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We consider an evaporating Schwarzschild black hole in a framework in which the spectral dimension of spacetime varies continuously from four at large distances to a number smaller than three at small distances, as suggested by various approaches to quantum gravity. We demonstrate that the evaporation stops when the horizon radius reaches a scale at which spacetime becomes effectively 3-dimensional, and argue that an observer remaining outside the horizon cannot probe the properties of the black hole at smaller scales. This result is universal in the sense that it does not depend on the details of the effective dimension as a function of the diffusion time. Observers falling into the black hole can resolve smaller scales, as can external observers in the presence of a cosmological constant. Even in these cases, though, we obtain an absolute bound $D \geq 2$ on the effective dimension that can be seen in any such attempt to measure the properties of the black hole.

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I. INTRODUCTION

General relativity describes spacetime as a smooth d-dimensional manifold. But while this picture has proven remarkably successful, it is quite plausible that it will break down at very small scales. A quantum theory of gravity must be, in some sense, a theory of the quantization of spacetime, and there is no reason to expect that a smooth classical description will hold to all scales. A central task of quantum gravity is to investigate alternative small scale descriptions.

Even without a smooth manifold structure, it is often possible to define an effective dimension of spacetime. The spectral dimension [1, 2], for instance, is determined by the rate of a diffusion process, and exists for any space on which a random walk can be defined. Such a definition seems tailor-made for thermodynamic applications, such as the process of black hole evaporation considered in the present paper.

Although it is risky to make too strong a claim without a definitive quantum theory of gravity, evidence from a number of different approaches suggests that the spectral dimension and similar generalized dimensions flow from four at large distance scales to two near the Planck scale [3, 4]. In particular, Causal Dynamical Triangulations (CDT) — a Lorentzian lattice approach to the gravitational path integral — yields a spectral dimension of spacetime, determined numerically, of the form [1]

$$\text{CDT:} \qquad D_{\rm IR} = 4.0 \pm 0.1 \qquad D_{\rm UV} = 1.80 \pm 0.25\,. \tag{1}$$

Here $D_{\rm IR}$ is the spectral dimension in the limit of infinite diffusion time, corresponding to an effective dimension at very large distances, while $D_{\rm UV}$ is the spectral dimension for short diffusion times, giving an effective dimension at very small distances.

It is evident that the spacetime dimension at large scales is compatible with four, but the dimension at small scales is smaller than four at greater than 5σ significance. Moreover, (1) is consistent with the suggestion that the small scale structure is effectively 2-dimensional. A reasonably good fit, valid for arbitrary values of the diffusion time σ, is [1]

$$D_{\rm spec}(\sigma) = a - \frac{b}{c+\sigma} \tag{2}$$

where $D_{\rm IR} = a$ and $D_{\rm UV} = a - b/c$. The constants b and c can be rescaled arbitrarily through changes of the lattice spacing, but their ratio and the constant a are both universal.

On the other hand, a recent study using an alternative lattice approach known as Euclidean Dynamical Triangulations (EDT) leads to a rather different result for the small scale dimension [5]:

$$\text{EDT:} \qquad D_{\rm IR} = 4.0 \pm 0.3 \qquad D_{\rm UV} = 1.46 \pm 0.06 \tag{3}$$

In fact, the result (3) may suggest $D_{\rm UV} = 3/2$, a result that is also compatible with (1). Amusingly, this is precisely the value for which the Bekenstein–Hawking entropy of a d-dimensional Schwarzschild black hole,

$$S_{\rm BH} \sim E^{(d-2)/(d-3)}\,, \tag{4}$$

coincides with the entropy of a d-dimensional CFT,

$$S_{\rm CFT} \sim E^{(d-1)/d}\,. \tag{5}$$

It is thus of interest to see what an evaporating Schwarzschild black hole has to say about this issue.

