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Charged Magnetic Brane Correlators and Twisted Virasoro algebras[1]

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Abstract

Prior work using gauge/gravity duality has established the existence of a quantum critical point in the phase diagram of 3+1-dimensional gauge theories at finite charge density and background magnetic field. The critical theory, obtained by tuning the dimensionless charge density to magnetic field ratio, exhibits nontrivial scaling in its thermodynamic properties, and an associated nontrivial dynamical critical exponent. In the present work, we analytically compute low energy correlation functions in the background of the charged magnetic brane solution to 4+1-dimensional Einstein-Maxwell-Chern-Simons theory, which represents the bulk description of the critical point. Results are obtained for neutral scalar operators, the stress tensor, and the U(1)-current. The theory is found to exhibit a twisted Virasoro algebra, constructed from a linear combination of the original stress tensor and chiral U(1)-current. The effective speed of light in the IR is renormalized downward for one chirality, but not the other, by finite density, a behavior that is consistent with a Luttinger liquid description of fermions in the lowest Landau level. The results obtained here do not directly shed light on the mechanism driving the phase transition, and we comment on why this is so.

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1 Introduction and summary of results

Holography provides a powerful tool for the study of strongly interacting fermion systems, including strongly coupled gauge theories. Non-relativistic extensions to systems at finite temperature, charge density, and external magnetic field, have applications to physical problems in many areas, including in condensed matter, heavy ion collisions, and astrophysics [1, 2, 3] . Using holography, thermodynamic functions, correlators, and transport properties may be calculated in terms solutions to classical gravity equations. A common setting for many of these systems is provided by RG flows in 3+1-dimensional gauge theories, with or without supersymmetry. The gravity dual to a wide class of these gauge theories is governed by 4+1-dimensional Einstein-Maxwell-Chern-Simons theory, whose only free parameter is the Chern-Simons coupling k, which is related to the number of chiral fermions in the dual gauge theory.

In the presence of an external magnetic field, the RG flow from 3+1-dimensional gauge theory in the UV takes us, in the IR, to an effective 1+1-dimensional CFT which is dominated by massless fermions in the lowest Landau level. These lowest Landau level fermions are only free to move parallel to the magnetic field lines, which leads to the 1+1-dimensional character of the low energy theory. This picture is borne out on the gravity side, where the RG flow is governed by the purely magnetic brane solution which interpolates between AdS_5 in the UV and AdS_3 (or BTZ when the temperature is small) in the IR [4, 5]. Holographic calculations of two-point correlators of the U(1) current and stress tensor further confirm the existence of two Virasoro algebras and a single chiral current algebra in the IR [6]. The common central charge of the Virasoro algebras equals the Brown-Henneaux value [7] for the AdS_3 near horizon geometry which, via the Cardy formula, accounts for the specific heat coefficient[2] $\hat{s}/\hat{T}$ at low $\hat{T}$. Such a specific heat is consistent with a 1+1-dimensional Luttinger liquid description [8] of the fermionic theory. The chirality of the current algebra is given by the sign of k, while its level is proportional to $|k|$. Thus, in this regime, a consistent picture has emerged in which the behavior of thermodynamic quantities and correlators are found to nicely match.

As the magnetic field is lowered (or equivalently as the charge density is increased), the lowest Landau level finds itself increasingly populated. Eventually, we expect qualitative changes to occur, both due to occupancy of higher fermionic Landau levels as well as bosonic Landau levels, along with any other strong interaction effects at finite density. Numerical and analytical holographic calculations, carried out in the dual Einstein-Maxwell-Chern-Simons theory, confirm this expectation and reveal a rich phase structure as a function of the Chern-Simons coupling k [9, 10, 11] (see [12, 13] for other examples of holographic phase

[2]In the boundary CFT only dimensionless combinations have an invariant physical meaning. Dividing by a suitable power of the charge density ρ, the dimensionless ratios for the magnetic field B, the entropy density s, and the temperature T will respectively be denoted by $\hat{B} = B/\rho^{2/3}$, $\hat{s} = s/\rho$, and $\hat{T} = T/\rho^{1/3}$.

transitions involving magnetic fields and finite density.) Specifically, for $k > 1/2$, the low $\hat{T}$ specific heat coefficient was derived exactly as a function of the magnetic field $\hat{B}$, and is given by,

$$\frac{\hat{s}}{\hat{T}} = \frac{\pi}{6} \frac{\hat{B}^3}{\hat{B}^3 - \hat{B}_c^3} \qquad\qquad \hat{B} > \hat{B}_c \tag{1.1}$$

While the constancy of $\hat{s}/\hat{T}$ as $\hat{T} \to 0$ is still as in Luttinger liquid theory, its dependence on $\hat{B}$ is not. As the magnetic field approaches the critical value $\hat{B}_c$ (which depends only on k), a quantum critical point was discovered at $\hat{B} = \hat{B}_c$. The scaling behavior of $\hat{s}$ at $\hat{B} = \hat{B}_c$ was obtained numerically in [10], and subsequently established by analytic calculations [11], both for $k > 3/4$. As reviewed in Appendix A, the picture was completed since by a numerical calculation of the scaling in the range $1/2 < k < 3/4$. Assembling these data, we have,

$$\hat{s} \sim \hat{T}^{\alpha} \qquad\qquad \alpha = \begin{cases} \frac{1}{3} & \frac{3}{4} \leq k \\ \\ \frac{1-k}{k} & \frac{1}{2} < k \leq \frac{3}{4} \end{cases} \tag{1.2}$$

This result signals further strong deviations from canonical Luttinger liquid behavior. The full scaling function, accounting for variations in both $\hat{T}$ and $\hat{B}$ in the critical region, was also computed analytically in [11] for the range $k > 1$, and agrees with the numerical results of [10].

Remarkably, as discussed in [10], the critical exponent $1/3$ arises also in the Hertz/Millis theory [14, 15, 16] of metamagnetic quantum phase transitions, in which a variation in the magnetic field across a non-zero critical value produces non-analytic behavior in the specific heat, but no change in symmetry. Metamagnetic quantum criticality has been observed [17] in a wealth of recent experiments on Strontium Ruthenates, where the entropy density has been mapped out through an extensive region of the temperature and magnetic field parameter space. Interestingly, as the critical point is approached the specific heat coefficient is found to diverge linearly in $1/(B - B_c)$, just as in (1.1).

The purpose of the present paper is dual. On the one hand, we wish to extend to the case of finite charge density the holographic calculations of various correlators which have been evaluated in [6] for vanishing charge density, and understand the relation with the low $\hat{T}$ thermodynamics already established for finite charge density. On the other hand, if the holographic calculations of thermodynamics are really relevant to metamagnetic quantum phase transitions, then obtaining the corresponding holographic correlators should allow one to address transport problems in metamagnetic materials close to quantum criticality in a model with only a single free parameter k. The results of this paper will constitute a modest first step in both directions, and will be summarized below.

1.1 Summary of results

In this paper, 2-point correlators will be evaluated in the IR regime for a neutral scalar, for the U(1) current, and for the stress tensor, in the presence of the charged magnetic brane solution. The method of overlapping expansions may be justified for small momenta, and will be used throughout.

Calculating the 2-point correlators of a neutral scalar in the IR regime, we identify effective left- and right-moving excitations with different propagation speeds. While left-movers propagate at the speed of light (with respect to the standard boundary metric) as they do for the purely magnetic brane, the speed of the right-movers is less than the speed of light in the presence of non-zero charge density. We compare these holographic predictions for the dynamics of chiral excitations with those of Luttinger liquid theory [8], in which the speed is also predicted to be renormalized downward. Finally, we argue in favor of a relation between the entropy density scaling of (1.2) for $1/2 < k < 3/4$ and the dynamical scaling properties of the scalar two-point functions. We leave unexplained, however, the precise nature of the crossover to the critical exponent $1/3$, and the precise value of the coupling k at which this cross-over should occur.

Calculating the 2-point functions of the U(1) current $\mathcal{J}_\pm$ and the stress tensor $\mathcal{T}_{\pm\pm}$ reveals that their IR behavior is governed by two *Virasoro algebras, one of which is twisted with the chiral current algebra.* The positive chirality correlators are given by the following expressions, valid for all $k > 1/2$,

$$\begin{aligned}
\left\langle \mathcal{J}_+(x^+)\,\mathcal{J}_+(y^+)\right\rangle &\sim -\frac{kc}{2\pi^2}\,\frac{1}{(x^+-y^+)^2}\\
\left\langle \mathcal{J}_+(x^+)\,\mathcal{T}_{++}(y^+)\right\rangle &\sim +\frac{kc\mu}{2\pi^2}\,\frac{1}{(x^+-y^+)^2}\\
\left\langle \mathcal{T}_{++}(x^+)\,\mathcal{T}_{++}(y^+)\right\rangle &\sim -\frac{kc\mu^2}{2\pi^2}\,\frac{1}{(x^+-y^+)^2}+\frac{c}{8\pi^2}\,\frac{1}{(x^+-y^+)^4}
\end{aligned} \tag{1.3}$$

where c is the Brown-Henneaux central charge, and μ is the chemical potential. The non-vanishing negative chirality correlator is given by,

$$\left\langle \mathcal{T}_{--}(x^-)\,\mathcal{T}_{--}(y^-)\right\rangle \sim \frac{c}{8\pi^2}\,\frac{1}{(x^--y^-)^4} \tag{1.4}$$

These correlators have higher order IR corrections, whose quantitative evaluation would require going beyond the overlapping expansion method. Qualitatively, however, we know that some of these higher order corrections arise from double and higher trace operators which are generated by the RG flow towards the IR [18, 19], just as they were shown to arise in the purely magnetic case in [6]. Upon forming the combination

$$\mathcal{T}^{(0)}_{++} = \mathcal{T}_{++} + \mu\mathcal{J}_+ \tag{1.5}$$

the chiral current algebra of $\mathcal{J}_+$ decouples from the Virasoro algebra of $\mathcal{T}^{(0)}_{++}$, and we are left with two decoupled Virasoro algebras, and one decoupled chiral current algebra, just as we had in the case of the purely magnetic case. These results are valid for all $k > 1/2$, and constitute perhaps the most important result of this paper. The net effect of turning on a non-zero charge density (or equivalently, a non-zero chemical potential μ) is simply to *twist* the positive chirality Virasoro algebra by the admixture of $\mu\mathcal{J}_+$, as well as renormalizing the speed of propagation. These results confirm the presence of a unique underlying CFT for all values of the magnetic field, but offer no immediate insight into the dynamical scaling properties of the entropy density in (1.2).

1.2 Organization

The remainder of this paper is organized as follows. In section 2, we review the Einstein-Maxwell-Chern-Simons theory and the salient properties of its charged magnetic brane solutions. In section 3, we calculate the 2-point correlators of a neutral scalar field of arbitrary mass, and discuss some of their immediate physical consequences in section 4. In section 5, we outline the general structure of the 2-point correlators of the U(1) current and the stress tensor. In section 6 we set up the technical calculation of the linear fluctuations in the gauge field and metric using overlapping expansions in light-cone gauge for the near region to LMN and Fefferman-Graham gauge in the far region. The detailed calculations of the twisted Virasoro terms in the current and stress tensor 2-point functions are derived in sections 7 and 8. A brief discussion of some open questions is presented in section 9. In Appendix A we review the behavior of the low temperature specific heat exponent as a function of k. In Appendix B, perturbative and WKB approximations to the scalar correlators are developed. In Appendix C, the full linearized fluctuation equations around the charged magnetic brane solution are derived in light-cone gauge, while the gauge transformations required in the overlap region are constructed in Appendix D.

2 Review of the charged magnetic brane solution

The action of five-dimensional Einstein-Maxwell-Chern-Simons theory with a negative cosmological constant and Chern-Simons coupling k was given in [6]. Here, we will need the corresponding Bianchi identity $dF = 0$ and field equations,

$$\begin{aligned} 0 &= d \star F + kF \wedge F \\ R_{MN} &= 4g_{MN} + \frac{1}{3} F^{PQ} F_{PQ} g_{MN} - 2F_{MP} F_N{}^P \end{aligned} \tag{2.1}$$

Throughout, we will take $k \geq 0$, since a sign reversal of k is equivalent to a parity transformation. For the supersymmetric value $k = 2/\sqrt{3}$, this system is a consistent truncation for all supersymmetric compactifications of Type IIB or M-theory to AdS_5 (see [20, 21, 22]). Solutions of (2.1) are guaranteed to be solutions of the full 10 or 11 dimensional supergravity field equations, and to provide holographic duals to the infinite class of supersymmetric field theories dual to these more general supersymmetric AdS_5 compactifications.

2.1 Boundary stress tensor and current

We will be considering asymptotically AdS_5 solutions, for which the metric and gauge field admit a Fefferman-Graham expansion [23]. Introducing a radial coordinate ρ, defined such that the AdS_5 boundary is located at $\rho = \infty$, the metric takes the asymptotic form

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + g_{\mu\nu}(\rho, x) dx^\mu dx^\nu \\ g_{\mu\nu}(\rho, x) &= \rho g^{(0)}_{\mu\nu}(x) + g^{(2)}_{\mu\nu}(x) + \frac{1}{\rho} g^{(4)}_{\mu\nu}(x) + \frac{\ln \rho}{\rho} g^{(\ln)}_{\mu\nu}(x) + \cdots \end{aligned} \tag{2.2}$$

while the gauge field takes the asymptotic form,

$$\begin{aligned} A &= A_\mu(\rho, x) dx^\mu \\ A_\mu(\rho, x) &= A^{(0)}_\mu(x) + \frac{1}{\rho} A^{(2)}_\mu(x) + \cdots \end{aligned} \tag{2.3}$$

Here $x^M = (\rho, x^\mu)$, with $\mu = +, -, 1, 2$. The coefficients $g^{(2)}_{\mu\nu}$, $g^{(\ln)}_{\mu\nu}$, and the trace of $g^{(4)}_{\mu\nu}$ are fixed by the Einstein equations to be local functionals of the conformal boundary metric $g^{(0)}_{\mu\nu}$.

The boundary stress tensor and current [24, 25] are defined in terms of the variation of the on-shell action with respect to $g^{(0)}_{\mu\nu}$ and $A^{(0)}_\mu$ respectively (see [6]). In terms of the Fefferman-Graham data the result is,

$$\begin{aligned} 4\pi G_5 T_{\mu\nu}(x) &= g^{(4)}_{\mu\nu}(x) + \text{local} \\ 2\pi G_5 J_\mu(x) &= A^{(2)}_\mu(x) + \text{local} \end{aligned} \tag{2.4}$$

Indices are raised and lowered using the conformal boundary metric $g^{(0)}_{\mu\nu}$. The local terms denote tensors constructed locally from $g^{(0)}_{\mu\nu}$ and $A^{(0)}_{\mu}$. In this paper, as in [6], we are interested in computing two-point correlation functions of operators at non-coincident points, to which the local terms in (2.4) do not contribute. Henceforth, we will drop all local terms.

2.2 Ansatz, Reduced field equations, and symmetries

The charged magnetic brane solution was constructed in [11]. It belongs to a class of solutions characterized by a constant background magnetic field, rotation invariance around this magnetic field, and translation invariance in all boundary directions x^μ. The Ansatz is conveniently expressed in the so-called "LMN" gauge,

$$\begin{aligned} ds^2 &= \frac{dr^2}{L^2 - MN} + M(dx^+)^2 + 2L dx^+ dx^- + N(dx^-)^2 + e^{2V} dx^i dx^i \\ A &= b x^1 dx^2 + A_+ dx^+ + A_- dx^- \end{aligned} \tag{2.5}$$

The index $i = 1, 2$ labels the coordinates of the plane orthogonal to the magnetic field. The magnetic field strength $b = F_{12}$ is constant, and by rescaling $x^{1,2}$, may be set to a convenient value, $b = \sqrt{3}$. The functions $L, M, N, V, A_\pm$ depend only upon the holographic coordinate r. The reduced field equations were given in [11], with the identifications $E = A'_+, P = -A'_-$,

$$\begin{array}{ll} \text{M1} & \Big((NE + LP)e^{2V}\Big)' + 2kbP = 0 \qquad (2.6) \\ \text{M2} & \Big((LE + MP)e^{2V}\Big)' - 2kbE = 0 \\ \text{E1} & L'' + 2V'L' + 4(V'' + V'^2)L - 4PE = 0 \\ \text{E2} & M'' + 2V'M' + 4(V'' + V'^2)M + 4E^2 = 0 \\ \text{E3} & N'' + 2V'N' + 4(V'' + V'^2)N + 4P^2 = 0 \\ \text{E4} & f(V')^2 + f'V' - 6 + \frac{1}{4}(L')^2 - \frac{1}{4}M'N' + b^2 e^{-4V} + MP^2 + 2LEP + NE^2 = 0 \\ \text{fV} & f'' + 4f'V' + 2fV'' + 4f(V')^2 - 24 = 0 \end{array}$$

Throughout, primes will denote derivatives with respect to the holographic coordinate, and we will use the abbreviation,

$$f = L^2 - MN \tag{2.7}$$

Notice that LMN gauge of (2.5) differs from Fefferman-Graham gauge of (2.3) only in the structure of the term proportional to dr^2. Finally, under constant $SL(2, \mathbf{R})$ transformations of the coordinates $x^\pm$,

$$\begin{pmatrix} \tilde{x}^+ \\ \tilde{x}^- \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \qquad\qquad \Lambda \in SL(2, R) \tag{2.8}$$

the Ansatz (2.5) is invariant provided the fields transform as follows,

$$\begin{pmatrix} \tilde{A}_+ \\ \tilde{A}_- \end{pmatrix} = \Lambda^t \begin{pmatrix} A_+ \\ A_- \end{pmatrix} \qquad\qquad \begin{pmatrix} \tilde{M} & \tilde{L} \\ \tilde{L} & \tilde{N} \end{pmatrix} = \Lambda^t \begin{pmatrix} M & L \\ L & N \end{pmatrix} \Lambda \tag{2.9}$$

while the field V and the combination f are invariant.

