Wilson Dirac spectrum for QCD with dynamical quarks

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The Wilson Dirac Spectrum for QCD with Dynamical Quarks

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Abstract

All microscopic correlation functions of the spectrum of the Hermitian Wilson Dirac operator with any number of flavors with equal masses are computed. In particular, we give explicit results for the spectral density in the physical case with two light quark flavors. The results include the leading effect in the discretization error and are given for fixed index of the Wilson Dirac operator. They have been obtained starting from chiral Lagrangians for the generating function of the Dirac spectrum. Microscopic correlation functions of the real eigenvalues of the Wilson Dirac operator are computed following the same approach.

I. INTRODUCTION

The deep chiral limit of QCD with two quark flavors and the intimately related nature of spontaneous chiral symmetry breaking is of direct phenomenological interest. Also for studies beyond the standard model such as in QCD like theories with many light flavors or where the fermions are outside the fundamental representation, the deep chiral limit is central. By a remarkable series of numerical and analytic developments it is now possible to access the chiral limit by means of lattice QCD. The work presented in this paper is an attempt to facilitate the next step to the deep chiral limit by offering an exact analytic understanding of the average behavior of the small eigenvalues of the Wilson Dirac operator at nonzero lattice spacing, a. The behavior of these eigenvalues is essential for chiral symmetry breaking [1–3] as well as for the stability of lattice QCD computations [4].

We consider the eigenvalue density of the Wilson Dirac operator in the microscopic scaling limit [6–9] where the product of the eigenvalues and the four-volume, V, as well as the product a^2V are kept fixed. This part of spectrum is uniquely determined by [5] global symmetries, their breaking and the γ_5-Hermiticity of the Wilson Dirac operator

$$D_W^\dagger = \gamma_5 D_W \gamma_5. \tag{1}$$

Because of this Hermiticity relation, the eigenvalues of the Hermitian Wilson Dirac operator,

$$D_5 \equiv \gamma_5 (D_W + m), \tag{2}$$

are real. In addition to correlations of these eigenvalues, we will also analyze the real eigenvalues of D_W in the microscopic limit.

In a recent letter [5] and a longer follow-up [10] we have shown how the *quenched* microscopic Wilson Dirac spectrum can be obtained from the chiral Lagrangian including order a^2-effects for Wilson fermions. Although the supersymmetric method used in [5, 10] can be applied to any number of flavors, the proliferation of terms makes the method only practical for use in the quenched case. Already for one dynamical flavor it becomes rather tedious to deal with analytically [11].

In this paper we follow a different path, the graded eigenvalue method, that results in simple expressions for any number of flavors with equal quark mass. It is also possible to write down compact expressions for all spectral correlation functions. The graded eigenvalue

method is based on the observation that the order a^2 terms in the (graded) chiral Lagrangian can be linearized at the expense of extra Gaussian integrations. This results in compact expressions for microscopic Wilson Dirac spectra for any number of flavors and all correlation functions. The method was originally developed to describe transitions between different universality classes of Random Matrix Theories [12–15]. The result obtained this way is an expression in terms of diffusion in superspace where a^2 plays the role of time.

All results in this paper are given for fixed index of the Wilson Dirac operator, defined for a given gauge field configuration by

$$\nu = \sum_k \mathrm{sign}(\langle k|\gamma_5|k\rangle). \tag{3}$$

Here, $|k\rangle$ denotes the k'th eigenstate of the Wilson Dirac operator. The microscopic eigenvalue density for fixed ν gives detailed information on the effect of a nonzero lattice spacing on the would be topological zero modes at zero lattice spacing.

The study of the spectrum also casts new light [5, 10] on the additional low energy constants of the chiral Lagrangian which is the backbone of Wilson chiral perturbation theory as developed in [16–22] (reviews of effective field theory methods at finite lattice spacings can be found in [23, 24]). By a match of the two-flavor results presented in this paper to the microscopic spectrum of the Wilson Dirac operator on the lattice, the value of the low energy constants can be measured. The spectrum of the Hermitian Wilson Dirac operator in the p-regime of Wilson chiral Perturbation Theory has been discussed in [25] and the results at next to leading order have been fitted to lattice data in [26].

The paper is organized as follows. Starting from a chiral Lagrangian for spectra of the Hermitian Wilson Dirac operator at nonzero lattice spacing, we derive compact expressions for all spectral correlation functions for any number of flavors. In the second part of this paper, we obtain expressions for the distribution of the chiralities over the real eigenvalues of the Wilson Dirac operator. Some technical details involving Efetov-Wegner terms are discussed in Appendix A, and in Appendix B we give explicit expressions for partition functions in terms of an integral over a diffusion kernel.

II. SPECTRAL PROPERTIES OF THE HERMITIAN WILSON DIRAC OPERATOR

The generating function for p-point spectral correlation functions of the eigenvalues of the Hermitian Wilson Dirac operator for QCD with N_f dynamical quarks in the sector of gauge field configurations with index ν is given by (recall that $D_5 = \gamma_5(D_W + m)$)

$$Z^{\nu}_{N_f+p|p} = \left\langle \det{}^{N_f}(D_5) \prod_{k=1}^{p} \frac{\det(D_5 + z_k)}{\det(D_5 + z_k{}' - i\epsilon_k\gamma_5)} \right\rangle. \tag{4}$$

The average is over gauge field configurations with index ν weighted by the Yang-Mills action. For $p = 0$ this is just the N_f flavor partition function. We will evaluate this generating function in the microscopic limit where $V \to \infty$ with

$$mV, \quad z_k V, \quad z_k{}' V, \quad a^2 V \tag{5}$$

kept fixed. The axial masses z_k are required when we apply the graded method to obtain p-point eigenvalue correlation functions of the Hermitian Wilson Dirac operator. For example, from the graded generating function $\mathcal{Z}^{\nu}_{N_f+1|1}(m, z, z'; a)$ we can obtain the spectral resolvent

$$G^{\nu}_{N_f+1|1}(z, m; a) = \lim_{z' \to z} \frac{d}{dz} \mathcal{Z}^{\nu}_{N_f+1|1}(m, z, z'; a), \tag{6}$$

and the density of eigenvalues, $\rho^{\nu}_5(\lambda^5, m; a)$, of D_5 follows from

$$\rho^{\nu}_5(\lambda^5, m; a) = \left\langle \sum_k \delta(\lambda^5_k - \lambda^5) \right\rangle_{N_f} = \frac{1}{\pi} \mathrm{Im}[G^{\nu}_{N_f+1|1}(-\lambda^5)]_{\epsilon \to 0}. \tag{7}$$

To derive expressions for the correlation functions in the microscopic limit we will rely on the graded eigenvalue method. Before deriving the general result, we will first consider the case $p = 0$, which is just the N_f-flavor partition function.

The chiral Lagrangian for Wilson chiral perturbation theory to $O(a^2)$ was derived in [16–18]. In [5] we obtained the microscopic partition function for fixed index ν also to order a^2, by decomposing the partition function according to

$$Z_{N_f}(m, \theta; a) \equiv \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z^{\nu}_{N_f}(m; a). \tag{8}$$

In the microscopic domain, the partition function reduces to a unitary matrix integral

$$Z^{\nu}_{N_f}(m, z; a) = \int_{U(N_f)} dU \, \det{}^{\nu} U \, e^{S[U]}, \tag{9}$$

where the action $S[U]$ for degenerate quark masses is given by

$$\begin{aligned} S &= \frac{m}{2}\Sigma V \mathrm{Tr}(U+U^\dagger) + \frac{z}{2}\Sigma V \mathrm{Tr}(U-U^\dagger) \\ &\quad -a^2 V W_6 [\mathrm{Tr}\left(U+U^\dagger\right)]^2 - a^2 V W_7 [\mathrm{Tr}\left(U-U^\dagger\right)]^2 - a^2 V W_8 \mathrm{Tr}(U^2+U^{\dagger 2}). \end{aligned} \tag{10}$$

In addition to the chiral condensate, Σ, the action also contains the low energy constants W_6, W_7 and W_8 as parameters. Since the W_6 and W_7 terms can be eliminated at the expense of an extra integration [10], we will only consider the W_8 term in the remainder of this paper. For the reasons discussed in section VII of [10] we consider only $W_8 > 0$. To simplify our notation below, we will absorb the factor VW_8 into a^2 and the factor $V\Sigma$ into m, z_k and λ^5

$$a^2 V W_8 \to a^2, \qquad mV\Sigma \to m, \qquad z_k V\Sigma \to z_k \quad \text{and} \quad \lambda^5 V\Sigma \to \lambda^5. \tag{11}$$

