Flavor Δ(54) in SU(5) supersymmetric model

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Flavor $\Delta(54)$ in $SU(5)$ SUSY Model

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Abstract

We design a supersymmetric $SU(5)$ GUT model using $\Delta(54)$, a finite non-abelian subgroup of $SU(3)_f$. Heavy right handed neutrinos are introduced which transform as a three-dimensional representation of our chosen family group. The model successfully reproduces the hierarchical structures of the Standard Model and the CKM mixing matrix. It then provides predictions for the light neutrinos with a normal hierarchy and masses such that $m_{\nu,1} \approx 5 \times 10^{-3}\ eV$, $m_{\nu,2} \approx 1 \times 10^{-2}\ eV$, and $m_{\nu,3} \approx 5 \times 10^{-2}\ eV$. We also provide predictions for masses of the heavy neutrinos, and corrections to the tri-bimaximal matrix that fit within experimental limits, e.g., a reactor angle of -7.31^o. A simple modification to our model is introduced at the end and is shown to also produce predictions that fall well within those limits.

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1 Introduction

The origins of the mass structure in the Standard Model (SM) is currently without explanation. However, current neutrino oscillation data provides clues that a finite non-abelian symmetry may be responsible. The oscillation evidence that we speak of comes in the form of the lepton mixing matrix ($\mathcal{U}_{mnsp}$) [1], which plays the same role as the CKM matrix ($\mathcal{U}_{ckm}$) for the quarks [2]. The most promising and current theoretical fit of $\mathcal{U}_{mnsp}$ is called the tri-bimaximal lepton mixing matrix ($\mathcal{U}_{\text{tri-bi}}$) [3], and it is in this form one is clearly lead to the possibility of a non-abelian finite group being the key to solving what is often referred to as the Flavor Problem.

In this paper, we postulate a finite subgroup of an $SU(3)_f$ family group: $\Delta(54)$[1] is responsible for the masses and mixing data observed [2]. We reach the goal of creating a model by way of the Froggatt-Nielsen (FN) formalism, which is an effective field theory suppressed by some mass scale [9]. The mass scale of the FN formalism allows for the introduction of a single parameter that controls the perturbative nature of the theory. The model is ambitious in that we try to only use this one parameter throughout. Now, these details and more of the model building process are laid out in several sections, which we now summarize.

In order that we produce experimentally viable results, it is essential to keep in mind all experimental data and constraints. To this end, Section 2 serves a dual role as a summary of the data to be reproduced and a discussion on how it should be accomplished. For the sake of organization, the section splits the phenomenology into quark and lepton sectors. We list in each what it is we want to reproduce from experimental results and how it can be done.

With these constraints, in Section 3, we give a closer look at $\Delta(54)$ and determine how it may be implemented. We also discuss how and why we split the matter content into specific representations of our flavor group. Then, under these choices, we make use of a toy model to demonstrate how we satisfy the constraints found in the previous section.

The fourth section contains the model which, as a final result, can be summarized as coming from $SU(5) \otimes \Delta(54) \otimes Z_3^u \otimes Z_2^d \otimes Z_2$. The underlying assumption of our model is that we have supersymmetry at this scale. We, thus, show the super-potential for our theory and take a look at the contributions to each sector. Here we find that the success of $\Delta(54)$: it can reproduce all known data in these sectors, and as for the neutrino sector, the superpotential terms are quite elegant and produce predictions. Specifically, we find a normal hierarchy structure with the neutrino masses being: $m_{\nu,1} \approx 5 \times 10^{-3}\ eV$, $m_{\nu,2} \approx 1 \times 10^{-2}\ eV$, and $m_{\nu,3} \approx 5 \times 10^{-2}\ eV$. As for the angles, we find a reactor angle of $\theta_{13} \approx -7.31^o$, with a post-dicted solar angle of $\theta_\odot \approx 34.36^0$, and an atmospheric angle of $\theta_{atm} \approx -45.15^o$.

The final section includes a simple modification to the model discussed above. We explore the alteration and show that it, too, may provide a viable model by taking a specific example and listing its predictions for the angles of the lepton mixing matrix.

2 Phenomenological Constraints

The goal is to produce phenomenologically correct Yukawa matrices for the quark sector and at the same time produce viable neutrino masses and to leading order the tri-bimaximal lepton

[1] We choose $\Delta(54)$ because at the time in which this work was done a model for quarks and leptons charged under this flavor group had not been explored.

[2] In the literature one can find many possible finite groups as the origins of the large mixing angles in the lepton sector, e.g., for A_4 [4], S_4 [5], $\Delta(27)$ [6], and $PSL_2(7)$ [7]. There are also more general attempts to solving the flavor problem, of which we list a few [8]

mixing matrix. As for the charged leptons, the choice of an $SU(5)$ GUT will automatically produce a Yukawa from the down-quark sector. The focus of this section is then the phenomenology involved in each matter sector and how to consolidate the data into mass matrices.

2.1 Quark sector

Current experiments allow for only two but important pieces of data. These come in the form of the approximate masses for the quarks and the quark mixing matrix known as the CKM matrix.

It is well known that by extrapolating mass data to the unification scale one can parametrize all masses in terms of the Cabibbo angle $\lambda_c \approx .226$, producing the hierarchical structure

$$\frac{m_e}{m_\tau} \approx \mathcal{O}(\lambda_c^{4,5}), \qquad \frac{m_\mu}{m_\tau} \approx \mathcal{O}(\lambda_c^2), \qquad m_\tau \sim m_b, \tag{2.1}$$

$$\frac{m_d}{m_b} \approx \mathcal{O}(\lambda_c^{4,5}), \qquad \frac{m_s}{m_b} \approx \mathcal{O}(\lambda_c^2), \qquad \frac{m_b}{m_t} \approx \mathcal{O}(\lambda_c^3), \tag{2.2}$$

$$\frac{m_u}{m_t} \approx \mathcal{O}(\lambda_c^8), \qquad \frac{m_c}{m_t} \approx \mathcal{O}(\lambda_c^4), \tag{2.3}$$

Included above is the relation between mass of the tau lepton and bottom quark which are approximately equal and the intra-family hierarchy, both the last relations in Eq. (2.1) and Eq. (2.2) respectively.

The choice of an $SU(5)$ model will guarantee the lepton masses and down-type quark masses are in fact related and so ensure that the mass of the tau and bottom quark are identical. So that with an $SU(5)$ model we will try to reproduce, in the form of eigenvalues of two Yukawa matrices, all the information found in Eq. (2.1) - (2.3).

The last experimental piece of data at our disposal is the CKM matrix. It is a mixing matrix composed roughly out of differences in angles that occurs from diagonalizing the Yukawa matrices of both quark sectors. The information contained there to third order approximation is

$$\mathcal{U}_{ckm} \approx \mathcal{O}\begin{pmatrix} 1 & \lambda_c & \lambda_c^3 \\ -\lambda_c & 1 & A\lambda_c^2 \\ \lambda_c^3 & -A\lambda_c^2 & 1 \end{pmatrix} , \tag{2.4}$$

A is the appropriate parameter found in the Wolfenstein prescription. Because of the very nature of its origins there is a limit in how much information we can derive about the structure of the quark Yukawas. Nevertheless, there are clues as to the texture structures and if we add to these the necessary eigenvalues required we can limit the possible choices for Yukawa matrices [10].

Taking all these constraints, and following guidelines found in [10] we find that at the very minimum we would need

$$Y^{(2/3)} \approx \mathcal{O}\begin{pmatrix} 0 & \lambda_c^6 & \lambda_c^{\geq 4} \\ \lambda_c^6 & \lambda_c^4 & \lambda_c^{\geq 2} \\ \lambda_c^{\geq 4} & \lambda_c^{\geq 2} & 1 \end{pmatrix} , \qquad Y^{(-1/3)} \approx \mathcal{O}\begin{pmatrix} 0 & \lambda_c^3 & \lambda_c^{\geq 3} \\ \lambda_c^3 & \lambda_c^2 & \lambda_c^{\geq 2} \\ \lambda_c^{\geq 1} & \lambda_c^{\geq 0} & 1 \end{pmatrix} , \tag{2.5}$$

assuming that coefficients are of $\mathcal{O}(1)$ and the Yukawa matrices are labeled by their corresponding quark charges. It should be noted that the above is a bit misleading, at least one of the $(2,3)$

positions must be λ_c^2. Now, the model building will have to satisfy the hard texture constraints given above and fall within the limits placed.

We do so not using the Cabbibo angle as our expansion parameter for the whole model but instead $\delta \approx .20$. There is some arbitrariness to this, the only constraint being that $\delta > .182$ [3], but we have chosen the stated value so that the mass relations are consistent at energies of the GUT scale of $2 \times 10^{16}\ GeV$ and its value must remain close to the Cabibbo angle if we expect Eq. (2.5) to remain true.

2.2 Lepton sector

Unification via $SU(5)$ will automatically produce information about the charged leptons once the down-quark Yukawas are known. So we will only concentrate on both the neutral leptons and heavy neutrinos.

In terms of experimental data the lepton sector does not share the same richness as the quark sector, but we do have available to us two key pieces of data[4]. First, experimental results have given us the mass squared differences [11]

$$\Delta m_{21}^2 \approx 7.59^{+.19}_{-.21} \times 10^{-5}\ eV^2, \qquad |\Delta m_{23}^2| \approx 2.43 \pm .13 \times 10^{-3}\ eV^2\ , \tag{2.6}$$

notice that the second relation does not allow us to determine the exact hierarchical structure. Nevertheless, a useful constraint that can be derived from the above is

$$29.6 \leq \frac{|\Delta m_{23}^2|}{\Delta m_{21}^2} \leq 34.7\ , \tag{2.7}$$

the average value being ≈ 32.0.

The second piece of experimental data comes in the form of the lepton mixing matrix $\mathcal{U}_{mns}$, which we shall assume to be approximately the tri-bimaximal matrix

$$\mathcal{U}_{mnsp} \approx U_{\text{tri-bi}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}\ . \tag{2.8}$$

The see-saw mechanism requires the existence of the regular neutral lepton Yukawa matrix $Y^{(0)}$ and an invertible Majorana matrix Y_{maj} [13]. These together are needed for the light neutrino mass approximation of

$$\mathcal{Y}_\nu \approx -\frac{v^2}{\mathcal{M}}\ Y^{(0)}\ \left(Y_{maj}\right)^{-1}\ Y^{(0)\ T}\ , \tag{2.9}$$

where v is the usual electroweak vacuum value and $\mathcal{M}$ is the mass scale of the Majorana term of the heavy right handed neutrinos. We can then diagonalize the above by $\mathcal{U}_{mns}$, i.e.,

[3]One can understand this limit by using the mass relation for the up-quark found in Eq. 2.3. We do not wish that it go as δ^7, which occurs when the value of δ is as small as it can be, viz., $\delta \approx .182$.

