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# Non-perturbative Unitarity of Gravitational Higgs Mechanism

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## Abstract

In this paper we discuss massive gravity in Minkowski space via gravitational Higgs mechanism, which provides a non-perturbative definition thereof. Using this non-perturbative definition, we address the issue of unitarity by studying the full nonlinear Hamiltonian for the relevant metric degrees of freedom. While perturbatively unitarity is not evident, we argue that no negative norm state is present in the full nonlinear theory.

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# 1 Introduction and Summary

The gravitational Higgs mechanism gives a non-perturbative and fully covariant definition of massive gravity [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The graviton acquires mass via spontaneous breaking of the underlying general coordinate reparametrization invariance by scalar vacuum expectation values.<sup>4</sup> In this paper, motivated by the results of [12] for the gravitational Higgs mechanism in de Sitter space, we study the issue of unitarity of the gravitational Higgs mechanism in the Minkowski background within the setup of [5].

Within the *perturbative* framework, unitarity requires that, in order not to propagate a negative norm state at the quadratic level in the action, the graviton mass term be of the Fierz-Pauli form [39]. Furthermore, higher order terms should be such that they do not introduce additional degrees of freedom that would destabilize the background, and difficulties in achieving this to all orders in perturbative expansion have been known for quite some time (see, *e.g.*, [40], [27], and references therein). In this regard, in order to circumvent the aforesaid difficulties within the perturbative expansion, [36] proposed an order-by-order construction, albeit in a special decoupling limit, such that higher-than-second-order time derivatives in the equations of motion are absent.

Here we propose a different approach. Following [12], our key observation is that perturbation theory appears to be inadequate, among other things, for the purposes of addressing the issue of unitarity.<sup>5</sup> Thus, while the theories of [5] reproduce the Fierz-Pauli action at the quadratic level, according to [14], at higher orders in perturbative expansion they do *not* reduce in the decoupling limit to the theories studied in [36] (and, according to [14], the same holds for the models discussed in [9]). However, since the definition of [5] is intrinsically non-perturbative, we can test the stability of the Minkowski background in the full nonlinear theory. In this paper we perform a non-perturbative Hamiltonian analysis for the relevant metric degrees of freedom and argue that the full nonlinear theory appears to be free of ghosts.

Our main result is that, in the gravitational Higgs mechanism, *non-perturbatively*, the Hamiltonian appears to be bounded from below in the Minkowski background.<sup>6</sup> We argue that this is indeed the case<sup>7</sup> by studying full nonlinear Hamiltonian for the relevant conformal and helicity-0 longitudinal modes with no spatial dependence,

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<sup>4</sup>For earlier and subsequent related works, see, *e.g.*, [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38], and references therein.

<sup>5</sup>For instance, in the de Sitter case, this becomes evident from the fact that in the full nonlinear theory no enhanced local gauge symmetry (or ghost) is present for any value of the Hubble parameter, while its appearance in the perturbative framework already at the quadratic order appears to be a mere artifact of linearization [12].

<sup>6</sup>We arrived at the same conclusion in [12] in the de Sitter case.

<sup>7</sup>Here we should note that in this regard we only analyze in detail the simplest example with higher (namely, four) derivative couplings in the scalar sector (23); however, based on the fact that in the gravitational Higgs mechanism diffeomorphisms are broken spontaneously, we believe our conclusions should hold in the general case as well, albeit we do not have a proof of this statement.

which is the dimensionally reduced diagonal Ansatz of [12]. We show, however, that within the same Ansatz, depending on the choice of the field parametrization, a perturbative decomposition in terms of the conformal and helicity-0 modes invariably leads to a) equations of motion with higher-than-second-order time derivatives in agreement with the claim of [36] or b) a ghost already at the quadratic level. We emphasize that the results of [36, 14, 38] are obtained in the context of an *intrinsically perturbative* field parametrization, which parametrization does not appear to possess a non-perturbative generalization, and all “no-go” results stemming therefrom appear to be mere artifacts of perturbative expansion.

We also revisit the gravitational Higgs mechanism in the simplest case with no higher derivative couplings in the scalar sector, first discussed in [2], which does *not* correspond to the Fierz-Pauli mass term at the quadratic level. Nonetheless, non-perturbatively, even this case appears to “resum” into a theory with a positive definite Hamiltonian.<sup>8</sup> We argue that this is the case using our dimensionally reduced diagonal Ansatz. We also reproduce the same result using the full Hamiltonian analysis of [11], thereby validating our Ansatz.