Eq. (9) is a supersymmetric extension of Wilson chiral perturbation theory with partial quenched quarks with masses z_k and z'_k which gives us access to the spectrum of the Dirac operator. Up to a normalization factor, the term proportional to W_8 in the action (10) can be rewritten as

$$\begin{aligned} e^{-a^2 \mathrm{Tr}(U^2+U^{\dagger 2})} &= e^{-2N_f a^2 - a^2 \mathrm{Tr}(U-U^\dagger)^2}, \\ &= c e^{-2N_f a^2} \int d\sigma e^{\mathrm{Tr}\sigma^2/16a^2 + \frac{1}{2}\mathrm{Tr}\sigma(U-U^\dagger)}, \end{aligned} \tag{12}$$

where σ is anti-hermitian, and c is a normalization constant. In a diagonal representation of σ denoted by $S \equiv \mathrm{diag}(is_1, \cdots, is_{N_f})$ the partition function with index ν is thus given by (up to a normalization constant)

$$\begin{aligned} Z^\nu_{N_f}(m;a) &= e^{-2N_f a^2} \int ds_k \Delta^2(\{s_k\}) \int_{U(N_f)} dU \, \det{}^\nu U e^{-\sum_k s_k^2/16a^2} e^{\frac{1}{2}\mathrm{Tr}U^\dagger(m+S) + \frac{1}{2}\mathrm{Tr}U(m-S)} \\ &= \int ds_k \Delta^2(\{s_k\}) e^{-\sum_k s_k^2/16a^2} \det{}^\nu(m-S) \, \tilde{Z}^\nu_{N_f}(\{(m^2+s_k^2)^{1/2}\}; a=0). \end{aligned} \tag{13}$$

The Vandermonde determinant is defined by

$$\Delta(x_1, \cdots, x_p) = \prod_{k>l}^{p} (x_k - x_l), \tag{14}$$

and an explicit expression for the partition function at $a = 0$ is given in Eq. (29). The expression for the partition function will be discussed in more detail for $N_f = 1$ and 2 in Appendix B.

The fermionic partition function has been written as an integral over a diffusion kernel times the partition function at $a = 0$. Next we will show that exactly the same is true for the graded generating functional obtained in [5]. The generating function of the microscopic p-point spectral correlation functions of the Hermitian Wilson Dirac operator with index ν is given by

$$Z^{\nu}_{N_f+p|p}(\mathcal{M},\mathcal{Z};a) \;=\; \int dU\ \mathrm{Sdet}(iU)^{\nu} e^{\frac{i}{2}\mathrm{Trg}(\mathcal{M}[U-U^{-1}])+\frac{i}{2}\mathrm{Trg}(\mathcal{Z}[U+U^{-1}])+a^2\mathrm{Trg}(U^2+U^{-2})}, \quad (15)$$

where $\mathcal{M} \equiv \mathrm{diag}(m_1 \ldots m_{N_f+2p})$ and $\mathcal{Z} \equiv \mathrm{diag}(z_1 \ldots z_{N_f+2p})$, and the integration is over $Gl(N_f+p|p)/U(p)$, see [27]. We use the convention that $\mathrm{Trg}\, A = \mathrm{Tr}[A_f] - \mathrm{Tr}[A_b]$, with A_f the fermion-fermion block of A, and A_b its boson-boson block. The definition of Sdet follows form the relation $\mathrm{Sdet}A = \exp[\mathrm{Trg}\ \mathrm{logA}]$. Notice that in comparison to (9) the integration over U has been rotated by i so that the convergence of the bosonic integrals is assured [5, 10]. A similar rotation is necessary when computing the spectrum of the Dirac operator at nonzero chemical potential [28]. The common origin is the non-Hermiticity of the Dirac operator. The manipulations from (9) to (13) can be repeated for the graded partition function. We start from the identity

$$\begin{aligned} e^{a^2\mathrm{Trg}(U^2+U^{\dagger 2})} \;&=\; e^{-2N_f a^2+a^2\mathrm{Trg}(U+U^{\dagger})^2}, \\ &=\; c e^{-2N_f a^2}\int d\sigma e^{\mathrm{Trg}\sigma^2/16a^2+\frac{i}{2}\mathrm{Trg}\sigma(U+U^{\dagger})}, \end{aligned} \quad (16)$$

where σ is an $(N_f+p|p)$ graded "Hermitian" matrix (see (21)) and c is an integration constant. After shifting integration variables $\sigma \to \sigma - \mathcal{Z}$ we obtain (the normalization constants will be fixed at the end of the calculation)

$$Z^{\nu}_{N_f+p|p}(\mathcal{M},\mathcal{Z};\hat{a}) \;=\; e^{-2N_f a^2}\int d\sigma \int dU\ \mathrm{Sdet}^{\nu}(iU) e^{\mathrm{Trg}(\sigma-\mathcal{Z})^2/16a^2+\frac{i}{2}\mathrm{Trg}(\sigma+\mathcal{M})U+\frac{i}{2}\mathrm{Trg}(\sigma-\mathcal{M})U^{-1}}. \quad (17)$$

We will evaluate this partition function for the $(p+N_f|p)$ graded diagonal matrix

$$\mathcal{Z} = \mathrm{diag}(\epsilon,\cdots,\epsilon,z_1,\cdots,z_p,z_1',\cdots,z_p'), \quad (18)$$

and it is understood that the limit $\epsilon \to 0$ is taken at the end of the calculation.

The partition function (9) and the generating function (15) satisfy the relation

$$Z^{-\nu}_{N_f}(m,z;a) \;=\; Z^{\nu}_{N_f}(m,-z;a), \quad (19)$$

$$Z^{-\nu}_{N_f+p|p}(\mathcal{M},\mathcal{Z};a) \;=\; Z^{\nu}_{N_f}(\mathcal{M},-\mathcal{Z};a). \quad (20)$$

For this reason we only consider the case $\nu \geq 0$ below. For $z = 0$ ($\mathcal{Z} = 0$) the generating function does not depend on the sign of ν.

The graded matrix σ has the structure

$$\sigma = \begin{pmatrix} i\sigma_f & \alpha \\ \beta & \sigma_b \end{pmatrix} \tag{21}$$

with Hermitian $(p + N_f) \times (p + N_f)$ matrix $\sigma_f^\dagger = \sigma_f$, Hermitian $p \times p$ matrix $\sigma_b^\dagger = \sigma_b$, and α and β are matrices in the Grassmann algebra. The matrix σ can be diagonalized by a super-unitary transformation [29]

$$\sigma = uSu^{-1}, \quad \text{with} \quad S \equiv \begin{pmatrix} is & 0 \\ 0 & t \end{pmatrix}, \tag{22}$$

where $s = \mathrm{diag}(s_1, \ldots, s_{p+N_f})$ and $t = \mathrm{diag}(t_1, \ldots, t_p)$ contain the real eigenvalues s_k and t_k.

Next we transform to the eigenvalues of σ and the super-unitary matrix U as integration variables. The measure is given by

$$d\sigma = B^2_{N_f+p|p}(S) \prod_{k=1}^{N_f+p} ds_k \prod_{k=1}^{p} dt_k \, du. \tag{23}$$

with Berezinian

$$B_{N_f+p|p}(S) = \frac{\prod_{k>l}^{N_f+p}(is_k - is_l) \prod_{k>l}^{p}(t_k - t_l)}{\prod_{k=1}^{p} \prod_{l=1}^{N_f+p}(t_k - is_l)}. \tag{24}$$

The measure du is the superinvariant Haar measure.

For degenerate quarks masses, $\mathcal{M}_k = m$, the integral over u can be performed by a graded generalization [13, 30] of the Itzykson-Zuber formula,

$$\int due^{\mathrm{Trg}(\rho-\xi)^2/2\tau} = B_0 + \frac{1}{(2\pi\tau)^{(N_f+2p)/2}} \frac{e^{(1/2\tau)\mathrm{Trg}(X^2+R^2)} \det e^{-R_k^b X_l^b/\tau} \det e^{-R_k^f X_l^f/\tau}}{B_{N_f+p|p}(X) B_{N_f+p|p}(R)}. \tag{25}$$

Here, X and R are the diagonal representation of ξ and ρ, in this order. B_0 contains the contributions due to the boundary terms. They result from the product of the infinity due the singularities in the measure and the vanishing result due to the Grassmann integration after changing to eigenvalues as integration variables. These contributions can be worked out by expanding the eigenvalues of σ in powers of the nilpotent terms (see Appendix A). They do not contribute to the spectral correlators discussed below and will be further analyzed in a future publication.