2.3 Charged magnetic brane solution

The basic charged magnetic brane solution satisfies the consistent restriction,

$$N = P = 0 \tag{2.10}$$

As a result, the functions $L = L_0$ and $V = V_0$ satisfy the reduced field equations E1, E4, and fV of the purely magnetic background solution [6]. The E4 equation is a constraint whose derivative vanishes by virtue of equations E1 and fV. Although no analytical solution is known, the relevant parameters governing the small and large r asymptotics are available numerically, and this will suffice for our purpose. The remaining functions $A_+ = A_0$ and $M = M_0$ were solved for in [11], and are given by,

$$\begin{aligned} A_0(r) &= \frac{c_V c_E}{kb} \, e^{2kb\psi(r)} \\ M_0(r) &= L_0(r) \left(-\frac{\alpha_\infty}{2b} - 4kb \int_\infty^r \frac{dr' A_0(r')^2}{L_0(r')^2 \, e^{2V_0(r')}} \right) \end{aligned} \tag{2.11}$$

where ψ is defined by,

$$\psi(r) = \int_\infty^r \frac{dr'}{L_0(r') \, e^{2V_0(r')}} \tag{2.12}$$

The parameters c_V, c_E, and α_∞ entering these functions specify the asymptotic behavior of these functions, as we will now spell out.

2.4 Asymptotic AdS$_5$ behavior

As $r \to \infty$, the solution approaches AdS$_5$, and we have the following asymptotics,

$$\begin{aligned} L_0(r) &= 2(r - r_0) - \frac{3 \ln r}{c_V^2 \, r} + \mathcal{O}(r^{-1}) \\ e^{2V_0(r)} &= c_V(r - r_0) + \mathcal{O}(r^{-1}) \\ A_0(r) &= \frac{c_V c_E}{kb} - \frac{c_E}{r} + \mathcal{O}(r^{-2}) \\ M_0(r) &= -\frac{\alpha_\infty}{b} \, r + \mathcal{O}(r^0) \\ \psi(r) &= -\frac{1}{2c_V \, r} + \mathcal{O}(r^{-2}) \end{aligned} \tag{2.13}$$

The interpretation of these constants was given in [11]. The constants c_V and r_0 characterize the purely magnetic solution. The asymptotics $E_0(r) \sim c_E/r^2$ of the electric field $E_0 = A_0'$ shows that c_E is proportional to the electric charge density ρ of the dual state, while the asymptotics of A_0 gives its chemical potential μ. The precise relations are as follows,

$$\rho = 4c_E \left(\frac{2b}{\alpha_\infty} \right)^{\frac{1}{2}} \qquad\qquad \mu = \frac{c_V c_E}{kb} \tag{2.14}$$

where the coefficient α_∞ arises from the asymptotics of the metric function M_0.

2.5 Asymptotic near-horizon behavior

As $r \to 0$, the solution approaches a family of exact solutions for which V is constant,

$$\begin{aligned} F &= b dx^1 \wedge dx^2 + q\, r^{k-1} dr \wedge dx^+ \\ ds^2 &= \frac{dr^2}{4b^2} - \left(\alpha_0 r + \frac{2q^2 r^{2k}}{k(2k-1)} \right) (dx^+)^2 + 4br dx^+ dx^- + dx^i dx^i \end{aligned} \tag{2.15}$$

The corresponding functions are given by,

$$\begin{aligned} L_0(r) &= 2br + \mathcal{O}(r^{1+\sigma}) \qquad\qquad \sigma = -\frac{1}{2} + \frac{\sqrt{57}}{6} \\ e^{2V_0(r)} &= 1 + 2r^\sigma + \mathcal{O}(r^{2\sigma}) \\ A_0(r) &= \frac{q}{k}\, r^k + \mathcal{O}(r^{k+\sigma}) \\ M_0(r) &= -\alpha_0 r + \mathcal{O}(r^{1+\sigma}) \\ \psi(r) &= \frac{\ln r}{2b} + \psi_0 + \mathcal{O}(r^\sigma) \end{aligned} \tag{2.16}$$

The constant ψ_0 is a property of the purely magnetic solution only. On the other hand, q and α_0 are associated with the presence of non-zero charge density.

The metric appearing in the (r, x^+, x^-) directions is the same as a 2+1-dimensional Schrodinger solution [26, 27], or alternatively a null-warped solution [28].

2.6 Relations between asymptotic parameters and regularity

The key asymptotic data in the large and small r asymptotics are related as follows,

$$\begin{aligned} q &= \frac{c_V c_E}{b}\, e^{2kb\psi_0} \\ \alpha_0 &= \alpha_\infty - 16 c_V^2 c_E^2 J(k) \qquad\qquad J(k) = \frac{1}{2k} \int_0^\infty \frac{dr\, e^{4kb\psi(r)}}{L_0(r)^2 e^{2V_0(r)}} \end{aligned} \tag{2.17}$$

They link the large r and small r behavior of A_0 and M_0. The integral $J(k)$ is convergent for $k > 1/2$, and is positive, so that we must have $\alpha_0 < \alpha_\infty$ as long as $q \neq 0$.

Regularity of the solution requires $M_0(r) \leq 0$ for all $r > 0$.footnoteMore precisely, if this condition is not obeyed then the zero temperature solutions studied here cannot be obtained as limits of smooth, finite temperature solutions. When $q \neq 0$, the near-horizon metric (2.15) is regular only when $k > 1/2$, a condition we will assume throughout. Since $M_0(r)/L_0(r) + \alpha_\infty/(2b)$ vanishes at $r = \infty$, and is monotonically increasing for decreasing r, regularity also requires $0 \leq \alpha_0$. The relation between α_∞ and α_0 was shown in [11][3] to be governed by the (normalized) magnetic field $\hat{B} = B/\rho^{2/3}$, where $B = 2b/c_V$,

$$\frac{\alpha_0}{\alpha_\infty} = 1 - \frac{\hat{B}_c^3}{\hat{B}^3} \qquad\qquad \hat{B}_c^3 = \frac{12J(k)}{c_V} \tag{2.18}$$

Thus, regularity of the charged magnetic brane solution forces $\hat{B} \geq \hat{B}_c$, with the critical limit $\hat{B} = \hat{B}_c$ corresponding to a quantum phase transition [11]. For $\hat{B} < \hat{B}_c$ a new branch of solutions opens up; these were studied numerically in [11], but no analytical results are currently available.

2.7 The action of $SL(2, \mathbf{R})$

$SL(2, \mathbf{R})$ transformations which preserve the consistent restriction $P = N = 0$ of (2.10) may be parametrized as follows,

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ -\lambda_0 \lambda & \lambda_0^{-1} \end{pmatrix} \tag{2.19}$$

They will map charged magnetic brane solutions with different parameters into one another. Denoting the transformed quantities by tildes, as in (2.9), we find,

$$\begin{aligned} \tilde{q} &= \lambda_0 \, q \\ \tilde{\alpha}_0 &= \lambda_0^2(\alpha_0 + 4b\lambda) \\ \tilde{\alpha}_\infty &= \lambda_0^2(\alpha_\infty + 4b\lambda) \end{aligned} \tag{2.20}$$

We see that the parameter λ_0 rescales the charge q, while the parameter λ changes the normalized magnetic field. In particular, all charged magnetic brane solutions may be obtained as $SL(2, \mathbf{R})$ maps of the critical solution for which $\alpha_0 = 0$, $\hat{B} = \hat{B}_c$, and $\alpha_\infty = \alpha_c = 16c_V^2 c_E^2 J(k)$, the magnetic field being given by,

$$1 - \frac{\hat{B}_c^3}{\hat{B}^3} = \frac{4b\lambda}{\alpha_c + 4b\lambda} \tag{2.21}$$

[3]The coefficients α_∞ and α_0 were denoted respectively by α and $\tilde{\alpha}$ in [11].

Turning on a non-zero temperature, as we did in [11], or non-zero momenta, as we will do here, will make these solutions physically inequivalent, as the $SL(2,\mathbf{R})$ transformation will now also change the temperature and/or the momenta. The practical advantage is that, without loss of generality, we can restrict to studying the solution corresponding to the critical magnetic field $\hat{B} = \hat{B}_c$, since the solution for other (finite) values of $\hat{B}$ may be derived from the critical solution by applying the above $SL(2,\mathbf{R})$ transformation. Thus, henceforth, we set $\alpha_0 = 0$, and $\alpha_\infty = \alpha_c$.

3 Correlators of a neutral scalar field

In the case of the purely magnetic background solution, the fluctuations of the gauge field decoupled from the metric fluctuations to linearized order, so that the correlator between the $U(1)$ current and the stress tensor vanished [6]. For the charged magnetic brane solution, no such decoupling takes place, a fact that renders the problem more involved. A simpler warm-up case is provided, however, by studying the fluctuations of a free neutral scalar field in the presence of the charged magnetic brane solution. Although the neutral scalar will only probe the metric of the background solution, and not its gauge field, its study will provide valuable information for scalar correlators that would be much more difficult to obtain for the full stress tensor and current correlators.

3.1 Reduced Field Equation

We consider a free neutral scalar field Φ with mass m. It obeys the wave equation,

$$\frac{1}{\sqrt{g}}\partial_M\left(\sqrt{g}g^{MN}\partial_N\Phi\right) - m^2\Phi = 0 \tag{3.1}$$

The background being invariant under translations in x^μ, and equation (3.1) being linear, it will suffice to consider plane wave solutions with momenta $p_\mu = (p_+, p_-, p_1, p_2)$. Since we will be interested only in the $x^\pm$-dependence, we set $p_1 = p_2 = 0$, which leaves,

$$\Phi(r, x^\pm) = \phi(r, p_\pm)\, e^{ipx} \tag{3.2}$$

where we continue to use the notations of [6], namely, $px = p_+x^+ + p_-x^-$, and $p^2 = p_+p_-$. In the background of the charged magnetic brane solution, the wave equation (3.1) reduces to,

$$L^2\phi'' + \frac{1}{K}(KL^2)'\phi' - \frac{2p^2}{L}\phi + \frac{p_-^2 M}{L^2} - m^2\phi = 0 \tag{3.3}$$

Throughout, we will often use the abbreviation $e^{2V} = K$. Since the metric is unperturbed in this system, we have dropped the subscript 0 on the functions L, M, K. No analytic solution is known to the reduced scalar equation (3.3), which comes as no surprise since the functions K, L, M themselves are not known analytically. Nonetheless, for small momenta (precise conditions will be spelled out later), we may use the method of overlapping expansions which was used already in [6] to calculate the entropy density of the charged magnetic brane solution for low temperature, as a function of $\hat{B}$. Following this method, we solve (3.3) in the low momentum limit, by matching the solutions in overlapping near and far regions, corresponding to small and large r respectively. As announced in section 2.7, we restrict to analyzing the critical solution corresponding to $\hat{B} = \hat{B}_c$, since all other solutions may be recovered by the $SL(2, \mathbf{R})$ transformations of section 2.7.

3.2 Solutions in the far region

In the far region, momentum dependence in (3.3) is to be neglected, which requires $p^2 \ll r$ and $p_-^2 \ll r$. We want the far region to extend down to $r \ll 1$ so that the far region will overlap non-trivially with the near region; this requires $p^2 \ll 1$ and $p_-^2 \ll 1$. For $k < 1$, there is, however, an additional requirement: since $M(r) \sim r^{2k}$ for $r \ll 1$, the above overlap condition will hold provided we have $p_-^2 r^{2k-2} \ll 1$ as well (this condition is automatic for $k > 1$ in view of the requirement $p_-^2 \ll 1$ already imposed earlier). Note that it is possible to satisfy all these conditions by making p_- small, and yet allowing p_+ to be of order 1.

Assuming that the above conditions are satisfied, all momentum dependence in (3.3) may be dropped, and we are left with the reduced equation in the far region,

$$L^2\phi'' + \frac{1}{K}(KL^2)'\phi' - m^2\phi = 0 \tag{3.4}$$

which involves only the data of the purely magnetic background solution. Clearly, no analytical solution for ϕ is available, since K and L themselves are known only numerically. The data we need from the far region, however, depend on only a small number of parameters which may be computed numerically, if so desired.

To see how this works, consider (3.4) for $r \gg 1$, where $L(r) \sim 2r$ and $K(r) \sim c_V r$,

$$4r^2\phi'' + 12r\phi' - m^2\phi = 0 \tag{3.5}$$

For $r \gg 1$, the general solution is given by,

$$\phi(r) = a_+ r^{\nu_+} + a_- r^{\nu_-} \qquad\qquad \nu_\pm = -1 \pm \sqrt{1 + m^2/4} \tag{3.6}$$

For $r \ll 1$, but still in the far region where momenta are dropped, we have $L(r) \sim 2br$ and $K(r) \sim 1$, so that (3.4) becomes,

$$12r^2\phi'' + 24r\phi' - m^2\phi = 0 \tag{3.7}$$

The general solution of (3.7) is

$$\phi(r) = b_+ r^{+\frac{\nu}{2}-\frac{1}{2}} + b_- r^{-\frac{\nu}{2}-\frac{1}{2}} \qquad\qquad \nu = \sqrt{1 + m^2/3} \tag{3.8}$$

By linearity of (3.4), there exists a linear relation between the coefficients $a_\pm$ and $b_\pm$, which we may express as follows,

$$\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = S \begin{pmatrix} b_+ \\ b_- \end{pmatrix} \qquad\qquad S = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \tag{3.9}$$

The correlator of the scalar field Φ is given by the ratio,

$$G = \frac{a_-}{a_+} = \frac{S_{-+} + S_{--}(b_-/b_+)}{S_{++} + S_{+-}(b_-/b_+)} \tag{3.10}$$

The matrix S depends only on the data of the purely magnetic solution, and on the mass m of the scalar, but not on any momenta. The ratio b_-/b_+ will be governed by the dynamics in the near region, and will depend on the momenta $p_\pm$.

3.3 Solutions in the near region: purely magnetic case

We begin by solving for the behavior of the scalar in the purely magnetic case, since this may be done analytically and simply. The near region is defined by $r \ll 1$, so that $L(r) \sim 2br$ and $K(r) \sim 1$. Equation (3.3) then takes the form,

$$12r^2\phi'' + 24r\phi' - \frac{p^2}{br}\phi - m^2\phi = 0 \tag{3.11}$$

By the change of variables $r = z^{-2}$, and the redefinition of the function $\phi(r) = z\tilde{\phi}(z)$, equation (3.11) is transformed into a modified Bessel equation,

$$z^2\tilde{\phi}'' + z\tilde{\phi}' - \left(\frac{p^2}{b^3}z^2 + \nu^2\right)\tilde{\phi} = 0 \tag{3.12}$$

Of its two linearly independent solutions, K_ν vanishes at the horizon $z = \infty$, while I_ν diverges there. Retaining only the regular solution, and expressing it in terms of $\phi(r)$, we obtain,

$$\phi(r) = \frac{2\sin\nu\pi}{\pi\sqrt{r}}K_\nu\left(\sqrt{\frac{p^2}{b^3 r}}\right) \tag{3.13}$$

In the $r \gg p^2$ part of the near region, ϕ is given by the following approximate behavior,

$$\phi(r) \sim \frac{1}{\Gamma(1-\nu)}\left(\frac{p^2}{4b^3}\right)^{-\frac{\nu}{2}} r^{+\frac{\nu}{2}-\frac{1}{2}} - \frac{1}{\Gamma(1+\nu)}\left(\frac{p^2}{4b^3}\right)^{+\frac{\nu}{2}} r^{-\frac{\nu}{2}-\frac{1}{2}} \tag{3.14}$$

This formula was derived using the familiar definition,

$$K_\nu(z) \;\; = \;\; \frac{\pi}{2\sin(\pi\nu)}\left(I_{-\nu}(z) - I_\nu(z)\right) \tag{3.15}$$

as well as the $z \to 0$ asymptotics of $I_\nu(z)$, given in (B.7) of Appendix B.