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There are various physical scenarios of interest. For instance, an observer could intend to probe microscopic distances with some high-energy scattering experiment. If the energy deposited in such an experiment gets concentrated in a sufficiently small region, then a black hole is created. However, we shall describe a different situation, in which the black hole exists already before the experiment is performed, so that we do not have to deal with the rather complicated process of black hole formation. We locate an observer outside a black hole, which for simplicity we assume to be spherically symmetric. She then performs some experiment permitting her to probe the scale of spherical shells concentrically surrounding the black hole. Heuristically, the black hole horizon hides the interior — and the related short-distance physics — from an outside observer. But as the black hole evaporates, its horizon shrinks, allowing the observer to probe smaller and smaller distances,[1] thereby gleaning some information about the effective dimension at small scales. An observer desperate for information about short-distance physics might even throw herself into the black hole to resolve the effective dimension at even smaller scales. The question we shall address is whether the dynamics puts any limit on either of these processes.

II. DILATON BLACK HOLE

We use 2-dimensional dilaton gravity to describe the (Euclidean) d-dimensional Schwarzschild black hole in various dimensions (see, e.g., [6]). This description has several advantages: it is simple; it captures the full classical and thermodynamical content of the theory [7]; and it allows a straightforward analytic continuation to arbitrary (even fractal or negative) dimensions.

The dilaton gravity action is given by

$$I = -\frac{1}{2G_2}\int \mathrm{d}^2x\sqrt{g}\left[XR - U(X)(\partial X)^2 - 2V(X)\right] + I_{\mathrm{b}}\,, \tag{6}$$

with gravitational coupling constant G_2 and a known boundary action I_{b} that is irrelevant to the current discussion. The dilaton has a higher-dimensional interpretation as the surface area; that is, $X(t,r)$ is the area of the $(d-2)$-sphere at fixed t and r (the orbit of the Killing vectors responsible for spherical symmetry).

The three terms in the bracket also have straightforward higher-dimensional meanings. Each represents a contribution to the d-dimensional Ricci scalar. The first term describes the intrinsic curvature of the 2-dimensional spacetime. The third term describes the intrinsic curvature of the $(d-2)$-sphere. The second term gives the contribution to curvature arising from the change of the area of the $(d-2)$-sphere as a function of time and radius. This is the term we shall modify by hand to accommodate an effective dimension that changes with the distance from the black hole.

For a d-dimensional Schwarzschild black hole, the potential $U(X)$ is given by

$$U(X) = -\frac{1}{X}\,\frac{d-3}{d-2}\,. \tag{7}$$

It is useful to define functions

$$Q(X) := Q_0 + \int^X \mathrm{d}X'\,U(X')\,, \tag{8}$$

$$w(X) := w_0 - 2\int^X \mathrm{d}X'\,e^{Q(X')}V(X')\,, \tag{9}$$

with some arbitrary integration constants Q_0 and w_0. The potential $V(X)$ can be obtained from the requirement that the model (6) have a flat ground state [6]:

$$V(X) \propto e^{-2Q(X)}\,U(X) \tag{10}$$

This requirement then yields

$$V(X) \propto X^{(d-4)/(d-2)}\,, \tag{11}$$

with a proportionality constant that sets the physical length scale.

The classical solutions of the field equations coming from the action (6) are then given by

$$X = X(r) \qquad \text{with } \partial_r X = e^{-Q(X)} \tag{12}$$

$$\mathrm{d}s^2 = \xi(r)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{\xi(r)} \tag{13}$$

$$\text{with } \xi(X) = w(X)e^{Q(X)}\Big(1 - \frac{4M}{w(X)}\Big)\,.$$

They are parametrized by a single constant of motion, the black hole mass M. The flat ground state property (10) implies $e^{Q(X)}w(X) = \text{const.}$, so that the Killing norm ξ is constant for vanishing black hole mass.

III. VARYING EFFECTIVE DIMENSION AND BLACK HOLE EVAPORATION

For fixed dimension d, the solution (12) with the potential (7) gives a dilaton

$$X \sim r^{d-2}\,. \tag{14}$$

As we show in appendix A, the spectral dimension determined from the corresponding dimensionally reduced d'Alembertian is d. This holds even if d is not an integer. A varying spectral dimension might thus reasonably correspond to a varying d in (7).

Let us suppose that over some relevant range of scales, the effective dimension is a strictly monotonic function of the diffusion time σ. This behavior occurs in both the CDT and EDT simulations described in the introduction; the CDT fit (2) is an example of such a functional dependence. We now make our two key working assumptions:

[1] By "distance" in this paper we always mean the radius of a $(d-2)$-sphere enveloping the center of the d-dimensional black hole.