Ignoring the contribution from B_0 we find the generating function

$$Z^{\nu}_{N_f+p|p}(\mathcal{M},\mathcal{Z};a) \;=\; \frac{e^{-2N_f a^2}}{(16\pi a^2)^{p+N_f/2}}\int dU\ \mathrm{Sdet}(iU)^{\nu}\int dsdt\frac{B_{N_f+p|p}(S)}{B_{N_f+p|p}(\mathcal{Z})}$$
$$e^{+\frac{i}{2}\mathrm{Trg}(S+m)U+\frac{i}{2}\mathrm{Trg}(S-m)U^{-1}}\times e^{(1/16a^2)\mathrm{Trg}(S^2+\mathcal{Z}^2)}\det e^{-S^b_k\mathcal{Z}^b_l/8a^2}\det e^{-S^f_k\mathcal{Z}^f_l/8a^2}. \quad (26)$$

The integral over U can be expressed in terms of the generating function for $a=0$. A convenient expression is obtained by observing that the masses S_k+m and S_k-m can be replaced by the same mass $i\sqrt{m^2-S_k^2}$ for both chiralities. A further simplification results from the symmetry of the integrand in the s_k and the t_k variables so that all terms in the Laplace expansion of the determinants from the Itzykson-Zuber integral give the same contribution. In terms of the $a=0$ partition function we thus obtain

$$Z^{\nu}_{N_f+p|p}(m,\mathcal{Z};a) \;=\; \frac{\prod_{k,l=1}^{p}(z'_k-z_l)\prod_k^p {z'_k}^{N_f}}{\Delta(\{z_l\})\Delta(\{z'_k\})\prod_k^p z_k^{N_f}}\frac{e^{-2N_f a^2}}{(16\pi a^2)^{(N_f+2p)/2}}\int dsdtB_{N_f+p|p}(S)\Delta(S/8a^2)$$
$$\times e^{\mathrm{Trg}[(S-\mathcal{Z})^2]/16a^2}\left(\frac{\prod_k(m-is_k)}{\prod_l(m-t_l)}\right)^{\nu}\tilde{Z}^{\nu}_{N_f+p|p}\left(\{(s_k^2+m^2)^{1/2}\},\{(m^2-t_l^2)^{1/2}\};a=0\right), \quad (27)$$

where Vandermonde determinant, $\Delta(S/8a^2)=\Delta(is_1/8a^2,\ldots,is_{N_f}/8a^2)$, and the prefactors result from the limit $\epsilon\to 0$.

We have succeeded in rewriting the a dependence of the generating function as an integral over the product of a diffusion kernel and the $a=0$ generating function given by [31, 32]

$$\tilde{Z}^{\nu}_{N_f+p|p}(x_1,\cdots,x_{N_f+2p};a=0) \quad (28)$$
$$=\; c\left(\frac{\prod_{k=N_f+p+1}^{N_f+2p}x_k}{\prod_{k=1}^{N_f+p}x_k}\right)^{\nu}\frac{\det[(x_k)^{l-1}\mathcal{I}_{\nu+l-1}(x_k)]}{\Delta(x_1^2,\cdots,x_{N_f+p}^2)\Delta(x^2_{N_f+p+1},\cdots,x^2_{N_f+2p})}$$

with

$$\mathcal{I}_q(x_k) \;=\; I_q(x_k), \qquad k=1,\cdots,N_f+p,$$
$$\mathcal{I}_q(x_k) \;=\; (-1)^q K_q(x_k), \qquad k=N_f+p+1,\cdots,N_f+2p. \quad (29)$$

We added a tilde to $\tilde{Z}^{\nu}_{N_f+p|p}$ because the mass-factors due to the zero modes have been amputated which is not the case for the partition function at $a\neq 0$.

The p point correlation function is obtained by differentiating with respect to the z_k and putting $z'_k=z_k$ afterwards. Only if all factors in the product $\prod(z'_k-z_k)$ are differentiated

do we get a nonzero result. The remaining factors from the Berezinian $B_{N_f+p|p}(\mathcal{Z})$ cancel. We thus find the correlator of p resolvents

$$G^{\nu}_{N_f+p|p}(z_1,\cdots,z_p;a) = \frac{e^{-2N_f a^2}}{Z^{\nu}_{N_f}(m;a)}\frac{1}{(16\pi a^2)^{(N_f+2p)/2}}\int dsdtB_{N_f+p|p}(S)\Delta(S/8a^2) \tag{30}$$
$$\times e^{\mathrm{Trg}[(S-\mathcal{Z})^2]/16a^2}\frac{\prod_k(m-is_k)^{\nu}}{\prod_l(m-t_l)^{\nu}}\tilde{Z}^{\nu}_{N_f+p|p}\left(\{(s_k^2+m^2)^{1/2}\},\{(m^2-t_l^2)^{1/2}\};a=0\right).$$

This result is universal in the sense that given the definition of ν according to Eq. (9), it is completely determined by the symmetries and Hermiticity properties of the QCD partition function [10]. Below we discuss explicit results for the microscopic spectral density for $N_f = 0, 1$ and 2. We also check that the partition function for N_f flavors reduces to previously derived results for $N_f = 0$ and $N_f = 1$.

A. Explicit Results for the Spectrum of D_5

In this section we discuss explicit results for the microscopic spectral density of D_5 for the quenched case and one and two dynamical flavors.

1. *The quenched case*

In this case the Berezinian is given by

$$B_{1|1}(S) = \frac{1}{t-is} \tag{31}$$

and the generating function reduces to

$$Z^{\nu}_{1|1}(m,z,z';a) = \frac{z'-z}{16a^2\pi}\int\frac{dsdt}{t-is}e^{-[(s+iz)^2+(t-z')^2]/16a^2}\frac{(m-is)^{\nu}}{(m-t)^{\nu}} \tag{32}$$
$$\times\tilde{Z}^{\nu}_{1|1}(\sqrt{m^2+s^2},\sqrt{m^2-t^2};a=0).$$

Here,

$$\tilde{Z}^{\nu}_{1|1}(x,y;a=0) = \frac{y^{\nu}}{x^{\nu}}[yK_{\nu+1}(y)I_{\nu}(x)+xK_{\nu}(y)I_{\nu+1}(x)]. \tag{33}$$

The resolvent is given by

$$G^{\nu}_{1|1}(z) = \frac{d}{dz}\bigg|_{z'=z}Z^{\nu}_{1|1}(m,z,z';a). \tag{34}$$

Only the term where the prefactor $z' - z$ is differentiated contributes to the resolvent. In the microscopic limit, this results in

$$G^{\nu}_{1|1}(z) \;=\; -\frac{1}{16a^2\pi}\int \frac{dsdt}{t-is} e^{-\frac{[s^2+t^2]}{16a^2}} \frac{(m-is-z)^{\nu}}{(m-t-z)^{\nu}} \times \tilde{Z}^{\nu}_{1|1}(\sqrt{m^2-(is+z)^2}, \sqrt{m^2-(t+z)^2}; a=0). \tag{35}$$

Note that we also shifted the integration variables s and t by $-iz$ and z', respectively. The effect of this shift is discussed in Appendix A.

The quenched microscopic eigenvalue density of D_5 follows from the imaginary part of the resolvent, cf. Eq. (7). We have checked numerically that this result coincides with the result obtained from a standard supersymmetric computation in [5]. See [5, 10] for plots of the quenched density.

In the $a \to 0$ limit at fixed m and z, the Gaussian integrals in Eq. (33) become δ-functions which can be integrated resulting in

$$\lim_{a\to 0} Z^{\nu}_{1|1}(m,z,z';a) = \left[\frac{(z-m)}{(z'-m)}\right]^{\nu} \tilde{Z}^{\nu}_{1|1}(\sqrt{m^2-z^2}, \sqrt{m^2-z'^2}; a=0). \tag{36}$$

The prefactor gives the contribution from the zero modes to the resolvent

$$\frac{\nu}{z-m}, \tag{37}$$

whereas the second factor gives the contribution of the nonzero modes for $a = 0$.

B. One Flavor

For $N_f = 1$ the generating function is given by

$$Z^{\nu}_{2|1} \;=\; \frac{e^{-2a^2}}{64a^3\pi^{3/2}} \frac{(z'-z)z'}{z} \int \frac{dtds_1ds_2(is_2-is_1)}{(t-is_1)(t-is_2)} e^{-[s_1^2+(s_2+iz)^2+(t-z')^2]/16a^2} \times \frac{(m-is_1)^{\nu}(m-is_2)^{\nu}}{(m-t)^{\nu}} \tilde{Z}^{\nu}_{2|1}((s_1^2+m^2)^{1/2}, (s_2^2+m^2)^{1/2}, (m^2-t^2)^{1/2}; a=0), \tag{38}$$

where

$$\tilde{Z}^{\nu}_{2|1}(x_1,x_2,x_3) = \frac{x_3^{\nu}}{x_1^{\nu}x_2^{\nu}(x_2^2-x_1^2)} \det \begin{pmatrix} I_{\nu}(x_1) & x_1 I_{\nu+1}(x_1) & x_1^2 I_{\nu+2}(x_1) \\ I_{\nu}(x_2) & x_2 I_{\nu+1}(x_2) & x_2^2 I_{\nu+2}(x_2) \\ (-1)^{\nu}K_{\nu}(x_3) & (-1)^{\nu+1}x_3 K_{\nu+1}(x_3) & (-1)^{\nu}x_3^2 K_{\nu+2}(x_3) \end{pmatrix}. \tag{39}$$

To obtain the resolvent we differentiate with respect to z' and put $z' = z$ after differentiation. An additional minus sign arises because z' is the bosonic source term. We thus find

$$G^{\nu}_{2|1}(z) = -\frac{e^{-2a^2}}{64a^3\pi^{3/2}Z^{\nu}_1(m;a)}\int\frac{ds_1ds_2dt(is_2-is_1)}{(t-is_1)(t-is_2)}e^{-[s_1^2+(s_2+iz)^2+(t-z)^2]/16a^2} \times\frac{(m-is_1)^{\nu}(m-is_2)^{\nu}}{(m-t)^{\nu}}\tilde{Z}^{\nu}_{2|1}((s_1^2+m^2)^{1/2},(s_2^2+m^2)^{1/2},(m^2-t^2)^{1/2};a=0), \tag{40}$$

where the s_2 and t integration contours are shifted such that s_2+iz and $t-z$ run over the real axis. The resolvent is normalized with respect to the one flavor partition function given in Eq. (100). As for the quenched case we have checked numerically that the one flavor microscopic density, which follows from the imaginary part of $G^{\nu}_{2|1}$, is identical to the result [11] obtained from chiral perturbation theory using the standard supersymmetric method. We refer to [11] for plots.

