Here F_0 and G_0 are absorbed by rescaling the t and η coordinates. Defining $\bar{r} = \ln \rho$, it may be rewritten as

$$ds^2 = \frac{-3/\Lambda}{\eta^2} (-\bar{r}^2 dt^2 + d\bar{r}^2 + d\theta^2 + d\eta^2). \quad (15)$$

In order to see what these solutions look like geometrically, we can reexpress them in a more conventional form by introducing the areal radial coordinate, *e.g.*, $r^2 = \rho^2 G$,¹

$$ds^2 = \frac{1}{H^2(\eta)} \left[-(-M - \Lambda_3 r^2) dt^2 + \frac{dr^2}{-M - \Lambda_3 r^2} + r^2 d\theta^2 + d\eta^2 \right]. \quad (16)$$

Here the constant C_1 is absorbed by rescaling the t -coordinate. The signs of Λ_3 and M and the corresponding form of the function H are as explained above. Note that the value of Λ_3 can be set to $-1, 0, +1$ by rescaling the coordinates according to its sign. Therefore, we see that the case of $\Lambda_3 < 0$ is the BTZ black string written in Eq. (5), and that the cases of $\Lambda_3 > 0$ and $\Lambda_3 = 0$ are nothing but dS_3 (de Sitter) and M_3 (flat) foliations of the same pure AdS_4 spacetime, respectively.

ii) $\Lambda > 0$; In this case the spacetime is asymptotically dS_4 . The metric can be rewritten as in Eq. (16) as well. However, the case of $\Lambda_3 < 0$ is not allowed since the function H cannot be real valued (*e.g.*, $|C_2|^2 = \Lambda/(12\Lambda_3)$ cannot be positive). Therefore, the solutions are nothing but dS_3 and M_3 foliations of the same dS_4 spacetime.

Now let us consider the case that the geometry is warped not only along the η -direction, but also along the ρ -direction (*i.e.*, $g \neq \text{constant}$). By solving all other field equations, we finally obtain

$$H = \sqrt{(-\Lambda/3)C} \sin \frac{\eta - \eta_0}{\sqrt{C}} + \frac{A}{D + \ln \rho}, \quad F = \frac{B}{(\ln \rho + D)^2}, \quad G = \frac{C}{\rho^2 (\ln \rho + D)^2}. \quad (17)$$

Here we assumed $B, C > 0$. By defining $r = \ln \rho + D$ and rescaling $\sqrt{B/C}t \rightarrow t$, $(\eta - \eta_0)/\sqrt{C} \rightarrow \eta$, $A/\sqrt{C} \rightarrow A$, the metric in Eq. (6) can be expressed as²

$$ds^2 = \frac{1}{(\sqrt{-\Lambda/3}r \sin \eta + A)^2} (-dt^2 + dr^2 + d\theta^2 + r^2 d\eta^2). \quad (18)$$

This metric looks like a new solution, but it turns out to be the pure AdS_4 . Defining $x_3 = r \sin \eta$, $x_1 = r \cos \eta$ and $\theta = x_2$, we find

$$ds^2 = \frac{-3/\Lambda}{(x_3 - x_{30})^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2), \quad (19)$$

which is the M_3 foliation of pure AdS_4 manifold. To conclude, even if we considered a more general class of warped solutions in the form of Eq. (6), we found that there are no new solutions other than the known BTZ black string solutions.

III. SOLUTIONS WITHOUT WARPING FACTOR

In this section, let us consider the case that the geometry is not warped, *i.e.*, $H = 1$ in Eq. (6). Here we consider more general case in which the ρ and θ coordinates are not necessarily radial and angular coordinates, respectively, *e.g.*, $\rho \rightarrow z$ and $\theta \rightarrow x$.

$$ds^2 = -f_0(z)dt^2 + f(z)dz^2 + f_1(z)dx^2 + f_2(z)dy^2. \quad (20)$$

Note that these spacetimes can be regarded as a uniform foliation of 3-dimensional spacetimes along the y -direction. Without loss of generality, we may set $f(z) = f_0(z)f_1(z)f_2(z)$ by redefining the z coordinate. Thus, the most general form of the metric is given by:³

$$ds^2 = -e^{2U}(dt^2 - e^{2V}dz^2) + e^{2(V-W)}dx^2 + e^{2W}dy^2, \quad (21)$$

¹ It should be point out that this coordinate transformation is not defined well for the case of $\Lambda_3 = 0$. However, the metric in Eq. (16) is well-defined in the limit of $\Lambda_3 \rightarrow 0$ with $-M > 0$, and describes the pure AdS_4 geometry.