3.3.1 IR behavior of 2-point correlator: purely magnetic case

The far and near regions overlap when $p^2 \ll r \ll 1$, so that the solution (3.8) valid in the far region must match the solution (3.14) in the near region. We see that their functional dependence on r indeed coincides, which allows us to extract the key ratio,

$$\frac{b_-}{b_+} = -\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{p^2}{4b^3}\right)^\nu \tag{3.16}$$

Assuming a fixed value of ν away from the positive integers, and that $p^2 \ll 1$, we find $b_-/b_+ \ll 1$. The scalar two-point function of (3.10) may then be expanded in powers of b_-/b_+, and the leading non-trivial momentum dependence identified,

$$\begin{aligned} G(p) &= \frac{S_{-+}}{S_{++}} + \frac{\det(S)}{(S_{++})^2} C_\nu p^{2\nu} + \mathcal{O}(p^{4\nu}) \\ C_\nu &= -\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)\, 2^{2\nu} b^{3\nu}} \end{aligned} \tag{3.17}$$

The first term in $G(p)$ produces a delta function in position space, which is local, and may be dropped. The determinant factor $\det(S)$ is proportional to the Wronskian of the differential equation (3.4), evaluates to $\det(S) = 3\nu(c_V)^{-1}(\nu_+ - \nu_-)^{-1}$, and thus never vanishes. As a result, the leading IR behavior of the scalar Green function is given by $p^{2\nu}$.

3.4 Solutions in the near region: charged magnetic brane case

In the near region $r \ll 1$ we have $K \sim 1$, $L \sim 2br$, and M is given by

$$M(r) = -a_M r^{2k} \qquad\qquad a_M = \frac{2q^2}{k(2k-1)} \tag{3.18}$$

where we are restricting to the critical solution and set $\alpha_0 = 0$. All momentum dependence in (3.3) now needs to be retained, resulting in the equation,

$$12r^2\phi'' + 24r\phi' - \frac{p^2}{br}\phi - \frac{p_-^2 a_M}{12} r^{2k-2}\phi - m^2\phi = 0 \tag{3.19}$$

Changing variables to $r = 1/z^2$, and redefining $\phi(r) = z\tilde{\phi}(z)$, transforms (3.19) into,

$$z^2\tilde{\phi}'' + z\tilde{\phi}' - \left(\frac{p^2}{b^3}z^2 + \nu^2 + \frac{p_-^2 a_M}{36} z^{4-4k}\right)\tilde{\phi} = 0 \tag{3.20}$$

Here, ν is given by (3.8). For $q = a_M = 0$, this equation coincides with (3.12), which has already been solved earlier. For $p_+ = 0$, but $p_- \neq 0$, equation (3.20) may be transformed into a modified Bessel equation, and solved analytically as well, for all k.

Henceforth, we will assume $q, p_\pm \neq 0$. Rescaling z and the scalar field as follows,

$$z = \frac{b^{3/2}x}{p} \qquad\qquad \tilde{\phi}(z) = \varphi(x) \tag{3.21}$$

transforms (3.20) into,

$$x^2\varphi'' + x\varphi' - \left(x^2 + \nu^2 + \xi^2 x^{4-4k}\right)\varphi = 0 \tag{3.22}$$

Thus, for fixed k and ν, and up to the above rescaling of z, the scalar field equation in the near region intrinsically depends only on a single combination ξ of the momenta, given by,

$$\xi = \xi_0 (p_+)^{k-1} (p_-)^k \qquad\qquad \xi_0 = q \left(\frac{3^{1-3k}}{2k(2k-1)} \right)^{\frac{1}{2}} \tag{3.23}$$

Let $\varphi(x)$ denote the solution to (3.22) that is smooth at the horizon, $x \to \infty$. In the overlap region, we have $p^2 \ll r \ll 1$ which translates to $p \ll x \ll 1$ in terms of the coordinate x, and the function $\varphi(x)$ takes the simplified form,

$$\varphi(x) \sim c_-(\xi) x^\nu + c_+(\xi) x^{-\nu} \tag{3.24}$$

The dependence on k and ν of coefficients $c_\pm(\xi)$ has not been exhibited here. Transforming back to the holographic coordinate r, we have,

$$\phi(r) \sim c_-(\xi) \left(\frac{p}{b^{3/2}} \right)^\nu r^{-\frac{\nu}{2}-\frac{1}{2}} + c_+(\xi) \left(\frac{p}{b^{3/2}} \right)^{-\nu} r^{\frac{\nu}{2}-\frac{1}{2}} \tag{3.25}$$

3.4.1 IR behavior of 2-point correlator: general form

In the overlap region, where we have $p^2 \ll r \ll 1$, the functional form (3.25) of $\phi(r)$ in the near region coincides with the functional form (3.8) in the far region. Matching the corresponding coefficients gives,

$$\frac{b_-}{b_+} = C_\nu(\xi) p^{2\nu} \qquad\qquad C_\nu(\xi) = \frac{c_-(\xi)}{c_+(\xi)\, b^{3\nu}} \tag{3.26}$$

Again, since $p^2 \ll 1$ we have $b_-/b_+ \ll 1$, so that the two-point correlator for the scalar may be reliably approximated as follows,

$$G(p) = \frac{S_{-+}}{S_{++}} + \frac{\det(S)}{(S_{++})^2} C_\nu(\xi) p^{2\nu} + \mathcal{O}(p^{4\nu}) \tag{3.27}$$

The $\xi \to 0$ limit reduces to the purely magnetic case so that we have $C_\nu(0) = C_\nu$ with C_ν given in (3.17). Also, the $p_+ \to 0$ limit, while keeping p_- fixed, should be smooth. For $k > 1$, this limit implies $\xi \to 0$, and again leads to the purely magnetic case. For $k < 1$, the $p_+ \to 0$ limit implies $\xi \to \infty$, and smoothness of G then requires,

$$C_\nu(\xi) \sim \xi^{\frac{\nu}{1-k}} \qquad\qquad \text{as} \qquad \xi \to \infty \tag{3.28}$$

The precise coefficient can be computed analytically but will not be needed here. The remaining p_- dependence of b_-/b_+ is then of the form $(p_-)^{\nu/(1-k)}$, as may also be seen directly from scaling arguments on the original equation (3.20) with $p^2 = p_+ = 0$.

Beyond these asymptotic limits, it does not appear possible to solve equation (3.22) for the near region behavior of the scalar field analytically for generic value of k. Besides the trivial $k = 1$ case, there is one special value $k = 3/4$ where a complete solution may be obtained in terms of Whittaker functions.

3.5 The solvable special case of $k = 3/4$

For the special value $k = 3/4$, equation (3.22) is related to the Whittaker equation [29]. The solution to (3.22) which vanishes for large x is given by,

$$\varphi(x) = e^{-x} x^{\nu} U\left(\frac{1}{2} + \nu + \frac{\xi^2}{2}, 1 + 2\nu, 2x\right) \tag{3.29}$$

where U is the confluent hypergeometric function in the notation of [29]. In the overlap region, its behavior simplifies to,

$$\varphi(x) \sim \frac{x^{\nu}}{\Gamma(1+2\nu)\Gamma(\frac{1}{2} - \nu + \frac{\xi^2}{2})} - \frac{x^{-\nu}}{\Gamma(1-2\nu)\Gamma(\frac{1}{2} + \nu + \frac{\xi^2}{2})} \tag{3.30}$$

where we have omitted an immaterial multiplicative constant common to both terms on the right hand side of (3.30). In terms of the holographic coordinate r, we find,

$$\phi(r) \sim \frac{(2pb^{-3/2})^{\nu}\, r^{-\frac{1}{2}-\frac{\nu}{2}}}{\Gamma(1+2\nu)\Gamma(\frac{1}{2} - \nu + \frac{\xi^2}{2})} - \frac{(2pb^{-3/2})^{-\nu}\, r^{-\frac{1}{2}+\frac{\nu}{2}}}{\Gamma(1-2\nu)\Gamma(\frac{1}{2} + \nu + \frac{\xi^2}{2})} \tag{3.31}$$

As a result, we find the following formula,

$$\frac{b_-}{b_+} = C_\nu(\xi)\, p^{2\nu} \qquad\qquad C_\nu(\xi) = -\frac{\Gamma(1-2\nu)\Gamma\left(\frac{1}{2} + \nu + \frac{\xi^2}{2}\right)}{\Gamma(1+2\nu)\Gamma\left(\frac{1}{2} - \nu + \frac{\xi^2}{2}\right)}\left(\frac{4}{b^3}\right)^{\nu} \tag{3.32}$$

One may check that the $\xi \to 0$ limit of the above expression reproduces the general result $C_\nu(0) = C_\nu$ of (3.17), by using the duplication formula for the Γ-function.

There are three more special cases, $k = 5/8$, $k = 7/8$, and $k = 3/2$ where the solutions to the differential equation (3.22) are expressible in terms of Heun functions. As of yet, we have found insufficient information on the asymptotic behavior of these functions to make use of them in the generation of the scalar two-point correlator for these values of k.

3.6 Comparing expansions with the exact $k = 3/4$ solution

Beyond obtaining the exact near-horizon solution for $k = 3/4$ it is possible to derive the functional form of the expansion in powers of ξ of $C_\nu(\xi)$, through a combination of perturbative and WKB methods. These calculations are carried out in Appendix B, and their results may be summarized as follows. The perturbative expansion in ξ is valid when $\xi \ll 1$, while the WKB expansion holds when $\hbar \ll 1$,

$$\begin{aligned} &\frac{1}{2} < k < 1 \qquad \text{in powers of} \qquad \xi^2 \sim \frac{p_-^{2k}}{p_+^{2-2k}} \\ &\frac{1}{2} < k < 1 \qquad \text{in powers of} \qquad \hbar \sim \left(\frac{p_+^{1-k}}{p_-^{k}}\right)^{1/(2k-1)} \end{aligned} \tag{3.33}$$

Both expansions may be applied to the $k = 3/4$ special case, where we have an exact solution. For $k = 3/4$, the above expansion parameters become,

$$\xi^2 \sim \frac{p_-^{3/2}}{p_+^{1/2}} \qquad\qquad \hbar = \frac{1}{\xi^2} \tag{3.34}$$

The exact solution of (3.32) for $k = 3/4$,

$$C_\nu(\xi) \sim \frac{\Gamma\left(\frac{1}{2} + \frac{\xi^2}{2} + \nu\right)}{\Gamma\left(\frac{1}{2} + \frac{\xi^2}{2} - \nu\right)} \tag{3.35}$$

clearly admits a Taylor series expansion in powers of ξ^2 for small ξ. It admits an expansion for large ξ, which may be computed using the following series expansion of the Γ-function,

$$\frac{\Gamma(y+2\nu)}{\Gamma(y)} = y^{2\nu}\left(1 + \frac{\nu(2\nu-1)}{y} + \mathcal{O}(y^{-2})\right) \qquad\qquad y = \frac{1}{2} + \frac{\xi^2}{2} - \nu \tag{3.36}$$

showing that the expansion parameter is $\hbar \sim 1/\xi^2$ for large ξ. This result matches the prediction of WKB given in (3.33), and (3.34).

Note that, contrarily to the case of the current correlators in the purely magnetic brane solution, the corrections in powers of ξ, either for large ξ or for small ξ, do not readily admit an interpretation in terms of double trace deformations of the action and operators, because the expansions in powers of ξ or $1/\xi$ are not given by geometric series. It would be interesting to investigate whether the structure seen here can be accounted for by multi-trace operator perturbations, or requires a novel paradigm.

We close with a discussion of the case $k > 1$. The combination ξ^2 now vanishes uniformly as $p_\pm \to 0$. It is multiplied by z^{4-4k}, a factor which becomes singular at $z = 0$ when $k > 1$. This singularity is beyond the domain of validity of the near-horizon region, in which we are considering equation (3.22). Thus, it is more suitable to start from equation (3.19), with $r \ll 1$ and to carry out a power expansion in $p_-^2 a_M$. The lowest order contribution to the series is just the purely magnetic solution.

4 Physical Applications

In this section we discuss two immediate physical applications of the calculations of scalar correlators, given in the previous section. A first concerns the connection between the critical scaling law of the entropy density of (1.2). A second deals with the velocities of the chiral excitation modes, their dependence on the magnetic field, and similar effects encountered in Luttinger liquid theory.

4.1 Dynamical scaling exponent and entropy scaling form

We begin by commenting on the dynamical scaling exponent that emerges in the near-horizon geometry, and its relation with the thermodynamic scaling relation (1.2) for the entropy density. The metric of the near-horizon geometry at the critical point is of the Schrödinger type and given by,

$$ds^2 = \frac{dr^2}{4r^2} - \frac{2q^2 r^{2k}}{k(2k-1)}(dx^+)^2 + 4br dx^+ dx^- + dx^i dx^i \tag{4.1}$$

As shown already in [11], it is invariant under the following scale transformations,

$$r \to \lambda r \qquad\qquad x^+ \to \lambda^{-k} x^+ \qquad\qquad x^- \to \lambda^{k-1} x^- \tag{4.2}$$

This dynamical scaling symmetry is reflected in the IR behavior of the correlators of the scalar field through the dependence on only the combination,

$$\xi \sim (p_-)^k (p_+)^{k-1} \tag{4.3}$$

which is invariant under the scalings of (4.2). Reverting to coordinates $x^+ \to t$ and $x^- \to x_3$, and frequency/momentum $p_+ \to \omega$ and $p_- \to \kappa$, used in [11] to study thermodynamics, we readily identify the standard dynamical scaling exponent, defined by $\omega \sim \kappa^z$, as

$$z = \frac{k}{1-k} \tag{4.4}$$

Scaling arguments alone then predict the scaling relation for the entropy density $\hat{s}$ as a function of $\hat{T}$ in d space-dimensions, $\hat{s} \sim \hat{T}^{d/z}$. Assuming that the effective IR theory is indeed a 1+1-dimensional CFT, as well as the value of the dynamical scaling z of (4.4) would lead to

$$\hat{s} \sim \hat{T}^\alpha \qquad\qquad \alpha = \frac{1-k}{k} \tag{4.5}$$

Comparison with (1.2) reveals agreement with the full gravity calculation only for $1/2 < k < 3/4$, but disagreement for $k > 3/4$. Thus, the key problem in explaining (1.2) becomes to understand the transition across $k = 3/4$. The near-horizon expansion in powers of ξ will also result when we will study the 2-point correlators of the current and stress tensor.

4.2 Speed of chiral excitations

The scalar correlator computed in the background of the charged magnetic brane may be used to calculate the velocities at which the low momentum scalar excitations propagate in the dual CFT. For $k > 1$, the low momentum limit $|p_\pm| \ll 1$ uniformly corresponds to $\xi \ll 1$. For $1/2 < k < 1$ the limit is not uniform, however, and we need to distinguish two regimes, loosely characterized by $|p_-| \ll |p_+|$ and $|p_-| \gg |p_+|$, and more precisely specified by $\xi \ll 1$ and $\xi \gg 1$ respectively. Up to local terms, and up to an overall constant, the Green function of (3.27) in both regimes is given as follows,

$$\begin{aligned} &\xi \ll 1 \qquad && G(p) \sim (p_+ p_-)^\nu \\ &\xi \gg 1 \qquad && G(p) \sim (p_-)^{\frac{\nu}{1-k}} \left(1 + c_1 (p_+)^{\frac{1-k}{2k-1}} (p_-)^{-\frac{k}{2k-1}} + \mathcal{O}(\hbar^2)\right) \end{aligned} \tag{4.6}$$

Since the second line arises only in the interval $1/2 < k < 1$, it is instructive to specialize to the solvable value $k = 3/4$, where $G(p)$ is given by,

$$G(p) \sim (p_-)^{4\nu} + c_1 (p_+)^{\frac{1}{2}} (p_-)^{4\nu - \frac{3}{2}} + \cdots \tag{4.7}$$

Taking the Fourier transform gives the propagator as follows. For $1/2 < k < 1$ and x^+ small compared to x^-, as well as for $k > 1$ and both $x^\pm$ large, we have,

$$G(x^\pm) \sim \frac{1}{(x^+)^{\nu+1} (x^-)^{\nu+1}} \tag{4.8}$$

For $1/2 < k < 1$ and x^+ large compared to x^-, we have,

$$G(x^\pm) \sim \frac{1}{(x^+)^{\frac{3}{2}} (x^-)^{4\nu - \frac{1}{2}}} \tag{4.9}$$

where the leading $(p_-)^{4\nu}$ term in (4.7) is local and may be dropped.