The essence of our results is well-illustrated by the following simple “toy” example:

$$\mathcal{H} = \frac{\mu^2}{\sqrt{p^2 + \mu^2}} \, , \quad (1)$$

where  $\mathcal{H}$  is the Hamiltonian,  $p$  is the canonical conjugate momentum, and  $\mu$  is a parameter. Non-perturbatively, this Hamiltonian is positive definite. On the other hand, in the “weak-field”, small  $p^2$  approximation, perturbatively there is a ghost as the kinetic term proportional to  $p^2$  has a wrong sign:  $\mathcal{H} = \mu - p^2/2\mu + \dots$ ; simply put, in the above example the weak-field approximation is invalid in the regime where the fake perturbative “ghost”, which is merely an artifact of linearization, would destabilize the background. The same appears to be the case in the gravitational Higgs mechanism examples we study in this paper.

To summarize, *non-perturbatively*, the gravitational Higgs mechanism appears to be free of ghosts.<sup>9</sup>

The rest of the paper is organized as follows. In Sections 2 and 3 we discuss the gravitational Higgs mechanism in Minkowski background, which results in massive gravity with the Fierz-Pauli mass term for the appropriately tuned cosmological constant. In Section 4 we derive the Hamiltonian for the relevant metric modes

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<sup>8</sup>So does a continuous set smoothly connecting this case to the aforesaid Fierz-Pauli case.

<sup>9</sup>However, in this paper we do not attempt to address the question of whether there is any superluminal propagation of signals or the related issue of causality. In this regard, we emphasize that the recent “no-go” results of [41] are obtained in the context of the aforesaid *intrinsically perturbative* field parametrization, directly rely on the results of [36], and do not apply to the full non-perturbative definition of the gravitational Higgs mechanism. To see if there is any superluminal propagation of signals in the full non-perturbative theory, it appears that one might have to develop some new non-perturbative methods, which is clearly beyond the scope of this paper. The non-Fierz-Pauli model of [2] is the “least non-perturbative” and might provide a fruitful testing ground in this context.

and show that it is bounded from below. In Section 5 we revisit the simplest case without higher derivative couplings in the scalar sector and show that there too the Hamiltonian is positive definite, in spite of the quadratic truncation not being of the Fierz-Pauli form.

## 2 Minkowski Solutions

Consider the induced metric for the scalar sector:

$$Y_{MN} = Z_{AB} \nabla_M \phi^A \nabla_N \phi^B . \quad (2)$$

Here  $M = 0, \dots, (D-1)$  is a space-time index, and  $A = 0, \dots, (D-1)$  is a global index. We will choose the scalar metric  $Z_{AB}$  to be the Minkowski metric:

$$Z_{AB} = \eta_{AB} . \quad (3)$$

Let

$$Y \equiv Y_{MN} G^{MN} . \quad (4)$$

The following action, albeit not the most general<sup>10</sup>, will serve our purpose here:

$$S_Y = M_P^{D-2} \int d^D x \sqrt{-G} [R - V(Y)] , \quad (5)$$

where *a priori* the “potential”  $V(Y)$  is a generic function of  $Y$ .

The equations of motion read:

$$\nabla^M (V'(Y) \nabla_M \phi^A) = 0 , \quad (6)$$

$$R_{MN} - \frac{1}{2} G_{MN} R = V'(Y) Y_{MN} - \frac{1}{2} G_{MN} V(Y) , \quad (7)$$

where prime denotes derivative w.r.t.  $Y$ . Multiplying (6) by  $Z_{AB} \nabla_S \phi^B$  and contracting indices, we can rewrite the scalar equations of motion as follows:

$$\partial_M \left[ \sqrt{-G} V'(Y) G^{MN} Y_{NS} \right] - \frac{1}{2} \sqrt{-G} V'(Y) G^{MN} \partial_S Y_{MN} = 0 . \quad (8)$$

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<sup>10</sup>One can consider a more general setup where the scalar action is constructed not just from  $Y$ , but from  $Y_{MN}$ ,  $G_{MN}$  and  $\epsilon_{M_0 \dots M_{D-1}}$ , see, *e.g.*, [5, 8, 9, 10, 12]. However, a simple action containing a scalar function  $V(Y)$  suffices to capture all qualitative features of gravitational Higgs mechanism. In particular, if this function is quadratic as in (23), the cosmological constant  $\Lambda$  must be negative in the context of the Minkowski background (but not in the de Sitter case – see [12]); however, generically there is no restriction on  $\Lambda$ , which can be positive, negative or zero even in the context of the Minkowski background, once we allow cubic and/or higher order terms in  $V(Y)$ , or consider non-polynomial  $V(Y)$ . As a side remark, let us note that no choice of polynomial  $V(Y)$  constructed from  $Y$  only reduces in the decoupling limit to the theories studied in [36] (*cf.* [14]).

Since the theory possesses full diffeomorphism symmetry, (8) and (7) are not all independent but linearly related due to Bianchi identities. Thus, multiplying (7) by  $\sqrt{-G}$ , differentiating w.r.t.  $\nabla^N$  and contracting indices we arrive at (8).