² We point out that the above metric becomes identical to the four-dimensional analogy of the solution found in Ref. [16] if we perform triple Wick rotations; $r = i\tau$, $t = ix$ and $\eta = i\sqrt{-\Lambda/3}b_0y$.

³ This choice of metric functions makes the field equation, $R_{ab} - \Lambda g_{ab} = 0$, be decoupled.

where U , V and W are functions of z only. Solving the Einstein equation gives

$$\begin{aligned} e^{2U} &= \frac{\lambda}{\Lambda} \left(\frac{d}{12\lambda} \right)^{\frac{1}{3}} \left(\frac{2c}{1+d e^{-cz}} \right)^{\frac{2}{3}} e^{-\frac{1}{3}(2a+c)z}, \\ e^{2V} &= \left(\frac{d}{12\lambda} \right)^{\frac{2}{3}} \left(\frac{2c}{1+d e^{-cz}} \right)^{\frac{4}{3}} e^{\frac{2}{3}(a-c)z}, \\ e^{2W} &= \left(\frac{d}{12\lambda} \right)^{\frac{1}{3}} \left(\frac{2c}{1+d e^{-cz}} \right)^{\frac{2}{3}} e^{\frac{1}{3}(b+a-c)z}, \end{aligned} \quad (22)$$

with a constraint equation

$$a^2 + \frac{b^2}{3} = c^2, \quad (23)$$

where a, b, c, d and λ are integration constants. This solution was found in Ref. [18]. Note that, when $c = 0$ and so $a = b = 0$, the solution reduces to the pure AdS_4 or dS_4 metric depending on the sign of Λ , respectively. Henceforth, we consider $c \neq 0$ case. However, the solution for $c = 0$ can be obtained as well by taking the $c \rightarrow 0$ limit in the solutions with $c \neq 0$.

Let us introduce a new coordinate defined as

$$\xi = \left(\frac{2c}{1+d e^{-cz}} \right)^{1/3} \quad \text{or} \quad e^{-cz} = \frac{1}{d} \left(\frac{2c}{\xi^3} - 1 \right). \quad (24)$$

By suitable rescalings⁴ of the coordinates t , x and y , the metric in Eq. (22) can be rewritten as

$$\begin{aligned} ds^2 &= -\frac{(-\Lambda)\xi^2}{3} \left(1 - \frac{K^3}{\xi^3} \right) \left| 1 - \frac{K^3}{\xi^3} \right|^{\frac{2(p-1)}{3}} dt^2 + \frac{d\xi^2}{\frac{(-\Lambda)\xi^2}{3} \left(1 - \frac{K^3}{\xi^3} \right)} \\ &\quad + \xi^2 \left| 1 - \frac{K^3}{\xi^3} \right|^{\frac{1-p+q}{3}} dx^2 + \xi^2 \left| 1 - \frac{K^3}{\xi^3} \right|^{\frac{1-p-q}{3}} dy^2. \end{aligned} \quad (25)$$

Here $K = \sqrt[3]{2c}$, $p = a/c$ and $q = b/c$, hence

$$p^2 + \frac{q^2}{3} = 1. \quad (26)$$

Note that these solutions are characterized by two integral constants K , p and the sign of q . We point out that the metric in Eq. (25) is valid for both signs of Λ .

For $\Lambda < 0$, the spacetime is asymptotically anti-de Sitter ($\xi \gg K$) and has a static boundary, allowing us to define ADM quantities. By using the Hamiltonian formalism in Refs. [19, 20], the gravitational mass and tension densities⁵ associated with t , x and y translation symmetries for stationary solutions can be obtained as

$$M \equiv \frac{(-\Lambda)K^3}{24\pi G} p, \quad \tau_1 \equiv -\frac{(-\Lambda)K^3}{48\pi G} (p - q), \quad \tau_2 \equiv -\frac{(-\Lambda)K^3}{48\pi G} (p + q), \quad (27)$$

respectively. We can see that the solutions denoted by Eq. (25) are characterized by these three ADM parameters restricted on the plane given by

$$M + \tau_1 + \tau_2 = 0. \quad (28)$$

Furthermore, actual values of these ADM parameters will be more restricted due to physical conditions for them although such conditions are not well known for asymptotically AdS spacetimes. Note that, in Eq. (25), the spacetime

⁴ $t \rightarrow |\Lambda| (2|d|^p / (3|\lambda|))^{1/3} t$, $x \rightarrow (12|\lambda| / |d|^{p-\sqrt{3}q})^{1/6} x$, $y \rightarrow (12|\lambda| / |d|^{p+\sqrt{3}q})^{1/6} y$.