[4]Cosmological data also provide limits on the sum of neutrino masses and the size of the most massive neutrino [12]

$$\sum m_{\nu,i} < (.17 - 2.0)\ eV,\ i = 1,2,3\ , \qquad .04 < m_{\nu,\text{heaviest}} < (.07 - .70)\ eV\ .$$

$$\mathcal{Y}_\nu = \mathcal{U}_{mnsp}\, m_\nu\, \mathcal{U}_{mnsp}^T\ . \tag{2.10}$$

The diagonal term m_ν will in general contain three different eigenvalues (masses) and after selecting these eigenvalues we can produce the light neutrino matrix from the tri-bimaximal matrix:

$$m_\nu = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} \quad \Rightarrow \quad \mathcal{Y}_\nu = \begin{pmatrix} \Delta_1 & \Delta_2 & \Delta_2 \\ \Delta_2 & \Delta_3 & \Delta_1 + \Delta_2 - \Delta_3 \\ \Delta_2 & \Delta_1 + \Delta_2 - \Delta_3 & \Delta_3 \end{pmatrix}\ , \tag{2.11}$$

in which we have that

$$\Delta_1 = \frac{1}{6}(4m_1 + 2m_2), \quad \Delta_2 = \frac{1}{6}(-2m_1 + 2m_2), \quad \Delta_3 = \frac{1}{6}(m_1 + 2m_2 + 3m_3)\ . \tag{2.12}$$

Thus the eigenvalues as functions of entries of $\mathcal{Y}_\nu$ are given as

$$m_1 = \Delta_1 - \Delta_2, \quad m_2 = \Delta_1 + 2\Delta_2, \quad m_3 = 2\Delta_3 - \Delta_1 - \Delta_2\ . \tag{2.13}$$

3 Model building with $\Delta(54)$

The focus of this section is to describe in some detail the strategy taken to produce our model. We begin with an attempt to familiarize ourselves with $\Delta(54)$ by having a quick look at its salient features. A complement to this section, i.e. with a more mathematical description of this group, can be found in Appendices A-B.

In brief, Appendix A contains a comparison of the group itself to that of a similar group $\Delta(27)$, which has been investigated as a flavor group [6], while Appendix B contains some of the mathematical information regarding the group $\Delta(54)$ that a reader would want to know for this paper.

Our model makes use of a supersymmetric $SU(5)$ GUT theory. This, of course, has a direct impact on how we build a theory under our flavor group. Now, although for the most part the choice of GUT is somewhat arbitrary, an $SU(5)$ theory has a method of unifying the charged lepton and down-type quark masses in a simple elegant way. Our choice means that we must place matter into specific representations under $SU(5)$ [14], these are:

$$\overline{N} \sim \mathbf{1}, \quad L,\ \overline{d} \sim \overline{\mathbf{5}}, \quad Q,\ \overline{u},\ \overline{e} \sim \mathbf{10}, \quad . \tag{3.1}$$

The L and Q are the $SU(2)$ weak doublets and the remainder particles are the right handed weak singlets.

3.1 $\Delta(54)$ as a flavor group

A glance at the appendix shows that the group has both two and three-dimensional representations. This translates into many options for assigning representations to the matter content. Although all options can be explored, we wish to limit them, and for an $SU(5)$ theory this can be done by examining the mass of the top quark.

The origins of its mass is at tree-level, since its value seems close to that of vacuum expectation value (vev) of the Higgs particle. Ensuring this result satisfactorily for the three-dimensional representation is very difficult if not impossible to do. To see that this is indeed the case, let's for the moment describe what would happen if we used such a three-dimensional representation.

First, our model assumes that the top quark mass comes from the product of two ten-dimensional representations of $SU(5)$. Let's assume that under our flavor group the $X \sim \mathbf{10}$ transforms as any of the four three-dimensional representations, i.e. $\mathbf{3}_1,\ \mathbf{3}_2,\ \overline{\mathbf{3}}_1,\ \overline{\mathbf{3}}_2$.[5] Then the interaction term responsible for mass produces no singlets but instead, schematically, a direct sum of three-dimensional representations

$$X \cdot X \sim \left(\overline{\mathbf{3}}_1 \oplus \overline{\mathbf{3}}_1\right)_S \oplus \overline{\mathbf{3}}_{2,A}\ , \tag{3.1}$$

with the subscripts S for the symmetric combination and A for the anti-symmetric one. The bar should be understood as the complex conjugate of whichever $\mathbf{3}$ is taken for X. In order to get a singlet term we must have a flavon ϕ which transforms as either a $\mathbf{3}_1$ or a $\overline{\mathbf{3}}_1$ depending on the representation chosen for the $\mathbf{10}$ so that via the FN mechanism

$$\frac{g}{M}\phi\ X \cdot X\ , \tag{3.2}$$

is the mass operator for the top quark, where g is a coupling constant, M is the mass scale for the mechanism, and we have suppressed the Higgs. In order to explain the mass of the top properly the vev of the flavon field must be the same order as the mass scale, i.e., $\langle\phi\rangle \sim M$. In terms of model building this fact is difficult to explain and it can be difficult to control the interaction terms involving ϕ. These difficulties are enough to make us avoid the use the three-dimensional representation of $\Delta(54)$ to describe the up-quarks.

We have chosen instead to have the top quark be a singlet under the flavor group, i.e. $X_3 \sim 1$, while the two remaining flavors together form a two-dimensional representation $(X_1, X_2)^T \sim \mathbf{2}_r$, $r = 1, 2, 3, 4$. Under this scheme we have a natural way to explain the mass of the top quark at tree-level: $X_3 X_3 H_u$, where H_u is the up-type Higgs field. So we take the approach that both quark sectors can be written in the same fashion just described. Our motivation for the choice of $\mathbf{2} \oplus \mathbf{1}$ structure is two-fold.

First, if we had chosen instead that the $\overline{\mathbf{5}}$ transform as $\mathbf{3}$ under $\Delta(54)$ it would be difficult to control the power in δ of any one entry in a Yukawa matrix without the danger of producing that same power in another. An issue math then arise that same power is lower than the power required. We refer the reader to Appendix B to confirm this. The second weaker reason is simply that the Yukawas of both quark sectors are similar by having structures which are copacetic with the use of two-dimensional representations. Texture zero structures that occur in both quark sectors are easily achievable and can be understood as coming from the vevs of the two-dimensional flavon.

We summarize our choice for the $SU(5)$ matter content under $\Delta(54)$:

[5]It is the case that the labels of the flavor group representation will be boldfaced just like for $SU(5)$. The difference between the two, besides some obvious cases like the $\mathbf{5}$ and $\overline{\mathbf{5}}$, is that those of the flavor group will usually contain subscripts. These should not be confused with family indices, as in the case of $SU(5)$ singlets, which should be understood by context.

$$
\begin{aligned}
(\mathbf{10}_1,\ \mathbf{10}_2)^T,\ \mathbf{10}_3 &\overset{\Delta(54)}{\sim} \mathbf{2}_p,\ \mathbf{1}, \\
(\overline{\mathbf{5}}_1,\ \overline{\mathbf{5}}_2)^T,\ \overline{\mathbf{5}}_3 &\overset{\Delta(54)}{\sim} \mathbf{2}_r,\ \mathbf{1}, \quad p,\ r = \{1,\ 2,\ 3,\ 4\}, \\
(\mathbf{1}_1,\ \mathbf{1}_2,\ \mathbf{1}_3)^T &\overset{\Delta(54)}{\sim} \mathbf{3}_s \text{ or } \overline{\mathbf{3}}_s, \quad s = 1,\ 2\ ,
\end{aligned} \tag{3.3}
$$

included above is the case where $p = r$. We now investigate the type of Yukawa matrices we can produce based on our choice of representations. All the possibilities for the up-quark and down-quark Yukawas are summarized with just two matrices respectively

$$
\left(\begin{array}{c|c}
\begin{array}{c}\mathbf{2}_p \otimes \mathbf{2}_p \\ = \\ (\mathbf{2}_p \oplus \mathbf{1})_S \oplus \mathbf{1}_{1,A}\end{array} & \rotatebox{-90}{$\mathbf{2}_p \otimes \mathbf{1} = \mathbf{2}_p$} \\
\hline
\mathbf{1} \otimes \mathbf{2}_p = \mathbf{2}_p & \mathbf{1} \otimes \mathbf{1} = \mathbf{1}
\end{array}\right), \quad \text{and} \quad
\left(\begin{array}{c|c}
\begin{array}{c}\mathbf{2}_p \otimes \mathbf{2}_r \\ = \\ \mathbf{2}_{s'} \oplus \mathbf{2}_{s''}\end{array} & \rotatebox{-90}{$\mathbf{2}_p \otimes \mathbf{1} = \mathbf{2}_p$} \\
\hline
\mathbf{1} \otimes \mathbf{2}_r = \mathbf{2}_r & \mathbf{1} \otimes \mathbf{1} = \mathbf{1}
\end{array}\right), \tag{3.4}
$$

where $s', s'' = \{1,\ 2,\ 3,\ 4\}$. The up Yukawa must always necessarily be the left case. While for the down it may be either the right case when $p \neq r \neq s' \neq s''$, or the left when $p = r$.

Recall that at the end of Section 2.1 it was mentioned that we shall try to reproduce the texture structure and constraints of Eq. (2.5). In order to show how this can be accomplished we will make use of a toy model that uses two matter fields, $\chi,\ \psi$, and two flavons θ_1 and θ_2. The goal is then to show how to obtain the texture structure we seek from matrices constructed in the fashion shown by Eq. (3.4).