We are interested in finding solutions of the form:

$$\phi^A = m \delta^A_M x^M , \quad (9)$$

$$G_{MN} = \eta_{MN} , \quad (10)$$

where  $m$  is a mass-scale parameter. The equations of motion (7) imply that

$$Y_* \equiv D m^2 \quad (11)$$

is the solution of the following equation

$$V(Y_*) = \frac{2}{D} Y_* V'(Y_*) , \quad (12)$$

which determines the mass scale  $m$ .

### 3 Massive Gravity

In this section, following [5], we study linearized fluctuations in the background given by (9) and (10). Since diffeomorphisms are broken spontaneously, the equations of motion are invariant under the full diffeomorphism invariance. The scalar fluctuations  $\varphi^A$  can therefore be gauged away using the diffeomorphisms:

$$\delta\varphi^A = \nabla_M \phi^A \xi^M = m \delta^A_M \xi^M . \quad (13)$$

However, once we gauge away the scalars, diffeomorphisms can no longer be used to gauge away any of the graviton components  $h_{MN}$  defined as:

$$G_{MN} = \eta_{MN} + h_{MN} . \quad (14)$$

Moreover, we will use the notation  $h \equiv \eta^{MN} h_{MN}$ .

After setting  $\varphi^A = 0$ , we have

$$Y_{MN} = m^2 \eta_{MN} , \quad (15)$$

$$Y = Y_{MN} G^{MN} = m^2 [D - h + \dots] = Y_* - m^2 h + \dots , \quad (16)$$

where the ellipses stand for higher order terms in  $h_{MN}$ .

Due to diffeomorphism invariance, the scalar equations of motion (6) are related to (7) via Bianchi identities. We will therefore focus on (7). Let us first rewrite it as follows:

$$\begin{aligned} R_{MN} - \frac{1}{2} G_{MN} R = \\ m^2 [\eta_{MN} V'(Y) - G_{MN} V'(Y_*)] - \frac{1}{2} G_{MN} [V(Y) - V(Y_*)] . \end{aligned} \quad (17)$$

Linearizing the r.h.s. of this equation, we obtain:

$$R_{MN} - \frac{1}{2}G_{MN}R = \frac{M^2}{2}[\eta_{MN}h - \zeta h_{MN}] + \dots, \quad (18)$$

where

$$M^2 \equiv m^2 V'(Y_*) - 2m^4 V''(Y_*) , \quad (19)$$

$$\zeta M^2 \equiv 2m^2 V'(Y_*) . \quad (20)$$

This corresponds to adding a graviton mass term of the form

$$-\frac{M^2}{4} [\zeta h_{MN} h^{MN} - h^2] \quad (21)$$

to the Einstein-Hilbert action, and the Fierz-Pauli combination corresponds to taking  $\zeta = 1$ . This occurs for a special class of potentials with

$$V'(Y_*) = -\frac{2}{D} Y_* V''(Y_*) . \quad (22)$$

Thus, as we see, we can obtain the Fierz-Pauli combination of the mass term for the graviton if we tune *one* combination of couplings. In fact, this tuning is nothing but the tuning of the cosmological constant – indeed, (22) relates the cosmological constant to higher derivative couplings.

Thus, consider a simple example:

$$V = \Lambda + Y + \lambda Y^2 . \quad (23)$$

The first term is the cosmological constant, the second term is the kinetic term for the scalars (which can always be normalized such that the corresponding coefficient is 1 by normalizing the scalars  $\phi^A$  accordingly), and the third term is a four-derivative term. We then have:

$$Y_* = -\frac{D}{2(D+2)} \lambda^{-1} , \quad (24)$$

which relates the mass parameter  $m$  to the higher derivative coupling  $\lambda$ :

$$m^2 = Y_*/D = -\frac{1}{2(D+2)} \lambda^{-1} , \quad (25)$$

and the graviton mass is given by:

$$M^2 = -\frac{2}{(D+2)^2} \lambda^{-1} . \quad (26)$$

Note that we must have  $\lambda < 0$ . Moreover, we have:

$$\Lambda = \frac{D^2 + 4D - 8}{4(D+2)^2} \lambda^{-1} . \quad (27)$$

So, the cosmological constant in this case must be negative, which is due to the choice of the potential (23); however, as we already emphasized above, for generic choices of the potential there is no restriction on the cosmological constant, which can be positive, negative or zero.

## 4 Is there a Ghost?

The purpose of this section is to argue that the full nonlinear theory of massive gravity in Minkowski space via gravitational Higgs mechanism is free of ghosts. We will do this by studying the full nonlinear action for the relevant modes, which we identify next. In particular, we will argue that no negative norm state is present for these modes.