⁵ We point out that the ADM densities here are defined for unit coordinate intervals, not for unit proper distances, as in Refs. [19, 20].

described by $-\infty < \xi \leq 0$ is the same as the spacetime described by $0 \leq \xi < \infty$ with opposite sign of K . Thus we restrict the range of ξ to be nonnegative. Suppose K is negative, *e.g.*, $\bar{K} = -K > 0$. Then, Eq. (27) shows that the corresponding ADM parameters are the same for both (K, p, q) and $(-K, -p, -q)$. Although the metrics look different very much for these two cases, this invariance indicates that both cases actually describe the same geometry. In fact, this equivalence can be explicitly shown by the coordinate transformation $\xi^3 + \bar{K}^3 \rightarrow \xi^3$ with $K \rightarrow \bar{K}$. Note that we may restrict the value of K be nonnegative using such symmetry. For given values of M , τ_1 and τ_2 , the integral constants appeared in Eq. (27) can be obtained by

$$K^3 = \frac{24\pi G}{(-\Lambda)} \sqrt{\frac{2}{3}(M^2 + \tau_1^2 + \tau_2^2)}, \quad p = \frac{M}{\sqrt{\frac{2}{3}(M^2 + \tau_1^2 + \tau_2^2)}}, \quad q = \frac{\tau_1 - \tau_2}{\sqrt{\frac{2}{3}(M^2 + \tau_1^2 + \tau_2^2)}}. \quad (29)$$

The topology of the extra two dimensional space depends on how the exponents of g_{xx} and g_{yy} components behave. If $(1 - p \pm q)/3 \neq 1$, the range of x and y could be unbounded and these two dimensional space are simply flat for a given ξ . Both directions or either one could also be compactified. On the other hand, if $(1 - p + q)/3 = 1$ or $(1 - p - q)/3 = 1$, x or y should be an angle coordinate centered at $\xi = K$, respectively, describing a stringlike object rather than a two brane object. In this case the solutions are described by the mass density and a single tension.

The case of $p = 1$ (*i.e.*, $\tau_1 = \tau_2 = -M/2$) with negative Λ corresponds to the well-known solutions appeared in Refs. [21–24],

$$ds^2 = -\frac{(-\Lambda)\xi^2}{3} \left(1 - \frac{K^3}{\xi^3}\right) dt^2 + \frac{d\xi^2}{\frac{(-\Lambda)\xi^2}{3} \left(1 - \frac{K^3}{\xi^3}\right)} + \xi^2(dx^2 + dy^2). \quad (30)$$

This solution is a black 2-brane metric⁶ in which the event horizon locates at $\xi = K$ and a spacelike curvature singularity exists at $\xi = 0$. The double Wick rotations of this solution for t and x coordinates corresponds to $p = -1/2$ and $q = \pm 3/2$ (*i.e.*, $(\tau_1, \tau_2) = (-2M, M)$ or $(M, -2M)$), which maybe called as four-dimensional static bubble.⁷ The metric for $q = 3/2$ becomes

$$ds^2 = -\frac{(-\Lambda)}{3} \xi^2 dt^2 + \frac{d\xi^2}{\frac{(-\Lambda)}{3} \xi^2 \left(1 - \frac{K^3}{\xi^3}\right)} + \xi^2 \left(1 - \frac{K^3}{\xi^3}\right) dx^2 + \xi^2 dy^2. \quad (31)$$

Here, the conical singularity at $\xi = K$ can be removed by assigning a suitable periodicity on the coordinate x (*e.g.*, $x \equiv x + \frac{4\pi}{K\sqrt{-3\Lambda}}$). Thus, the range of the coordinate ξ runs from K to infinity, the geometry is regular everywhere, and its Penrose diagram is given in Fig. 1. The geometrical properties for other cases will be analyzed in detail below.

The curvature squared for the metric becomes

$$R^{abcd}R_{abcd} = 4\Lambda^2 \left\{ 1 + \frac{1}{6}(1+p)(2p-1)^2 \left(\frac{K^6}{\xi^6} - 1 \right) + \frac{1}{6}(1-p)(2p+1)^2 \left[\left(\frac{\xi^3}{K^3} - 1 \right)^{-2} - 1 \right] \right\}. \quad (32)$$

Thus we see that curvature singularities occur at $\xi = 0$ and $\xi = K$ generically. However, the curvature singularity exists only at $\xi = 0$ for $p = 1$ and at $\xi = K$ for $p = -1$. These curvature singularities turn out to be mostly naked as will be explained in more detail below. The cases where the curvature singularity is enclosed by an event horizon are only for $p = 1$ with negative cosmological constant.