The small a limit for fixed m and z can be obtained by first shifting $s_2 \to s_2 - iz$ and $t \to t+z$ and then expand the nonexpential factors in Eq. (39). The integration measure can be expanded as

$$\frac{(is_2-is_1+z)}{(t+z-is_1)(t-is_2)} = [1+\frac{1}{z}(is_2-t)]\frac{1}{t-is_2}+\cdots. \tag{41}$$

For small a the partition function $\tilde{Z}_{2|1}((s_1^2+m^2)^{1/2},((s_2-iz)^2+m^2)^{1/2},(m^2-(t+z)^2)^{1/2};a=0)$ (denoted by $\tilde{Z}_{2|1}$ below) can be expanded to first order in s_k and t

$$m^{-\nu}\tilde{Z}^{\nu}_1(m;a=0)+s_1\left.\frac{d}{ds_1}\tilde{Z}_{2|1}\right|_{s_1=s_2=t=0}+s_2\left.\frac{d}{ds_2}\tilde{Z}_{2|1}\right|_{s_1=s_2=t=0}+t\left.\frac{d}{dt}\tilde{Z}_{2|1}\right|_{s_1=s_2=t=0}. \tag{42}$$

The term linear in s_1 does not contribute to leading order in a and the remaining terms can be written as

$$\begin{aligned} &m^{-\nu}\tilde{Z}^{\nu}_1(m;a=0)+(t-is_2)\left.\frac{d}{dt}\tilde{Z}_{2|1}\right|_{s_1=s_2=t=0} \\ &= m^{-\nu}\tilde{Z}^{\nu}_1(m;a=0)+(t-is_2)\left.\frac{d}{dz'}\right|_{z'=z}\tilde{Z}_{2|1}(m,\sqrt{m^2-z^2},\sqrt{m^2-z'^2}). \end{aligned} \tag{43}$$

The linear term in s_1 in the expansion of the prefactor

$$\frac{(m-is_1)^{\nu}(m-z-is_2)^{\nu}}{(m-z-t)^{\nu}} = \frac{m^{\nu}(m-z)^{\nu}}{(m-z')^{\nu}}\left[-\nu\frac{is_1}{m}+\nu\frac{t-is_2}{m-z}\right]+\cdots. \tag{44}$$

also vanishes after integration. Combining the contributions from Eqs. (41-44) we obtain the small a limit

$$G^{\nu}(z) = \frac{\nu}{z-m}+\tilde{G}^{\nu}(m,z;a=0)+\frac{1}{z}, \tag{45}$$

where $\tilde{G}^\nu(m,z;a=0)$ is the resolvent of the nonzero eigenvalues. The additional $1/z$ term will be canceled by Efetov-Wegner terms which contribute to the real part of the resolvent.

C. Two Flavors

In this section we write out the $N_f = 2$ generating function with dynamical quark masses m and index ν. With the Berezinian given by

$$B_{3|1}(s) = \frac{(is_2 - is_1)(is_3 - is_1)(is_3 - is_2)}{(t - is_1)(t - is_2)(t - is_3)} \tag{46}$$

we find the generating function

$$\begin{aligned} Z^\nu_{3|1} \;=\; & -\frac{e^{-4a^2}}{\pi^2(16a^2)^3}\frac{z'^2(z'-z)}{z^2}\int \frac{ds_1 ds_2 ds_3 dt (is_2 - is_1)^2(is_3 - is_1)(is_3 - is_2)}{(t - is_1)(t - is_2)(t - is_3)} \\ & \times e^{-[s_1^2+s_2^2+(s_3+iz)^2+(t-z')^2]/16a^2}\frac{(m - is_1)^\nu (m - is_2)^\nu (m - is_3)^\nu}{(m-t)^\nu} \\ & \times \tilde{Z}^\nu_{3|1}((s_1^2+m^2)^{1/2},(s_2^2+m^2)^{1/2},(s_3^2+m^2)^{1/2},(m^2-t^2)^{1/2};a=0). \end{aligned} \tag{47}$$

The partition function for $a = 0$ is given by

$$\begin{aligned} \tilde{Z}^\nu_{3|1}(x_1,x_2,x_3,x_4;a=0) \;=\; & 2\frac{x_4^\nu}{x_1^\nu x_2^\nu x_3^\nu}\frac{1}{(x_3^2-x_2^2)(x_3^2-x_1^2)(x_2^2-x_1^2)} \\ \times \det & \begin{pmatrix} I_\nu(x_1) & x_1 I_{\nu+1}(x_1) & x_1^2 I_{\nu+2}(x_1) & x_1^3 I_{\nu+3}(x_1) \\ I_\nu(x_2) & x_2 I_{\nu+1}(x_2) & x_2^2 I_{\nu+2}(x_2) & x_2^3 I_{\nu+3}(x_2) \\ I_\nu(x_3) & x_3 I_{\nu+1}(x_3) & x_3^2 I_{\nu+2}(x_3) & x_3^3 I_{\nu+3}(x_3) \\ (-1)^\nu K_\nu(x_4) & x_4(-1)^{\nu+1}K_{\nu+1}(x_4) & x_4^2(-1)^{\nu+2}K_{\nu+2}(x_4) & x_4^3(-1)^{\nu+3}K_{\nu+3}(x_4) \end{pmatrix}. \end{aligned} \tag{48}$$

The resolvent is obtained by differentiating the factor $z' - z$ with respect to z at $z' = z$. This results in

$$\begin{aligned} G^\nu_{3|1}(z;a) \;=\; & \frac{e^{-4a^2}}{\pi^2(16a^2)^3 Z^\nu_{N_f=2}(m;a)}\int ds_1 ds_2 ds_3 dt \frac{(is_2 - is_1)^2(is_3 - is_1)(is_3 - is_2)}{(t - is_1)(t - is_2)(t - is_3)} \\ & \times e^{-\frac{1}{16a^2}[s_1^2+s_2^2+(s_3+iz)^2+(t-z)^2]}\frac{(m - is_1)^\nu (m - is_2)^\nu (m - is_3)^\nu}{(m-t)^\nu} \\ & \times \tilde{Z}^\nu_{3|1}((s_1^2+m^2)^{1/2},(s_2^2+m^2)^{1/2},(s_3^2+m^2)^{1/2},(m^2-t^2)^{1/2};a=0), \end{aligned} \tag{49}$$

where the s_3 and t integration contours are shifted such that $s_3 + iz$ and $t - z$ run over the real axis. The resolvent has been normalized with respect to the two-flavor partition function defined in Eq. (103).

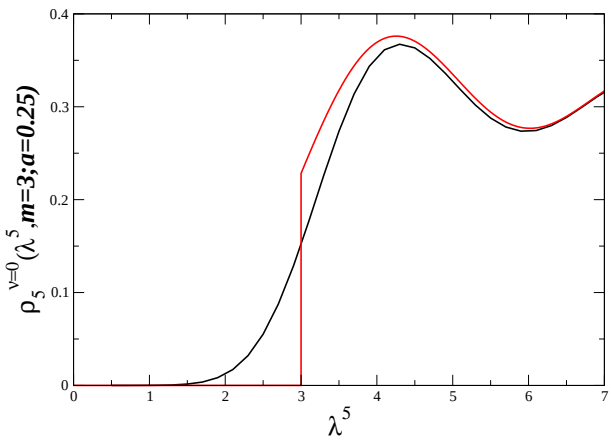

FIG. 1: Spectral density of D_5 for $N_f = 2$, $\nu = 0$, $m = 3$ for $a = 0.25$ (black curve) and $a = 0$ (red curve).

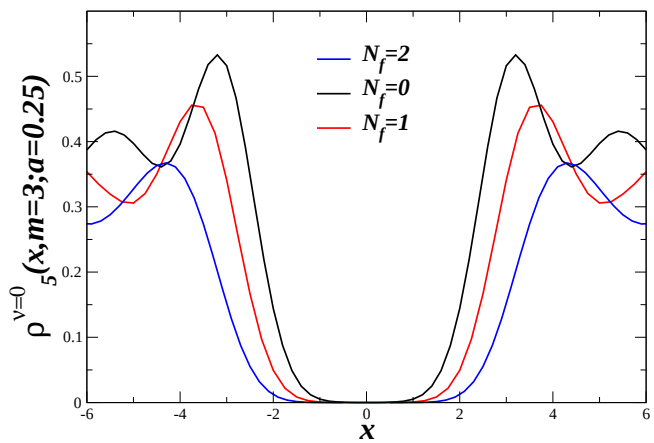

FIG. 2: The spectral density of the eigenvalues of D_5 in the sector with index $\nu = 0$ plotted for $N_f = 0, 1$ and 2. The value of $a = 0.25$ and $m = 3$. The increasing repulsion from the origin for larger N_f is clearly visible.