1. The diffusion time σ — specifically, the diffusion time necessary to capture information about the transverse space of constant r and t — is itself a strictly monotonic function of the dilaton X.

2. The potential (7) remains valid even when the dimension depends on the scale; that is,

$$U(X) = -\frac{1}{X}\,\frac{D(X)-3}{D(X)-2}\,. \tag{15}$$

The first assumption is motivated by the observation that σ and X both determine the scale: small diffusion times and small values of the dilaton field both correspond to small distances, while large σ and X both correspond to large distances. The second comes from the interpretation of the second term in the action (6), as described in the paragraph below that equation, and from the results of appendix A. We check this assumption for a particular potential in appendix B, and show that the $D(X)$ in (15) is in good agreement with the spectral dimension over the whole range of X, with a maximal deviation of 15% (see Fig. 2 in appendix B).

For the sake of concreteness, we shall assume that the fit (2) is valid and that the diffusion time is a monotonic increasing function f of the dilaton X:

$$D(X) = a - \frac{b}{c+f(X)} \tag{16}$$

In appendix B, we provide a simple toy model with linear f, which allows us to elucidate certain aspects of the black hole evaporation. Our main conclusions are independent of these specific choices, however; all that matters is that $D(X)$ is some strictly monotonic function of X that goes to four for $X\to\infty$ and to some value smaller than three for $X\to 0$.

Again we determine $V(X)$ from the flat ground state requirement (10). This choice ensures that the theory allows 2-dimensional flat spacetime as solution. These choices above should be considered as working assumptions; others are conceivable. However, we believe these assumptions are sufficiently well-motivated to warrant a study of their consequences.

We make now our key observation. The function $V(X)$ vanishes if $U(X)$ vanishes, which happens precisely for $D(X)=3$, regardless of the detailed properties of the effective dimension as a function of the dilaton. The vanishing of V implies, in turn, that w' is zero. Since surface gravity κ is given by [7, 8]

$$\kappa = \frac{1}{2}\,w'\Big|_{X=X_h} \propto V(X_h) \tag{17}$$

where X_h is the value of the dilaton evaluated at the horizon, it follows that the black hole is extremal if $V(X_h)=0$. Consequently, the black hole stops evaporating once the horizon drops to a scale for which $D(X_h)=3$.

As a byproduct, we also learn that specific heat must turn positive before the black hole horizon drops to the critical size at which $D(X_h)=3$. To see this, note first that the temperature increases monotonically as long as the effective dimension $D(X)$ is sufficiently close to four. In that region we recover the standard result that Schwarzschild black holes have negative specific heat. Since the temperature drops to zero smoothly as $D(X_h)\to 3$, it must have a maximum at some value X_h^c, with $3 < D(X_h^c) < 4$. In the region $X_h < X_h^c$, the specific heat is therefore positive. Such a behavior might be expected on general grounds for quantum-corrected Schwarzschild black holes.

A further byproduct is that a curvature singularity necessarily appears behind the horizon. This can be shown as follows. The 2-dimensional Ricci scalar is [6]

$$\begin{aligned} R &= 4Me^{-Q(X)}U'(X) \\ &\propto \frac{(D(X)-2)(D(X)-3)-XD'(X)}{X^2(D(X)-2)^2}\,. \end{aligned} \tag{18}$$

The curvature diverges at $X=0$ and $D(X)=2$. Both loci are always within the black hole region, according to the results above. While the existence of a curvature singularity might have been anticipated on general grounds from singularity theorems, it is not clear that they apply to a situation in which the effective dimension varies.

Interestingly, the result (18) implies that not even an observer falling into a black hole is able to resolve scales of the surface area corresponding to an effective dimension smaller than two. Thus, even if the effective dimension near $X=0$ were given by, say, $D_{\mathrm{UV}}=3/2$, no observer would encounter this value before reaching the singularity.[2] Instead, we establish the result that no observer, inside or outside the horizon, can see a value $D(X)<2$.