IV. PROPERTIES OF THE SOLUTIONS WITH NEGATIVE COSMOLOGICAL CONSTANT

Let us consider the case of negative cosmological constant ($\Lambda < 0$). It can be easily seen that all geometries in Eq. (25) become pure AdS₄ as $\xi \rightarrow \infty$. Thus the spatial infinity $\xi = \infty$ is the timelike AdS boundary.

⁶ As mentioned above, when one of the x and y coordinates is compactified properly, this metric becomes the black string discovered by Lemos [21].

⁷ This solution appeared as the so called AdS soliton in Ref. [23].

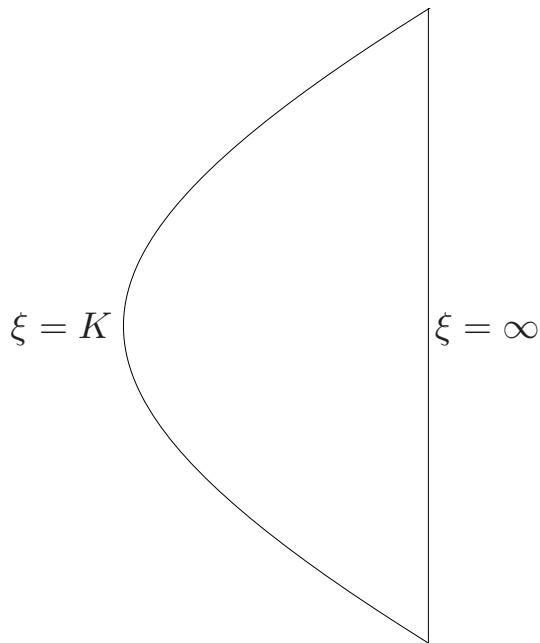


FIG. 1: Penrose diagram of the bubble solution

For the case of $p = -1$, the metric becomes

$$\begin{aligned}
 ds^2 = & -\frac{(-\Lambda)\xi^2}{3} \left(1 - \frac{K^3}{\xi^3}\right) \left|1 - \frac{K^3}{\xi^3}\right|^{-4/3} dt^2 + \frac{d\xi^2}{\frac{-\Lambda\xi^2}{3}(1 - K^3/\xi^3)} \\
 & + \xi^2 \left|1 - \frac{K^3}{\xi^3}\right|^{2/3} (dx^2 + dy^2).
 \end{aligned} \tag{33}$$

The Penrose diagram of this metric for the range $K \leq \xi < \infty$ is given in Fig. 2. Namely, the curvature singularity at $\xi = K$ surface is naked. Note that ξ plays the role of a time coordinate if $0 < \xi < K$. Since $\xi = 0$ surface is not a curvature singularity in this case, the spacetime will be extended even for negative ξ . Allowing negative values of ξ , we can see that the spacetime covered by $-\infty < \xi \leq K$ is, in fact, equivalent to the case of $p = 1$ with $0 \leq \xi < \infty$.⁸ Therefore, we do not need to consider the region with $\xi < K$ in this case. We point out that the mass density of this solution in Eq. (27) is negative and that two tension densities are equal and positive.

For the cases other than $p = \pm 1, \pm 1/2$, there are two curvature singularities at $\xi = 0$ and $\xi = K$ as can be seen in Eq. (32). The causal structure of this spacetime can be seen in Fig. 3. The singularity at $\xi = K$ divides the solution into two disjoint regions, namely, a time dependent one described by $\xi \in (0, K)$ and a static one by $\xi \in (K, \infty)$.

The static spacetime is asymptotically Anti-de Sitter and bounded by a naked singularity inside at $\xi = K$. The geometry in x and y spatial directions is interesting as we approach to the $\xi = K$ surface. Whether its spatial distance shrinks or expands depends on the exponents in Eq. (25). For a given mass density the sign of this exponents solely depends on the value of the tension density τ_i . For instance, the spatial distance in x shrinks to zero if $\tau_1 > -M/2$ for positive mass density and if $\tau_1 > M$ for negative mass density and *vice versa* for the spatial distance in y . For $\tau_1 = -M/2 (= \tau_2)$, both directions are regular. Note that $\tau_1 + \tau_2 (= -M)$ should be conserved for a given M . For $M > 0$, thus, if one of the spatial distance diverges, then the other should shrink as $\xi \rightarrow K$. For $M < 0$, on the other hand, both distances shrink if $M < \tau_1 < -2M$. The x direction still shrinks, but y direction diverges if $\tau_1 > -2M$. If $\tau_1 < M$, the x direction diverges while the y direction shrinks. For $\tau_1 = -2M$ (and so $\tau_2 = M$), the x direction shrinks but the y direction is regular. These properties are similar to the case of hypercylindrical spacetimes in the absence of cosmological constant in five dimensions.