As is manifest from the factorized form of these contributions, the propagation of left-movers and right-movers are independent from one another. To calculate the velocity of propagation of each chiral mode, we need to focus on the normalization of the boundary metric which, for the critical solution $\hat{B} = \hat{B}_c$, is given by,

$$ds^2 = 4dx^+ dx^- + \beta (dx^+)^2 + dx^i dx^i \qquad\qquad \beta = -\frac{\alpha_c}{b} \tag{4.10}$$

Thus we see that the propagators of (4.8) and (4.9) were in fact expressed in an oblique coordinate system. To find the physical velocities of propagation, we need to convert to a set of standard coordinates, $\tilde{x}^\pm$, defined by

$$\tilde{x}^+ = x^+ \qquad\qquad \tilde{x}^- = x^- + \beta x^+ \tag{4.11}$$

in which the boundary metric is given by,

$$ds^2 = 4d\tilde{x}^+ d\tilde{x}^- + dx^i dx^i \tag{4.12}$$

Introducing cartesian coordinates t, x_3 by $\tilde{x}^\pm = (x_3 \pm t)/2$, the speed of light is set to 1. The propagation of the chiral modes follows the singularities of the propagator. For the mode $x^+ = 0$, we have $\tilde{x}^+ = x_3 + t = 0$, corresponding to a left-moving chiral excitation propagates at the speed of light. For the mode $x^- = 0$, we have $\tilde{x}^- - \beta\tilde{x}^+ = 0$, or $(1-\beta)x_3 = (1+\beta)t$, which corresponds to a right-moving chiral excitation propagating at the speed

$$v = \frac{1+\beta}{1-\beta} \tag{4.13}$$

Since $\beta < 0$, the right-mover propagate slower than light. We may obtain a formula for the velocity in terms of intrinsic scale invariant physical observables in the boundary CFT by expressing α_∞ as follows, $\alpha_\infty = 4c_E^2 c_V^3 \hat{B}^3/3$, using the formulas of (2.17) and (2.18). Using next the expression for the chemical potential μ and its scale invariant form $\hat{\mu} = \mu/\rho^{1/3}$, we further express $c_V c_E$ in terms of μ, resulting in $\alpha_\infty = 8bk^2\hat{\mu}^2\hat{B}^2$, and the velocity,

$$v = \frac{1 - 2k^2\hat{\mu}^2\hat{B}^2}{1 + 2k^2\hat{\mu}^2\hat{B}^2} \tag{4.14}$$

4.3 Comparison with Luttinger liquid theory

Luttinger liquid theory extends the Fermi liquid picture in 1+1 dimensions by including all (marginal) four-Fermi interactions. We begin by reviewing the Luttinger liquid construction. As a finite charge density is turned on, a Fermi surface develops, which in 1+1 dimensions consists of just two points with opposite momenta $\pm k_F$, as depicted in Figure 1.

At those points, we have chiral fermionic degrees of freedom. Assuming the presence of just a single species, (spinless fermions) we may describe these with the complex fields ψ_L and ψ_R. Their Hamiltonian contains the standard free part,

$$H_0 = v_F \int dx \left(\psi_L^\dagger (i\partial_x + k_F)\psi_L + \psi_R^\dagger (i\partial_x - k_F)\psi_R \right) \tag{4.15}$$

where v_F is the Fermi velocity. In Luttinger liquid theory, all (marginal) four-Fermi interactions are included in the Hamiltonian as well,

$$H_{\text{int}} = 2\pi v_F \int dx \left(g_2 \left(\psi_L^\dagger\psi_L\right)\left(\psi_R^\dagger\psi_R\right) + g_4 \left(\psi_L^\dagger\psi_L\right)^2 + g_4 \left(\psi_R^\dagger\psi_R\right)^2 \right) \tag{4.16}$$

where g_2 and g_4 are dimensionless couplings. The Luttinger model considered here is left-right symmetric, but not Lorentz invariant. It may be solved by bosonization to a single free bosonic excitation, whose velocity is renormalized to,

$$v = v_F \sqrt{(1+g_4)^2 - g_2^2} \tag{4.17}$$

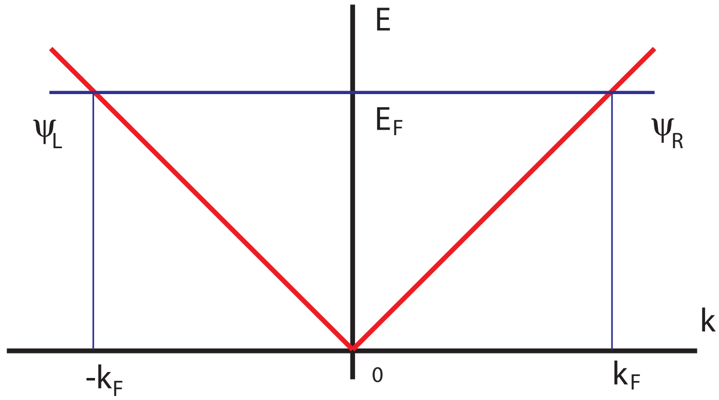

Figure 1: Fermi "surface" and 1+1-dimensional chiral fermions ψ_L and ψ_R

Qualitatively, this effect is similar to the change in velocity we have found for the scalar excitations in the presence of the charged magnetic brane. There, however, the renormalization of velocity affects only one chirality, not both. The renormalization of velocity also affects the entropy density dependence on T, as the specific heat coefficient for the above Luttinger liquid is given by,

$$\frac{s}{T} = \frac{\pi}{3v} \tag{4.18}$$

Comparing this expression with the entropy density for the charged magnetic brane, we see that a decreased velocity leads to an increase in specific heat coefficient, an effect that is qualitatively consistent with the gravity picture.

5 Structure of current and stress tensor correlators

In this section, we will derive the general structure of the 2-point functions involving the U(1) current $\mathcal{J}$ and the stress tensor $\mathcal{T}$ in the state dual to the charged magnetic brane solution. We will show that the correlators may be organized in terms of a Taylor expansion in powers of the charge density parameter q of the background solution (defined in (2.17) and related formulas). This general structure will be confirmed by explicit calculations in subsequent sections.

5.1 Review of correlators in the purely magnetic solution

The 2-point correlators of $\mathcal{J}$ and $\mathcal{T}$ in the state dual to the purely magnetic background solution (for which $q = 0$) were evaluated to leading order in the approximation of small momenta [6]. The results are given as follows,[4]

$$\begin{aligned}
\langle \mathcal{J}_+(p)\, \mathcal{J}_+(-p)\rangle \Big|_{q=0} &\sim \frac{kc}{2\pi}\,\frac{p_+}{p_-} \\
\langle \mathcal{T}_{++}(p)\, \mathcal{T}_{++}(-p)\rangle \Big|_{q=0} &\sim \frac{c}{48\pi}\,\frac{p_+^3}{p_-} \\
\langle \mathcal{T}_{--}(p)\, \mathcal{T}_{--}(-p)\rangle \Big|_{q=0} &\sim \frac{c}{48\pi}\,\frac{p_-^3}{p_+}
\end{aligned} \tag{5.1}$$

All other correlators vanish for $q = 0$ in this approximation. In fact, since the purely magnetic background solution is invariant under charge conjugation, the mixed correlators $\langle \mathcal{J}\,\mathcal{T}\rangle$ in the dual state vanish exactly. The overall strength of the correlators is governed by,

$$c = \frac{b}{2G_3} \tag{5.2}$$

which is the Brown-Henneaux central charge for AdS_3 with radius $\ell_3 = b^{-1}$.

5.2 Extended symmetries for the charged magnetic brane solution

As q is turned on, the charge conjugation and 2-dimensional Lorentz symmetries of the purely magnetic background are spontaneously broken by the non-zero charge density q. Charge conjugation reverses the sign of q, while a Lorentz boost transforms q as a (light-like) Lorentz vector. The latter motivates the notation,

$$q = q_+ \tag{5.3}$$

[4]The properly normalized 2-dimensional current and stress tensor operators, which were denoted by $\hat{\mathcal{J}}$ and $\hat{\mathcal{T}}$ in [6], (see equations (4.23) and (6.18) of [6] for their precise definition) will be denoted respectively by $\mathcal{J}$ and $\mathcal{T}$ respectively here, i.e. without hats.

where the subscript + stands for a Lorentz index. *Extended transformations*, which transform the fields as well as the symmetry breaking parameter, will leave the background solution invariant, as always with spontaneous symmetry breaking. Concretely, transforming all Lorentz vector and tensor fields as well as the parameter q_+ under *extended Lorentz transformations* leaves the charged magnetic brane solution invariant. Similarly, the solution is invariant under *extended charge conjugation*. In the dual theory, these extended Lorentz and charge conjugation symmetries will be realized through Ward identities for the corresponding spontaneously broken symmetries. It is this invariance that we will use to organize the q_+-dependence of all correlators for the charged magnetic brane solution.

Extended charge conjugation symmetry allows the correlators $\langle \mathcal{J} \mathcal{T} \rangle$ to be non-vanishing functions which are odd in q_+ and permits q_+-dependent corrections to the correlators $\langle \mathcal{J} \mathcal{J} \rangle$ and $\langle \mathcal{T} \mathcal{T} \rangle$ which are even in q_+. Extended Lorentz invariance further restricts the correlators as follows. In momentum space, the correlators may be organized through their dependence on the combinations $q_+ p_-$ and $p^2 = p_+ p_-$ which are invariant under extended Lorentz symmetry, and respectively odd and even under extended charge conjugation. (In particular, these are the only combinations that will enter the full reduced field equations (C.5), as will be discussed later.)

5.3 The approximation

The full analytic calculation of the correlators for general values of momenta is at present out of reach, and we will instead construct an approximate solution with the help of overlapping expansions. Our aim is to extract the leading behavior of certain correlators in the limit of low momenta. It will be helpful to spell out here precisely what this approximation entails. In momentum space, the approximation consists in neglecting the following contributions:

1. purely local terms produced by polynomials in p_+ and p_-;
2. non-integer powers of p^2, such as p^{4k} and $p^{4\sigma+2}$ which arise from the near region solution;
3. positive integer powers of p^2 which arise from the far region solution;
4. positive integer powers of $q_+ p_-$ which arise from the far region solution for $q_+ \neq 0$.

Any term which is suppressed by a contribution of the above type relative to a leading term will be neglected. As we will see, these rules will allow us to isolate the leading long-distance behavior of certain correlators, and will amount to setting other correlators to zero. Within this approximation, the correlators of (5.1) in the purely magnetic background solution were derived in [6] by explicit calculation using the overlapping expansion methods. The corrections to (5.1) referred to in point 4. are absent here, while the corrections of point 3. are immaterial since their presence would produce only purely local terms.

5.4 General functional dependence on $p_\pm$ and q, and analyticity

Extended Lorentz symmetry restricts each 2-point correlator to be of the form,

$$(p_+)^{s_+}(p_-)^{s_-}F(q_+p_-, p_+p_-) \tag{5.4}$$

where $s_\pm$ are non-negative integers, $s_+ - s_-$ is the total spin of the correlator, and F is a function of the extended Lorentz invariants q_+p_- and $p^2 = p_+p_-$. Extended charge conjugation symmetry prohibits fractional powers of q_+, since these would produce non-trivial monodromy under $q_+ \to -q_+$. The $q_+ \to 0$ limit will be assumed to be smooth, since it physically corresponds to letting the charge density tend to zero, which is a smooth process. As a result, F must have a Taylor series expansion in powers of (q_+p_-).

In view of point 2. of the approximations made in section 5.3, we are to omit all fractional powers of p^2, leaving us to retain only integer powers of p^2. In view of points 1. and 3. of section 5.3, and the fact that $s_\pm \geq 0$, we may omit all non-negative integer powers of p^2. Putting all together, we find that F must take the form,

$$F(q_+p_-, p_+p_-) = \sum_{m=0}^{\infty} \sum_{n>0} F_{m,n}(q_+p_-)^m(p_+p_-)^{-n} \tag{5.5}$$

The sum over m is restricted to even (resp. odd) integers for correlators that are even (resp. odd) under extended charge conjugation. Finally, we retain only the leading long distance contribution, corresponding to the term in (5.5) with smallest m and largest n.

5.5 The trace of the stress tensor

The trace $\mathcal{T}$ of the full 4-dimensional stress tensor $\mathcal{T}_{\mu\nu}$ is governed by the trace anomaly and is thus a local function of the metric and gauge field. As a result, any 2-point correlator involving $\mathcal{T}$ is local and, within the approximations spelled out in section 5.3, effectively vanishes, a result we will denote by $\mathcal{T} \approx 0$. Concretely, $\mathcal{T}$ is given by,[5]

$$\mathcal{T} = g^{\mu\nu}\mathcal{T}_{\mu\nu} = 4\mathcal{T}_{+-} - 4\beta\mathcal{T}_{--} + \frac{8}{c_V}\mathcal{T}_K \tag{5.6}$$

where we have used the form of the conformal boundary metric $g_{\mu\nu}$ for the charged magnetic brane in the oblique coordinate system (2.5),

$$ds^2_\infty = dx^+dx^- + \beta(dx^+)^2 + \frac{c_V}{4}dx^idx^i \tag{5.7}$$

[5] Here, and throughout, we set $\mathcal{T}_{11} = \mathcal{T}_{22} = \mathcal{T}_K$, and $\beta = -\alpha_c/(4b)$ with α_c given in (2.13). By construction, the holographic stress tensor is symmetric, so that we have $\mathcal{T}_{+-} = \mathcal{T}_{-+}$.

Furthermore, it will be proven in section 7.2 that the 2-point correlator of $\mathcal{T}_K$ with any one of the operators $\mathcal{J}_\pm$, $\mathcal{T}_{\pm\pm}$, $\mathcal{T}_{+-}$, and $\mathcal{T}_K$ vanishes within the approximations of section 5.3, so that we also have $\mathcal{T}_K \approx 0$. Combining this result with $\mathcal{T} \approx 0$ and (5.6), we find that

$$\mathcal{T}_{+-} - \beta \mathcal{T}_{--} \approx 0 \tag{5.8}$$

since its 2-point correlator with any one of the operators $\mathcal{J}_\pm$, $\mathcal{T}_{\pm\pm}$, $\mathcal{T}_{+-}$, and $\mathcal{T}_K$ vanishes within the approximations of section 5.3. Result (5.8) should come as no surprise: putting the metric ds^2_∞ in standard form by changing coordinates to,

$$\tilde{x}^+ = x^+ \qquad\qquad \tilde{x}^- = x^- + \beta x^+ \tag{5.9}$$

condition (5.8) translates to the customary vanishing trace condition $\tilde{\mathcal{T}}_{+-} \approx 0$.

The trace condition (5.8) leads to relations between the correlators for various components of the stress tensor. Some care is needed, however, in keeping these relations consistent with the approximations of section 5.3. For example, to order q^0 the correlators of $\mathcal{T}_{--}$ with $\mathcal{T}_{+-}$ and $\mathcal{T}_{++}$ vanish, and thus it follows from (5.8) that the correlators of $\mathcal{T}_{+-}$ with $\mathcal{T}_{+-}$ and $\mathcal{T}_{++}$ vanish at least to order q^2, all within the approximations of section 5.3. The correlator of $\mathcal{T}_{+-}$ with $\mathcal{T}_{--}$ to order q^2 is proportional to $\beta p_-^3/p_+$. Contrarily to naive appearances, this correlator vanishes within the approximation of section 5.3. Indeed, since we have $\beta = -\alpha_c/b \sim q^2$, the correlator exhibits a factor of $(q_+ p_-)^2$, and must be set to zero. Analogous arguments hold for the correlators of $\mathcal{T}_{+-}$ with the currents, and lead to their vanishing. Collecting the results obtained in this way, we have,

$$\begin{aligned} \langle \mathcal{T}_{+-}(p) \mathcal{T}_{\pm\pm}(-p) \rangle &\sim 0 \\ \langle \mathcal{T}_{+-}(p) \mathcal{T}_{+-}(-p) \rangle &\sim 0 \\ \langle \mathcal{T}_{+-}(p)\, \mathcal{J}_\pm (-p) \rangle &\sim 0 \end{aligned} \tag{5.10}$$

Higher order corrections in q_+ will be accompanied by higher powers in p_- and are to be neglected as well. Thus (5.10), and thus the operator relation $\mathcal{T}_{+-} \approx 0$, will be valid to all orders in q_+ within the approximations of section 5.3.