Let us note that, once we gauge away the scalars, the full nonlinear action becomes:

$$S_G = M_P^{D-2} \int d^D x \sqrt{-G} \left[ R - \tilde{V}(\eta^{MN} G_{MN}) \right] , \quad (28)$$

where  $\tilde{V}(\zeta) \equiv V(m^2 \zeta)$ .

To identify the relevant modes in the full nonlinear theory, let us note that in the linearized theory the potentially “troublesome” mode is the longitudinal helicity-0 mode  $\rho$ . However, we must also include the conformal mode  $\omega$  as there is kinetic mixing between  $\rho$  and  $\omega$ . In fact,  $\rho$  and  $\omega$  are not independent but are related via Bianchi identities. Therefore, in the linearized language one must look at the modes of the form

$$h_{MN} = \eta_{MN} \omega + \nabla_M \nabla_N \rho . \quad (29)$$

Furthermore, based on symmetry considerations, namely, the  $SO(D-1)$  invariance in the spatial directions,<sup>11</sup> we can focus on field configurations independent of spatial coordinates [12]. Indeed, for our purposes here we can compactify the spatial coordinates on a torus  $T^{D-1}$  and disregard the Kaluza-Klein modes. This way we reduce the  $D$ -dimensional theory to a classical mechanical system, which suffices for our purposes here. Indeed, with proper care (see [12]), if there is a negative norm state in the uncompactified theory, it will be visible in its compactified version, and vice-versa.

Let us therefore consider field configurations of the form:

$$G^{MN} = \text{diag}(g(t) \eta^{00}, f(t) \eta^{ii}) , \quad (30)$$

where  $g(t)$  and  $f(t)$  are functions of time  $t$  only. The action (28) then reduces as follows:

$$S_G = -\kappa \int dt g^{-\frac{1}{2}} f^{-\frac{D-1}{2}} \left\{ \gamma g U^2 + \tilde{V}(g + \Omega) \right\} , \quad (31)$$

where

$$\kappa \equiv M_P^{D-2} W_{D-1} , \quad (32)$$

$$\gamma \equiv (D-1)(D-2) , \quad (33)$$

$$U \equiv \frac{1}{2} \partial_t \ln(f) , \quad (34)$$

$$\Omega \equiv (D-1)f , \quad (35)$$

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<sup>11</sup>Indeed, negative norm states cannot arise from purely space-like components or spatial derivatives, and are due to time-like components and/or time derivatives.



and  $W_{D-1}$  is the volume in the spatial dimensions (*i.e.*, the volume of  $T^{D-1}$ ). Note that  $g$  is a Lagrange multiplier. The goal is to integrate out  $g$  and obtain the corresponding action for  $f$ . It is then this action that we should test for the presence of a negative norm state.

The equation of motion for  $g$  reads:

$$\tilde{V}(g + \Omega) - 2g\tilde{V}'(g + \Omega) = \gamma g U^2 . \quad (36)$$

The following discussion can be straightforwardly generalized to general  $\tilde{V}$ . However, for our purposes here it will suffice to consider quadratic  $\tilde{V}$  corresponding to (23). We then have:

$$3\lambda m^2 g^2 + \left[1 + 2\lambda m^2 \Omega + \frac{\gamma}{m^2} U^2\right] g - \left\{\frac{\Lambda}{m^2} + \Omega [1 + \lambda m^2 \Omega]\right\} = 0 . \quad (37)$$

We can therefore express  $g$  in terms of  $f$  and  $\partial_t \ln(f)$ :

$$6\lambda m^2 g = - \left[1 + 2\lambda m^2 \Omega + \frac{\gamma}{m^2} U^2\right] + \sqrt{\left[1 + 2\lambda m^2 \Omega + \frac{\gamma}{m^2} U^2\right]^2 + 12\lambda m^2 \left\{\frac{\Lambda}{m^2} + \Omega [1 + \lambda m^2 \Omega]\right\}} , \quad (38)$$

where the branch is fixed by the requirement that  $g \equiv 1$  when  $f \equiv 1$ . Substituting the so expressed  $g$  into (31), we obtain an action which is a nonlinear functional of  $f$  and  $\partial_t \ln(f)$ .

For our purposes here it is more convenient to work with the canonical variable  $q$ , where

$$q \equiv \ln(f) , \quad (39)$$

$$\Omega = (D - 1)e^q , \quad (40)$$

$$U = \frac{1}{2}\partial_t q , \quad (41)$$

and the action reads:

$$S_G = \int dt L = -\kappa \int dt g^{-\frac{1}{2}} e^{-\frac{D-1}{2}q} \left\{ \gamma g U^2 + \tilde{V}(g + \Omega) \right\} , \quad (42)$$

where  $L$  is the Lagrangian. This action corresponds to a classical mechanical system with a lagrange multiplier  $g$ .