3.2 A quark sector toy model

We start with the notation that is used in this toy model and throughout other sections from now on. So far we have decided that the representations of the matter content will be split into $\mathbf{2} \oplus \mathbf{1}$ flavor representations for reasons explained in the section before. So in order to distinguish matter that transforms as a $\mathbf{2}$ from that transforming as a $\mathbf{1}$, our convention uses an underline for doublets and no such underline for singlets, e.g., we write, for the left handed quark $SU(2)$ doublet

$$
\underline{Q} \equiv \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \sim \mathbf{2}_2\ , \quad Q \equiv Q_3 \sim \mathbf{1}\ , \tag{3.5}
$$

where it is understood that the subscripts on the Q's denote flavor indices. As can be seen the notation will be cleaner than using subscripts or superscripts to denote the differences in representations. For the flavon fields the variable ϕ will be used for $\mathbf{3}$, θ for $\mathbf{2}$, and σ for either the $\mathbf{1}_1$ or the $\mathbf{1}$ representations. Any subscripts found on the flavons will aid in simply distinguishing among them.

Returning to our toy model, we shall assume that our fields should transform as shown in Table 1:

The second flavon will be used for the case where we want to show with clarity a quadratic term in flavons. For the purpose of brevity we will look at the Yukawa term for the down-type quarks, but when possible we will discuss the up-type quark Yukawa as well. The reason for looking at the down Yukawa is that it presents the most generic possible scheme since it allows both the case where $p = r$ and $p \neq r$.

Table 1: Matter content and flavons for the toy model with $p,\ r,\ s = \{1,\ 2,\ 3,\ 4\}$.

Matter	$SU(5)$	$\Delta(54)$	Flavons, $\langle vev \rangle$	$SU(5)$	$\Delta(54)$
$\underline{\psi},\ \psi$	$\overline{\mathbf{5}}$	$\mathbf{2}_r,\ \mathbf{1}$	$\theta_1,\ \begin{pmatrix} a & b \end{pmatrix}^T$	$\mathbf{1}$	$\mathbf{2}_s$
$\underline{\chi},\ \chi$	$\mathbf{10}$	$\mathbf{2}_p,\ \mathbf{1}$	$\theta_2,\ \begin{pmatrix} c & d \end{pmatrix}^T$	$\mathbf{1}$	$\mathbf{2}_s$

A Yukawa matrix for the down quark can be built from the flavon interacting with the terms $\underline{\chi}\ \underline{\psi}$, $\underline{\chi}\psi$, $\chi\underline{\psi}$, $\chi\psi$. Schematically the structure of the mass matrix is therefore

$$\begin{pmatrix} \underline{\chi\psi} & \underline{\chi}\psi \\ \chi\underline{\psi} & \chi\psi \end{pmatrix} , \tag{3.6}$$

following the same partitioning scheme as in Eq. (3.4). With the all the above in mind we shall now look at several cases involving different choices for relationships between the variables $p,\ r,\ s$. In each case we list the possible results and label them, only going up to quadratic order in flavon fields. Greek letters not previously defined are just coupling constants, and multiple such constants in front of a term indicate there are a number of different ways to get a flavor invariant. The first case where $p = r$ will be the case for our model and so we will spend some time pointing out its important features.

- $\mathbf{p = r}$. One should notice that this is the first case in Eq. (3.4). There are two possible choices we can take for the flavon; either $p = r = s$ or $p = r \neq s$.

  (*i*) $p = r = s$. The tree-level results allow for a non-zero term in the $(3, 3)$ position, useful in the case of the top quark. However this is not the only allowed contribution, since all the zero order contributions are

$$\mathcal{O}(\theta^0) :\ \alpha\ \underline{\chi}\ \underline{\psi} + \beta\ \chi\ \psi \mapsto \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix} . \tag{3.7}$$

  For a realistic model, we would not like the 2×2 locations occupied at this order. To avoid these results, we are lead to conclude that $\underline{\chi}\ \underline{\psi}$ must be charged under some symmetry that forbids it.

  For first order contributions in flavons we have:

$$\mathcal{O}(\theta) :\ \alpha\ \theta_1\ \underline{\chi}\ \underline{\psi} + \beta\ \theta_1\ \underline{\chi}\ \psi + \beta'\ \theta_1\ \chi\ \underline{\psi} \mapsto \begin{pmatrix} \alpha a & 0 & \beta b \\ 0 & \alpha b & \beta a \\ \beta' b & \beta' a & 0 \end{pmatrix} . \tag{3.8}$$

  The reader should notice how the vevs contribute to the entries above. A choice of $a = 0$ would mean that the $(1, 1)$ zero could be protected. For the up-quarks we could instead have $a = \delta^{\geq 2}$ and $b = 0$ in order to satisfy our texture constraint while hoping that symmetries disallow any 2×2 terms.

  A look at the Kronecker products reveals that the second-order in flavons can produce doublets and two types of singlets.

$$\mathcal{O}(\theta^2):\ \ (\alpha,\ \beta,\ \gamma)\,\theta_1\ \theta_2\ \underline{\chi}\ \underline{\psi}+\rho\theta_1\ \theta_2\ \underline{\chi}\ \psi+\rho'\theta_1\ \theta_2\ \chi\ \underline{\psi}+\sigma\theta_1\ \theta_2\ \chi\ \psi\mapsto \begin{pmatrix} \alpha bd & \beta bc+\gamma ad & \rho ac \\ \beta ad+\gamma bc & \alpha ac & \rho bd \\ \rho' ac & \rho' bd & \sigma(ad+bc) \end{pmatrix} . \tag{3.9}$$

The $(\alpha,\ \beta,\ \gamma)$ is there because the associated term contains three different ways to obtain a singlet, hence the three couplings (see Appendix B). It should be noted that there are in fact two different but equivalent ways to perform the product of the first term:

$$(\theta_1\ \underline{\chi})(\theta_2\ \underline{\psi}) \quad \text{and} \quad (\theta_1\ \theta_2)(\underline{\chi}\ \underline{\psi})\ . \tag{3.10}$$

Because they are equivalent, there will be no need to differentiate between them and we shall make no effort in the future to do so.

For the up-quarks, if we for the moment assumed only one flavon, say θ_1 with $b=0$, we see that we respect the $(1,1)$ zero while the $(2,2)$ can be filled in. Via the FN mechanism we are allowed to have that $a\approx\delta^2$ so that we can produce the textures allowed in Eq. (2.5). Note that this is indeed realized in our model.

(ii) $p=r\neq s$. The tree-level results should remain the same. The results differ from above in that there are no possible first-order interactions.

The zeroth order terms are given by

$$\mathcal{O}(\theta^0):\ \alpha\underline{\chi}\ \underline{\psi}+\beta\chi\ \psi\mapsto \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix} . \tag{3.11}$$

The second-order results follows much in the same way as the case where $p=r=s$;

$$\mathcal{O}(\theta^2):\ \ (\alpha,\ \beta)\,\theta_1\ \theta_2\ \underline{\chi}\ \underline{\psi}+\gamma\theta_1\ \theta_2\ \chi\ \psi\mapsto \begin{pmatrix} 0 & \alpha ad+\beta bc & 0 \\ \alpha bc+\beta ad & 0 & 0 \\ 0 & 0 & \gamma(ad+bc) \end{pmatrix} . \tag{3.12}$$

Once again there is an ambiguity about how to perform the product of the first term. Direct calculation for all possible cases shows again that the ambiguity is irrelevant because each product is equivalent. Notice that there are only two couplings, which show that there are only two ways to produce singlets for this case.

- $\mathbf{p}\neq\mathbf{r}$. Now we have the second case of Eq. (3.4). Before we go on to discuss the two possible choices, looking at Table 4, we find that

$$\mathbf{2}_p\otimes\mathbf{2}_r=\mathbf{2}_{s'}\oplus\mathbf{2}_{s''},\quad p\neq r\neq s'\neq s''\ . \tag{3.13}$$

The above has direct implications at tree-level since now there is only one result we can have and that is

$$\mathcal{O}(\theta^0):\ \alpha\chi\,\psi \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}. \tag{3.14}$$

As for the first order, a flavon can only transform as either the $\mathbf{2}_{s'}$ or the $\mathbf{2}_{s''}$. The specifics will depend on the representations, but the results will be in one of four sets of possible combinations where in each set only one matrix would be chosen:

$$\mathcal{O}(\theta):\ \alpha\theta_1\,\underline{\chi}\,\underline{\psi} \mapsto \begin{matrix} \begin{pmatrix} \alpha a & 0 & 0 \\ 0 & \alpha b & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{or} & \begin{pmatrix} 0 & \alpha b & 0 \\ \alpha a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \alpha a & 0 & 0 \\ 0 & \alpha b & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{or} & \begin{pmatrix} 0 & \alpha a & 0 \\ \alpha b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}, \tag{3.15}$$

where we list only two sets for brevity and the other two can be obtained by interchanging a and b. The "or" is because there are two possible choices for representation of θ_1, a theme that continues at second order:

$$\mathcal{O}(\theta^2):\ \alpha\theta_1\,\theta_2\,\underline{\chi}\,\underline{\psi}+\gamma\theta_1\,\theta_2\,\chi\,\psi \mapsto \begin{matrix} \begin{pmatrix} \alpha ac & 0 & 0 \\ 0 & \alpha bd & 0 \\ 0 & 0 & \gamma(ad+bc) \end{pmatrix} & \text{or} & \begin{pmatrix} 0 & \alpha bd & 0 \\ \alpha ac & 0 & 0 \\ 0 & 0 & \gamma(ad+bc) \end{pmatrix} \\ \begin{pmatrix} \alpha ac & 0 & 0 \\ 0 & \alpha bd & 0 \\ 0 & 0 & \gamma(ad+bc) \end{pmatrix} & \text{or} & \begin{pmatrix} 0 & \alpha ac & 0 \\ \alpha bd & 0 & 0 \\ 0 & 0 & \gamma(ad+bc) \end{pmatrix} \end{matrix}, \tag{3.16}$$

where to get the other set of matrices one needs only to interchange the roles of the ac terms with bd.

The above provides a small glimpse into the workings of two-dimensional representations. Although not discussed above one can tell which entries provide texture zeros by clever choice of vevs. With an understanding of the texture structure that $\Delta(54)$ can produce, we are now ready to discuss our model.

3.3 Some remarks

We had mentioned in the beginning of the section that we would let the right handed neutrinos transform as $\mathbf{3}_1$ of our flavor group. The choice is somewhat arbitrary, we could have easily chosen the representation $\overline{\mathbf{3}}_1$, $\mathbf{3}_2$, or $\overline{\mathbf{3}}_2$. Regardless, their Clebsch-Gordan (CG) coefficients are similar enough so that any choice would do with no clear advantage of one over the other.