IV. ADDING A COSMOLOGICAL CONSTANT

In the derivation above, the assumption (10) of a flat ground state was crucial. Let us relax this assumption to allow for de Sitter or anti-de Sitter ground states. This is of interest in part because our present Universe appears to have a positive cosmological constant [9, 10]. On more theoretical grounds, 3-dimensional general relativity has

[2] Recall that $D(X)$ is essentially a spectral dimension measured with a diffusion time determined by the area of the space of constant r and t. One could imagine a *local* observation made by a freely falling observer at a much smaller scale (with the caveat that such a process might create another black hole). As mentioned at the end of the introduction, such an observation would not be easily described in the framework of spherically symmetric dimensional reduction, and our arguments do not say anything about the possible outcome. But such a measurement would also not capture the properties of the black hole, which are our main interest here.

no black hole solutions unless a negative cosmological constant is present [11, 12], and one might worry that the critical dimension of three derived above may merely reflect this fact.

Given a dilaton gravity model (6) obtained from the dimensional reduction of D-dimensional Einstein gravity, we can add a cosmological constant by a simple shift of the potential $V(X)$ [6, 7],

$$V_\Lambda(X) = -\lambda^2 e^{-2Q(X)}\, U(X) + \Lambda N(X) X\,. \tag{19}$$

Here λ is a dimensionful constant that sets the physical scale, Λ is the cosmological constant, and $N(X) > 0$ is a D-dependent normalization. Some standard choices are $N(X) = D(X)(D(X)-1)$ and $N(X) = 1$. Our conclusions below are independent of the precise choice of $N(X)$, as long as it contains no zeros or singularities for $D \geq 2$. We assume that this is the case. With this change, Hawking evaporation no longer stops at $D(X) = 3$, but continues until some other value of D, which is determined by the condition $V_\Lambda(X) = 0$, i.e.,

$$\frac{D(X)-3}{D(X)-2} = \frac{\Lambda}{\lambda^2}\, X^2 N(X) e^{2Q(X)}\,. \tag{20}$$

The key observation is that the right hand side of (20) is always positive (negative) for a positive (negative) cosmological constant. This means that (20) must have a solution with $D(X) > 3$ for $\Lambda > 0$, and with $D(X) < 3$ for $\Lambda < 0$.

Moreover, it is clear that for $\Lambda < 0$ a solution must exist with $D(X) > 2$. Indeed, for any finite negative value of Λ the right-hand side is bounded from below, while the left-hand side is unbounded from below as $D \to 2$ from above. Hence if one plots the two sides as functions of X, the curves must intersect at some $X = X_c$ such that $D(X_c)$ lies between two and three.

Thus, for $\Lambda > 0$, Hawking evaporation stops at some critical value X^c with $D(X^c) > 3$, while for $\Lambda < 0$, the evaporation stops at a critical value with $2 < D(X^c) < 3$. Clearly, if Λ is tiny, the critical dimension at which evaporation stops is very close to three. If Λ is large and negative, on the other hand, an external observer can resolve distance scales small enough to correspond to a dimension smaller than three. It remains true, however, that $D = 2$ is an absolute bound for any observer, inside or outside the horizon, regardless of the value of the cosmological constant. This bound is insensitive to the details of any of our choices — the relationship between spectral dimension and diffusion time, the relationship between the diffusion time and the dilaton field, the normalization of the cosmological constant — as long as these respect the plausible monotonicity properties we introduced above.

We conclude that no observer — outside or inside the black hole — is capable of resolving (radial) distances that correspond to an effective dimension smaller than two. These results provide independent evidence in favor of the proposal [3] that quantum gravity should be effectively 2-dimensional at small distance scales.

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Appendix A: Dimensional reduction and the spectral dimension

Consider any space in which a diffusion process can be defined. Such a process is characterized by a heat kernel $K(\mathbf{x}', \mathbf{x}; \sigma)$. The spectral dimension is the dimension measured by the rate of diffusion [1, 2],

$$D_{\text{spec}}(\sigma) = -2\sigma\, \frac{\mathrm{d}}{\mathrm{d}\sigma}\, \ln K(\mathbf{x}, \mathbf{x}; \sigma)\,. \tag{A1}$$

For a flat d-dimensional space, the heat kernel is [13]

$$K(\mathbf{x}', \mathbf{x}; \sigma) = (4\pi\sigma)^{-d/2} \exp\left\{-\frac{|\mathbf{x}' - \mathbf{x}|^2}{4\sigma}\right\} \tag{A2}$$

and it is easily checked that $D_{\text{spec}} = d$.