Next, the conjugate momentum is given by

$$p = \frac{\partial L}{\partial(\partial_t q)} = -\kappa e^{-\frac{D-1}{2}q} \times \left\{ \frac{1}{2}g^{-\frac{1}{2}}\hat{g}\gamma U^2 - \frac{1}{2}g^{-\frac{3}{2}}\hat{g}\tilde{V}(g + \Omega) + g^{-\frac{1}{2}}\hat{g}\tilde{V}'(g + \Omega) + g^{\frac{1}{2}}\gamma U \right\} , \quad (43)$$

where

$$\hat{g} \equiv \frac{\partial g}{\partial(\partial_\tau q)} . \quad (44)$$

Using (36), (43) simplifies to

$$p = -\kappa e^{-\frac{D-1}{2}q} g^{\frac{1}{2}} \gamma U , \quad (45)$$

and the Hamiltonian is given by

$$\mathcal{H} = p \partial_t q - L = -\kappa g^{-\frac{1}{2}} e^{-\frac{D-1}{2}q} \left[ \gamma g U^2 - \tilde{V}(g + \Omega) \right] . \quad (46)$$

We can now see if this Hamiltonian is bounded from below.

First, using (36), we have:

$$\mathcal{H} = 2\kappa g^{\frac{1}{2}} e^{-\frac{D-1}{2}q} \tilde{V}'(g + \Omega) = 2m^2 \kappa g^{\frac{1}{2}} e^{-\frac{D-1}{2}q} \left[ 1 + 2\lambda m^2 (g + \Omega) \right] . \quad (47)$$

Using (38), we can rewrite this Hamiltonian as follows:

$$\mathcal{H} = \frac{2}{3} m^2 \kappa g^{\frac{1}{2}} e^{-\frac{D-1}{2}q} [X - Z] , \quad (48)$$

where

$$X \equiv \sqrt{\left[ 1 + 2\lambda m^2 \Omega + \frac{\gamma}{m^2} U^2 \right]^2 + 12\lambda m^2 \left\{ \frac{\Lambda}{m^2} + \Omega [1 + \lambda m^2 \Omega] \right\}} , \quad (49)$$

$$Z \equiv \frac{\gamma}{m^2} U^2 - 4\lambda m^2 \Omega - 2 . \quad (50)$$

The presence of a ghost would imply that the Hamiltonian is unbounded from below for large values of  $U^2$  (recall that  $U^2$  contains the “kinetic” term). However, it is not difficult to show that this Hamiltonian suffers from no such pathology. Indeed, we can rewrite it as follows:

$$\begin{aligned} \mathcal{H} &= \frac{2}{3} m^2 \kappa g^{\frac{1}{2}} e^{-\frac{D-1}{2}q} \frac{X^2 - Z^2}{X + Z} = \\ &= 2m^2 \kappa g^{\frac{1}{2}} e^{-\frac{D-1}{2}q} \frac{4\lambda\Lambda - 1 + 2(\gamma/m^2) U^2 [1 + 2\lambda m^2 \Omega]}{X + Z} , \end{aligned} \quad (51)$$

which in the large  $U^2$  limit reads:

$$\mathcal{H} = 2m^2 \kappa g^{\frac{1}{2}} e^{-\frac{D-1}{2}q} [1 + 2\lambda m^2 \Omega] + \mathcal{O}(1/U^2) . \quad (52)$$

Furthermore, from (38) we have

$$6\lambda m^2 g = \frac{X^2 - Q^2}{X + Q} = 12 \frac{\lambda\Lambda + \lambda m^2 \Omega [1 + \lambda m^2 \Omega]}{X + Q} = \mathcal{O}(1/U^2) , \quad (53)$$

where

$$Q \equiv \frac{\gamma}{m^2} U^2 + 2\lambda m^2 \Omega + 1 . \quad (54)$$

So, in the large  $U^2$  limit the Hamiltonian actually vanishes.

Note that the above argument implicitly assumes that  $\Omega$  is bounded from above. This is indeed the case as  $g$  must be at least non-negative, which implies that

$$\lambda\Lambda + \lambda m^2 \Omega [1 + \lambda m^2 \Omega] \leq 0 , \quad (55)$$

and  $\Omega$  is bounded as follows (note that we must have  $\Omega \geq 0$ ):

$$\max \left( 0 , -\frac{1 - \sqrt{1 - 4\lambda\Lambda}}{2\lambda m^2} \right) \leq \Omega \leq -\frac{1 + \sqrt{1 - 4\lambda\Lambda}}{2\lambda m^2} , \quad (56)$$

and we must further have

$$\Lambda \geq \frac{1}{4\lambda} , \quad (57)$$

which is always satisfied due to (27) as  $\lambda < 0$ .