As for the choice of two-dimensional representations for the matter content, there is some arbitrariness to this too. A look at Appendix B, focusing on the CG coefficients, will reveal that all two-dimensional representations under the case $\mathbf{2}_r \otimes \mathbf{2}_r$ have the same result. The only interesting feature occurs in the $\mathbf{2}_p \otimes \mathbf{2}_r$ with $p \neq r$ case. In terms of model building, one could make use of the fact that such a product produces two different two-dimensional representations. Even though this could be exploited in a clever fashion, the author has found that using the same two-dimensional representation throughout requires less flavons and leads, consequently, to a simpler model.

Finally, now that we have opted to use the same **2** for our model, which one should be used? Looking at Appendix B shows that taking the product of $\mathbf{2}_1 \times \mathbf{3}_1$ produces CG coefficients that contain powers of $\omega = e^{2\pi i/3}$. The same is true for the cases involving $\mathbf{2}_2$ and $\mathbf{2}_3$ with the sole exception of $\mathbf{2}_4$. It should be possible to absorb the ω into coupling constants, thus in effect we have no real advantage of using one representation over another. However, for the sake of clarity and simplicity we choose instead to use $\mathbf{2}_4$ and avoid the issue altogether.

4 The $SU(5) \otimes \Delta(54)$ model

The model has a supersymmetric background, and we assume that we are above unification scale of $SU(5)$ GUT. The matter content found in the standard model fits into $SU(5)$ representations as

$$X \sim \mathbf{10}\ , \quad \Psi \sim \overline{\mathbf{5}}\ , \quad \overline{N} \sim \mathbf{1}\ . \tag{4.1}$$

For reasons discussed in Section 3.1 we chose to have both $\overline{\mathbf{5}}$ and the **10**, in two and one-dimensional representations but kept the heavy neutrinos as three-dimensional, i.e.,

$$(X_1, X_2)^T \equiv \underline{\chi} \sim \mathbf{2}_4, \quad X_3 \equiv \chi \sim \mathbf{1}; \quad (\Psi_1, \Psi_2)^T \equiv \underline{\psi} \sim \mathbf{2}_4, \quad \Psi_3 \equiv \psi \sim \mathbf{1}; \quad \overline{N} \sim \mathbf{3}_1\ . \tag{4.2}$$

Remember that the top quark mass was a motivation for using the singlet and doublet structure for the **10**. Aside from these assignments there are other charges that we have given these fields, namely the $Z_3^u \otimes Z_2^d \otimes Z_2$ charges. The superscripts indicate that these charges are primarily given to those fields that contain the associated right handed particle.

As we will show soon, the quark and charged lepton sectors are populated mainly by three extra fields:

$$\theta_u \sim \mathbf{2}_4, \quad \theta_d \sim \mathbf{2}_4, \quad \sigma \sim \mathbf{1}\ , \tag{4.3}$$

The letters as subscripts remind us that these fields are charged under a cyclic symmetry (Z_n) with a superscript of that same letter.

On the other hand, the neutral lepton sector is primarily populated by just two three-dimensional flavons:

$$\phi \sim \overline{\mathbf{3}}_1, \quad \phi' \sim \overline{\mathbf{3}}_1\ , \tag{4.4}$$

once again indicating the appropriate $\Delta(54)$ charge. The final ingredients are the Higgs fields which include both the five- and forty-five- dimensional representations of $SU(5)$.

We may now present the super-potential, but without all the clutter of coupling constants,

$$W_{model} = W^u + W^d + W^\nu_{dirac} + W^\nu_{majorana}\ , \tag{4.5}$$

where

$$
\begin{aligned}
W^u &\approx \chi\chi\ H_u + (\theta_u\ \underline{\chi})\chi\ H_u + \theta_d^2(\theta_u\ \underline{\chi})\chi\ H_u + (\theta_u\ \underline{\chi})(\theta_u\ \underline{\chi})\ H_u + \theta_d^2(\theta_u\ \underline{\chi})(\theta_u\ \underline{\chi})\ H_u\ , \\
W^d &\approx \chi\psi\ H_d + (\theta_u\ \underline{\chi})\psi\ H_d + \theta_d^2(\theta_u\ \underline{\chi})\psi\ H_d + \chi(\theta_d\ \underline{\psi})\ H_d + (\theta_u\ \underline{\chi})(\theta_d\ \underline{\psi})\ H_d + (\theta_d\ \underline{\psi})(\sigma\ \underline{\chi}\ H_d^{45})\ , \\
W^\nu_{dirac} &\approx \phi\psi\overline{N}\ H_u + (\phi'\ \underline{\psi})\overline{N}\ H_u\ , \\
\frac{W^\nu_{majorana}}{\mathcal{M}} &\approx \phi^2\ \overline{NN} + \phi'^2\ \overline{NN}\ .
\end{aligned}
\tag{4.6}
$$

The value $\mathcal{M}$ is the Majorana mass scale that is to be determined at a later time. We have listed only terms that contribute to lowest order in their respective matrix entries. The parentheses have no bearing on how to take products under our flavor group, they merely indicate that distinct fields have the correct cyclic charges to be neutral under those charges. For a summary of the field content and their charges look at Table 2.

It should be stated that in Table 2 we could have included another cyclic symmetry Z_2^n. For this symmetry the $\overline{N}$ would be odd and so would the ϕ and ϕ' flavons. All other fields could in principle remain neutral. The model however, does not seem to require the extra symmetry and so we leave this symmetry out of the table.

The next three subsections will contain some of the finer details of our model. The first two subsections include a look at the vevs of the new fields we have introduced and a detailed look on how each of the super-potential terms populate their matrices. The last section presents the final results of our model. These phenomenological results include the masses for both light and heavy neutrinos as well as the expected corrections to the tri-bimaximal matrix.

4.1 Flavon content and vacuum expectation values

The vacuum expectation values of the flavon fields go as

$$
\langle\sigma\rangle \sim c, \quad \langle\theta\rangle \sim (a_1,\ a_2)^T, \quad \langle\phi\rangle \sim (b_1,\ b_2,\ b_3)^T\ , \tag{4.7}
$$

where the exact vevs can be found in the table discussed above. As said in the introduction, we make use of the FN mechanism, which means that each flavon vev will be suppressed by an effective mass scale (M) of some gauged interaction at much higher energies. The suppressed vevs then are postulated to go as

$$
\frac{c}{M} \approx \delta^{m+1}, \quad \frac{a_1}{M} \approx \delta^2, \quad \frac{a_2'}{M} \approx \delta, \quad \frac{b_1}{M}, \frac{b_1'}{M} \approx \delta^n, \qquad m \geq 0,\ n > 0\ , \tag{4.8}
$$

where m and n are integers. The value of m can be determined from the relative size of $v_{5,d}$, the vev of the H_d, to the vev $v_{45,d}$ of H_d^{45} by way of

$$
\left\langle \sigma H_d^{45} \right\rangle \propto \delta^m v_{45,d} \approx v_{5,d}\ . \tag{4.9}
$$

For $v_{45,u}$, the vev of H_u^{45}, we assume that $v_{5,u} \geq v_{45,u} \geq v_{45,d}$. The implicit assumption above is that $v_{45,d} \geq v_{5,d}$, otherwise we may lose our perturbative power by having a singlet with a vev that is greater or equal to the FN scale M. Finally, we must mention the relative size between $v_{5,u}$ to that of $v_{5,d}$, we expect

$$
cot(\beta) \equiv \frac{v_{5,d}}{v_{5,u}} \propto \mathcal{O}(\delta^3)\ , \tag{4.10}
$$

Table 2: Field content and charges of our model with $\omega = e^{\frac{2\pi \mathrm{i}}{3}}$. There could be another symmetry Z_2^n but it is found unnecessary.

Matter	$SU(5)$	$\Delta(54)$	Z_3^u	Z_2^d	Z_2
$\overline{N}$	$\mathbf{1}$	$\mathbf{3}_1$	1	1	1
$\underline{\psi},\ \psi$	$\overline{\mathbf{5}}$	$\mathbf{2}_4,\ \ \mathbf{1}$	$1,\ \ 1$	$-1,\ \ 1$	$1,\ \ 1$
$\underline{\chi},\ \chi$	$\mathbf{10}$	$\mathbf{2}_4,\ \ \mathbf{1}$	$\omega,\ \ 1$	$1,\ \ 1$	$1,\ \ 1$
Higgs					
$H_u,\ H_d$	$\mathbf{5},\ \overline{\mathbf{5}}$	$\mathbf{1},\ \ \mathbf{1}$	$1,\ \ 1$	$1,\ \ 1$	$1,\ \ 1$
$H_u^{45},\ H_d^{45}$	$\mathbf{45},\ \overline{\mathbf{45}}$	$\mathbf{1},\ \ \mathbf{1}$	$\omega^2,\ \ \omega$	$1,\ \ 1$	$-1,\ \ -1$
Flavons, $\langle vev\rangle$					
$\theta_u,\ \begin{pmatrix}a_1 & 0\end{pmatrix}^T$	$\mathbf{1}$	$\mathbf{2}_4$	ω^2	1	1
$\theta_d,\ \begin{pmatrix}0 & a_2'\end{pmatrix}^T$	$\mathbf{1}$	$\mathbf{2}_4$	1	-1	1
$\phi,\ \begin{pmatrix}b_1 & b_1 & 0\end{pmatrix}^T$	$\mathbf{1}$	$\overline{\mathbf{3}}_1$	1	1	1
$\phi',\ \begin{pmatrix}b_1' & 0 & 0\end{pmatrix}^T$	$\mathbf{1}$	$\overline{\mathbf{3}}_1$	1	-1	1
Singlets , $\langle vev\rangle$					
$\sigma,\ \ c$	$\mathbf{1}$	$\mathbf{1}$	1	1	-1

which would satisfy the intra-family relationship m_b/m_t.

As for the value of n, it may be determined by the size of the baryon asymmetry our model predicts from leptogenesis constraints on the lightest of the heavy neutrinos, M_1 [15]. Current approximate bounds limit the mass of $M_1 > 10^8\ GeV$ and, as we shall see at the end of this section, this limit will restrict our possible choices for n such that $n = 1,\ 2,\ 3$.

4.2 Quark Yukawas

The purpose of this section is to explore in detail the results written in Equation (4.6) for the quark sectors. We shall limit our investigation to demonstrating the origins of all Yukawa textures and the necessary coupling constants. Each super-potential contains terms that produce the leading order contribution to their Yukawa matrix. All other terms, including those which are of $\mathcal{O}(\delta^8)$ and higher for the up-quarks and $\mathcal{O}(\delta^5)$ for the down-quarks, will be neglected.