⁸ We use the change of variable $\xi^3 - K^3 = -\xi'^3$.

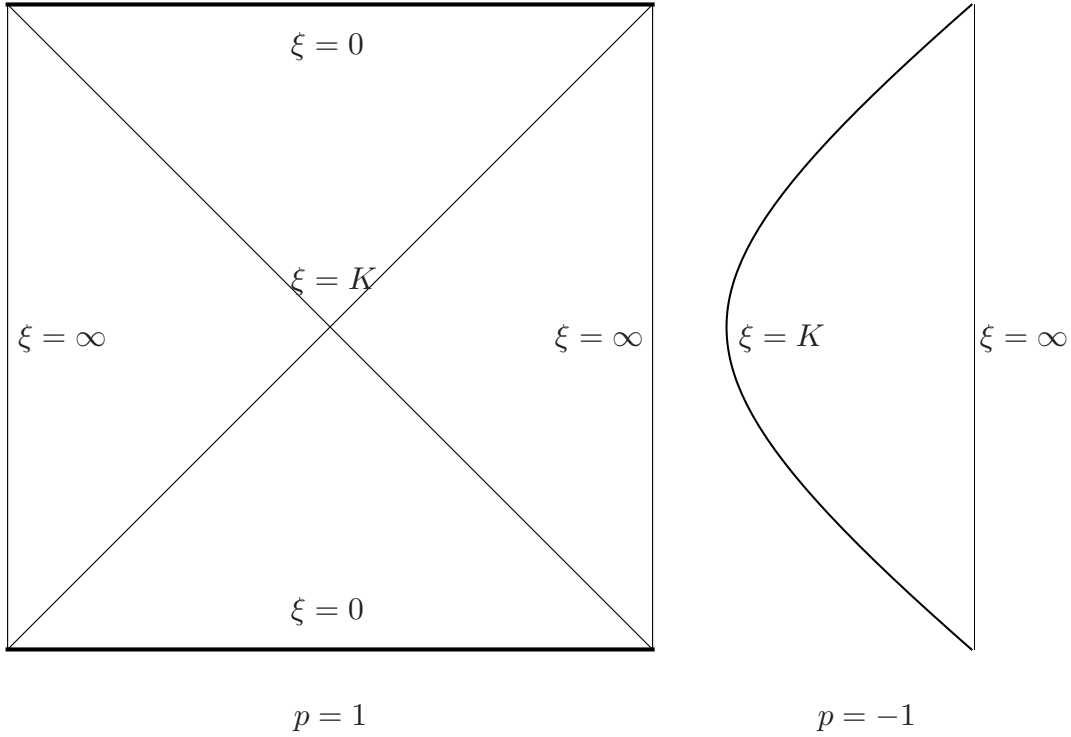


FIG. 2: Penrose diagram of the solution with $p = \pm 1$. Thick lines denote curvature singularities.

It is interesting to see the behaviors of spatial distances. At $t = \text{constant}$ and $\xi = \text{constant}$, the spatial lengths in x and y directions with unit coordinate distance are

$$L_x \equiv \sqrt{g_{xx}} = \xi \left(1 - \frac{K^3}{\xi^3}\right)^{\frac{1-p+q}{6}} \quad \text{and} \quad L_y \equiv \sqrt{g_{yy}} = \xi \left(1 - \frac{K^3}{\xi^3}\right)^{\frac{1-p-q}{6}}, \quad (34)$$

respectively. As $\xi \rightarrow K$, they shrink to zero or finite size, or diverge to infinity depending on the values of p and q . The total volume becomes

$$V_{xy} \equiv L_x L_y = \xi^2 \left(1 - \frac{K^3}{\xi^3}\right)^{\frac{1-p}{3}}. \quad (35)$$

Thus we see that it always shrinks to zero as $\xi \rightarrow K$ except for the case of $p = 1$ since $p \leq 1$ in Eq. (26).

The time dependent spacetime for $0 \leq \xi < K$ is interesting. Since ξ plays the role of time, we define a cosmological proper time as follows,

$$\bar{t} = \int_0^\xi \sqrt{g_{\xi'\xi'}} d\xi' = \frac{2}{\sqrt{3|\Lambda|}} \left[\frac{\pi}{2} - \tan^{-1} \left(\sqrt{\frac{K^3}{\xi^3} - 1} \right) \right]. \quad (36)$$