5.6 The current component $\mathcal{J}_-$

It will be shown by explicit calculation in section 7.5 that the following relation holds in all 2-point correlators within the approximations of the overlapping expansion method,

$$\mathcal{J}_- - Q_+ \mathcal{T}_{--} \approx 0 \qquad\qquad Q_+ = \frac{k\alpha_c}{c_V c_E} \tag{5.11}$$

This relation is analogous to (5.8). As a result, the fate of the 2-point correlators involving $\mathcal{J}_-$, under the approximations of section 5.3, is analogous to the fate of the correlators involving the $\mathcal{T}_{+-}$ component of the stress tensor, and we deduce the relations,

$$\begin{aligned} \langle \mathcal{J}_-(p)\, \mathcal{J}_\pm(-p)\, \rangle &\sim 0 \\ \langle \mathcal{J}_-(p) \mathcal{T}_{\pm\pm}(-p) \rangle &\sim 0 \\ \langle \mathcal{J}_-(p) \mathcal{T}_{+-}(-p) \rangle &\sim 0 \end{aligned} \tag{5.12}$$

Equivalently, the operator relation $\mathcal{J}_- \approx 0$ will be valid to all orders in q_+, within the approximations of section 5.3.

5.7 Conservation of current and stress tensor

Gauge invariance and translation invariance in x^μ guarantee that the current $\mathcal{J}_\mu$ and the stress tensor $\mathcal{T}_{\mu\nu}$ must be conserved,[6]

$$\begin{aligned} \partial_+ \mathcal{J}_- + \partial_- \mathcal{J}_+ - 2\beta\, \partial_- \mathcal{J}_- &= 0 \\ \partial_+ \mathcal{T}_{-+} + \partial_- \mathcal{T}_{++} - 2\beta\, \partial_- \mathcal{T}_{-+} &= 0 \\ \partial_+ \mathcal{T}_{--} + \partial_- \mathcal{T}_{+-} - 2\beta\, \partial_- \mathcal{T}_{--} &= 0 \end{aligned} \tag{5.13}$$

Throughout we set to zero the momenta $p_i = 0$, so that no derivates ∂_i occur and $\mathcal{T}_K$ does not enter. In view of the relations $\mathcal{T}_{+-} \approx 0$ and $\mathcal{J}_- \approx 0$ derived in the preceding two subsection, these conservation relations reduce to *chiral conservation* equations,

$$\begin{aligned} \partial_- \mathcal{J}_+ &\approx 0 \\ \partial_- \mathcal{T}_{++} &\approx 0 \\ \partial_+ \mathcal{T}_{--} &\approx 0 \end{aligned} \tag{5.14}$$

These equations hold when inserted into any 2-point correlator, and within the approximations of section 5.3.

The equations $\mathcal{T}_{+-} \approx 0$, $\mathcal{J}_- \approx 0$, together with the chiral conservation equations imply the general structure of the remaining 2-point correlators. To see how this works, we consider first the case $q_+ = 0$, and use these equations to derive the general structure of (5.1). For example, by Lorentz invariance, the 2-point correlator of $\mathcal{J}_+$ must be of the form,

$$\langle \mathcal{J}_+(p)\, \mathcal{J}_+(-p) \rangle \Big|_{q=0} \sim \frac{p_+}{p_-} f(p^2) \tag{5.15}$$

where f depends only on the Lorentz invariant $p^2 = p_+ p_-$. Following the approximations of point 2. of section 5.3, we neglect all fractional powers of p^2, so that f has a Laurent

[6] More precisely, the divergence of the current is nonzero in general due to the chiral anomaly, but since the anomaly is purely local, it may be set to zero consistently within the present framework.

expansion in positive and negative integer powers of p^2. Strictly positive powers produce local terms, which are to be discarded following point 1. Strictly negative powers are prohibited by the chiral conservation equation $p_- \mathcal{J}_+(p) = 0$ of (5.14), which implies that f must be constant. The value of the constant is derived by explicit calculation [6]. The same argument may be adapted to the remaining correlators of (5.1), using the chiral conservation of the stress tensor $p_- \mathcal{T}_{++}(p) = p_+ \mathcal{T}_{--}(p) = 0$ within the approximations of section 5.3.

5.8 Structure of correlators for the charged magnetic brane

We will now derive the general structure of the 2-point correlators of $\mathcal{J}$ and $\mathcal{T}$, constrained by extended charge conjugation and extended Lorentz symmetry, in the approximation of section 5.3, and under the (mild) assumption that the $q_+ \to 0$ limit is smooth and leads to the general form (5.5) for all 2-point correlators.

The relations $\mathcal{T}_{+-} \approx 0$ and $\mathcal{J}_- \approx 0$ lead to the vanishing of all 2-point functions involving these operators, as was already expressed in (5.10) and (5.12). The only remaining non-chiral correlator is $\langle \mathcal{T}_{++}(p) \mathcal{T}_{--}(-p) \rangle$; its general form is given by (5.4) and (5.5) with $s_+ = s_- = 0$. The chiral conservation equations $p_+ \mathcal{T}_{--} \approx p_- \mathcal{T}_{++} \approx 0$ readily exclude all terms with $n \geq 1$, so that this correlator must vanish,

$$\langle \mathcal{T}_{++}(p) \mathcal{T}_{--}(-p) \rangle \sim 0 \tag{5.16}$$

The remaining correlators are all chiral.

For negative chirality, only a single 2-point correlator, namely $\langle \mathcal{T}_{--}(p) \mathcal{T}_{--}(-p) \rangle$, remains in view of $\mathcal{J}_- \approx 0$ and (5.12). Its general structure is given by (5.4) and (5.5) with $s_+ = 0$ and $s_- = 4$. Chiral conservation $p_+ \mathcal{T}_{--} \approx 0$ forces us to restrict to $n = 1$. On the other hand, all terms with $m > 0$ are to be omitted by point 4. of the approximations of section 5.3. Hence only the $m = 0$, $n = 1$ term survives, which is independent of q_+ and thus must coincide with the result from the purely magnetic case,

$$\langle \mathcal{T}_{--}(p) \, \mathcal{T}_{--}(-p) \rangle \sim \frac{c}{48\pi} \frac{p_-^3}{p_+} \tag{5.17}$$

For positive chirality, we find the following 2-point correlators,

$$\begin{aligned}
\langle \mathcal{J}_+(p) \, \mathcal{J}_+(-p) \rangle &\sim \frac{kc}{2\pi} \frac{p_+}{p_-} \\
\langle \mathcal{J}_+(p) \, \mathcal{T}_{++}(-p) \rangle &\sim \gamma_1 \frac{q_+ p_+}{p_-} \\
\langle \mathcal{T}_{++}(p) \, \mathcal{T}_{++}(-p) \rangle &\sim \frac{c}{48\pi} \frac{p_+^3}{p_-} + \gamma_2 \frac{q_+^2 p_+}{p_-}
\end{aligned} \tag{5.18}$$

where γ_1 and γ_2 are two coefficients which are not determined by general arguments, and which will have to be obtained by explicit calculation. The terms proportional to c on the first and last lines correspond to the $q_+ = 0$ contribution from (5.1).

To prove the form of the correlator $\langle \mathcal{J}_+ \mathcal{J}_+ \rangle$ in (5.18), we set $s_+ = 2$ and $s_- = 0$ in (5.4), and use chiral conservation $p_- \mathcal{J}_+(p) \approx 0$ to restrict the expansion of F in (5.5) to $n \leq 2$ and $n \leq m + 1$. The $n = 1$ contribution is local for $m \geq 1$, while the $n = 2$ contribution is always local.[7] Hence only the $n = 1$, $m = 0$ contribution remains, which is independent of q_+ and coincides with the first line of (5.1) which yields the first line of (5.4).

The form of the correlator $\langle \mathcal{J}_+ \mathcal{T}_{++} \rangle$ in (5.18) is proven in analogous fashion. Its general form is given by (5.4) and (5.5) with $s_+ = 3, s_- = 0$. Chiral conservation of either $\mathcal{J}_+$ or $\mathcal{T}_{++}$ forces $n \leq 3$ and $n \leq m + 1$. Extended charge conjugation requires m to be odd, so that we must actually have $m \geq 1$ and $n \geq 2$. The contribution $n = 3$ is local, leaving only $n = 2$ with $m = 1$, with all $m \geq 3$ contributions to be omitted in view of point 4. of section 5.3. This is precisely the form given in the second line of (5.18).

The form of the correlator $\langle \mathcal{T}_{++} \mathcal{T}_{++} \rangle$ in (5.18) is given by (5.4) and (5.5) with $s_+ = 4$ and $s_- = 0$. Chiral conservation of $\mathcal{T}_{++}$ forces $n \leq 4$ and $n \leq m+1$. Extended charge conjugation requires m to be even, which restricts to $m = 0, 2$. All contributions with $m - n \geq 1$ are to be omitted in view of point 4. of section 5.3, while the $n = 4$ contributions, as well as the $n = m = 2$ contributions are local. This leaves only the contribution $m = 0$, $n = 1$ corresponding to the first term of the correlator in (5.18), and the contribution $m = 2$, $n = 3$ corresponding to the second term in (5.18). Since the first term is independent of q_+, its normalization is provided by the result of (5.1).

Explicit evaluation in section 8 will confirm the general form of the 2-point correlators derived above, and will determine the values of the coefficients as follows,

$$\gamma_1 = -\frac{c_V c_E}{4\pi G_3 q_+} \qquad\qquad \gamma_2 = \frac{c_V^2 c_E^2}{4\pi G_3 k b (q_+)^2} \tag{5.19}$$

The coefficients are independent of q_+ since c_E/q_+ is. We will also show in section 8 that the 2-point correlators in (5.18) remain the dominant IR contribution throughout $k > 1/2$.

5.9 Structure for general value of $\hat{B} \geq \hat{B}_c$ and twisting

The value of the normalized magnetic field $\hat{B}$, which was expressed in terms of the parameters of the charged background solution in (2.18), is found to enter the general structure of the 2-point correlator only through γ_1 and γ_2. In particular, for the purely magnetic solution we have $\hat{B} = \infty$ and $q_+ = 0$, so that $\gamma_1 = \gamma_2 = 0$, but for finite $\hat{B}$ we have $\gamma_1, \gamma_2 \neq 0$. Thus, the two Virasoro algebras and the chiral U(1)-current algebra, which were identified for the

[7] Note that a contribution in $(p_-)^{-n}$ with $n \geq 1$ is local, even though it is not polynomial in $p_\pm$.

purely magnetic case, persist for all $\hat{B} \geq \hat{B}_c$. As the charge density is turned on, however, the current and stress tensor operators become *twisted*. In the positive chirality sector, this may be seen by diagonalizing the system of 2-point functions of $\mathcal{J}_+$ and $\mathcal{T}_{++}$, by setting,

$$\begin{aligned} \mathcal{J}_+ &= \mathcal{J}_+^{(0)} \\ \mathcal{T}_{++} &= \mathcal{T}_{++}^{(0)} - \mu_+ \mathcal{J}_+^{(0)} \qquad\qquad \mu_+ = \frac{c_V c_E}{kb} = -\frac{2\pi}{kc}\gamma_1 q_+ \end{aligned} \tag{5.20}$$

If $\mathcal{J}_+^{(0)}$ and $\mathcal{T}_{++}^{(0)}$ obey the 2-point functions of the purely magnetic solution of (5.1), then $\mathcal{J}_+$ and $\mathcal{T}_{++}$ defined by (5.20) will obey (5.18). Note that this result crucially depends upon the precise value taken by γ_2, and its relation to γ_1, given by,

$$\gamma_2 = \frac{2\pi}{kc}\,\gamma_1^2 \tag{5.21}$$

Note that the twisting of (5.20) is possible because both $\mathcal{J}_+^{(0)}$ and $\mathcal{T}_{++}^{(0)}$ are chirally conserved.

6 Linearized metric and gauge field fluctuations

In this section, we begin the process of calculating the two-point correlators of the $U(1)$ current $\mathcal{J}$ and the stress tensor $\mathcal{T}$. To do so, the full Einstein-Maxwell-Chern-Simons equations are linearized around the charged magnetic background solution. Identifying the source and vev components in the Fefferman-Graham expansion of these solutions, we derive the vevs of the current and stress tensor using (2.4), and from there extract (the non-local parts of) all two-point functions. Since the background solution is invariant under translations in x^μ, we solve for plane waves with fixed momenta p_μ for $\mu = +, -, 1, 2$. For simplicity, we restrict to the most interesting case where $p_1 = p_2 = 0$, leaving only the momenta $p_\pm$.

6.1 Parametrization of linear fluctuations

A general Ansatz for the plane wave fluctuations of the metric and gauge field is given by,

$$\begin{aligned} A &= A_C + a_M\, e^{ipx} dx^M \\ ds^2 &= ds^2_C + h_{MN}\, e^{ipx} dx^M dx^N \end{aligned} \tag{6.1}$$

We continue to use the notation $px = p_+ x^+ + p_- x^-$. Here, A_C and ds^2_C are the gauge potential and metric of the charged magnetic brane solution. In the gauge adopted in (2.5), referred to as LMN gauge, the charged background solution takes the form,

$$\begin{aligned} ds^2_C &= \frac{dr^2}{L_0(r)^2} + M_0(r)(dx^+)^2 + 2L_0(r) dx^+ dx^- + e^{2V_0(r)} dx^i dx^i \\ A_C &= b x^1 dx^2 + A_0(r) dx^+ \end{aligned} \tag{6.2}$$

where the functions L_0 and V_0 are those of the purely magnetic background solution, obeying equations E1, E4, and fV of (2.6) with $N = P = 0$, while A_0 and M_0 are given in (2.11). Translation invariance in x^μ of the solution (6.2) justifies our consideration of plane wave fluctuations in (6.1). The functions a_M and g_{MN} depend on the holographic radius, but are independent of x^μ. Invariance under rotations in the $x^{1,2}$ plane of the solution (6.2) leads us to require rotation invariance of the fluctuations of (6.1), which implies the following conditions,

$$a_i = h_{ir} = h_{i+} = h_{i-} = 0 \qquad\qquad h_{ij} = \delta_{ij} h_K \qquad\qquad i, j = 1, 2 \tag{6.3}$$

Even with the benefit of translation and rotation symmetry, the corresponding reduced field equations remain quite involved, and will not be presented here.

To obtain workable reduced equations we choose a gauge. It turns out that a convenient gauge choice is not global, but is rather obtained by patching together the light-cone gauge for the near region and the LMN gauge for the far region. In the overlap region, a gauge transformation is required to patch the two gauge choices together. In the extreme UV limit of the far region, we will make a further gauge transformation to Fefferman-Graham gauge, which is required to properly extract the current and stress tensor data.

6.1.1 Near-region: light-cone gauge

In the near region, defined by $r \ll 1$, we will adopt light-cone gauge, and denote the corresponding fluctuation fields with tildes, namely $\tilde{a}_M$ and $\tilde{h}_{MN}$. These fields satisfy the usual light-cone gauge conditions,

$$\tilde{a}_- = \tilde{h}_{r-} = \tilde{h}_{+-} = \tilde{h}_{--} = 0 \tag{6.4}$$

in addition to the consequences of invariance under translations in x^μ, and rotations in x^1, x^2 as in (6.3). The full gauge field and metric in light-cone gauge become,

$$\begin{aligned}
\tilde{A} &= A_C + \left(\tilde{a}_r dr + \tilde{a}_+ dx^+\right) e^{ipx} \\
d\tilde{s}^2 &= ds_C^2 + \left(\tilde{h}_{rr} dr^2 + 2\tilde{h}_{r+} dr dx^+ + \tilde{h}_{++}(dx^+)^2 + \tilde{h}_K dx^i dx^i\right) e^{ipx}
\end{aligned} \tag{6.5}$$

where the remaining fields $\tilde{a}_r, \tilde{a}_+, \tilde{h}_{rr}, \tilde{h}_{r+}, \tilde{h}_{++}$, and $\tilde{h}_K$ depend only on r.

6.1.2 Far region: LMN gauge

In the far region all momentum-dependence in the reduced field equations is to be neglected. This specification always requires $p_+ p_- \ll r$. As we have discussed for the case of scalar fluctuations in section 3.2, however, depending on the value of k, additional restrictions on r may be needed in order to ensure that the far region will extend non-trivially into the near region. We will exhibit these conditions as we proceed forward. We stress, however, that even if the reduced field equations do not involve momenta in the far region, their boundary conditions will be given by p-dependent coefficients.