The microscopic eigenvalue density of D_5 with two light flavors of mass m is then given by

$$\rho_5^{\nu,N_f=2}(\lambda^5, m; a) = \frac{1}{\pi}\mathrm{Im}[G^{\nu}_{3|1}(-\lambda^5)]_{\epsilon\to 0}. \tag{50}$$

In Fig. 1 we show the two-flavor microscopic spectral density of D_5 as a function of λ^5 for $\nu = 0$ and $m = 3$ and compare the result for $a = 0.25$ and $a = 0$. The area below the two curves is the same within our numerical accuracy. In Fig. 2 we compare the two-flavor result to the one-flavor result and the quenched result with the same parameters.

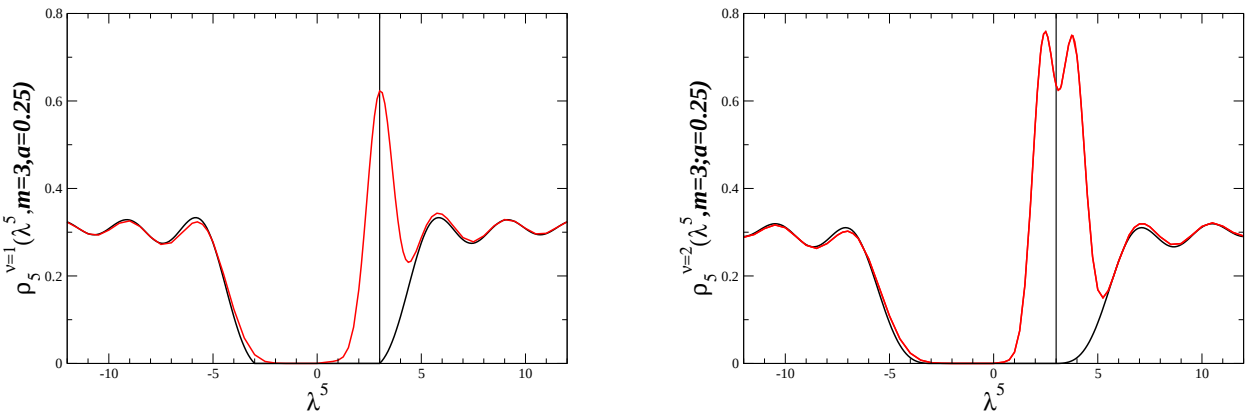

FIG. 3: The effect of the index on the spectral density of the Hermitian Wilson Dirac operator for two flavors. Results for index $\nu = 1$ (left) and $\nu = 2$ (right) are shown for $m = 3$ and $a = 0.25$ (red curve) and $a = 0$ (black curve). The vertical black line marks the position of the ν fold δ-function due to the exact topological modes at $a = 0$. Note that the primary effect of a for small a is to smear out the δ-function.

The effect of non-zero index ν for $N_f = 2$ is displayed in Fig. 3. Note that the two flavor eigenvalue density is positive definite (the square of the Wilson-fermion determinant is real and positive) and that the spectral gap cannot close completely on the microscopic scale due to the repulsion from the origin (the square of the Wilson-fermion determinant vanishes quadratically as an eigenvalue of D_5 approaches zero). It would be most interesting to compare these analytical predictions to dynamical lattice data, such as those presented in [4].

For small a at fixed m and z we can write

$$\frac{(is_3 - is_1 + z)(is_3 - is_2 + z)}{(t - is_1 + z)(t - is_2 + z)(t - is_3)} = [1 + \frac{2}{z}(is_3 - t)]\frac{1}{t - is_3}. \tag{51}$$

The constant term contributes to the real part of the resolvent, and as in the one-flavor case, we expect that it will be canceled by contributions from the Efetov-Wegner terms. The $a \to 0$ limit of the $N_f = 2$ resolvent is obtained by expanding the pre-exponential factors in Eq. (49) as in the one-flavor case.

III. THE DISTRIBUTION OF REAL MODES

We now consider the eigenvalues of the usual Wilson Dirac operator D_W. For small nonzero values of the lattice spacing a, the eigenvalues, λ^W, of D_W spread into a narrow band around the imaginary axis of the complex eigenvalue plane. The eigenvalues in the complex plane make up complex conjugate pairs $\lambda^W, (\lambda^W)^*$ or are exactly real. See [33] for a derivation of these properties of the Wilson Dirac operator. In this section we analyze the microscopic spectral correlation functions for the real eigenvalues of D_W.

The generating function for the p-point correlation function with N_f dynamical quarks in the sector of gauge field configurations with index ν takes the form

$$Z^\nu_{N_f+p|p} = \left\langle \det{}^{N_f}(D_W + m_f) \prod_{k=1}^{p} \frac{\det(D_W + m_k)}{\det(D_W + m'_k - i\epsilon\gamma_5)} \right\rangle. \tag{52}$$

The spectral resolvent for the one point function is

$$\Sigma^\nu_{N_f+1|1}(m, m_f; a) = \lim_{m' \to m} \frac{d}{dm} \mathcal{Z}^\nu_{N_f+1|1}(m_f, m, m'; a). \tag{53}$$

To be precise, the one point function that corresponds to this resolvent is the distribution of the chiralities, $\mathrm{sign}(\langle k|\gamma_5|k\rangle)$, over the real modes, $\lambda^W_k \in \mathbb{R}$,

$$\rho^\nu_\chi(\lambda^W, m_f; a) \equiv \left\langle \sum_{\lambda^W_k \in \mathbb{R}} \delta(\lambda^W_k + \lambda^W)\, \mathrm{sign}(\langle k|\gamma_5|k\rangle) \right\rangle_{N_f}. \tag{54}$$

In the sector with index ν as defined in eq. (3) we have

$$\int d\lambda^W \; \rho^\nu_\chi(\lambda^W, m_f; a) = \nu. \tag{55}$$

The fact that the results obatined below from Wilson chiral perturbation theory satisfy this demonstrates that the sectors defined in eq. (8) correspond to a Dirac operator D_W with index ν as defined in eq. (3) [10].

The distribution of the chiralities over the real modes, can be obtained from the imaginary part of the spectral resolvent (see section II of [10])

$$\rho^\nu_\chi(\lambda^W, m_f; a) \;=\; \frac{1}{\pi} \mathrm{Im}[\Sigma^\nu_{N_f+1|1}(m_f, m = \lambda^W; a)]_{\epsilon\to 0}. \tag{56}$$

The p-point spectral resolvent is given by

$$\begin{aligned} \Sigma^\nu_{N_f+p|p}(m_1, \ldots, m_p, m_f; a) \;=\; & \lim_{m'_1 \to m_1} \cdots \lim_{m'_p \to m_p} \frac{d}{dm_1} \cdots \frac{d}{dm_p} \\ & \times \mathcal{Z}^\nu_{N_f+p|p}(m_f, m_1, \ldots, m_p, m'_1, \ldots, m'_p; a). \end{aligned} \tag{57}$$

As in the case of the one-point function, the discontinuities across the real axis give the p-point density correlation functions.

As was discussed in [5], the generating function for the correlation functions (57) is given by

$$Z^{\nu}_{N_f+p|p}(\mathcal{M};a) \;=\; \int dU\ \mathrm{Sdet}(iU)^{\nu} e^{\frac{i}{2}\mathrm{Trg}(\mathcal{M}[U-U^{-1}])+a^2\mathrm{Trg}(U^2+U^{-2})}, \tag{58}$$

which is just the generating function (15) for $\mathcal{Z}=0$. The mass matrix corresponding to (52) is given by the $(p+N_f|p)$ graded diagonal matrix

$$\mathcal{M} \equiv \mathrm{diag}(m_f,\cdots,m_f,m_1,\cdots,m_p,m_1',\cdots,m_p'). \tag{59}$$

The first N_f entries are the physical masses and need not be identical.

In order to derive the p-point function the we start with the identity (instead of the identity (16))

$$\begin{aligned} e^{a^2\mathrm{Trg}(U^2+U^{-1\,2})} \;&=\; e^{2N_f a^2+a^2\mathrm{Trg}(U-U^{-1})^2}, \\ &=\; c e^{2N_f a^2}\int d\sigma e^{\mathrm{Trg}\sigma^2/16a^2+\frac{i}{2}\mathrm{Trg}\sigma(U-U^{-1})}, \end{aligned} \tag{60}$$

where σ is an $(N_f+p|p)$ graded Hermitian matrix (see Eq. (21)) and c a normalization constant.. After a shift of σ by $\mathcal{M}$ we obtain

$$Z^{\nu}_{N_f+p|p}(\mathcal{M};a) \;=\; e^{2N_f a^2}\int d\sigma \int dU\ \mathrm{Sdet}^{\nu}(iU) e^{\mathrm{Trg}(\sigma-\mathcal{M})^2/16a^2+\frac{i}{2}\mathrm{Trg}\,\sigma(U-U^{-1})}. \tag{61}$$