The super-potential contributions making the up Yukawa matrix are given by

$$\begin{aligned} W^u \ \approx\ & \chi\chi\ H_u + \alpha(\theta_u\ \underline{\chi})\chi\ H_u + \beta\theta_d^2(\theta_u\ \underline{\chi})\chi\ H_u + \rho(\theta_u\ \underline{\chi})(\theta_u\ \underline{\chi})\ H_u + \\ & \gamma\theta_d^2(\theta_u\ \underline{\chi})(\theta_u\ \underline{\chi})\ H_u\ . \end{aligned} \tag{4.11}$$

It should be stated that the $SU(5)$ algebra requires that any contribution to the H_u^{45} from the **10** must be anti-symmetric in flavor space. As a result, the only anti-symmetric combination

$(\underline{\chi}\ \underline{\chi})_A$ produces a $\mathbf{1}_1$. Since there are no flavon $\mathbf{1}_1$ singlets, there are no devastating low order contributions and the only contributions that can survive would be corrections to the Yukawa matrices, e.g., the lowest order correction is $\theta_u^3(\sigma\ \underline{\chi}\ \underline{\chi}\ H_u^{45})$.

In Eq. (4.11) the Greek letters $\alpha,\ \beta,\ \rho,\ \gamma$ are couplings which also aid in identifying where each term contributes to the up Yukawa matrix:

$$Y_5^{(2/3)} \approx \mathcal{O}\begin{pmatrix} 0 & \gamma\delta^6 & \beta\delta^4 \\ \gamma\delta^6 & \rho\delta^4 & \alpha\delta^2 \\ \beta\delta^4 & \alpha\delta^2 & 1 \end{pmatrix} . \tag{4.12}$$

The down-quark sector is a bit more complex, for we include both contributions due to the regular Higgs H_d and the H_d^{45}. Both contributions will be added to produce a single Yukawa matrix, and so below we only include those terms that are leading in their sum. Primes on Greek letters are for the couplings that occur in this case, and so the terms we have are

$$\begin{aligned} W^d \quad \approx \quad & \chi\psi\ H_d + \alpha'(\theta_u\ \underline{\chi})\psi\ H_d + \beta'\theta_d^2(\theta_u\ \underline{\chi})\psi\ H_d + \beta''\chi(\theta_d\ \underline{\psi})\ H_d \\ & +(\gamma',\ \gamma'')(\theta_u\ \underline{\chi})(\theta_d\ \underline{\psi})\ H_d + \rho'\ (\theta_d\ \underline{\psi})(\sigma\ \underline{\chi}\ H_d^{45})\ , \end{aligned} \tag{4.13}$$

with $(\gamma',\ \gamma'')$ meaning that there are two ways to produce singlets, each with their own couplings. In terms of δ, we have

$$Y_5^{(-1/3)} \approx \mathcal{O}\begin{pmatrix} 0 & \gamma'\delta^3 & \beta'\delta^4 \\ \gamma''\delta^3 & 0 & \alpha'\delta^2 \\ \beta''\delta & 0 & 1 \end{pmatrix} , \quad Y_{45}^{(-1/3)} \approx \mathcal{O}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho'\delta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.14}$$

Finally, with all the above results, one can construct the Yukawa matrices from the well known results of $SU(5)$ GUT models [16]:

$$\begin{aligned} Y^{(2/3)} &= Y_5^{(2/3)}, \\ Y^{(-1/3)} &= Y_5^{(-1/3)} + Y_{45}^{(-1/3)}, \\ Y^{(-1)} &= Y_5^{(-1/3)T} - 3\cdot Y_{45}^{(-1/3)}\ . \end{aligned} \tag{4.15}$$

4.3 Neutrino masses

A similar procedure as outlined in [17] is followed here. We postulate the addition of two new terms to the super-potential of the MSSM:

$$W^\nu = LH_uY^{(0)}\overline{N} + \mathcal{M}\overline{N}Y_{maj}\overline{N}. \tag{4.16}$$

The Majorana term also comes with a mass scale $\mathcal{M}$ which we suppose can come from a higher energy scale. The matrices $Y^{(0)}$ and Y_{maj} are the same Yukawa and Majorana matrices as defined in Section 2.2 and seen in Eq. (2.9). We designed the model to produce the above with the assumptions that the flavors of $\overline{\mathbf{N}}$ together form a $\mathbf{3}_1$ under our flavor group. To accomplish the task, we employed the use of two three-dimensional representations ϕ and ϕ', whose details can be found in Table 2.

Our model, Eq. (4.6), produces the Dirac term

$$W^{\nu}_{dirac} \approx \phi\psi\overline{N}\ H_u + (\phi'\ \underline{\psi})\overline{N}\ H_u\ , \tag{4.17}$$

rewritten here for convenience. The resulting Yukawa matrix is

$$Y^{(0)} \approx \frac{1}{M}\begin{pmatrix} 0 & 0 & b_1' \\ 0 & b_1' & 0 \\ b_1 & b_1 & 0 \end{pmatrix} . \tag{4.18}$$

In the above there are no coupling constants included, because they are of $\mathcal{O}(1)$ and can be simply absorbed by their respective vevs. In principle, it would be possible to get tri-bimaximal mixing in the case that $\mathcal{O}(b_1') \neq \mathcal{O}(b_1)$. However, if this is the case, and it is carried through to the Majorana matrix, then the light neutrino matrix $\mathcal{Y}_\nu$ would contain entries that are sums of various powers in δ. A somewhat simple calculation will show that this is true.

In cases like these, it is difficult to diagonalize by $\mathcal{U}_{tri-bi}$, since either careful cancellations are needed in the various powers in δ or some explanation for the complexity of the coupling constants should be given. To avoid such a complication, it is found best to assume that $\mathcal{O}(b_1') = \mathcal{O}(b_1)$. In fact, its found that much more elegant results can arise when one assumes that $b_1' = b_1$ and so this is the assumption we shall make.

The Majorana contributions terms, found in Eq. (4.6), are

$$\frac{W^{\nu}_{majorana}}{\mathcal{M}} \approx \phi^2\ \overline{NN} + \phi'^2\ \overline{NN}\ . \tag{4.19}$$

The Majorana matrix is then

$$\frac{Y_{maj}}{\delta^{2n}} \approx \begin{pmatrix} \alpha & \sigma & \rho \\ \sigma & \alpha & \rho \\ \rho & \rho & \beta \end{pmatrix} + \begin{pmatrix} \alpha' & 0 & 0 \\ 0 & 0 & \rho' \\ 0 & \rho' & 0 \end{pmatrix} . \tag{4.20}$$

The unprimed Greek letters correspond to couplings for the ϕ and primed letters for ϕ'. Do not confuse these parameters for those written down in the quark sector. Just as before, they are coupling constants resulting from the number of ways one can get a singlet term. Notice that the vevs of the flavons are included, but found within δ^{2n} by Eq. (4.8). The best choices for the parameters above seem to be

$$\alpha = \sigma = 0, \quad \rho = -\beta = \rho' = 1, \quad |\alpha'| = .100 \pm .004\ . \tag{4.21}$$

The parameter α' can control the value of ratio of the mass squared differences found in Eq. (2.7). The choice of $|\alpha'| = .1$ produces exactly the ratio of 32 that fits current data.

4.4 Phenomenological Results

We have successfully produced Yukawa matrices with entries of the same order as we had sought in Eq. (2.5). We have even produced a set of matrices for the neutrinos that together produce a light neutrino matrix that can be diagonalized by the tri-bimaximal matrix. Here we take things a step further and try to reproduce the SM results and find values for neutrino sector.

The first step is to reproduce the results of the SM extrapolated to the energy scale of 2×10^{16} GeV [18]. We have seen that for the quark sector, based on our super-potential terms, there are ten parameters to be determined. One of these parameters is found to be irrelevant and so left equal to one (the $(1,3)$ and $(3,1)$ entries of the up Yukawas). We are then left with nine that are chosen such that they reproduce masses and the CKM matrix which means only seven constraints and so two free parameters. The last two parameters are chosen such that they at the same time respect the mass of the down-quark (due to higher order corrections) and also fit the limits of the experimental results for the solar angle of the lepton mixing matrix. Our model has some sensitivity to the values of the final parameters which explains the errors we placed on the predicted angles.

As for the neutrinos, we have discussed these free parameters and because of the constraints imposed both by data and the tri-bimaximal matrix, we have only one parameter (what we called α' in the neutrino analysis).

Quark Sector:

$$Y^{(2/3)} \approx \begin{pmatrix} 0 & 1.1\delta^6 & \delta^4 \\ 1.1\delta^6 & \delta^4 & -1.8\delta^2 \\ \delta^4 & -1.8\delta^2 & 1 \end{pmatrix} , \quad m_u \approx v_{5,u} \begin{pmatrix} 2.7\delta^8 & & \\ & 2.3\delta^4 & \\ & & 1 \end{pmatrix} , \tag{4.22}$$

and

$$Y^{(-1/3)} \approx \begin{pmatrix} 0 & .5\delta^3 & .5\delta^4 \\ -.3\delta^3 & .5\delta^2 & -.6\delta^2 \\ -.5\delta & 0 & 1 \end{pmatrix} , \quad m_d \approx v_{5,d} \begin{pmatrix} .6\delta^4 & & \\ & .5\delta^2 & \\ & & 1 \end{pmatrix} . \tag{4.23}$$

Diagonalization also reproduces a CKM matrix (U_{ckm}) consistent with data extrapolated to the GUT scale.