When all momentum dependence in the reduced equations is neglected, the fluctuations will be analyzed in LMN gauge, where they correspond to x^μ-independent fluctuations of the functions $L, M, N, V, A_\pm$ of (2.5). LMN gauge will be convenient here, because this linear fluctuation problem was considered and solved already in Section 5 and Appendix B of [11]. The fluctuation fields in LMN gauge will be denoted by the original letters a_M and h_{MN}, and are subject to the following gauge conditions,

$$a_r = h_{r+} = h_{r-} = 0 \qquad\qquad h_{rr} = -\frac{2}{L_0^3} h_{+-} + \frac{M_0}{L_0^4} h_{--} \tag{6.6}$$

in addition to the consequences of invariance under translations in x^μ, and rotation in x^1, x^2 as in (6.3). The relation for h_{rr} is imposed to guarantee that the form of the dr^2 term is preserved by the fluctuations. Thus, the gauge field and metric in LMN gauge become,

$$\begin{aligned} A &= A_C + \Big(a_+ dx^+ + a_- dx^-\Big) e^{ipx} \\ ds^2 &= ds_C^2 + \Big(h_{rr} dr^2 + h_{++}(dx^+)^2 + 2h_{+-} dx^+ dx^- + h_{--}(dx^-)^2 + h_K dx^i dx^i\Big) e^{ipx} \end{aligned} \tag{6.7}$$

The remaining functions $a_\pm, h_{rr}, h_{\pm\pm}, h_{+-}$, and h_K depend on r, but are independent of x^μ.

6.2 Reduced field equations in the near region

In the near region, where $r \ll 1$, we adopt light-cone gauge. The reduced equations for fluctuations in light-cone gauge around the full charged magnetic background solution are derived in Appendix B. For $r \ll 1$, it is appropriate to use the approximation,[8]

$$\begin{aligned} V_0(r) &= 0 & A_0(r) &= \frac{qr^k}{k} \\ L_0(r) &= 2br & M_0(r) &= -\frac{2q^2 r^{2k}}{k(2k-1)} \end{aligned} \tag{6.8}$$

so that the general light-cone equations (C.1), (C.2), (C.3), (C.4), and especially (C.5) greatly simplify. The reduced equations may be presented most symmetrical in terms of the functions $\tilde h_K$ and $\tilde c(r) = 4r\tilde a(r)$, and we have,

$$\begin{aligned} 0 &= r^2 \tilde h_K'' + 2r\tilde h_K' - \frac{4}{3}\tilde h_K - \frac{p^2}{4b^3 r}\tilde h_K - \frac{q^2 p_-^2 r^{2k-2}}{72k(2k-1)}\tilde h_K + \frac{iqp_- r^{k-1}}{b^3}\tilde c_r \\ 0 &= r^2 \tilde c_r'' + 2r\tilde c_r' - k(k-1)\tilde c_r - \frac{p^2}{4b^3 r}\tilde c_r - \frac{q^2 p_-^2 r^{2k-2}}{72k(2k-1)}\tilde c_r - \frac{iqp_- r^{k-1}}{b^3}\tilde h_K \end{aligned} \tag{6.9}$$

Note that the equations only depend on the combinations $p^2 = p_+ p_-$ and qp_-. For later use, we also record the relations giving the remaining fields in the near region,

$$\begin{aligned} \tilde h_{rr} &= -\frac{\tilde h_K}{6r^2} & \tilde h_{++} &= \frac{288 r^2}{p_-^2}\Big(r^2 \tilde h_K'' + 2r\tilde h_K' - \frac{4}{3}\tilde h_K + \frac{i}{b^3} qp_- r^k \tilde a_r\Big) \\ \tilde h_{r+} &= -\frac{4ibr}{p_-}\tilde h_K' & \tilde a_+ &= \frac{24 ibr^2}{p_-}\left((r\tilde a_r)' + k\tilde a_r\right) \end{aligned} \tag{6.10}$$

Rescaling r by p^2,

$$r = \frac{p^2 x}{b^3} \qquad \tilde h_K(r) = \hat h_K(x) \qquad \tilde c_r(r) = \hat c_r(x) \tag{6.11}$$

[8] In keeping with the discussion of section 2.7, we restrict attention to the critical background solution for which $\hat B = \hat B_c$, and $\alpha_0 = 0$, so that no linear term in M_0 is present as $r \to 0$. The solutions for $\hat B > \hat B_c$ may then be reconstructed by applying an $SL(2,R)$ transformation to the critical solution.

reveals that the equations in terms of the rescaled variable x,

$$
\begin{aligned}
0 &= x^2\hat{h}_K'' + 2x\hat{h}_K' - \frac{4}{3}\hat{h}_K - \frac{1}{4x}\hat{h}_K - \frac{1}{4}\xi^2 x^{2k-2}\hat{h}_K + i\xi_1\xi\, x^{k-1}\hat{c}_r \\
0 &= x^2\hat{c}_r'' + 2x\hat{c}_r' - k(k-1)\hat{c}_r - \frac{1}{4x}\hat{c}_r - \frac{1}{4}\xi^2 x^{2k-2}\hat{c}_r - i\xi_1\xi\, x^{k-1}\hat{h}_K
\end{aligned}
\tag{6.12}
$$

intrinsically only depend on the combination ξ, which was already introduced in (3.23) for the scalar fluctuation problem. Here, we have used the notation $\xi_1^2 = 2k(2k-1)/3$.

On the one hand, for $k > 1$, we automatically have $\xi \to 0$ in the limit of small momenta $p_\pm$, no matter how this limit is being taken. Also, the ξ-dependent terms in (6.12) are well-behaved as $x \to 0$. On the other hand, for $1/2 < k < 1$, the combination ξ exhibits a directional singularity in the limit of small momenta $p_\pm$, so that ξ may tend to zero in one limit, to ∞ in another, or to any finite value. Also, for $1/2 < k < 1$, the ξ-dependent terms in (6.12) diverge as $x \to 0$. Finally, it is unlikely that equations (6.12) can be solved for finite ξ, beyond the type of perturbative results that were derived for the scalar problem in Appendix A. To avoid these three complications, we will henceforth restrict to the case where $k \geq 1$, and proceed to solve it in the limit of low momenta, which implies $\xi \ll 1$.

6.3 Solutions in the near region

Assuming $k \geq 1$ and $\xi \ll 1$, equations (6.15) decouple from one another,

$$
\begin{aligned}
0 &= r^2\tilde{h}_K'' + 2r\tilde{h}_K' - \frac{4}{3}\tilde{h}_K - \frac{p^2}{4r}\tilde{h}_K \\
0 &= r^2\tilde{c}_r'' + 2r\tilde{c}_r' - k(k-1)\tilde{c}_r - \frac{p^2}{4r}\tilde{c}_r
\end{aligned}
\tag{6.13}
$$

and may be solved in terms of modified Bessel functions,

$$
\begin{aligned}
\tilde{h}_K(r) &= \frac{v_0}{\sqrt{r}}\left(I_{2\sigma+1}\left(\sqrt{\frac{p^2}{b^3 r}}\right) - I_{-2\sigma-1}\left(\sqrt{\frac{p^2}{b^3 r}}\right)\right) \\
\tilde{c}_r(r) &= \frac{c_0}{\sqrt{r}}\left(I_{2k-1}\left(\sqrt{\frac{p^2}{b^3 r}}\right) - I_{-2k+1}\left(\sqrt{\frac{p^2}{b^3 r}}\right)\right)
\end{aligned}
\tag{6.14}
$$

where v_0 and c_0 are independent of r, and σ was given in (2.16). As was the case for the scalar fluctuations, the above linear combinations of Bessel functions are required by regularity of $\tilde{h}$ and $\tilde{a}$ as $r \to 0$, and are proportional to $K_{2\sigma+1}$ and K_{2k-1} respectively.

6.4 Solutions in the overlap region (light-cone gauge)

The overlap region, defined by $p^2 \ll r \ll 1$, is contained in the near region so that the approximations of (6.8), as well as the reduced equations of (6.13) are valid. It is also contained in the far region $p^2 \ll r$, where all momentum dependence in the reduced field equations is to be neglected, so that equations (6.13) actually reduce to,

$$\begin{aligned} 0 &= r^2 \tilde{h}_K'' + 2r\tilde{h}_K' - \frac{4}{3}\tilde{h}_K \\ 0 &= r^2 \tilde{c}_r'' + 2r\tilde{c}_r' - k(k-1)\tilde{c}_r \end{aligned} \tag{6.15}$$

Its solutions are given by,

$$\begin{aligned} \tilde{h}_K(r) &= v_+ r^\sigma + v_- r^{-1-\sigma} \\ \tilde{a}_r(r) &= \frac{1}{4}\left(c_+ r^{k-2} + c_- r^{-k-1}\right) \end{aligned} \tag{6.16}$$

The coefficients $v_\pm$ and $c_\pm$ are independent of r, and may be determined in terms of v_0 and c_0 respectively by matching the solutions of (6.16) with those of (6.14) in the $p^2 \ll r$ limit. Here, we will only need the ratios of these coefficients,

$$\frac{v_-}{v_+} = -\frac{\Gamma(-2\sigma)}{\Gamma(2+2\sigma)}\left(\frac{p}{2}\right)^{4\sigma+2} \qquad\qquad \frac{c_-}{c_+} = -\frac{\Gamma(2-2k)}{\Gamma(2k)}\left(\frac{p}{2}\right)^{4k-2} \tag{6.17}$$

Clearly, to leading order in small momenta, the coefficients v_- and c_- are suppressed compared to v_+ and c_+, and the corresponding terms may effectively be dropped. In terms of the coefficients $v_\pm$, $c_\pm$, the remaining light-cone gauge functions are given as follows,

$$\begin{aligned} \tilde{h}_{rr} &= -\frac{\tilde{h}_K(r)}{6r^2} & \qquad \tilde{h}_{r+} &= -\frac{4ib}{p_-}\left(\sigma v_+ r^\sigma - (\sigma+1)v_- r^{-\sigma-1}\right) \\ \tilde{h}_{++} &= 0 & \qquad \tilde{a}_+ &= \frac{6ib(2k-1)}{p_-}c_+ r^k \end{aligned} \tag{6.18}$$

The cancellation of $\tilde{h}_{++}$ is brought about by the fact that its various terms in (6.10) cancel for the overlap region solution of $\tilde{h}_K$ of (6.16) and/or are of higher order in p^2/r.

6.5 Change to LMN gauge in overlap region

We have solved the linearized equations around the charged magnetic brane in the near region in light-cone gauge, and need to match the result with the solution in the far region in LMN gauge. The matching takes place in the overlap region $p^2 \ll r \ll 1$ where both approximations are simultaneously valid, and where a simplified solution holds. The general form of the solution in light-cone gauge will be denoted by,

$$
\begin{aligned}
\tilde{A}_M dx^M &= A_C + \tilde{a}_M(r)\, e^{ipx} dx^M \\
\tilde{H}_{MN} dx^M dx^N &= ds_C^2 + \tilde{h}_{MN}(r)\, e^{ipx} dx^M dx^N
\end{aligned} \tag{6.19}
$$

while its general form in LMN gauge will be denoted by,

$$
\begin{aligned}
A_M dx^M &= A_C + a_M(r)\, e^{ipx} dx^M \\
H_{MN} dx^M dx^N &= ds_C^2 + h_{MN}(r)\, e^{ipx} dx^M dx^N
\end{aligned} \tag{6.20}
$$

Since the background solution is the same in both gauges, the gauge transformation between the linear fluctuations may be carried out at the linearized level. This involves a diffeomorphism vector field U^M and a gauge transformation field Θ which take the form,

$$
\begin{aligned}
U^M(r,x) &= u^M(r)\, e^{ipx} \\
\Theta(r,x) &= \theta(r)\, e^{ipx}
\end{aligned} \tag{6.21}
$$

To convert the metric and gauge fluctuations $\tilde{H}_{MN}$ and $\tilde{A}_M$ of the light-cone gauge to H_{MN} and A_M of LMN gauge in the overlap region, we solve the equations,

$$
\begin{aligned}
H_{MN} &= \tilde{H}_{MN} + \nabla_M U_N + \nabla_N U_M \\
A_M &= \tilde{A}_M + U^K F_{KM} + \partial_M \Theta
\end{aligned} \tag{6.22}
$$

for U^M, Θ, as well as H_{MN} and A_M. The covariant derivatives ∇_M, and the field strength F_{KM} are with respect to the background solution, and so is the metric used to raise and lower indices. Substituting the plane wave forms of (6.19) and (6.20),

$$
\begin{aligned}
h_{MN} e^{ipx} &= \tilde{h}_{MN} e^{ipx} + \nabla_M U_N + \nabla_N U_M \\
a_M e^{ipx} &= \tilde{a}_M e^{ipx} + U^K F_{KM} + \partial_M \Theta
\end{aligned} \tag{6.23}
$$

Imposing rotation invariance, as spelled out in (6.3), on $\tilde{h}_{MN}$, $\tilde{a}_M$, h_{MN}, and a_M gives $u^1 = u^2 = 0$. Next, we impose the conditions for light-cone gauge of (6.4) on $\tilde{h}_{MN}$ and $\tilde{a}_M$, and for LMN gauge of (6.6) on h_{MN}, and a_M. Finally, we assign the expressions for the light-cone gauge solution in the overlap region of (6.16) and (6.18) to the remaining components of the fields $\tilde{h}_{MN}$ and $\tilde{a}_M$. The resulting equations are derived in (D.2), (D.3), and (D.4) of Appendix D.

6.6 Solutions in the overlap region (LMN gauge)

The corresponding solutions in the overlap region for the metric and gauge fluctuations in LMN gauge are constructed in Appendix D as well, and are of the following form,

$$
h_{++}(r) = s_{++} r + t_{++}
$$

$$
\begin{aligned}
h_{--}(r) &= s_{--}r + t_{--} \\
h_{+-}(r) &= s_{+-}r + t_{+-} \\
a_+(r) &= \sigma_+ + \tau_+ r^k \\
a_-(r) &= \sigma_- + \tau_- r^{-k}
\end{aligned}
\tag{6.24}
$$

with the coefficients given by,

$$
\begin{aligned}
s_{++} &= 4ibp_+u_0^- & t_{++} &= \frac{p_+p_-}{24b}s_{++} + \frac{p_+^3}{24bp_-}s_{--} \\
s_{--} &= 4ibp_-u_0^+ & t_{--} &= \frac{p_+p_-}{24b}s_{--} + \frac{p_-^3}{24bp_+}s_{++} \\
s_{+-} &= 0 & t_{+-} &= 2bu_0^r + \frac{1}{24b}\left(p_+^2 s_{--} + p_-^2 s_{++}\right) \\
\sigma_+ &= ip_+\theta_0 & \tau_+ &= (2k-1)\frac{6ibc_+}{p_-} + \frac{qp_+(k-1)}{4bp_-k}s_{--} + \frac{qp_-}{4bp_+}s_{++} \\
\sigma_- &= ip_-\theta_0 & \tau_- &= \frac{ip_-c_-}{4k}
\end{aligned}
\tag{6.25}
$$

Since $c_-/c_+ \sim p^{4k}$, we may effectively set $c_- = 0$ within the approximations of section 5.3. The metric fluctuations of the field h_K are given by,

$$
h_K(r) = v_+ r^\sigma + v_- r^{-1-\sigma} \tag{6.26}
$$

and decouple from all other fluctuations of both the metric and the gauge field. Since $v_-/v_+ \sim p^{4\sigma+2}$, we may effectively set $v_- = 0$ within the approximations of section 5.3.