In the dimensional reduction we have considered here, a d-dimensional spacetime is treated as if it had only two dimensions. If the lower dimensional model reflects the true physics, though, it must somehow capture the full spectral dimension. To see how this works, consider first the case of a flat spacetime, with a dimensionally reduced (Euclidean) d'Alembertian

$$\Delta = \partial_\tau^2 + \partial_r^2 + (\partial_r \ln X)\partial_r + \frac{1}{r^2}\tilde{\Delta}_{d-2} \tag{A3}$$

where the dilaton is the surface area, $X \sim r^{d-2}$, and $\tilde{\Delta}_{d-2}$ is the Laplacian on the $(d-2)$-sphere. The first two terms yield the intrinsic 2-dimensional d'Alembertian, while the last two terms capture information about the remaining $d-2$ dimensions. We can similarly split the dimension d into two parts, one corresponding to the 2-dimensional d'Alembertian and one coming from the dimensional dependence of the dilaton field on r:

$$d = 2 + \frac{\mathrm{d}(\ln X)}{\mathrm{d}(\ln r)} \tag{A4}$$

We shall now show that as long as the diffusion time σ is not too small, this is a good approximation of the spectral dimension D_{spec}, even if d is not an integer.

We are interested in a spherical reduction, in which only the zero angular momentum modes are present. The eigenvalues of $\tilde{\Delta}_{d-2}$ are $\ell(\ell + d - 3)$, so for these $\ell = 0$ modes, the last term in (A3) drops out. The operator (A3) is hermitian with respect to the integration measure

$$\mathrm{d}\mu = \mathrm{d}\tau\, \mathrm{d}r\, r^{d-2} \tag{A5}$$

and has nonsingular orthonormal eigenfunctions

$$f_{\omega k}(t, r) = \sqrt{\frac{k}{2\pi}}\, r^{\frac{3-d}{2}} e^{i\omega t} J_\nu(kr) \quad \text{with} \quad \nu = \tfrac{d-3}{2} \tag{A6}$$

with eigenvalues $-\omega^2 - k^2$. We can use these to evaluate the heat kernel for Δ (again with $\ell = 0$):

$$K_0(r', t', r, t; \sigma) = \int \mathrm{d}\omega\, \mathrm{d}k e^{-\sigma(k^2+\omega^2)} f^*_{\omega k}(t', r') f_{\omega k}(t, r)$$
$$= \frac{1}{\sqrt{2\pi\sigma}} (rr')^{\frac{3-d}{2}} e^{-(t'-t)^2/4\sigma} \frac{1}{2\sigma} e^{-(r'^2+r^2)/4\sigma} I_\nu\left(\frac{r'r}{2\sigma}\right) \tag{A7}$$

It may be checked that this is equivalent to the angular average of the flat space heat kernel (A2) over a $(d-2)$-sphere of fixed r and τ.

To obtain a spectral dimension, we need a logarithmic derivative of this quantity. For small σ, the argument of the modified Bessel function is large, and we can use the asymptotic behavior $I_\nu(z) \sim e^z/\sqrt{2\pi z}$. It is then easy to check that $D_{\mathrm{spec}} \sim 2$: in this limit, the heat kernel does not see the higher dimensional space. This means that the diffusion time must not be too small (as compared to $X^{2/(d-2)}$), since otherwise the s-waves are not sensitive to the higher dimensions, but merely feel the presence of the time and radial coordinates. This observation provides an independent motivation for our first working assumption in section III. For large σ, on the other hand, we can exploit the asymptotic behavior of the modified Bessel function at small argument,

$$I_\nu(z) \sim \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,, \tag{A8}$$

to approximate the heat kernel. We find

$$K_0(r, t, r, t; \sigma) \sim \sigma^{-\frac{d}{2}} e^{-r^2/2\sigma} \tag{A9}$$

giving the desired spectral dimension

$$D_{\mathrm{spec}} = d\,. \tag{A10}$$

For $d = 4$, the heat kernel (A7) can be expressed in terms of elementary functions, and the crossover between the "small σ" and "large σ" regimes can be investigated analytically. This crossover happens quite rapidly, at relatively small values: if we define a dimensionless variable $s = \sqrt{\sigma}/r$, we find that D_{spec} rises from very nearly two at $s = 0.4$ to very nearly four at $s = 4$ (see Fig. 1).