Thus, $\xi = 0$ and K corresponds to $\bar{t} = 0$ and $\pi/\sqrt{3|\Lambda|}$, respectively. In terms of the proper time and $t \rightarrow z$ the metric can be rewritten as

$$ds^2 = -d\bar{t}^2 + K^2 \left(\frac{\sin 2\beta\bar{t}}{2\beta} \right)^{2/3} \left[\left(\frac{\tan \beta\bar{t}}{\beta} \right)^{\frac{2(p-q)}{3}} dx^2 + \left(\frac{\tan \beta\bar{t}}{\beta} \right)^{\frac{2(q+p)}{3}} dy^2 + \left(\frac{\tan \beta\bar{t}}{\beta} \right)^{-\frac{4p}{3}} dz^2 \right]. \quad (37)$$

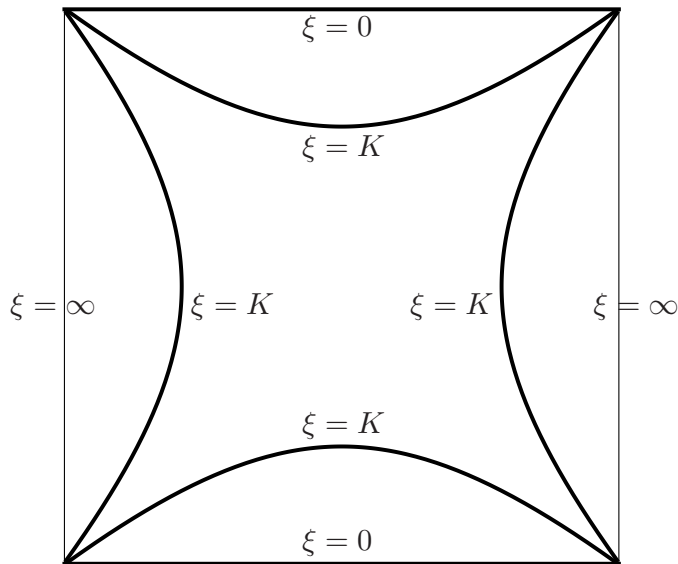


FIG. 3: Penrose diagram of the solution with negative cosmological constant with $p \neq \pm 1, \pm 1/2$. Thick lines denote curvature singularities.

Here $\beta = \sqrt{3|\Lambda|}/2$ and the coordinates x , y and z are scaled suitably. Note that this metric becomes exactly the Kasner solution as $\Lambda \rightarrow 0$ [25]. Therefore, this class of solutions we found is the extension of Kasner solution in the presence of negative cosmological constant. We point out that this AdS-Kasner solution starts from an initial singularity, but ends up with a final singularity, contrary to the case of Kasner solutions, in a finite time $(\pi/2\beta)$. Furthermore, the total spatial volume shrinks to zero at the future infinity due to the oscillating overall scale factor.

V. PROPERTIES OF THE SOLUTIONS WITH POSITIVE COSMOLOGICAL CONSTANT

In this section, we analyze the properties for the case of positive cosmological constant. For $\Lambda > 0$, we reexpress the metric in (25) as

$$\begin{aligned}
 ds^2 = & -\frac{\Lambda\xi^2}{3} \left(\frac{K^3}{\xi^3} - 1 \right) \left| \frac{K^3}{\xi^3} - 1 \right|^{\frac{2(p-1)}{3}} dt^2 + \frac{d\xi^2}{\frac{\Lambda\xi^2}{3} \left(\frac{K^3}{\xi^3} - 1 \right)} \\
 & + \xi^2 \left| \frac{K^3}{\xi^3} - 1 \right|^{\frac{1-p+q}{3}} dx^2 + \xi^2 \left| \frac{K^3}{\xi^3} - 1 \right|^{\frac{1-p-q}{3}} dy^2.
 \end{aligned} \tag{38}$$

First of all, we see that the metric with $K = 0$ corresponds to the pure de Sitter spacetime where the ξ -coordinate plays the role of time.⁹ If $K \neq 0$, we can set it to be ± 1 by rescaling coordinates. As described below Eq. (28) for the case of $\Lambda < 0$, we can also restrict the range of ξ and K as $0 \leq \xi < \infty$ and $K \geq 0$, respectively.

The causal structures of the spacetimes in this case can be easily understood since the only difference from the case of $\Lambda < 0$ is that the signs of g_{tt} and $g_{\xi\xi}$ are flipped. Consequently, the role of t and ξ coordinates are swapped and so the Penrose diagrams are same as those of the $\Lambda < 0$ case, but tilted by 90 degrees.