Lepton Sector: $SU(5)$ with H_d^{45} guarantees our successful reproduction of the masses

$$Y^{(-1)} \approx \begin{pmatrix} 0 & -.3\delta^3 & -.5\delta \\ .5\delta^3 & -1.5\delta^2 & 0 \\ .5\delta^4 & -.6\delta^2 & 1 \end{pmatrix} , \quad m_e \approx v_{5,d} \begin{pmatrix} .2\delta^4 & & \\ & 1.5\delta^2 & \\ & & 1 \end{pmatrix} . \tag{4.24}$$

As for the neutrinos, we have found that

$$Y^{(0)} \approx \delta^{2n} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} , \quad Y_{maj} \approx \delta^{2n} \begin{pmatrix} \alpha' & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix} , \quad |\alpha'| = .10 , \tag{4.25}$$

Using the light neutrino approximation and using $\alpha' = \pm .10$ we obtain

$$\mathcal{Y}_\nu \approx \frac{v_{5,u}^2}{2\mathcal{M}\Delta} \begin{pmatrix} 0 & \Delta & \Delta \\ \Delta & -1 & 1+\Delta \\ \Delta & 1+\Delta & -1 \end{pmatrix} , \quad m_\nu \approx \frac{v_{5,u}^2}{2\mathcal{M}\Delta} \begin{pmatrix} \Delta & & \\ & 2\Delta & \\ & & 2+\Delta \end{pmatrix} , \tag{4.26}$$

where we remind the reader that m_ν is the scale for light neutrino masses. The value of Δ is such that $\Delta \approx .22$ for $\alpha' = -.10$ and $\Delta \approx -.18$ for $\alpha' = .10$. We predict that the mass scale $\mathcal{M}$ is

$$\mathcal{M} \approx 3 \times 10^{15}\ GeV\ , \tag{4.27}$$

a value that is one order away from our GUT model scale. Results that follow are independent on the sign of α'. Both the corrections to the tri-bimaximal matrix and the masses of the light neutrinos (normal hierarchy) are predicted to be

$$\begin{array}{rcll} |\nu_e\rangle & \approx & .83|\nu_1\rangle + .57|\nu_2\rangle - .13|\nu_3\rangle, & m_{\nu,1} \approx 5 \times 10^{-3}\ eV \\ |\nu_\mu\rangle & \approx & -.47|\nu_1\rangle + .53|\nu_2\rangle - .71|\nu_3\rangle, & m_{\nu,2} \approx 1 \times 10^{-2}\ eV \\ |\nu_\tau\rangle & \approx & -.33|\nu_1\rangle + .64|\nu_2\rangle + .70|\nu_3\rangle, & m_{\nu,3} \approx 5 \times 10^{-2}\ eV \end{array}\ , \tag{4.28}$$

where we want to make it clear that

$$\frac{m_{\nu,2}}{m_{\nu,1}} = 2\ , \quad \frac{m_{\nu,3}}{m_{\nu,1}} = 10\ , \quad \text{and} \quad \sum_i m_{\nu,i}\ =\ 6.5 \times 10^{-2}\ eV\ . \tag{4.29}$$

While we predict that the masses for the heavier neutrinos are

$$M^{\nu}_{heavy} \approx \delta^{2n} \begin{pmatrix} 9.7 \times 10^{12}\ GeV & & \\ & 2.2 \times 10^{14}\ GeV & \\ & & 3.4 \times 10^{14}\ GeV \end{pmatrix}\ , \tag{4.30}$$

i.e., two masses are nearly degenerate. As mentioned earlier, the value of n could be chosen such that the masses are consistent with limits posed by leptogenesis responsible for the baryon asymmetry [15],

$$M_1 \equiv 9.7\ \delta^{2n} \times 10^{12}\ GeV > 10^8\ GeV\ \ \rightarrow\ \ n =\ 1,\ 2,\ 3\ . \tag{4.31}$$

Because the corrections for the tri-bimaximal matrix are obtained from diagonalization of the charged lepton Yukawa, care must be taken so that the angles obtained are well within experimental limits [11]:

$$|\theta_{13}|\ <\ 11.4^o, \quad \theta_\odot\ \approx\ 34.43^{+1.35\ o}_{-1.22}\ , \quad 36.8^o < -\theta_{atm} < 53.2^o\ . \tag{4.32}$$

With the above in mind, we predict (and postdict) that

$$\theta_{13}\ \approx\ -7.31^{+0.60\ o}_{-1.75}\ , \quad \theta_\odot\ \approx\ 34.46^{+1.02\ o}_{-1.52}\ , \quad \theta_{atm}\ \approx\ -45.15^{+0.04\ o}_{-0.10}\ . \tag{4.33}$$

We should mention that the reactor angle (θ_{13}) is somewhat large. The origin for this is the $(1,3)$ position of the charged lepton Yukawa, which leads to a rotation angle (from diagonalizing the Yukawa) "θ_{13}" that is comparable to the "θ_{12}" rotation angle. Now we can track the phases by following the guidelines given in [19], which provides methods for determining how many free phases there are and where in the Yukawas they may be located. We then find that the $(1,3)$ position for the charged lepton Yukawa could have a phase. So the reactor angle, being a sum of two comparable angles (as stated earlier) with a phase difference between them, could be such that in general $0^o \lesssim -\theta_{13} \lesssim 7.31^o$.

Table 3: Changes to the field charges from previous model.

Matter	$SU(5)$	$\Delta(54)$	Z_3^u	Z_2^d	Z_2
χ	$\mathbf{10}$	1	1	1	-1
ψ	$\overline{\mathbf{5}}$	1	1	1	-1
Flavons, $\langle vev \rangle$					
ϕ, $\begin{pmatrix} b_1 & b_1 & 0 \end{pmatrix}^T$	$\mathbf{1}$	$\overline{\mathbf{3}}_1$	1	1	-1

5 A possible modification

We present here a modification to our previous model that is based on the possibility that the flavor singlets of the matter content may be charged under the Z_2 of our previous model. Table 3 contains only the changes we expect to make to the model.

Notice that one of the **3** flavons that was previously neutral is now odd by necessity (unless we change the neutrino terms) under the Z_2 charge. As for the super-potential, the major changes are the terms that contribute to the 1×2 and 2×1 blocks of the Yukawa matrices, no changes are found for the neutrino terms.

$$W_{model} = W^u + W^d + W^{\nu}_{dirac} + W^{\nu}_{majorana} \ , \tag{5.1}$$

where

$$\begin{aligned}
W^u &\approx \chi\chi \ H_u + (\theta_u \ \underline{\chi})(\sigma \ \chi) \ H_u + \theta_d^2(\theta_u \ \underline{\chi})(\sigma \ \chi) \ H_u + (\theta_u \ \underline{\chi})(\theta_u \ \underline{\chi}) \ H_u + \theta_d^2(\theta_u \ \underline{\chi})(\theta_u \ \underline{\chi}) \ H_u \ , \\
W^d &\approx \chi\psi \ H_d + \theta_d^2(\underline{\chi} \ \psi \ H_d^{45}) + (\sigma \ \chi)(\theta_d \ \underline{\psi}) \ H_d + (\theta_u \ \underline{\chi})(\theta_d \ \underline{\psi}) \ H_d + (\theta_d \ \underline{\psi})(\sigma \ \underline{\chi} \ H_d^{45}) \ , \\
W^{\nu}_{dirac} &\approx \phi\psi\overline{N} \ H_u + (\phi' \ \underline{\psi})\overline{N} \ H_u \ , \\
\frac{W^{\nu}_{majorana}}{\mathcal{M}} &\approx \phi^2 \ \overline{N}\overline{N} + \phi'^2 \ \overline{N}\overline{N} \ .
\end{aligned} \tag{5.2}$$

The vev $M\delta^{m+1}$ of the σ field still depends heavily on the relative size of the two down-type Higgs' vevs. Since we cannot know for sure the value of these, all we can do is to write down the form of the Yukawa matrix as a function of m:

$$Y_5^{(2/3)} \approx \mathcal{O} \begin{pmatrix} 0 & \gamma\delta^6 & \beta\delta^{m+5} \\ \gamma\delta^6 & \rho\delta^4 & \alpha\delta^{m+3} \\ \beta\delta^{m+5} & \alpha\delta^{m+3} & 1 \end{pmatrix} , \tag{5.3}$$

and

$$Y_5^{(-1/3)} \approx \mathcal{O}\begin{pmatrix} 0 & \gamma'\delta^3 & 0 \\ \gamma''\delta^3 & 0 & 0 \\ \beta''\delta^{2+m} & 0 & 1 \end{pmatrix} \;, \quad Y_{45}^{(-1/3)} \approx \mathcal{O}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho'\delta^2 & \alpha'\delta^2 \\ 0 & 0 & 0 \end{pmatrix} \;. \tag{5.4}$$

We have decided to keep the same Greek letters as before because they still correspond to the same terms of our previous model with the sole exception of α' which now originates from the 45 Higgs. We take as a concrete example the case where $m = 1$:

$$Y^{(2/3)} \approx \mathcal{O}\begin{pmatrix} 0 & 2.5\delta^6 & \delta^6 \\ 2.5\delta^6 & 2.3\delta^4 & \delta^4 \\ \delta^6 & \delta^4 & 1 \end{pmatrix}, \quad Y^{(-1/3)} \approx \mathcal{O}\begin{pmatrix} 0 & .5\delta^3 & 0 \\ .6\delta^3 & .5\delta^2 & 1.3\delta^2 \\ -4\delta^3 & 0 & 1 \end{pmatrix} \;. \tag{5.5}$$

Leaving out many of the details and keeping all other results the same, the mixing angles for this case of our model become

$$\theta_{13} \;\approx\; -1.05^{+2.80}_{-1.16}{}^{\;o}, \quad \theta_{\odot} \;\approx\; 34.48^{+0.52}_{-1.25}{}^{\;o}, \quad \theta_{atm} \;\approx\; -\,44.47 \pm .01\;^{o}\;. \tag{5.6}$$

6 Conclusion

The goal of this paper was to create a model for an $SU(5)$ GUT that can reproduce all known data with the use of a flavor group $\Delta(54)$. We began with the SM in the form of mass hierarchies and one mixing matrix. With these in mind, we found constraints, Eq. (2.5), on the form of the texture structures the quark Yukawa matrices must have.

A look at the flavor group and the aid of a toy model allowed us to see how one can possibly reproduce these texture structures. The lepton sector, as far as neutrinos are concerned, was obtained with a minimalist approach of introducing only the fewest number of new flavons and fairly simple vev structures. From these principles we have succeeded in producing a viable model for neutrinos that can satisfy all constraints provided by experiments.

Finally, we provided a possible alternative that would be viable for more strict assumptions as to the relationship between the vevs of the H_d and H_d^{45}. The model should be considered in every respect as viable as the first one, but contains the bonus of needing less parameters.