6.7 Reduced field equations in the far region

In the far region, defined by $p^2 \ll r$, all momentum dependence in the field equations is to be omitted. The corresponding reduced field equations for the metric and gauge field fluctuations, in LMN gauge, coincide with the perturbative expansion of the fields L, M, N, V, E, P, obtained already in equations (5.35), (5.36) and (5.37) of [11]. Making the correspondence with the present notations, the expansion is as follows,

$$
\begin{aligned}
L(r,x) &= L_0(r) + \varepsilon L_1(r)\, e^{ipx} & L_1(r) &= h_{+-}(r) \\
M(r,x) &= M_0(r) + \varepsilon M_1(r)\, e^{ipx} & M_1(r) &= h_{++}(r) \\
N(r,x) &= \varepsilon N_1(r)\, e^{ipx} & N_1(r) &= h_{--}(r) \\
E(r,x) &= E_0(r) + \varepsilon E_1(r)\, e^{ipx} & E_1(r) &= a_+(r)' \\
P(r,x) &= \varepsilon P_1(r)\, e^{ipx} & P_1(r) &= -a_-(r)' \\
V(r,x) &= V_0(r) + \varepsilon V_1(r)\, e^{ipx} & 2e^{2V_0(r)}V_1(r) &= h_K(r) \\
f(r,x) &= f_0(r) + \varepsilon f_1(r)\, e^{ipx} & &
\end{aligned}
\tag{6.27}
$$

where ε is a formal expansion parameter. In [11], we had also introduced the gauge potentials A_1 and C_1, which are related to the fields E_1 and P_1 by $E_1 = A_1'$ and $P_1 = C_1'$; these potentials are related to the present fields by

$$\begin{aligned} A_1(r) &= a_+(r) - A_+^0 \\ -C_1(r) &= a_-(r) - A_-^0 \end{aligned} \tag{6.28}$$

where $A_\pm^0$ are constants. Here, f was defined in (2.7), and we have,

$$\begin{aligned} f_0 &= L_0^2 \\ f_1 &= 2L_0 L_1 - M_0 N_1 \end{aligned} \tag{6.29}$$

Maxwell's equations for the perturbation functions are given by,

$$\begin{aligned} &\text{M1} \qquad 0 = \Big(e^{2V_0}(E_0 N_1 + L_0 P_1)\Big)' + 2kbP_1 \\ &\text{M2} \qquad 0 = \Big(e^{2V_0}(2L_0 E_0 V_1 + L_0 E_1 + E_0 L_1 + M_0 P_1)\Big)' - 2kbE_1 \end{aligned} \tag{6.30}$$

while Einstein's equations are given by,

$$\begin{aligned} &\text{E1} \qquad 0 = L_1'' + 2V_0' L_1' + 2V_1' L_0' + 4L_1 \left(V_0'' + (V_0')^2\right) + 4L_0 \left(V_1'' + 2V_0' V_1'\right) - 4E_0 P_1 \\ &\text{E2} \qquad 0 = M_1'' + 2V_0' M_1' + 2V_1' M_0' + 4M_1 \left(V_0'' + (V_0')^2\right) + 4M_0 \Big(V_1'' + 2V_0' V_1'\Big) + 8E_0 E_1 \\ &\text{E3} \qquad 0 = N_1'' + 2V_0' N_1' + 4N_1 \left(V_0'' + (V_0')^2\right) \\ &\text{fV} \qquad 0 = f_1'' + 4V_0' f_1' + 4V_1' f_0' + 2V_1'' f_0 + 2V_0'' f_1 + 4(V_0')^2 f_1 + 8V_0' V_1' f_0 \\ &\text{E4} \qquad 0 = \left(6V_0'' + 12(V_0')^2\right) f_1 + \left(6V_1'' + 24V_0' V_1'\right) f_0 + 6V_1' f_0' + 6V_0' f_1' \\ &\qquad\qquad -32b^2 e^{-4V_0} V_1 + 8L_0 E_0 P_1 + 4N_1 E_0^2 \end{aligned} \tag{6.31}$$

We also have first integral equations, of which we will need the following,

$$\begin{aligned} e^{2V_0}(E_0 N_1 + L_0 P_1) &= -2kbC_1 \\ e^{2V_0}(2L_0 E_0 V_1 + L_0 E_1 + E_0 L_1 + M_0 P_1) &= 2kbA_1 \\ (N_1 M_0' - M_0 N_1') e^{2V_0} - 8kbA_0 C_1 &= 2\lambda_0 \\ (N_1 L_0' - L_0 N_1') e^{2V_0} &= \nu_0 \end{aligned} \tag{6.32}$$

The solution to these equations in the far region needs to be matched to the solution of (6.24), (6.25), and (6.26) that we have derived earlier for the overlap region (recall that the overlap region is contained in the far region, so that both solutions must match in the overlap region). Thus, in the overlap region, we must have,

$$L_1(r) = s_{+-} r + t_{+-} \qquad\qquad s_{+-} = 0$$

$$
\begin{aligned}
M_1(r) &= s_{++}r + t_{++} \\
N_1(r) &= s_{--}r + t_{--} \\
A_1(r) &= \sigma_+ + \tau_+ r^k \\
-C_1(r) &= \sigma_- + \tau_- r^{-k} \\
2e^{2V_0(r)}V_1(r) &= v_+ r^\sigma + v_- r^{-\sigma-1}
\end{aligned}
\tag{6.33}
$$

where the coefficients $s_{\pm\pm}, t_{\pm\pm}, s_{+-}, t_{+-}, \sigma_\pm$, and $\tau_\pm$ are related by the equations of (6.25). The coefficients $v_\pm$ are decoupled from all other fields. At low momenta, the coefficient v_- may effectively be set to 0.

6.8 Solutions in the far region

The system of linear equations of (6.30) and (6.31) for the far region turns out to be solvable by quadratures in the following sense. Their general solution may be obtained by (successive) quadratures in terms of the functions L_0 and V_0 which characterize the purely magnetic solution. In fact, L_0 itself may be obtained in terms of V_0 by quadratures. Thus, although V_0 is not (as of yet) known analytically, all other fluctuation functions are determined explicitly in terms of V_0. The system may be integrated iteratively in the following sequence,

$$
\begin{array}{ccccccccc}
\mathrm{E3} & \longrightarrow & \mathrm{M1} & \longrightarrow & (\mathrm{fV}, \mathrm{E4}) & \longrightarrow & \mathrm{M2} & \longrightarrow & \mathrm{E2} \\
N_1 & \longrightarrow & P_1 & \longrightarrow & (f_1, V_1) & \longrightarrow & E_1 & \longrightarrow & M_1
\end{array}
\tag{6.34}
$$

The top line gives the order in which the equations should be solved, and the bottom line gives the corresponding functions obtained. Equations fV and E4 are coupled and must be solved together; their solution is obtained by solving a single auxiliary third order linear differential to which two solutions are a priori known for symmetry reasons. The full solution in the far region was obtained in [11].

In preparation for the calculation of correlators, we will review here the solution for the functions N_1 and P_1, which do not require the more complicated solution for V_1 and f_1. These solutions were obtained already in [11]. Inspection of equation E3 for N_1 in (6.31) and equation E1 of (2.6) shows that one solution for N_1 is proportional to L_0, while the other (linearly independent) solution is proportional to the function L_0^c, defined by,

$$
L_0^c(r) = L_0(r) \int_\infty^r \frac{dr'}{L_0(r')^2 e^{2V_0(r)}}
\tag{6.35}
$$

The function L_0^c involves data of the purely magnetic background solution only, and has the following asymptotics,

$$
\begin{aligned}
&r \to 0 && L_0^c(r) = -\frac{1}{2b} + \mathcal{O}(r^\sigma) \\
&r \to \infty && L_0^c(r) = -\frac{1}{4c_V(r-r_0)} + \frac{3\ln r}{16c_V^3\, r^3} + \mathcal{O}(r^{-3})
\end{aligned}
\tag{6.36}
$$

Thus, the general solution for N_1 is a linear combination of L_0 and L_0^c. Since we have $L_0(r) = 2br$ for small r, we see that the asymptotic behavior of $N_1(r)$ is indeed of the form given already in (6.33) for the overlap region. Identifying the coefficients gives,

$$N_1(r) = \frac{L_0(r)}{2b} s_{--} - 2bL_0^c(r)t_{--} \tag{6.37}$$

Since this solution does not involve the charge density parameter q, it coincides with the corresponding fluctuation solution for the purely magnetic case of [6].

Next, the function P_1, and thus A_-, may be obtained by solving equation M1. It is actually more convenient to solve instead the first integral equation on the third line of (6.32), since this equation gives directly $C_1 = A_-^0 - A_-$ in terms of data that we have already obtained. The result is as follows,

$$A_-(r) = \sigma_- + \frac{\lambda_0}{4kbA_0(r)} + \frac{A_0(r)}{4b} s_{--} - b\frac{A_0 L_0^c}{L_0}(r)t_{--} - \frac{M_0}{4kL_0A_0} t_{--} \tag{6.38}$$

Using $A_0(r) \sim qr^k/k$ as $r \to 0$, we match the asymptotics of this expression with the one in the overlap region given in (6.24). This accounts for the presence of the additive constant σ_- in (6.38), and solves for $\lambda_0 = 4bq\tau_-$. Since the coefficient $\tau_- \sim c_- \sim p^{4k}$ is suppressed within our approximation, we will set

$$\tau_- = \lambda_0 = 0 \tag{6.39}$$

in (6.38), and throughout the remainder of this paper.

7 Current and stress tensor correlators: general set-up

In this section, we will begin the calculation of the actual 2-point correlators of the current $\mathcal{J}$ and the stress tensor $\mathcal{T}$ in the state dual to the charged magnetic solution. A number of general implications will be derived from the structure of the sources and the solutions in the far and overlap regions, which will form the basis for the computation of all correlators in later sections.

7.1 Identification of sources and vevs

Sources and vevs may be identified from the $r \to \infty$ boundary asymptotics of the metric and gauge fluctuations, following (2.2) and (2.3),

$$\begin{aligned}
L_1(r) &\sim 4rg_{+-}^{(0)} + g_{+-}^{(2)} + \frac{1}{4r} g_{+-}^{(4)} + \cdots \\
M_1(r) &\sim 4rg_{++}^{(0)} + g_{++}^{(2)} + \frac{1}{4r} g_{++}^{(4)} + \cdots \\
N_1(r) &\sim 4rg_{--}^{(0)} + g_{--}^{(2)} + \frac{1}{4r} g_{--}^{(4)} + \cdots \\
A_1(r) &\sim A_+^{(0)} + \frac{1}{4r} A_+^{(2)} + \cdots \\
-C_1(r) &\sim A_-^{(0)} + \frac{1}{4r} A_-^{(2)} + \cdots \\
2V_1 e^{2V_0(r)} &\sim 4rg_{ii}^{(0)} + g_{ii}^{(2)} + \frac{1}{4r} g_{ii}^{(4)} + \cdots \qquad i = 1, 2
\end{aligned} \tag{7.1}$$

The factors of 4 in the normalization of r have been included to ensure that the conformal boundary metric has the proper customary normalization, given by,

$$ds_0^2 = dx^+ dx^- + \beta(dx^+)^2 + \frac{c_V}{4} dx^i dx^i \qquad \beta = -\frac{\alpha}{b} \tag{7.2}$$

which requires the following relation between r and the Fefferman-Graham coordinate ρ,

$$\rho = 4(r - r_0) + \frac{\ell_1}{r} - \frac{3 + 6 \ln r}{c_V^2 r} + \cdots \tag{7.3}$$

where r_0 and ℓ_1 are constants which will not be needed here. The current and stress tensor vevs are given in terms of $g_{\mu\nu}^{(4)}$ and $A_\mu^{(2)}$ by equation (2.4). As in [6], we will prefer to deal with normalized 2-dimensional current and stress tensor, obtained by integrating over the $x^{1,2}$ space with coordinate volume V_2, and effective 3-dimensional Newton constant $G_3 = G_5/V_2$. The expressions for the vevs of these *normalized current and stress tensor* are given by,

$$\begin{aligned}
16\pi G_3 T_{\mu\nu}(x) &= c_V g_{\mu\nu}^{(4)}(x) + \text{local} \\
8\pi G_3 J_\mu(x) &= c_V A_\mu^{(2)}(x) + \text{local}
\end{aligned} \tag{7.4}$$

We close this subsection by making a key observation. *The sources to the metric and gauge field fluctuations, $g^{(0)}_{\mu\nu}$ and $A^{(0)}_\mu$ are independent of the background charge density q_+.* Indeed, the sources are external fields whose strength may be dialed from the outside without reference to the value of q_+. This important observation will be used throughout.

7.2 Fluctuations in V_1 and correlators with $\mathcal{T}_K$

In this section, we will derive the 2-point correlators involving the stress tensor component $\mathcal{T}_K = \mathcal{T}_{11} = \mathcal{T}_{22}$ conjugate to the metric variations in V_1. Using the general result that the trace of the full stress tensor $\mathcal{T}$, defined in (5.6) is governed by the trace anomaly, and has local correlators with all operators, we will derive the result of (5.8) for the two-dimensional trace part $\mathcal{T}_{+-} - \beta\mathcal{T}_{--} \approx 0$.

The solutions in which $\mathcal{T}_K$ is sourced are governed by the fluctuation equations for f_1, V_1. These equations were solved in Appendix B of [11], and the general solution takes the form,

$$V_1 = \zeta_t V_1^t + \zeta_d V_1^d + \zeta_n V_1^n + V_1^p \tag{7.5}$$

where the scripts t, d, n refer to the three homogeneous solutions for which $N_1 = P_1 = 0$, while p stands for the inhomogeneous solution arising from $N_1, P_1 \neq 0$. Without loss of generality, the latter may be normalized so that V_1^p vanishes at $r = \infty$ and $r \to 0$. The mode $V_1^n(r)$ behaves as $r^{-1-\sigma}$ for $r \to 0$, matches onto the coefficient v_- in the overlap region, and is thus to be omitted within our approximation of section 5.3, so that $\zeta_n = 0$.

In summary, the stress tensor $\mathcal{T}_K$ is sourced only by the *dilation mode* $V_1^d(r)$, so that we set $\zeta_t = \zeta_n = N_1 = P_1 = 0$. The remaining fluctuations for the dilation mode are given by,

$$\begin{aligned} L_1^d(r) &= rL_0(r)' - L_0(r) \\ M_1^d(r) &= rM_0(r)' - M_0(r) \\ E_1^d(r) &= rE_0(r)' - E_0(r)/2 \\ V_1^d(r) &= rV_0(r)' \end{aligned} \tag{7.6}$$

This is an exact solution to all orders in q_+. Since $N_1(r) = 0$, and $M_1^d(r) \sim r^{2k}$ as $r \to 0$, the metric matching conditions in the overlap region imply that we must have,

$$s_{\pm\pm} = t_{\pm\pm} = 0 \tag{7.7}$$

Since $L_1^d(r) \sim r^{1+\sigma}$, $E_1^d(r) \sim r^{k-1}$, and $V_1^d(r) \sim r^\sigma$ as $r \to 0$, we may always match these solutions of the far region in the overlap region by adjusting u_0^r and c_+ in (6.25), and v_+ in (6.26) without creating any relations between the sources and the expectation values. Finally, since $V_1^d(\infty) = 1$, the coefficient ζ_d is given by the source $g^{(0)}_{ii}$ for $i = 1, 2$. As a result, the expectation values of the operators $\mathcal{T}_{--}$, $\mathcal{T}_{+-}$, $\mathcal{T}_{++}$, $\mathcal{J}_+$, $\mathcal{J}_-$, and $\mathcal{T}_K$ are all local functions of $g_{ii}^{(0)}$. The corresponding correlators vanish, and by locality of the total trace $\mathcal{T}$, so do the correlators with the combination $\mathcal{T}^{+-} + \beta\mathcal{T}^{++}$, and we have,

$$\begin{aligned}
\langle \mathcal{J}_\pm(p)\, \mathcal{T}_K(-p)\rangle = \ \langle \mathcal{J}_\pm(p)\ (\mathcal{T}_{+-} - \beta\mathcal{T}_{--})\,(-p)\rangle &= 0 \\
\langle \mathcal{T}_{\pm\pm}(p)\, \mathcal{T}_K(-p)\rangle = \langle \mathcal{T}_{\pm\pm}(p)\ (\mathcal{T}_{+-} - \beta\mathcal{T}_{--})\,(-p)\rangle &= 0 \\
\langle \mathcal{T}_{+-}(p)\, \mathcal{T}_K(-p)\rangle = \langle \mathcal{T}_{+-}(p)\ (\mathcal{T}_{+-} - \beta\mathcal{T}_{--})\,(-p)\rangle &= 0 \\
\langle \mathcal{T}_K(p)\, \mathcal{T}_K(-p)\rangle = \ \langle \mathcal{T}_K(p)\ (\mathcal{T}_{+-} - \beta\mathcal{T}_{--})\,(-p)\rangle &= 0
\end{aligned} \tag{7.8}$$

These results form the basis for the derivation of the relations (5.10).

7.3 Expansion of N_1 in powers of q_+

The solution for the field $N_1(r)$ was constructed in (6.37),

$$N_1(r) = \frac{L_0(r)}{2b} s_{--} - 2bL_0^c(r)t_{--} \tag{7.9}$$

The functions $L_0(r)$ and $L_0^c(r)$ are data of the purely magnetic background solution, and thus independent of q_+. Therefore, all q_+ dependence is contained in the coefficients s_{--} and t_{--}, which themselves admit a Taylor expansion in q_+. In fact, a Taylor expansion holds for all components of the tensors $s_{\mu\nu}$ and $t_{\mu\nu}$, and we have,[9]

$$\begin{aligned}
s_{\mu\nu} &= s_{\mu\nu}^{[0]} + s_{\mu\nu}^{[1]} + s_{\mu\nu}^{[2]} + \cdots \\
t_{\mu\nu} &= t_{\mu\nu}^{[0]} + t_{\mu\nu}^{[1]} + t_{\mu\nu}^{[2]} + \cdots
\end{aligned} \tag{7.10}$$

It is important to notice that, even though N_1 is a metric fluctuation, its expansion in powers of q_+ may contain odd powers through the gauge field sources.