The case of $p = 1$ is shown in Fig. 4. The curvature singularity locates at $\xi = 0$, which is naked. The spacetime region in $0 < \xi < K$ is static, but the region in $K < \xi < \infty$ becomes time-dependent, describing a Kasner spacetime. We point out that the spacetime with $p = -1/2$ is indeed the same as that with $p = 1$. In the case of $\Lambda < 0$, both

⁹ $\xi = e^{\sqrt{\Lambda/3}t}$ and $t = z/\sqrt{\Lambda/3}$ give the coordinate chart which covers only half the full de Sitter spacetime.

metrics are related through the double Wick rotations, $\sqrt{-\Lambda/3}t \leftrightarrow ix$, while in the case of $\Lambda > 0$ the two metrics are simply related by renaming $\sqrt{\Lambda/3}t \leftrightarrow x$.

The Penrose diagram for the case of $p = -1$ is given by Fig. 5. The metric in the range of $\xi < K$ turns out to be equivalent to the metric with $p = 1$ as in the case of $\Lambda < 0$. The left hand side in the Penrose diagram describes a cosmological spacetime starting from an initial singularity and ending up with a de Sitter spacetime. The right hand side corresponds to a cosmological spacetime starting from a de Sitter metric which collapses to a singularity in the future.

Finally, the causal structures for the cases other than $p = \pm 1, \pm 1/2$ can be seen in the Penrose diagram in Fig. 6. Explicit form of the metric describing the $K \leq \xi < \infty$ is given by

$$ds^2 = -d\bar{t}^2 + K^2 \left(\frac{\sinh 2\beta\bar{t}}{2\beta} \right)^{2/3} \left[\left(\frac{\tanh \beta\bar{t}}{\beta} \right)^{\frac{2(p-q)}{3}} dx^2 + \left(\frac{\tanh \beta\bar{t}}{\beta} \right)^{\frac{2(q+p)}{3}} dy^2 + \left(\frac{\tanh \beta\bar{t}}{\beta} \right)^{-\frac{4p}{3}} dz^2 \right]. \quad (39)$$

Here $\beta = \sqrt{3\Lambda}/2$ and the coordinates x, y and z are scaled suitably. The comoving time \bar{t} is related to the ξ coordinate by

$$\bar{t} = \int_K^\xi \sqrt{|g_{\xi'\xi'}|} d\xi' = \frac{1}{\beta} \log \left(\sqrt{\frac{\xi^3}{K^3} + 1} + \sqrt{\frac{\xi^3}{K^3} - 1} \right). \quad (40)$$

This class of solutions is indeed the extension of the Kasner solution in the presence of positive cosmological constant.

VI. SUMMARY AND DISCUSSIONS

We have investigated string or branelike solutions of four-dimensional vacuum Einstein equations in the presence of cosmological constant. In the case of warped stringlike solutions, we have shown that the BTZ black string is the only solution in the form of metrics given in Eq. (6).¹⁰ The general solutions for the metric form (20) in the

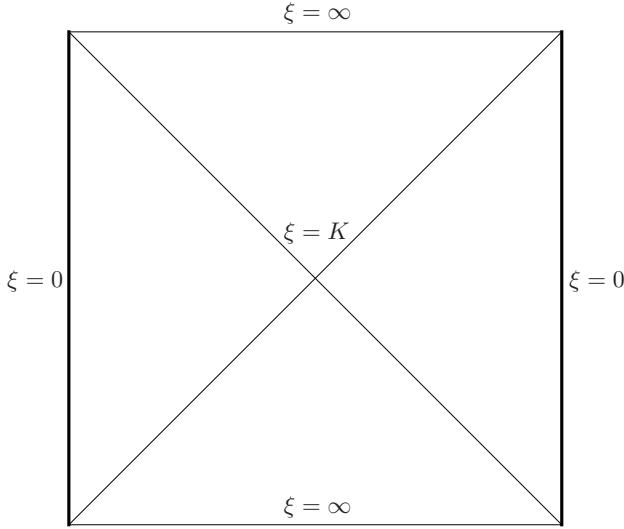


FIG. 4: Penrose diagram of the solution with $p = 1, -1/2$ in the case of $\Lambda > 0$.

¹⁰ We mention that a warped BTZ black 2-brane solution is recently found even in five-dimensional gravity with a Gauss-Bonnet term [26].



FIG. 5: Penrose diagram of the solution with $p = -1$ in the case of $\Lambda > 0$.