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Appendix

A A comparison of $\Delta(54)$ with $\Delta(27)$

There is a great deal of similarities between these two groups, but $\Delta(27)$ has has been used as a flavor group in a number of investigations. Likely this has been the case because, as we shall show here, its structure is not as complex as that of $\Delta(54)$. The richness in its structure actually starts with its presentations which shares all the same features of $\Delta(27)$ but with the addition of two conjugations and two second-order elements. To see this let's look at the presentation for $\Delta(27)$ [20]:

$$\begin{aligned} \Delta(27) \sim (Z_3 \otimes Z_3) \rtimes Z_3 : \qquad a^3 \;=\; c^3 \;=\; d^3 \;=\; 1, \\ cd \;=\; dc, \\ aca^{-1} \;=\; c^{-1}d^{-1}, \;\; ada^{-1} \;=\; c\;. \end{aligned} \tag{A.1}$$

We clearly see that there are three third-order elements and as expected two of them commute. A look at Table 4 shows that the group includes nine one-dimensional and two three-dimensional representations. Now the presentation of $\Delta(54)$ [21] is:

$$\begin{aligned} \Delta(54) \sim (Z_3 \otimes Z_3) \rtimes S_3 : \;\; a^3 \;=\; b^2 \;=\; (ab)^2 \;=\; c^3 \;=\; d^3 \;=\; 1, \\ cd \;=\; dc, \\ aca^{-1} \;=\; c^{-1}d^{-1}, \;\; ada^{-1} \;=\; c, \\ bcb^{-1} \;=\; d^{-1}, \qquad bdb^{-1} \;=\; c^{-1}\;. \end{aligned} \tag{A.2}$$

It is clear from the above that $\Delta(54)$ has not only third-order operators but also second-order ones, which adds to its complexity. As a result, looking at Appendix B, one sees that it has not only one and three-dimensional representations but also two-dimensional representations.

A summary of these facts and a quick description of the Kronecker products is contained in Table 4 found below.

Table 4: Summary of some of the differences between $\Delta(27)$ and $\Delta(54)$. The values $r,\ s,\ p,\ t = 1,2,3,4$

$\Delta(27)$	$\Delta(54)$
nine **1**- and two **3**-dimensional reps.	two **1**-, four **2**-, and four **3**-dimensional reps.
	$\mathbf{2}_p \otimes \mathbf{2}_r = \mathbf{2}_s \oplus \mathbf{2}_t, p \neq r \neq s \neq t$
$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}$	$\mathbf{2}_r \otimes \mathbf{2}_r = (\mathbf{1} \oplus \mathbf{2}_r)_S \oplus \mathbf{1}_{1,A}$
$\mathbf{3} \otimes \overline{\mathbf{3}} = \sum^9 \mathbf{1}$	$\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}$
	$\mathbf{3} \otimes \overline{\mathbf{3}} = (\mathbf{1} \text{ or } \mathbf{1}_1) \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{2}_4$

B Flavor symmetry $\Delta(54)$

The flavor group under consideration is a special case of $\Delta(6n^2)$, where $n = 3$. A complete study of $\Delta(6n^2)$ can be found in Ref. [21]. From this source we may obtain the character table, Kronecker products, and the Clebsch-Gordan coefficients. We list some results here, specifically the character tables and Kronecker products.

B.1 Character table

The character table reveals a rich structure behind this group. One clearly sees that there are one, two and three-dimensional representations. Notice that the three-dimensional representations are complex, where the conjugates are indicated by a bar.

Character table of $\Delta(54)$										
$n=3$	$1C_1$	$1C_1^{(1)}$	$1C_1^{(2)}$	$6C_1$	$6C_2^{(0)}$	$6C_2^{(1)}$	$6C_2^{(2)}$	$9C_3^{(0)}$	$9C_3^{(1)}$	$9C_3^{(2)}$
$\mathbf{1}$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_1$	1	1	1	1	1	1	1	-1	-1	-1
$\mathbf{2}_1$	2	2	2	2	-1	-1	-1	0	0	0
$\mathbf{2}_2$	2	2	2	-1	-1	2	-1	0	0	0
$\mathbf{2}_3$	2	2	2	-1	-1	-1	2	0	0	0
$\mathbf{2}_4$	2	2	2	-1	2	-1	-1	0	0	0
$\mathbf{3}_1$	3	3ω	$3\omega^2$	0	0	0	0	1	ω^2	ω
$\bar{\mathbf{3}}_1$	3	$3\omega^2$	3ω	0	0	0	0	1	ω	ω^2
$\mathbf{3}_2$	3	3ω	$3\omega^2$	0	0	0	0	-1	$-\omega^2$	$-\omega$
$\bar{\mathbf{3}}_2$	3	$3\omega^2$	3ω	0	0	0	0	-1	$-\omega$	$-\omega^2$

Table 5: $\omega = e^{\frac{2\pi i}{3}}$.

B.2 Kronecker products

In order to build a theory with invariant quantities, it's necessary to know how products of representations break down into irreducible representations:

$$
\begin{aligned}
\mathbf{1}_1 \otimes \mathbf{1}_1 &= \mathbf{1} \\
\mathbf{1}_1 \otimes \mathbf{2}_1 &= \mathbf{2}_1 \\
\mathbf{1}_1 \otimes \mathbf{2}_2 &= \mathbf{2}_2 \\
\mathbf{1}_1 \otimes \mathbf{2}_3 &= \mathbf{2}_3 \\
\mathbf{1}_1 \otimes \mathbf{2}_4 &= \mathbf{2}_4 \\
\mathbf{1}_1 \otimes \mathbf{3}_1 &= \mathbf{3}_2 \\
\mathbf{1}_1 \otimes \bar{\mathbf{3}}_1 &= \bar{\mathbf{3}}_2 \\
\mathbf{1}_1 \otimes \mathbf{3}_2 &= \mathbf{3}_1 \\
\mathbf{1}_1 \otimes \bar{\mathbf{3}}_2 &= \bar{\mathbf{3}}_1
\end{aligned}
\qquad
\begin{aligned}
\mathbf{2}_1 \otimes \mathbf{2}_1 &= (\mathbf{1} + \mathbf{2}_1)_S + (\mathbf{1}_1)_A \\
\mathbf{2}_1 \otimes \mathbf{2}_2 &= \mathbf{2}_3 + \mathbf{2}_4 \\
\mathbf{2}_1 \otimes \mathbf{2}_3 &= \mathbf{2}_2 + \mathbf{2}_4 \\
\mathbf{2}_1 \otimes \mathbf{2}_4 &= \mathbf{2}_2 + \mathbf{2}_3 \\
\mathbf{2}_1 \otimes \mathbf{3}_1 &= \mathbf{3}_1 + \mathbf{3}_2 \\
\mathbf{2}_1 \otimes \bar{\mathbf{3}}_1 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2 \\
\mathbf{2}_1 \otimes \mathbf{3}_2 &= \mathbf{3}_1 + \mathbf{3}_2 \\
\mathbf{2}_1 \otimes \bar{\mathbf{3}}_2 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2
\end{aligned}
\qquad
\begin{aligned}
\mathbf{2}_2 \otimes \mathbf{2}_2 &= (\mathbf{1} + \mathbf{2}_2)_S + (\mathbf{1}_1)_A \\
\mathbf{2}_2 \otimes \mathbf{2}_3 &= \mathbf{2}_1 + \mathbf{2}_4 \\
\mathbf{2}_2 \otimes \mathbf{2}_4 &= \mathbf{2}_1 + \mathbf{2}_3 \\
\mathbf{2}_2 \otimes \mathbf{3}_1 &= \mathbf{3}_1 + \mathbf{3}_2 \\
\mathbf{2}_2 \otimes \bar{\mathbf{3}}_1 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2 \\
\mathbf{2}_2 \otimes \mathbf{3}_2 &= \mathbf{3}_1 + \mathbf{3}_2 \\
\mathbf{2}_2 \otimes \bar{\mathbf{3}}_2 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2
\end{aligned}
$$

$$\begin{aligned}
\mathbf{2}_3 \otimes \mathbf{2}_3 &= (\mathbf{1}+\mathbf{2}_3)_S + (\mathbf{1}_1)_A \\
\mathbf{2}_3 \otimes \mathbf{2}_4 &= \mathbf{2}_1 + \mathbf{2}_2 \\
\mathbf{2}_3 \otimes \mathbf{3}_1 &= \mathbf{3}_1 + \mathbf{3}_2 \\
\mathbf{2}_3 \otimes \bar{\mathbf{3}}_1 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2 \\
\mathbf{2}_3 \otimes \mathbf{3}_2 &= \mathbf{3}_1 + \mathbf{3}_2 \\
\mathbf{2}_3 \otimes \bar{\mathbf{3}}_2 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2
\end{aligned}
\qquad
\begin{aligned}
\mathbf{2}_4 \otimes \mathbf{2}_4 &= (\mathbf{1}+\mathbf{2}_4)_S + (\mathbf{1}_1)_A \\
\mathbf{2}_4 \otimes \mathbf{3}_1 &= \mathbf{3}_1 + \mathbf{3}_2 \\
\mathbf{2}_4 \otimes \bar{\mathbf{3}}_1 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2 \\
\mathbf{2}_4 \otimes \mathbf{3}_2 &= \mathbf{3}_1 + \mathbf{3}_2 \\
\mathbf{2}_4 \otimes \bar{\mathbf{3}}_2 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2
\end{aligned}$$

$$\begin{aligned}
\mathbf{3}_1 \otimes \mathbf{3}_1 &= (\bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_1)_S + (\bar{\mathbf{3}}_2)_A \\
\mathbf{3}_1 \otimes \bar{\mathbf{3}}_1 &= \mathbf{1} + \mathbf{2}_1 + \mathbf{2}_2 + \mathbf{2}_3 + \mathbf{2}_4 \\
\mathbf{3}_1 \otimes \mathbf{3}_2 &= \bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_2 + \bar{\mathbf{3}}_2 \\
\mathbf{3}_1 \otimes \bar{\mathbf{3}}_2 &= \mathbf{1}_1 + \mathbf{2}_1 + \mathbf{2}_2 + \mathbf{2}_3 + \mathbf{2}_4 \\
\mathbf{3}_2 \otimes \mathbf{3}_2 &= (\bar{\mathbf{3}}_1 + \bar{\mathbf{3}}_1)_S + (\bar{\mathbf{3}}_2)_A \\
\mathbf{3}_2 \otimes \bar{\mathbf{3}}_2 &= \mathbf{1} + \mathbf{2}_1 + \mathbf{2}_2 + \mathbf{2}_3 + \mathbf{2}_4
\end{aligned}$$

B.3 Clebsch-Gordan Coefficients

We first must define a vector space of each of the irreducible representations. These will demonstrate how a vector transforms under the generators a, b, and c of the irreducible representations:

$$\mathbf{3}_1 \; : \; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}_a , \; \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}_b , \; \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \\ x_3 \end{pmatrix}_c , \qquad \bar{\mathbf{3}}_1 \; : \; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}_a , \; \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}_b , \; \begin{pmatrix} \omega^2 x_1 \\ \omega x_2 \\ x_3 \end{pmatrix}_c ,$$

$$\mathbf{3}_2 \; : \; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}_a , \; \begin{pmatrix} -x_3 \\ -x_2 \\ -x_1 \end{pmatrix}_b , \; \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \\ x_3 \end{pmatrix}_c , \qquad \bar{\mathbf{3}}_2 \; : \; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}_a , \; \begin{pmatrix} -x_3 \\ -x_2 \\ -x_1 \end{pmatrix}_b , \; \begin{pmatrix} \omega^2 x_1 \\ \omega x_2 \\ x_3 \end{pmatrix}_c ,$$

$$\mathbf{2}_1 \; : \; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_a , \; \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_b , \; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_c , \qquad \mathbf{2}_2 \; : \; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_a , \; \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_b , \; \begin{pmatrix} \omega^2 x_1 \\ \omega x_2 \end{pmatrix}_c ,$$

$$\mathbf{2}_3 \; : \; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_a , \; \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_b , \; \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_c , \qquad \mathbf{2}_4 \; : \; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_a , \; \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_b , \; \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_c ,$$

$$\mathbf{1}_1 \; : \; (x) \mapsto (x)_a , \; (-x)_b , \; (x)_c \; .$$

With the above mappings defined, it becomes possible find the outcomes of taking the product of any two representations. The list below is not exhaustive, but we include those that are important to this paper.