Now let us consider the generalization to non-integer dimension. The dimension entered our derivation only in the behavior of the dilaton, $X \sim r^{d-2}$, and in the r dependence of the potential term in (A3). But nothing in the derivation required that d be an integer. Hence if the dilaton behaves as $X \sim r^{d-2}$ and the potential term has a lowest eigenvalue of zero, the spectral dimension is d, whether d is an integer or not. Equivalently, $D_{\mathrm{spec}} = 2 + \mathrm{d}(\ln X)/\mathrm{d}(\ln r)$, just as suggested in (A4). For a spectral dimension that varies slowly with scale — more precisely, one that varies slowly compared to the low-lying eigenfunctions of (A3) — this should remain a good approximation. As we show in appendix B below, this certainly seems to be the case in a simple, explicit model, in which the spectral dimension calculated by (A4) and the effective dimension in the dilaton potential agree well over a very large range of values of X.

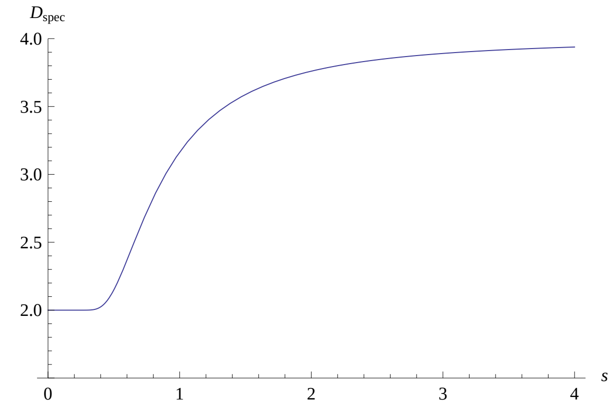

FIG. 1: Spectral dimension D_{spec} for $d = 4$ as function of s

Appendix B: Paper-and-pencil example

A simple dilaton gravity model that realizes the features discussed in the main text can be obtained by choosing

$$D(X) = 4 - \frac{5c/2}{c+X} \tag{B1}$$

for the effective dimension in (15). By construction, we have $D_{\mathrm{IR}} = 4$ and $D_{\mathrm{UV}} = 3/2$, as in the EDT result (3). The action is then given by (6) with potentials

$$U(X) = -\frac{X - 3c/2}{2X^2 - cX/2} \qquad V(X) = -X^5 \frac{X - 3c/2}{(X - c/4)^6}\,, \tag{B2}$$

and a convenient choice of integration constants yields

$$w(X) = \exp\{-Q(X)\} = 2X^3(X - c/4)^{-5/2}\,. \tag{B3}$$

The deformed Schwarzschild black hole is then described by the line element (13) with Killing norm

$$\xi(X) = 1 - \frac{2M\,(X - c/4)^{5/2}}{X^3}\,. \tag{B4}$$

The dilaton field evaluated at the horizon, X_h, can be expressed in terms of the black hole mass M by solving $\xi(X_h) = 0$ numerically. Applying the general results of [7, 8], we obtain a Hawking temperature

$$T_{\mathrm{H}} = \left.\frac{w'(X)}{4\pi}\right|_{X=X_h} = \frac{1}{4\pi} \frac{X_h^3 - 3cX_h^2/2}{(X_h - c/4)^{7/2}} \tag{B5}$$
$$= \frac{1}{8\pi M}\left(1 + \frac{5c}{8M^2} + \mathcal{O}(c^2/M^4)\right)$$