The next step is to decompose $\sigma = uSu^{-1}$ with S a diagonal graded matrix (see Eq. (22)) and perform the integration over u by a supersymmetric generalization of the Itzykson-Zuber integral. We find

$$\begin{aligned} Z^{\nu}_{N_f+p|p}(\mathcal{M};a) \;=\; &\frac{e^{2N_f a^2}}{(16\pi a^2)^{(N_f+2p)/2}}\int dsdt \frac{B_{N_f+p|p}(S)}{B_{N_f+p|p}(\mathcal{M})}\det e^{-\mathcal{M}^b_k S^b_l/8a^2}\det e^{-\mathcal{M}^f_k S^f_l/8a^2} \\ &\times e^{\mathrm{Trg}[S^2+\mathcal{M}^2]/16a^2}\frac{\prod_k (is_k)^{\nu}}{\prod_l (t_l)^{\nu}}\tilde{Z}^{\nu}_{N_f+p|p}\left(\left\{\sqrt{(is_k)^2}\right\},\left\{\sqrt{(t_l)^2}\right\};a=0\right). \end{aligned} \tag{62}$$

For degenerate dynamical quarks the above expression can be further simplified.

$$\begin{aligned} \det e^{-\mathcal{M}^f_k S^f_l/8a^2} \;=\; &\Delta(m_1,\cdots,m_{N_f})\Delta(S^f_1/8a^2,\cdots,S^f_{N_f}/8a^2)e^{-m(S^f_1+\cdots+S^f_{N_f})/8a^2} \\ &\times\det[e^{-m_k S^f_l/8a^2}]_{k,l=N_f+1,\cdots,N_f+p} + \quad \text{permutations of } \ S^f_k. \end{aligned} \tag{63}$$

All permutations of the S_k give the same contribution. For $m'_k \to m_{N_f+k}$ we obtain

$$\frac{\Delta(m_1,\cdots,m_{N_f})}{B_{N_f+p|p}(\mathcal{M})} \to \prod_{k=1}^{p}(m'_k - m_{N_f+k}). \tag{64}$$

The final expression for the generating function with degenerate quark masses is given by

$$\begin{aligned} Z^{\nu}_{N_f+p|p}(\mathcal{M};a) &= \frac{\prod_{k=1}^{p}(m'_k - m_{N_f+k})e^{2N_f a^2}}{(16\pi a^2)^{(N_f+2p)/2}} \int dsdt B_{N_f+p|p}(S)\Delta(S_1^f/8a^2,\cdots,S_{N_f}^f/8a^2) \\ &\times e^{(1/16a^2)\mathrm{Trg}[(S-\mathcal{M})^2]}\frac{\prod_k (is_k)^{\nu}}{\prod_l (t_l)^{\nu}}\tilde{Z}^{\nu}_{N_f+p|p}\left(\left\{\sqrt{(is_k)^2}\right\},\left\{\sqrt{(t_l)^2}\right\};a=0\right). \end{aligned} \tag{65}$$

In order to obtain nonzero contributions to the spectral resolvent (57) all m_k in the prefactor have to be differentiated. Below we give the explicit expressions in a couple of cases relevant for current lattice simulations. This expression is only valid if the sign ϵ is the same for all bosonic quark. To extract spectral correlation functions we also need expressions for which the sign of ϵ is different for different bosonic flavors. This issue will be addressed in a future publication.

A. The quenched case

The quenched one-point function $\rho^{\nu}_{\chi}(\lambda^W;a)$ follows from

$$\begin{aligned} \Sigma^{\nu}_{1|1}(m;a) &= -\frac{1}{16a^2\pi}\int \frac{dsdt}{t-is}e^{-\frac{s^2+t^2}{16a^2}}\frac{(is+m)^{\nu}}{(t+m)^{\nu}} \\ &\times \tilde{Z}^{\nu}_{1|1}(\sqrt{(is+m)^2},\sqrt{(t+m)^2};a=0), \end{aligned} \tag{66}$$

after using (56). The explicit form of $Z^{\nu}_{1|1}$ at $a=0$ is given in Eq. (33).

The two-point function in the quenched case is given by

$$\begin{aligned} \Sigma^{\nu}_{2|2}(m_1,m_2;a) &= \frac{1}{(16\pi a^2)^2}\int ds_1ds_2dt_1dt_2 \\ &\times \frac{(is_2+m_2-is_1-m_1)(t_2+m_2-t_1-m_1)}{(t_1-is_1)(t_2+m_2-is_1-m_1)(t_1+m_1-is_2-m_2)(t_2-is_2)} \\ &\times e^{-\frac{1}{16a^2}[s_1^2+s_2^2+t_1^2+t_2^2]}\frac{(is_1+m_1)^{\nu}(is_2+m_2)^{\nu}}{(t_1+m_1)^{\nu}(t_2+m_2)^{\nu}} \\ &\times \tilde{Z}^{\nu}_{2|2}(\sqrt{(is_1+m_1)^2},\sqrt{(is_2+m_2)^2},\sqrt{(t_1+m_1)^2},\sqrt{(t_2+m_2)^2};a=0). \end{aligned} \tag{67}$$

Here the $a=0$ partition function takes the form

$$\tilde{Z}^{\nu}_{2|2}(x_1,x_2,x_3,x_4;a=0) \;=\; 2\frac{x_3^\nu x_4^\nu}{x_1^\nu x_2^\nu}\frac{1}{(x_2^2-x_1^2)(x_4^2-x_3^2)} \tag{68}$$

$$\times\det\begin{pmatrix} I_\nu(x_1) & x_1 I_{\nu+1}(x_1) & x_1^2 I_{\nu+2}(x_1) & x_1^3 I_{\nu+3}(x_1) \\ I_\nu(x_2) & x_2 I_{\nu+1}(x_2) & x_2^2 I_{\nu+2}(x_2) & x_2^3 I_{\nu+3}(x_2) \\ (-1)^\nu K_\nu(x_3) & x_3(-1)^{\nu+1}K_{\nu+1}(x_3) & x_3^2(-1)^{\nu+2}K_{\nu+2}(x_3) & x_3^3(-1)^{\nu+3}K_{\nu+3}(x_3) \\ (-1)^\nu K_\nu(x_4) & x_4(-1)^{\nu+1}K_{\nu+1}(x_4) & x_4^2(-1)^{\nu+2}K_{\nu+2}(x_4) & x_4^3(-1)^{\nu+3}K_{\nu+3}(x_4) \end{pmatrix}.$$

The two-point correlation function contains a term due to self-correlations,

$$R_2(x,y)=\left\langle\sum_{k,l}\frac{1}{x+\lambda_k}\frac{1}{y+\lambda_l}\right\rangle=\left\langle\sum_{k}\frac{1}{x+\lambda_k}\frac{1}{y+\lambda_k}\right\rangle+\left\langle\sum_{k\neq l}\frac{1}{x+\lambda_k}\frac{1}{y+\lambda_l}\right\rangle. \tag{69}$$

This term can be rewritten as

$$\left\langle\sum_{k}\frac{1}{x+\lambda_k}\frac{1}{y+\lambda_k}\right\rangle=\frac{1}{y-x}\left\langle\sum_{k}\frac{1}{x+\lambda_k}-\frac{1}{y+\lambda_k}\right\rangle \tag{70}$$

which is singular for $y\to x$ if x and y are on opposite sides of the cut of the resolvent. It can be shown in general terms [34] that such singular terms are due to Efetov-Wegner terms and are not included in the expression (67). They will be analyzed in a future publication. The two-point spectral correlation function

$$\begin{aligned}\rho^\nu_\chi(\lambda_1^W,\lambda_2^W;a) \;=\; &\left\langle\sum_{\lambda_k^W,\lambda_l^W\in\mathbb{R}}\delta(\lambda_k^W+\lambda_1^W)\,\mathrm{sign}(\langle k|\gamma_5|k\rangle)\delta(\lambda_l^W+\lambda_2^W)\,\mathrm{sign}(\langle l|\gamma_5|l\rangle)\right\rangle \\ &-\left\langle\sum_{\lambda_k^W\in\mathbb{R}}\delta(\lambda_k^W+\lambda_1^W)\,\mathrm{sign}(\langle k|\gamma_5|k\rangle)\right\rangle\left\langle\sum_{\lambda_k^W\in\mathbb{R}}\delta(\lambda_k^W+\lambda_2^W)\,\mathrm{sign}(\langle k|\gamma_5|k\rangle)\right\rangle\end{aligned} \tag{71}$$

can also be decomposed into sum of self-correlations and genuine two-point correlations

$$\begin{aligned}\rho^\nu_\chi(\lambda_1^W,\lambda_2^W;a) \;=\; &\delta(\lambda_1^W-\lambda_2^W)\left\langle\sum_{\lambda_k^W\in\mathbb{R}}\delta(\lambda_k^W+\lambda_1^W)\right\rangle \\ &+\left\langle\sum_{\lambda_k^W,\lambda_l^W\in\mathbb{R},k\neq l}\delta(\lambda_k^W+\lambda_1^W)\,\mathrm{sign}(\langle k|\gamma_5|k\rangle)\delta(\lambda_l^W+\lambda_2^W)\,\mathrm{sign}(\langle l|\gamma_5|l\rangle)\right\rangle \\ &-\left\langle\sum_{\lambda_k^W\in\mathbb{R}}\delta(\lambda_k^W+\lambda_1^W)\,\mathrm{sign}(\langle k|\gamma_5|k\rangle)\right\rangle\left\langle\sum_{\lambda_k^W\in\mathbb{R}}\delta(\lambda_k^W+\lambda_2^W)\,\mathrm{sign}(\langle k|\gamma_5|k\rangle)\right\rangle.\end{aligned} \tag{72}$$