- $x \otimes y$: $\mathbf{1}_1 \otimes \mathbf{1}_1 = \mathbf{1}$

$$x \otimes y = xy \; . \tag{B.1}$$

- $x \otimes y$: $\mathbf{1}_1 \otimes \mathbf{2}_r = \mathbf{2}_r, \;\; r = 1,\; 2,\; 3,\; 4.$

$$x \otimes y = \begin{pmatrix} xy_1 \\ -xy_2 \end{pmatrix} \ . \tag{B.2}$$

- $x \otimes y$: $\mathbf{2}_r \otimes \mathbf{2}_r = (\mathbf{1} \oplus \mathbf{2}_r)_S \oplus (\mathbf{1}_1)_A$

$$x \otimes y = \left[\frac{1}{\sqrt{2}}(x_1y_2 + x_2y_1) \oplus \begin{pmatrix} x_2y_2 \\ x_1y_1 \end{pmatrix} \right]_S \oplus \left[\frac{1}{\sqrt{2}}(x_1y_2 - x_2y_1) \right]_A \ . \tag{B.3}$$

- $x \otimes y$: $\mathbf{2}_1 \otimes \mathbf{2}_2 = \mathbf{2}_3 \oplus \mathbf{2}_4$

$$x \otimes y = \begin{pmatrix} x_2y_2 \\ x_1y_1 \end{pmatrix} \oplus \begin{pmatrix} x_1y_2 \\ x_2y_1 \end{pmatrix} \ . \tag{B.4}$$

- $x \otimes y$: $\mathbf{2}_1 \otimes \mathbf{2}_3 = \mathbf{2}_2 \oplus \mathbf{2}_4$

$$x \otimes y = \begin{pmatrix} x_2y_2 \\ x_1y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2y_1 \\ x_1y_2 \end{pmatrix} \ . \tag{B.5}$$

- $x \otimes y$: $\mathbf{2}_1 \otimes \mathbf{2}_4 = \mathbf{2}_2 \oplus \mathbf{2}_3$

$$x \otimes y = \begin{pmatrix} x_1y_2 \\ x_2y_1 \end{pmatrix} \oplus \begin{pmatrix} x_1y_1 \\ x_2y_2 \end{pmatrix} \ . \tag{B.6}$$

- $x \otimes y$: $\mathbf{2}_2 \otimes \mathbf{2}_3 = \mathbf{2}_1 \oplus \mathbf{2}_4$

$$x \otimes y = \begin{pmatrix} x_2y_2 \\ x_1y_1 \end{pmatrix} \oplus \begin{pmatrix} x_1y_2 \\ x_2y_1 \end{pmatrix} \ . \tag{B.7}$$

- $x \otimes y$: $\mathbf{2}_2 \otimes \mathbf{2}_4 = \mathbf{2}_1 \oplus \mathbf{2}_3$

$$x \otimes y = \begin{pmatrix} x_1y_1 \\ x_2y_2 \end{pmatrix} \oplus \begin{pmatrix} x_1y_2 \\ x_2y_1 \end{pmatrix} \ . \tag{B.8}$$

- $x \otimes y$: $\mathbf{2}_3 \otimes \mathbf{2}_4 = \mathbf{2}_1 \oplus \mathbf{2}_2$

$$x \otimes y = \begin{pmatrix} x_1y_2 \\ x_2y_1 \end{pmatrix} \oplus \begin{pmatrix} x_1y_1 \\ x_2y_2 \end{pmatrix} \ . \tag{B.9}$$

- $x \otimes y$: $\mathbf{2}_1 \otimes \mathbf{3}_1 = \mathbf{3}_1 \oplus \mathbf{3}_2$
  $\mathbf{2}_1 \otimes \overline{\mathbf{3}}_1 = \overline{\mathbf{3}}_1 \oplus \overline{\mathbf{3}}_2$

$$x \otimes y = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1y_1 + \omega^2 x_2y_1 \\ \omega x_1y_2 + \omega x_2y_2 \\ \omega^2 x_1y_3 + x_2y_3 \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} x_1y_1 - \omega^2 x_2y_1 \\ \omega x_1y_2 - \omega x_2y_2 \\ \omega^2 x_1y_3 - x_2y_3 \end{pmatrix} \ . \tag{B.10}$$

- $x \otimes y$: $\mathbf{2}_4 \otimes \mathbf{3}_1 = \mathbf{3}_1 \oplus \mathbf{3}_2$

$$x \otimes y = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1y_3 + x_2y_2 \\ x_1y_1 + x_2y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} x_1y_3 - x_2y_2 \\ x_1y_1 - x_2y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \ . \tag{B.11}$$

- $x \otimes y$: $\mathbf{2}_4 \otimes \overline{\mathbf{3}}_1 = \overline{\mathbf{3}}_1 \oplus \overline{\mathbf{3}}_2$

$$x \otimes y = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1y_2 + x_2y_3 \\ x_1y_3 + x_2y_1 \\ x_1y_1 + x_2y_2 \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} x_1y_2 - x_2y_3 \\ x_1y_3 - x_2y_1 \\ x_1y_1 - x_2y_2 \end{pmatrix} . \tag{B.12}$$

- $x \otimes y$: $\mathbf{3}_1 \otimes \mathbf{3}_1 = (\overline{\mathbf{3}}_1 \oplus \overline{\mathbf{3}}_1)_S \oplus (\overline{\mathbf{3}}_2)_A$

$$x \otimes y = \left[\begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} x_2y_3 + x_3y_2 \\ x_3y_1 + x_1y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix} \right]_S \oplus \left[\frac{1}{\sqrt{2}} \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \right]_A . \tag{B.13}$$

- $x \otimes y$: $\mathbf{3}_1 \otimes \overline{\mathbf{3}}_1 = \mathbf{1} \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{2}_4$

$$\begin{aligned} x \otimes y \quad = \quad & \frac{1}{\sqrt{3}} (x_1y_1 + x_2y_2 + x_3y_3) \;\oplus\; \frac{1}{\sqrt{3}} \begin{pmatrix} x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3 \\ \omega x_1y_1 + \omega^2 x_2y_2 + x_3y_3 \end{pmatrix} \\ & \oplus \frac{1}{\sqrt{3}} \begin{pmatrix} x_1y_2 + \omega^2 x_2y_3 + \omega x_3y_1 \\ x_3y_2 + \omega^2 x_2y_1 + \omega x_1y_3 \end{pmatrix} \;\oplus\; \frac{1}{\sqrt{3}} \begin{pmatrix} x_2y_1 + \omega^2 x_3y_2 + \omega x_1y_3 \\ x_2y_3 + \omega^2 x_1y_2 + \omega x_3y_1 \end{pmatrix} \\ & \oplus \frac{1}{\sqrt{3}} \begin{pmatrix} x_3y_2 + x_2y_1 + x_1y_3 \\ x_2y_3 + x_1y_2 + x_3y_1 \end{pmatrix} . \end{aligned} \tag{B.14}$$

- $x \otimes y$: $\mathbf{3}_1 \otimes \mathbf{3}_2 = \overline{\mathbf{3}}_2 \oplus \overline{\mathbf{3}}_2 \oplus \overline{\mathbf{3}}_1$

$$x \otimes y = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} x_2y_3 + x_3y_2 \\ x_3y_1 + x_1y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} . \tag{B.15}$$

- $x \otimes y$: $\mathbf{3}_1 \otimes \overline{\mathbf{3}}_2 = \mathbf{1}_1 \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{2}_4$

$$\begin{aligned} x \otimes y \quad = \quad & \frac{1}{\sqrt{3}} (x_1y_1 + x_2y_2 + x_3y_3) \;\oplus\; \frac{1}{\sqrt{3}} \begin{pmatrix} x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3 \\ -\omega x_1y_1 - \omega^2 x_2y_2 - x_3y_3 \end{pmatrix} \\ & \oplus \frac{1}{\sqrt{3}} \begin{pmatrix} x_1y_2 + \omega^2 x_2y_3 + \omega x_3y_1 \\ -x_3y_2 - \omega^2 x_2y_1 - \omega x_1y_3 \end{pmatrix} \;\oplus\; \frac{1}{\sqrt{3}} \begin{pmatrix} -x_2y_1 - \omega^2 x_3y_2 - \omega x_1y_3 \\ x_2y_3 + \omega^2 x_1y_2 + \omega x_3y_1 \end{pmatrix} \\ & \oplus \frac{1}{\sqrt{3}} \begin{pmatrix} -x_3y_2 - x_2y_1 - x_1y_3 \\ x_2y_3 + x_1y_2 + x_3y_1 \end{pmatrix} . \end{aligned} \tag{B.16}$$

- $x \otimes y$: $\overline{\mathbf{3}}_1 \otimes \overline{\mathbf{3}}_1 = (\mathbf{3}_1 \oplus \mathbf{3}_1)_S \oplus (\mathbf{3}_2)_A$

$$x \otimes y = \left[\begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} x_2y_3 + x_3y_2 \\ x_3y_1 + x_1y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix} \right]_S \oplus \left[\frac{1}{\sqrt{2}} \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \right]_A . \tag{B.17}$$

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