The Bekenstein–Hawking entropy is

$$S_{\rm BH} = \frac{2\pi X_h}{G_2} = 4\pi M^2 \left(1 - \frac{5c}{16M^2} + \mathcal{O}(c^2/M^4)\right) , \tag{B6}$$

where we set $G_2 = 2$ in order to obtain the entropy in 4-dimensional Planck units where $G_N = 1$. That this is the correct value of the 2-dimensional gravitational coupling constant G_2 can be seen, e.g., from Eqs. (5)-(8) in [14]. The specific heat is then

$$C = 2\pi \left.\frac{w'}{w''}\right|_{X=X_h} = 4\pi \frac{X_h(X_h - c/4)(X_h - 3c/2)}{-X_h^2 + 3cX_h + 3c^2/2} . \tag{B7}$$

For large black holes, those with $M^2 \gg c$, the standard Schwarzschild results are recovered. Thus, black hole evaporation initially follows rather precisely the semi-classical approximation. For smaller black holes, however, the thermodynamic properties deviate appreciably from the semiclassical results. In particular, in the interval $3c/2 < X_h < c(3 + \sqrt{15})/2$, the specific heat is positive. At $X_h = c(3 + \sqrt{15})/2$ it has a pole, indicating a Hawking–Page-like phase transition. In the limit $X_h \to 3c/2$, the black hole temperature and specific heat both drop to zero, in accordance with the third law. Specific heat scales linearly with temperature for small T, as in a degenerate Fermi gas. The Sommerfeld constant scales like $c^{3/2}$, $C/T|_{T\to 0} \sim c^{3/2}$.

From (B1), we see that the endpoint of Hawking evaporation corresponds to an effective dimension of $D = 3$, as expected from the general discussion. The entropy of the final extremal black hole is $S = 3\pi c/2$, and thus depends on microscopic details. A singularity occurs at $X = c/4$, corresponding to the universal result $D = 2$. This example not only realizes the general features discussed in the body of the paper, but also provides the first concrete dilaton gravity model for an evaporating Schwarzschild black hole with bounded Hawking flux, and indeed recovers the end state predicted in [15].

We can also compare the effective dimension (B1) to the estimate of the spectral dimension in appendix A.

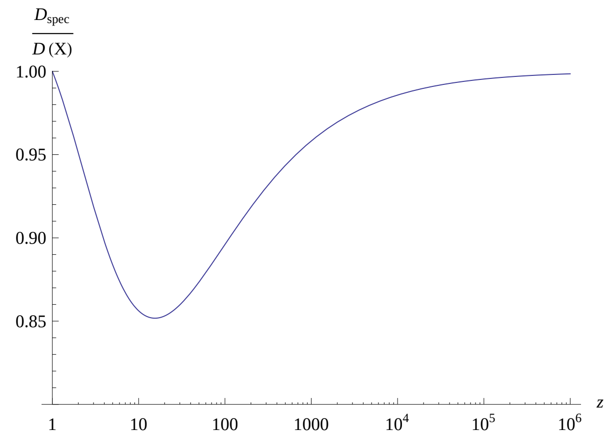

FIG. 2: Ratio $D_{\rm spec}/D(X)$

There, we showed that if the spectral dimension varies slowly, it is approximately given by equation (A4). From the solution for the dilaton (12) with the function $Q(X)$ as in (B3), we have

$$r = \frac{\sqrt{c}}{16}\left[\sqrt{z-1}\left(8 + \frac{9}{z} - \frac{2}{z^2}\right) - 15\arctan\sqrt{z-1}\right] , \tag{B8}$$

where $z = \frac{4}{c}X$ is a rescaled dilaton field, and the integration constant is chosen so that $r = 0$ at the singularity. Hence, again using (12), we have

$$D_{\rm spec} = \frac{1}{4}\frac{1}{(z-1)^2}\left[16z^2 - 7z + 6 - 15z^2\frac{\arctan\sqrt{z-1}}{\sqrt{z-1}}\right] . \tag{B9}$$

It is easy to check that $D_{\rm spec}$ approaches four for large X, and that it is nonsingular at $z = 1$ (i.e., $X = c/4$), approaching two. Figure 2 shows the ratio $D_{\rm spec}/D(X)$ as a function of z in a log-linear plot. The ratio is nearly one for the entire range, with a maximum deviation of about 15% around $z = 15$, supporting our heuristic arguments in the body of this paper.

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