An important observation is that the sign of the chirality drops out in the expression for the self-correlations so that the diagonal part of the two-point correlator gives the density

of real modes

$$\rho_\chi^\nu(\lambda_1^W, \lambda_2^W = \lambda_1^W; a) \;=\; \left\langle \sum_{\lambda_k^W \in \mathbb{R}} \delta(\lambda_k^W + \lambda_1^W) \right\rangle . \tag{73}$$

B. One dynamical flavor

With one dynamical flavor of mass m_f we have

$$\begin{aligned}\Sigma_{2|1}^\nu(m, m_f; a) \;=\; & -\frac{e^{2a^2}}{64a^3\pi^{3/2} Z_1^\nu(m_f; a)} \int \frac{ds_1 ds_2 dt (is_2 + m - is_1 - m_f)}{(t + m - is_1 - m_f)(t - is_2)} e^{-[s_1^2+s_2^2+t^2]/16a^2} \\ & \times \frac{(-is_1 - m_f)^\nu (is_2 + m)^\nu}{(t+m)^\nu} \tilde{Z}_{2|1}^\nu(\sqrt{(is_1+m_f)^2}, \sqrt{(is_2+m)^2}, \sqrt{(t+m)^2}; a = 0).\end{aligned} \tag{74}$$

Note that the one flavor theory has a sign problem and consequently the one-point function

$$\rho_\chi^\nu(\lambda^W, m_f; a) = \frac{1}{\pi} \mathrm{Im}[\Sigma_{2|1}^\nu(m - i\epsilon, m_f; a)]_{\epsilon \to 0} \tag{75}$$

changes sign at $\lambda^W = m_f$, see also [11] where this function was derived by a direct supersymmetry computation.

The $a \to 0$ limit at fixed m and m_f is obtained by expanding the pre-exponential factors to first order in the s_k and t. This results in

$$\Sigma_{2|1}^\nu(m, m_f; a) = \frac{\nu}{m} + \frac{1}{m - m_f} + \Sigma_{2|1}(m, m_f; a = 0). + \cdots . \tag{76}$$

The $1/(m - m_f)$ term is expected to cancel against the Efetov-Wegner terms.

C. Two dynamical flavors

Finally we give the explicit form of the distribution of the chiralities over the real eigenvalues of D_W in a sector with fixed index ν for the physically relevant case of two light flavors with mass m_f. The spectral resolvent can be expressed as

$$\begin{aligned}\Sigma_{3|1}^\nu(m, m_f; a) \;=\; & \frac{e^{4a^2}}{\pi^2 (16a^2)^3 Z_{N_f=2}^\nu(m_f; a)} \int ds_1 ds_2 ds_3 dt \\ & \times \frac{(is_2 - is_1)^2 (is_3 + m - is_1 - m_f)(is_3 + m - is_2 - m_f)}{(t + m - is_1 - m_f)(t + m - is_2 - m_f)(t - is_3)} \\ & \times e^{-\frac{1}{16a^2}[s_1^2+s_2^2+s_3^2+t^2]} \frac{(is_1 + m_f)^\nu (is_2 + m_f)^\nu (is_3 + m)^\nu}{(t+m)^\nu} \\ & \times \tilde{Z}_{3|1}^\nu(\sqrt{(is_1+m_f)^2}, \sqrt{(is_2+m_f)^2}, \sqrt{(is_3+m)^2}, \sqrt{(t+m)^2}; a = 0),\end{aligned} \tag{77}$$

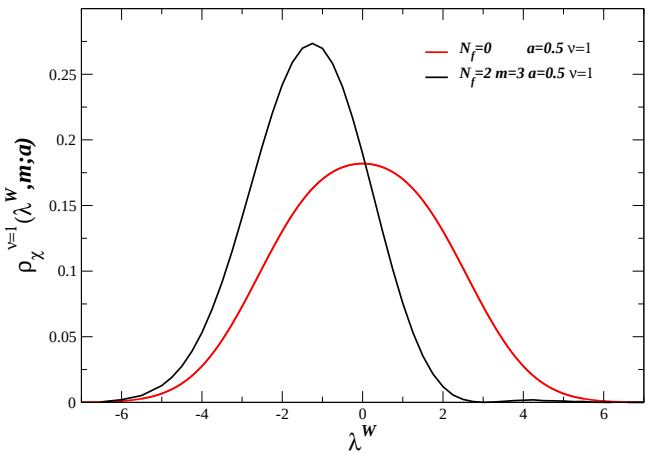

FIG. 4: The distribution of the chiralities over the real modes for $N_f = 0$ (red) and 2 (black). The repulsion away from the quark mass is clear in the unquenched curve. The fact that the area below the two curves equals ν suggests that the sectors defined in eq. (8) correspond to a Dirac operator D_W with index ν as defined in eq. (3).

where the two flavor partition function in the prefactor is given by Eq. (103) and the explicit form of $Z^{\nu}_{3|1}$ at $a = 0$ is given in (48).

A plot of the distribution of the chiralities over the real modes, ρ^{ν}_{χ}, is given in Fig. 4.

IV. CONCLUSIONS

We have obtained analytical expressions for all microscopic spectral correlation functions of the Wilson Dirac operator for any number of flavors with equal quark mass. In particular, we have computed the microscopic spectrum of the Hermitian Wilson Dirac operator in the physically relevant two flavor case and the distribution of the chiralities over the real eigenvalues of the Wilson Dirac operator. The results were obtained from a chiral Lagrangian for the generating function of the Wilson-Dirac spectrum using the graded eigenvalue method. We have also given expressions for an arbitrary number of flavors as well as higher order correlation functions. We have checked that our results for zero and one flavor are in complete agreement with a previous calculation based on a brute force supersymmetric computation. Since these results are based on a chiral Lagrangian that follows from the global symmetries of the lattice QCD partition function they can also be derived from a chiral random matrix

theory for the Wilson Dirac operator with the same global symmetries. This enables us to derive additional results using random matrix techniques which we hope to address in a future paper.

The new results, which have been obtained for a γ_5-Hermitian Wilson Dirac operator, give the leading order effect of the lattice discretization on the spectrum of the Wilson Dirac operator also in the physically relevant two flavor case. The analytical understanding of the smallest eigenvalues of the Wilson Dirac operator can be used to optimize the choices of parameters in lattice QCD for which the simulation is stable. Our results a offer a direct way to measure the low energy constants of Wilson chiral perturbation theory.

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Appendix A. EFETOV-WEGNER TERMS

In this Appendix we illustrate the effect of Efetov-Wegner terms for the Gaussian super-integral

$$Z(z_1, z_2) = \int d\sigma e^{\mathrm{Trg}(\sigma-\mathcal{Z})^2/16a^2}, \tag{78}$$

where σ and $\mathcal{Z}$ are the $(1|1)$ supermatrices

$$\sigma = \begin{pmatrix} a & \chi \\ \rho & b \end{pmatrix}, \qquad \mathcal{Z} = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}, \tag{79}$$

and $d\sigma$ is the integral over the matrix elements of σ. Clearly, the integral does not depend on z_1 and z_2 so that after a proper normalization of the measure we have

$$Z(z_1, z_2) = 1. \tag{80}$$

The supermatrix σ can be diagonalized by

$$\sigma = u \begin{pmatrix} is & 0 \\ 0 & t \end{pmatrix} u^{-1} \tag{81}$$

with

$$u = \exp \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}. \tag{82}$$

We first perform the integral by transforming to the eigenvalues of σ as integration variables and then perform the integral over u by a supersymmetric generalization of the Itzykson-Zuber integral. This results in (Actually this is a special case of Eq. (26) where the partition function for $a = 0$ is put equal to unity.)

$$Z(z_1, z_2) = \frac{(z_2 - z_1)}{16\pi a^2} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \frac{1}{t - is} e^{-[(s+iz_1)^2 + (t-z_2)^2]/16a^2}. \tag{83}$$

Using polar coordinates we obtain

$$Z(z_1, z_2) = \frac{z_2 - z_1}{16\pi a^2} \int dr d\phi e^{i\phi + i\theta} e^{-[r^2 + z_2^2 - z_1^2 - 2r\sqrt{z_2^2 - z_1^2}\cos\phi]/16a^2}, \tag{84}$$

with

$$e^{i\theta} = \frac{z_2 + z_1}{\sqrt{z_2^2 - z_1^2}}. \tag{85}$$

The integral over ϕ is a modified Bessel function

$$Z(z_1, z_2) = \frac{(z_2 - z_1)e^{i\theta}}{8a^2} \int dr I_1(r\sqrt{z_2^2 - z_1^2}/8a^2) e^{-[r^2 + z_2^2 - z_1^2]/16a^2}. \tag{86}$$

Using that

$$\int_0^{\infty} dx e^{-\alpha x^2} I_1(\beta x) = \frac{1}{\beta}(e^{\beta^2/4\alpha} - 1) \tag{87}$$

we obtain

$$Z(z_1, z_2) = -e^{-[z_2^2 - z_1^2]/16a^2} + 1, \tag{88}$$

which disagrees with Eq. (80). The missing contributions are the Efetov-Wegner terms which arise due to nilpotent terms at the singularity of the measure. Below we will evaluate these terms by regularizing the singularity.