Thus, as we see, there appears to be no ghost in the full nonlinear theory. The key ingredient here is the parametrization of the conformal and helicity-0 longitudinal modes. We have been working with (30), while the results of [36, 14] apply to (29). The difference between the two is that (30) has no derivatives. Thus, at the quadratic order, the second derivatives introduced by the parametrization corresponding to (29) can be integrated by parts to arrive at an action containing only first derivatives of  $\omega$  and  $\rho$ . However, we have explicitly checked that already at the cubic level the second derivatives introduced by the parametrization corresponding to (29) cannot be integrated by parts, so the resulting action invariably includes terms with second derivatives of  $\rho$ . This is clearly problematic already at the cubic level and suggests that the parametrization corresponding to (29) cannot be used beyond the linearized approximation. Indeed, a nonlinear completion of (29) is given by:

$$G^{MN} = \eta^{MN} f + \nabla^M \nabla^N u . \quad (58)$$

where the covariant derivative is defined w.r.t. the metric  $\eta_{MN}$  (this choice does not affect our discussion here). Note, however, that such a parametrization of the metric is rather problematic in the context of the full nonlinear theory (28) as it introduces higher derivative terms in  $u$ , which should therefore not be used as the canonical variable in the full nonlinear theory. This suggests that our parametrization (30) is indeed adequate (while our results here suggest that the perturbative field parametrization underlying the results of [14] does not appear to be adequate in the full non-perturbative theory).

## 5 The Linear Potential Revisited

In this section we consider the linear potential

$$V(Y) = \Lambda + Y . \quad (59)$$

This example was discussed in [2] and worked out in detail in the Hamiltonian formalism in [11]. The potential (59) does not satisfy the condition (22) and, therefore, does not lead to the Fierz-Pauli action at the quadratic order; consequently, it propagates a negative norm excitation at that order in perturbation theory. Nonetheless, we will show that the full non-perturbative Hamiltonian is bounded from below. This is shown in two ways: In the dimensionally reduced parametrization of (30), and in the full Hamiltonian analysis of [11]. This supports our conclusion that the Ansatz (30) is fully adequate for analyzing unitarity.

Recalling the definitions of Section 4, we have  $\tilde{V}(x) = \Lambda + m^2 x$ , and the  $g$  equation of motion gives

$$g = (\Lambda + m^2 \Omega) / (m^2 + \gamma U^2), \quad (60)$$

thus, the Hamiltonian reads:

$$\mathcal{H} = 2\kappa m^2 e^{-\frac{D-1}{2}q} g^{1/2}, \quad (61)$$

which is in general positive definite (assuming  $g > 0$ , which is the case for  $f > -\Lambda/(D-1)m^2 = (D-2)/(D-1)$ , where we have used (11) and (12)). Therefore, as for the quadratic potential of the previous section, it appears that non-perturbatively there is no ghost. Furthermore, let us point out that perturbatively there is a ghost, since in a “weak-field”, small  $U^2$  expansion the kinetic term proportional to  $U^2$  has a wrong sign. This is similar to instructive toy examples discussed in Section 1 hereof as well as Appendix A of [12].

We arrive at the same result in full generality by introducing the lapse constraint into the gauge-fixed Hamiltonian derived for this potential in [11]. The result reads, in their notation,

$$\mathcal{H}_{fix} = \frac{\sqrt{\det h}}{8\pi G} \sqrt{\frac{1}{N^2} + \left( \frac{8\pi G}{\sqrt{\det h}} \right)^2 \mathcal{H}_i^{GR} \delta^{ij} \mathcal{H}_j^{GR}}, \quad (62)$$

which is positive definite, and reduces to (61) for the Ansatz (30) since, in that case,  $\mathcal{H}_i^{GR} = 0$ ,  $1/N^2 = g$  and  $\det h = 1/f^{D-1}$ .

In fact, there is nothing “special” about the linear potential (59) in terms of unitarity. The discussion in Section 4 is completely independent of (22) (with the exception of the reference to (27) immediately following (57), which, however, does not affect our discussion here – see below), which is the condition corresponding to having the Fierz-Pauli term at the quadratic order. In particular, the discussion in Section 4 and its conclusion that non-perturbatively there appears to be no negative norm state are valid for all values of  $\lambda$ , including those smoothly interpolating between (27) (which corresponds to the Fierz-Pauli term) and  $\lambda = 0$ , which corresponds to the linear potential (59) as long as (57) is satisfied along the interpolation path (and this condition does not pose an obstruction to such interpolation). Simply put, non-perturbatively there appears to be no negative norm state for a continuous

family of models arising out of the gravitational Higgs mechanism, and the linear potential point and the Fierz-Pauli point are not any special in this regard.