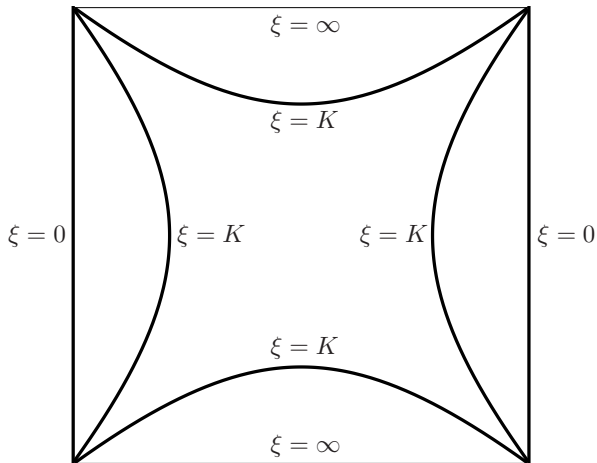


FIG. 6: Penrose diagram of the solution with $p \neq \pm 1, \pm 1/2$ in the case of $\Lambda > 0$.

case of non-warped stringlike configurations are given by Eq. (25). When the cosmological constant is negative, these non-warped stringlike configurations are characterized by ADM mass (M) and tension densities (τ_1, τ_2) with a constraint of $\tau_1 + \tau_2 = -M$. The case of equal tension densities (*i.e.*, $\tau_1 = \tau_2 = -M/2$) turns out to be the well known black 2-brane metric (30). The case that one of the tension densities is twice of the other with opposite sign (*e.g.*, $\tau_1 = -\tau_2/2 = M$) is the AdS soliton spacetime (31) whose geometry is regular everywhere. The cases other than these two possess a naked singularity. The time-dependent solutions shown in Fig. 3 can be regarded as the AdS extension of the well-known Kasner solutions. For the positive cosmological constant, on the other hand, the time-dependent metric turns out to be the Kasner-de Sitter spacetime [15] and the time independent solutions possess naked singularities.

It is worth considering the stability of solutions we described. In the absence of the cosmological constant, it was shown [27] that both the black string and the Kaluza-Klein bubble solutions in spacetime dimensions higher than or equal to five are unstable due to the Gregory-Laflamme instability [10]. One may expect that the black 2-brane metric in Eq. (30) is also unstable due to the Gregory-Laflamme instability occurring in generic brane or stringlike spacetimes. According to the Gubser-Mitra conjecture [28], however, this expectation might be wrong. The conjecture states that a black string or brane spacetime is stable if and only if it is thermodynamically stable. Being considered as a thermodynamic system, the black 2-brane in Eq. (30) gives $M = (8\pi^2 k_B^3 / 3G|\Lambda|^2 \hbar^3) T^3$ where the black hole temperature $T = (\hbar|\Lambda|/4\pi k_B)K$ is used. Thus the heat capacity for the black hole is always positive, indicating classical stability of this spacetime through the Gubser-Mitra conjecture. Indeed the classical linearized stability has been studied for this background spacetime, showing the absence of instability [29]. In addition, Chen, Schleich and Witt [30] showed that the warped AdS black string is also stable.

The static bubble solution in Eq. (31) turns out to be stable as well. This solution is a special case of $p = 2$ in Ref. [23], where Horowitz and Myers proposed the so-called new positive energy theorem. According to it, this metric is the ground state among all spacetime configurations having the same boundary behaviors. Thus this spacetime is presumably stable under perturbations. Actually, the similar solution in the case of $p = 3$ in Ref. [23], which is called as the AdS soliton spacetime in 5-dimensions, is shown to be stable under small perturbations [31].

We have seen that solutions for the class of metric form (20) are mostly singular except for several regular solutions for specific values of parameters. What the implications of this fact would be? One may simply discard all these singular solutions. Or, since these metrics are static vacuum solutions, they might be the end states of matter collapses. Note that the physical parameters characterizing the initial matter distribution in such processes could

be arbitrary as long as they satisfy suitable physical conditions. Therefore, the appearance of many naked singular solutions through such dynamical processes could be an example of violating the cosmic censorship conjecture.

On the other hand, however, there might be a third scenario. For the spacetime solutions having naked singularities at the present consideration, their stability behaviors are not known as yet explicitly. We speculate though that these four dimensional spacetimes are unstable, except for the case that the singularities are located at the past (future) infinity. In the case of 5-dimensions without having cosmological constant, these singular solutions are known to be unstable under small perturbations [5]. If this speculation is true, such instability probably drives the matter collapse to some stable states before they reach to singular states, avoiding a violation of the cosmic censorship conjecture. Actually, a recent numerical study showed that, although there exists an electro-vacuum solution containing a naked singularity, dynamical collapses of charged scalar fields in $(2+1)$ anti-de Sitter background do not produce such a singular solution [32].

In the presence of positive cosmological constant, it was shown that the Kasner-de Sitter spacetime is unstable because the tensor mode perturbations increase indefinitely at the initial anisotropic stage [15]. However, it was also shown that, for the spacetime with $p = 1$ is stable under the perturbation [33].

Acknowledgments

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