We regularize the integral (78) over the matrix elements of σ by introducing the factor

$$\theta(\sqrt{a^2 + b^2} - \epsilon). \tag{89}$$

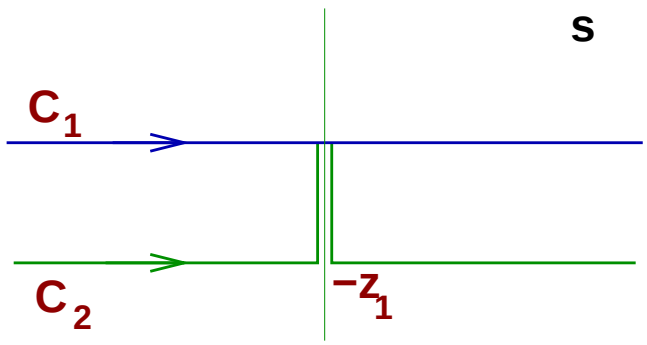

FIG. 5: Shifting the s-integration from C_1 to C_2 gives an additional contribution from the discontinuity across the imaginary s-axis.

Writing out the decomposition (81) we obtain

$$\begin{aligned} a &= s - i\alpha\beta(is - t), \\ b &= t + \alpha\beta(is - t) \end{aligned} \tag{90}$$

so that

$$\sqrt{a^2 + b^2} = \sqrt{s^2 + t^2} - \frac{\alpha\beta(is - t)^2}{\sqrt{s^2 + t^2}}, \tag{91}$$

and

$$\theta(\sqrt{a^2 + b^2} - \epsilon) = \theta(\sqrt{s^2 + t^2} - \epsilon) - \delta(\sqrt{s^2 + t^2} - \epsilon)\frac{\alpha\beta(is - t)^2}{\sqrt{s^2 + t^2}}. \tag{92}$$

The measure is given by

$$d\sigma = \frac{dsdtd\alpha d\beta}{(t - is)^2} \tag{93}$$

resulting in

$$Z(z_1, z_2) = \frac{1}{2\pi}\int \frac{dsdtd\alpha d\beta}{(t - is)^2}\theta((a^2 + b^2)^{1/2} - \epsilon)e^{[(is - z_1)^2 - (t - z_2)^2 - 2\alpha\beta(z_2 - z_1)(is - t)]/16a^2}. \tag{94}$$

Expanding the nilpotents in the exponent reproduces the result obtained from the Itzykson-Zuber integral which does not have to be regularized. We thus find

$$Z(z_1, z_2) = e^{[z_1^2 - z_2^2]/16a^2} + \frac{z_2 - z_1}{16\pi a^2}\int \frac{dsdt}{t - is}e^{[(is - z_1)^2 - (t - z_2)^2]/16a^2} = 1. \tag{95}$$

This is not the end of the story. Because of a discontinuity in the integrand we cannot simply shift the integration over s by $-iz_1$. There is an additional contribution from the discontinuity across the imaginary s-axis. We have that (see Fig. 5)

$$\int_{C_1} dsF(s) = \int_{C_2} dsF(s) \tag{96}$$

with

$$\int_{C_2} dsF(s) = \int_{-\infty}^{\infty} dsF(s - iz_1) + i\int_0^{z_1} dyF(-iz_1 + iy - \epsilon) - i\int_0^{z_1} dyF(-iz_1 + iy + \epsilon). \tag{97}$$

Applying this to the integral in Eq. (95) we obtain for the contribution of the vertical part of the integration contour

$$\begin{aligned} I_\Delta &\equiv \frac{z_2 - z_1}{16\pi a^2}\int dydt2\pi\delta(t - z_1 + y)e^{[y^2-(t-z_2)^2]/16a^2} \\ &= \frac{z_2 - z_1}{8a^2}\int_0^{z_1} dye^{[2y(z_1-z_2)-(z_1-z_2)^2]/16a^2}, \\ &= e^{-(z_1-z_2)^2/16a^2} - e^{[z_1^2-z_2^2]/16a^2}. \end{aligned} \tag{98}$$

The second term cancels against the Efetov-Wegner term.

The same derivation can be applied to the calculation of the quenched resolvent. The conclusion is that if we shift the z and z' dependence from the exponent to the $1/(t - is)$ factor, the Efetov-Wegner term is of the form $\exp(-(z-z')^2/16a^2)$ which does not contribute to the quenched resolvent.

Appendix B. DIFFUSIVE PARTITION FUNCTION

The N_f-flavor fermionic partition function was derived in Section II from the chiral Lagrangian. Including the normalization the N_f flavor partition function in the sector with index ν is given by

$$\begin{aligned} Z^\nu_{N_f}(m, \mathcal{Z}; a) &= \frac{e^{-2N_f a^2}}{(16\pi a^2)^{N_f/2}}\int \prod ds_k \Delta(\{is_k\})\Delta(\{is_k/8a^2\}) \\ &\times e^{-[(s_1+iz_1)^2+\cdots(s_{N_f}+iz_{N_f})^2]/16a^2}\prod_k (m - is_k)^\nu \tilde{Z}^\nu_{N_f}((m^2 + s_1^2)^{1/2}, \cdots, (m^2 + s_{N_f}^2)^{1/2}; a = 0). \end{aligned} \tag{99}$$

The normalization factor is such that we recover an identity for $a \to 0$.

For $N_f = 1$ we find

$$Z^{\nu}_{N_f=1}(m,z;a) = \frac{e^{-2a^2}}{\sqrt{16\pi a^2}} \int_{-\infty}^{\infty} ds e^{-(s+iz)^2/(16a^2)} (is-m)^{\nu} \frac{I_{\nu}((s^2+m^2)^{1/2})}{(s^2+m^2)^{\nu/2}}. \tag{100}$$

Using the identity

$$\left(\frac{is-m}{is+m}\right)^{\nu/2} I_{\nu}((s^2+m^2)^{1/2}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\nu\theta} e^{-m\sin\theta + s\cos\theta} \tag{101}$$

the integral over s becomes a simple Gaussian integral, and the one-flavor partition function can be rewritten as

$$Z^{\nu}_{N_f=1}(m,z;a) \;=\; e^{-2a^2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\nu\theta} \; e^{-m\sin\theta - iz\cos\theta + 4a^2\sin^2\theta} \tag{102}$$

This is indeed the expression for the one flavor partition derived in [5].

For $N_f = 2$ the normalization factor is given by $\mathcal{N} = 1/(\pi(16a^2)^2)$, so that the two flavor partition function reduces to

$$\begin{aligned} Z^{\nu}_{N_f=2}(m_1,m_2;a) \;=\; & \frac{e^{4a^2}}{\pi 8a^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds_1 ds_2 \frac{(is_1 - is_2)}{m_1 - m_2} e^{-\frac{1}{16a^2}[(s_1+im_1)^2 + (s_2+im_2)^2]} (is_1)^{\nu} (is_2)^{\nu} \\ & \times \tilde{Z}^{\nu}_2(((is_1)^2)^{1/2}, ((is_2)^2)^{1/2}; a=0), \end{aligned} \tag{103}$$

where

$$\tilde{Z}^{\nu}_2(x_1,x_2) = \frac{2}{x_1^{\nu} x_2^{\nu}(x_2^2 - x_1^2)} \det \begin{vmatrix} I_{\nu}(x_1) & x_1 I_{\nu+1}(x_1) \\ I_{\nu}(x_2) & x_2 I_{\nu+1}(x_2) \end{vmatrix}. \tag{104}$$

It is also instructive to work out the partition function for $N_f = -1$. Using the general expression (26) we obtain

$$Z^{\nu}_{N_f=-1}(m,z;a) = \frac{e^{2a^2}}{\sqrt{16\pi a^2}} \int_{-\infty}^{\infty} dt e^{-(t-z)^2/16a^2} \frac{(m^2-t^2)^{\nu/2}}{(t-m)^{\nu}} (-1)^{\nu} K_{\nu}((m^2-t^2)^{1/2}). \tag{105}$$

Using the identity

$$2\left(\frac{t-i\epsilon+m}{t-i\epsilon-m}\right)^{\nu/2} K_{\nu}((m^2-(t-i\epsilon)^2)^{1/2}) = \int_{-\infty}^{\infty} ds \; e^{-\nu s} e^{-im\sinh s - i(t-i\epsilon)\cosh s} \tag{106}$$

we obtain after performing the Gaussian integration over t and shifting the s-integration by πi

$$\begin{aligned} Z^{\nu}_{N_f=-1}(m,z;a) \;&=\; e^{2a^2} \int_{-\infty}^{\infty} ds \; e^{-\nu s} e^{im\sinh s + iz\cosh s - 4a^2\cosh^2 s} \\ &=\; \int_{-\infty}^{\infty} ds \; e^{-\nu s} e^{-im\sinh s - iz\cosh s - 2a^2\cosh(2s)}. \end{aligned} \tag{107}$$

which agrees with the bosonic part of the result obtained in [5].

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