The fact that non-Fierz-Pauli points appear to be unitary deserves further elaboration. Thus, in the gravitational Higgs mechanism we start with massless gravity with  $D(D-3)/2$  degrees of freedom coupled to  $D$  scalars, one of which is time-like. As was explained in [5], at the Fierz-Pauli point (22) the time-like scalar does not propagate, so in the scalar sector we have only  $(D-1)$  propagating space-like degrees of freedom, which are eaten in the gravitational Higgs mechanism producing  $(D+1)(D-2)/2 (= D(D-3)/2 + (D-1))$  degrees of freedom for the massive graviton, all of which are unitary. What about the non-Fierz-Pauli points? Here we have an extra time-like scalar degree of freedom, which in the perturbative language in the Higgs phase results in a ghost-like propagating degree of freedom, namely, the trace of the graviton  $h = h^M_M$ . However, *non-perturbatively* the Hamiltonian is positive-definite and there is no ghost, while if the Hamiltonian is expanded perturbatively, as we saw above, a ghost invariably appears. Simply put, the weak-field approximation is invalid in the regime where the fake perturbative “ghost”, which is merely an artifact of linearization, would destabilize the background.<sup>12</sup>

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## References

- [1] Z. Kakushadze and P. Langfelder, “Gravitational Higgs Mechanism”, Mod. Phys. Lett. A15 (2000) 2265, arXiv:hep-th/0011245.
- [2] G. ’t Hooft, “Unitarity in the Brout-Englert-Higgs Mechanism for Gravity”, arXiv:0708.3184 [hep-th].
- [3] Z. Kakushadze, “Gravitational Higgs Mechanism and Massive Gravity”, Int. J. Mod. Phys. A23 (2008) 1581, arXiv:0709.1673 [hep-th].
- [4] I. Oda, “Gravitational Higgs Mechanism with a Topological Term”, arXiv:0709.2419 [hep-th].
- [5] Z. Kakushadze, “Massive Gravity in Minkowski Space via Gravitational Higgs Mechanism”, Phys. Rev. D77 (2008) 024001, arXiv:0709.1673 [hep-th].

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<sup>12</sup>In this regard, let us note a difference between the gravitational Higgs mechanism and its gauge theory counterpart. In the latter scalar vacuum expectation values are constant, while in the former they depend linearly on space-time coordinates: (9). In fact, the background is not even static. It would take infinite energy to destabilize such a background. This is reminiscent to infinite-tension domain walls discussed in [42, 43].

- [6] Z. Kakushadze, “Massless Limit of Gravitational Higgs Mechanism”, *Int. J. Geom. Meth. Mod. Phys.* 05 (2008) 157, arXiv:0711.0386 [hep-th].
- [7] M. Maeno and I. Oda, “Massive Gravity in Curved Cosmological Backgrounds”, *Int. J. Mod. Phys. A* 24 (2009) 81, arXiv:0808.1394 [hep-th].
- [8] D.A. Demir and N.K. Pak, “General Tensor Lagrangian from Gravitational Higgs Mechanism”, *Class. Quantum. Grav.* 26 (2009) 105018, arXiv:0904.0089 [hep-th].
- [9] A.H. Chamseddine and V. Mukhanov, “Higgs for Graviton: Simple and Elegant Solution”, *JHEP* 1008 (2010) 011, arXiv:1002.3877 [hep-th].
- [10] I. Oda, “Higgs Mechanism for Gravitons”, arXiv:1003.1437 [hep-th]; “Remarks on Higgs Mechanism for Gravitons”, arXiv:1004.3078 [hep-th].
- [11] J. Kluson, “Hamiltonian Analysis of the Higgs Mechanism for Graviton”, arXiv:1005.5458 [hep-th].
- [12] A. Iglesias and Z. Kakushadze, “Massive Gravity in de Sitter Space via Gravitational Higgs Mechanism”, *Phys. Rev. D* 82 (2010) 124001, arXiv:1007.2385 [hep-th].
- [13] L. Alberte, A.H. Chamseddine and V. Mukhanov, “Massive Gravity: Resolving the Puzzles”, *JHEP* 1012 (2010) 023, arXiv:1008.5132 [hep-th].
- [14] L. Berezhiani and M. Mirbabayi, “Unitarity Check in Gravitational Higgs Mechanism”, arXiv:1010.3288 [hep-th].
- [15] L. Alberte, A.H. Chamseddine and V. Mukhanov, “Massive Gravity: Exorcising the Ghost”, arXiv:1011.0183 [hep-th].
- [16] M.J. Duff, “Dynamical Breaking of General Covariance and Massive Spin-2 Mesons”, *Phys. Rev. D* 12 (1975) 3969.
- [17] C. Omero and R. Percacci, “Generalized Nonlinear Sigma Models In Curved Space And Spontaneous Compactification”, *Nucl. Phys. B* 165 (1980) 351.
- [18] M. Gell-Mann and B. Zwiebach, “Space-Time Compactification Due To Scalars”, *Phys. Lett. B* 141 (1984) 333.
- [19] R. Percacci, “The Higgs Phenomenon in Quantum Gravity”, *Nucl. Phys. B* 353 (1991) 271.
- [20] M.B. Green and C.B. Thorn, “Continuing between Closed and Open Strings”, *Nucl. Phys. B* 367 (1991) 462.

- [21] W. Siegel, “Hidden Gravity in Open-String Field Theory”, Phys. Rev. D49 (1994) 4144, arXiv:hep-th/9312117.
- [22] M. Porrati, “Higgs Phenomenon for 4-D Gravity in Anti de Sitter Space”, JHEP 0204 (2002) 058, arXiv:hep-th/0112166.
- [23] N. Arkani-Hamed, H. Georgi and M.D. Schwartz, “Effective Field Theory for Massive Gravitons and Gravity in Theory Space”, Annals Phys. 305 (2003) 96, arXiv:hep-th/0210184.
- [24] A.H. Chamseddine, “Spontaneous Symmetry Breaking for Massive Spin-2 Interacting with Gravity”, Phys. Lett. B557 (2003) 247, arXiv:hep-th/0301014.
- [25] I.A. Bandos, J.A. de Azcarraga, J.M. Izquierdo and J. Lukierski, “Gravity, p-branes and a Space-time Counterpart of the Higgs Effect”, Phys. Rev. D68 (2003) 046004, arXiv:hep-th/0301255.
- [26] N. Arkani-Hamed, H.-C. Cheng, M.A. Luty and S. Mukohyama, “Ghost Condensation and a Consistent Infrared Modification of Gravity”, JHEP 0405 (2004) 074, arXiv:hep-th/0312099.
- [27] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, “Ghosts in Massive Gravity”, JHEP 0509 (2005), arXiv:hep-th/0505147.
- [28] N. Arkani-Hamed, H.-C. Cheng, M.A. Luty, S. Mukohyama and T. Wiseman, “Dynamics of Gravity in a Higgs Phase”, JHEP 0701 (2007) 036, arXiv:hep-ph/0507120.
- [29] M. Leclerc, “The Higgs Sector of Gravitational Gauge Theories”, Annals Phys. 321 (2006) 708, arXiv:gr-qc/0502005.
- [30] I. Kirsch, “A Higgs Mechanism for Gravity”, Phys. Rev. D72 (2005) 024001, arXiv:hep-th/0503024.
- [31] E. Kiritsis, “Product CFTs, Gravitational Cloning, Massive Gravitons and the Space of Gravitational Duals”, JHEP 0611 (2006) 049, arXiv:hep-th/0608088.
- [32] Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, “Spontaneous Lorentz Breaking and Massive Gravity”, Phys. Rev. Lett. 131101 (2007) 99, arXiv:hep-th/0703264.
- [33] M.V. Bebronne and P.G. Tinyakov, “Massive Gravity and Structure Formation”, Phys. Rev. D76 (2007) 084011, arXiv:0705.1301 [astro-ph].
- [34] R. Jackiw, “Lorentz Violation in Diffeomorphism Invariant Theory”, arXiv:0709.2348 [hep-th].

- [35] M.V. Bebronne and P.G. Tinyakov, “Black hole solutions in massive gravity”, JHEP 0904 (2009) 100, arXiv:0902.3899 [hep-th].
- [36] C. de Rham and G. Gabadadze, “Generalization of the Fierz-Pauli Action”, Phys. Rev. D82 (2010) 044020, arXiv:1007.0443 [hep-th].
- [37] S.F. Hassan, S. Hofmann and M. von Strauss, “Brane Induced Gravity, its Ghost and the Cosmological Constant Problem”, arXiv:1007.1263 [hep-th].
- [38] C. de Rham, G. Gabadadze and A.J. Tolley, “Resummation of Massive Gravity”, arXiv:1011.1232 [hep-th].
- [39] M. Fierz and W. Pauli, “On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field”, Proc. Roy. Soc. Lond. A173 (1939) 211.
- [40] D.G. Boulware and S. Deser, “Can gravitation have a finite range?”, Phys. Rev. D6 (1972) 3368.
- [41] A. Gruzinov, “All Fierz-Paulian massive gravity theories have ghosts or superluminal modes”, arXiv:1106.3972 [hep-th].
- [42] G. Dvali and M. Shifman, “Surviving on the Slope: Supersymmetric Vacuum in the Theories Where It Is Not Supposed to Be”, Phys. Lett. B454 (1999) 277, arXiv:hep-th/9901111.
- [43] Z. Kakushadze, “Localized (Super)Gravity and Cosmological Constant”, Nucl. Phys. B589 (2000) 75, arXiv:hep-th/0005217.