The relations between $s_{\pm\pm}$ and $t_{\pm\pm}$, given on the first two lines of (6.25) may similarly be expanded in powers of q_+. Since the coefficients of these relations are independent of q_+, the relations are found to hold order by order,

$$\begin{aligned}
t_{++}^{[n]} &= \frac{p_+p_-}{24b} s_{++}^{[n]} + \frac{p_+^3}{24bp_-} s_{--}^{[n]} \\
t_{--}^{[n]} &= \frac{p_+p_-}{24b} s_{--}^{[n]} + \frac{p_-^3}{24bp_+} s_{++}^{[n]}
\end{aligned} \tag{7.11}$$

Next, we use the key observation of the end of section 7.1: the sources $g_{\mu\nu}^{(0)}$ and $A_\mu^{(0)}$ are independent of q_+. In particular, the source of the field N_1 must be independent of q_+. But

[9]The order of expansion in powers of q_+ will be denoted by a square bracket superscript, while the Fefferman-Graham order of expansion in powers of ρ^{-1} will be denoted by superscript parentheses.

this source may be identified from the $r \to \infty$ asymptotics of the solution by (7.1) and, using $L_0(r) \sim 2br$ and $L_0^c(r) \sim -1/(2b)$ is found to be given by, $4g_{--}^{(0)} = s_{--}$. As a result we have,

$$\begin{aligned} s_{--}^{[0]} &= \frac{1}{4b} g_{--}^{(0)} \\ s_{--}^{[n]} &= 0 \qquad\qquad n \geq 1 \end{aligned} \tag{7.12}$$

and thus (7.11) simplifies as follows,

$$\begin{aligned} t_{++}^{[n]} &= \frac{p_+ p_-}{24b} s_{++}^{[n]} \qquad\qquad n \geq 1 \\ t_{--}^{[n]} &= \frac{p_-^3}{24bp_+} s_{++}^{[n]} \qquad\qquad n \geq 1 \end{aligned} \tag{7.13}$$

which implies $p_+^2 t_{--}^{[n]} = p_-^2 t_{++}^{[n]}$. The $r \to \infty$ asymptotics allows us to extract the expectation value of the normalized stress tensor component T_{--} and we find,

$$16\pi G_3 T_{--} = c_V g_{--}^{(4)} = 2b\, t_{--} \tag{7.14}$$

Note that all (even) orders in q_+ are allowed to contribute to t_{--}.

7.4 Expansion of A_- in powers of q_+

The solution for A_- obtained in (6.38), with the overlap region matching conditions of (6.39), is given by,

$$A_-(r) = \sigma_- + \frac{A_0(r)}{4b} s_{--} - b\frac{A_0 L_0^c}{L_0}(r) t_{--} - \frac{M_0}{4kL_0A_0} t_{--} \tag{7.15}$$

and also admits a Taylor expansion in powers of q_+. Let us now investigate how this expansion affects the source and vev for this field. To derive the $r \to \infty$ asymptotics of A_-, we make use of the fact that $L_0^c/L_0 = \mathcal{O}(r^{-2})$, as well as the following auxiliary asymptotics,

$$\begin{aligned} A_0(r) &\sim \frac{c_V c_E}{kb}\left(1 - \frac{kb}{c_V r}\right) + \mathcal{O}(r^{-2}) \\ \frac{M_0(r)}{L_0(r)} &\sim -\frac{\alpha_c}{2b} + \mathcal{O}(r^{-2}) \end{aligned} \tag{7.16}$$

As a result, we find the following equations for the source term $A_-^{(0)}$, and the vev term $A_-^{(2)}$,

$$\begin{aligned} A_-^{(0)} &= \sigma_- + \frac{c_V c_E}{12k} s_{--} + \frac{\alpha_c}{8c_V c_E} t_{--} \\ A_-^{(2)} &= -\frac{c_E}{b} s_{--} + \frac{\alpha_c kb}{2c_V^2 c_E} t_{--} \end{aligned} \tag{7.17}$$

Using the fact that the source $A_-^{(0)}$ is independent of q_+, and the earlier result (7.12), we obtain the following relations between the expansion coefficients in powers of q_+,

$$\begin{aligned}
A_-^{(0)} &= \sigma_-^{[0]} \\
0 &= \sigma_-^{[1]} + \frac{c_V c_E}{48k} g_{--}^{(0)} + \frac{\alpha_c}{8c_V c_E} t_{--}^{[0]} \\
0 &= \sigma_-^{[n]} + \frac{\alpha_c}{8c_V c_E} t_{--}^{[n-1]} \qquad\qquad n \geq 2
\end{aligned} \tag{7.18}$$

where we have used the fact that c_E and α_c/c_E are linear in q_+. The expectation value of the normalized current component $\mathcal{J}_-$ is given by,

$$8\pi G_3 J_- = c_V A_-^{(2)} = \frac{\alpha_c k b}{2c_V c_E} \, t_{--} \tag{7.19}$$

where we have omitted the local term proportional to s_{--} in deriving this result from (7.17).

7.5 Proportionality of $\mathcal{J}_-$ and $\mathcal{T}_{--}$

Comparing the formulas for the expectation values for $\mathcal{J}_-$ in (7.19) and for $\mathcal{T}_{--}$ in (7.14), we readily obtain a simple identity between these expectation values,

$$J_- = Q_+ T_{--} \qquad\qquad Q_+ = \frac{k\alpha_c}{2c_V c_E} \tag{7.20}$$

In deriving this result, we have omitted the fractional power dependence on p^2 by setting $c_- = v_- = 0$. Since the result is obtained by overlapping expansion methods, we are of course also omitting the integer power corrections in p^2 and $q_+ p_-$. In sum, relation (7.20) holds within the approximations of section 5.3. Since (7.20) holds for any sources, it is equivalent to the operator identity, $\mathcal{J}_- \approx Q_+ \mathcal{T}_{--}$ announced in (5.11). This equation is invariant under extended Lorentz symmetry, since Q_+ is linear in q_+.

8 Calculation of the correlators of $\mathcal{J}_+$ and $\mathcal{T}_{++}$

Using the general set-up of the preceding section, all correlators involving $\mathcal{J}_+$ and $\mathcal{T}_{++}$ will be evaluated in this section. The calculations will be organized along the lines laid out in section 5, and within the confines of the approximations discussed in section 5.3. Correlators involving $\mathcal{J}_+$ will be reached by turning on the single source $A_-^{(0)}$, while leaving the remaining sources $A_+^{(0)}$ and $g_{\pm\pm}^{(0)}$ turned off. Similarly, correlators involving $\mathcal{T}_{++}$ will be reached by turning on only the source $g_{--}^{(0)}$. The chiral correlators $\langle \mathcal{J}_+ \mathcal{J}_+ \rangle$, $\langle \mathcal{J}_+ \mathcal{T}_{++} \rangle$, and $\langle \mathcal{T}_{++} \mathcal{T}_{++} \rangle$ will be evaluated in this manner to all orders in q_+. We will also take the opportunity to check that correlators of $\mathcal{J}_+$ and $\mathcal{T}_{++}$ involving mixed chiralities, or the trace of the stress tensor, vanish as announced in (5.10) and (5.12).

8.1 Sourcing $\mathcal{J}^-$

We begin by sourcing only the current $\mathcal{J}^-$ by turning on $A_-^{(0)}$. This case is the simplest, and we will discuss the corresponding calculation in detail. The general solution for the reduced Maxwell fields $a_\pm(r)$ in the overlap region was derived in (6.25). Using (6.39), it may alternatively be expressed as follows,

$$
\begin{aligned}
a_+(r) &= \sigma_+ + \frac{k\tau_+}{q_+} A_0(r) \\
a_-(r) &= \sigma_-
\end{aligned}
\tag{8.1}
$$

Here, we have used the $r \to 0$ asymptotics of the function $A_0(r)$ in (2.16) to recast (6.24) in terms of A_0 in the overlap region. The corresponding Maxwell field strengths are given by,

$$
\begin{aligned}
E_1(r) &= \frac{k\tau_+}{q_+} E_0(r) \\
P_1(r) &= 0
\end{aligned}
\tag{8.2}
$$

We will now show that the form of the perturbations exhibited in (8.1) and (8.2), although initially derived for, and valid in, the overlap region only, actually provides an exact solution valid throughout the far region, under certain conditions. To establish this, we point out that the vanishing of P_1, together with the proportionality of $E_1(r)$ to $E_0(r)$, indicates that these perturbations may be viewed as the result of transforming q_+ by,

$$
q_+ \quad \to \quad q_+ + k\tau_+ \tag{8.3}
$$

Since τ_+ should be viewed as a linear perturbation, this shift is infinitesimal. It may be interpreted as a rescaling of q_+ and may be achieved by an $SL(2, \mathbf{R})$ transformation of (2.20) with $\lambda_0 = 1 + k\tau_+/q_+$ and $\lambda = 0$. Here, τ_+ actually depends on momenta and on source functions, but from the point of view of the differential equations in r for the far region, this extra dependence is inconsequential.

8.1.1 Solution in the far region

The coefficients $\sigma_\pm$ and τ_+ may be eliminated in terms of the sources[10] $A_\pm^{(0)}$ to the Maxwell field $A_\pm$. Using the large r asymptotics result for $A_0(r)$ given in (2.13), and the relation $p_+\sigma_- = p_-\sigma_+$ derived from (6.25), we find,

$$\begin{aligned} a_+(r) &= \frac{p_+}{p_-}A_-^{(0)} + \left(A_+^{(0)} - \frac{p_+}{p_-}A_-^{(0)}\right)\frac{kb}{c_V c_E}A_0(r) \\ a_-(r) &= A_-^{(0)} \end{aligned} \tag{8.4}$$

Since the solution (8.4) is effectively generated by the transformation of (8.3), we readily determine the solutions in the far region for the remaining functions, and we find,

$$\begin{aligned} 0 &= L_1 = N_1 = P_1 = V_1 = f_1 \\ M_1(r) &= m_0 L_0(r) + m_1 L_0^c(r) + \frac{2kb}{c_V c_E}\left(A_+^{(0)} - \frac{p_+}{p_-}A_-^{(0)}\right)M_0(r) \end{aligned} \tag{8.5}$$

The first two terms in M_1 provide homogeneous solutions to the equation E2 for $M_1(r)$, while the third term provides an inhomogeneous solution sourced by the perturbation $E_1(r)$. In partial summary, the fields of (8.4) and (8.5) solve the far region equations, and their Maxwell fields properly match onto those of the overlap solution. To make them into full fledged solutions, it remains to ensure that also the metric functions match with those of the solution in the overlap region, and satisfy the absence of metric sources as $r \to \infty$.

8.1.2 Matching the metric solutions in the overlap region

Matching with the overlap region solution determines the coefficients,

$$m_0 = \frac{s_{++}}{2b} \qquad\qquad m_1 = -2bt_{++} \tag{8.6}$$

and insisting on vanishing metric sources as $r \to \infty$ gives the following relations,

$$\begin{aligned} s_{--}^{[n]} &= 0 && n \geq 0 \\ s_{++}^{[n]} &= s_{++}^{[0]} = 0 && n \geq 2 \\ s_{++}^{[1]} &= \frac{2kb\alpha_c}{c_V c_E}\left(A_+^{(0)} - \frac{p_+}{p_-}A_-^{(0)}\right) \end{aligned} \tag{8.7}$$

[10]It will be instructive to temporarily keep both sources $A_\pm^{(0)}$ even though we will ultimately set $A_+^{(0)} = 0$.

With the help of these equations, we evaluate $t_{\pm\pm}^{[n]}$ using the overlap region solution of (7.11) and (7.12) for $g_{--}^{(0)} = 0$, and we find,

$$
\begin{aligned}
t_{++}^{[n]} &= t_{--}^{[n]} = 0 \qquad\qquad n \geq 2 \\
t_{++}^{[0]} &= t_{--}^{[0]} = 0 \\
t_{++}^{[1]} &= \frac{2kb\alpha_c}{c_V c_E} \frac{p_+ p_-}{24b} \left(A_+^{(0)} - \frac{p_+}{p_-} A_-^{(0)} \right) \\
t_{--}^{[1]} &= \frac{2kb\alpha_c}{c_V c_E} \frac{p_-^3}{24bp_+} \left(A_+^{(0)} - \frac{p_+}{p_-} A_-^{(0)} \right)
\end{aligned} \tag{8.8}
$$

Clearly, $t_{++}^{[1]}$ will provide only local terms in the expectation value T_{++} and may be omitted within our approximation. The term sourced by $A_-^{(0)}$ in $t_{--}^{[1]}$ similarly produces only local terms and may be omitted. Thus, if we set $A_+^{(0)} = 0$, then all contributions to $t_{\pm\pm}$ are local and may be omitted in the matching process in the overlap region. In conclusion, when $A_+^{(0)} = 0$, the expressions of (8.4) and (8.5) provide an exact solution to all orders in q_+, and match with the solution in the overlap region, up to local terms.

To close, it is interesting to see what happens when $A_+^{(0)} \neq 0$. There is now a non-local contribution to N_1, forced upon us by the matching of the solution in the far region with the solution in the overlap region. But this contribution, in turn, will source P_1 as well as V_1, f_1, L_1, E_1 and M_1. Therefore, when $A_+^{(0)} \neq 0$, the expressions of (8.4) and (8.5) are *incompatible* with the overlap solution, even within the approximations of section 5.3.

8.1.3 Calculation of correlators with $\mathcal{J}^-$

Putting together the result of section 8.1.2, under the assumption that $A_+^{(0)} = 0$, we obtain the following solution,

$$
\begin{aligned}
a_+(r) &= \frac{p_+}{p_-} A_-^{(0)} \left(1 - \frac{kb}{c_V c_E} A_0(r) \right) \\
a_-(r) &= A_-^{(0)} \\
M_1(r) &= -\frac{2kb}{c_V c_E} \frac{p_+}{p_-} A_-^{(0)} \left(M_0(r) + \frac{\alpha_c}{2b} L_0(r) \right)
\end{aligned} \tag{8.9}
$$

with the remaining fields $L_1 = N_1 = V_1 = 0$, up to local terms. This solution is valid to all orders in q_+, within the approximation of section 5.3. Using the $r \to \infty$ asymptotics of A_0 given in (2.13), and the following $r \to \infty$ asymptotics for M_0,

$$
M_0(r) + \frac{\alpha_c}{2b} L_0(r) = \quad \sim \quad \frac{c_V c_E^2}{kbr} \tag{8.10}
$$

we derive the asymptotics of the fields as $r \to \infty$, and we find,

$$\begin{aligned} a_+(r) &\sim \frac{p_+}{p_-} A_-^{(0)} \frac{kb}{c_V r} \\ a_-(r) &= A_-^{(0)} \\ M_1(r) &= -\frac{p_+}{p_-} A_-^{(0)} \frac{2c_E}{r} \end{aligned} \tag{8.11}$$

The expectation values of the current $\mathcal{J}^-$ with all components of $\mathcal{J}$ and $\mathcal{T}$ may be read off from (8.11) and $L_1 = N_1 = V_1 = 0$. First of all, a number of expectation values are found to vanish, $J_- = T_{--} = T_{+-} = T_K = 0$, within the approximations of section 5.3. This result implies that the corresponding correlators also vanish,

$$\begin{aligned} \Big\langle \mathcal{J}_-(p)\, \mathcal{J}^-(-p) \Big\rangle &= 0 \\ \Big\langle \mathcal{T}_{--}(p)\, \mathcal{J}^-(-p) \Big\rangle &= 0 \\ \Big\langle \mathcal{T}_{+-}(p)\, \mathcal{J}^-(-p) \Big\rangle &= 0 \\ \Big\langle \mathcal{T}_K(p)\, \mathcal{J}^-(-p) \Big\rangle &= 0 \end{aligned} \tag{8.12}$$

Converting the upper index to lower indices,

$$\mathcal{J}^- = 2\mathcal{J}_+ - 4\beta \mathcal{J}_- \tag{8.13}$$

we see that the results of (8.12) are all consistent with the correlators (5.12) predicted on general grounds.

The remaining correlators with $\mathcal{J}^-$ are governed by the following expectation values,

$$\begin{aligned} J_+ &= \frac{c_V}{8\pi G_3} A_+^{(2)} = \frac{kb}{2\pi G_3} \frac{p_+}{p_-} A_-^{(0)} \\ T_{++} &= \frac{c_V}{16\pi G_3} g_{++}^{(4)} = -\frac{c_V c_E}{2\pi G_3} \frac{p_+}{p_-} A_-^{(0)} \end{aligned} \tag{8.14}$$

As a result, we find the corresponding correlators,

$$\begin{aligned} \langle \mathcal{J}_+(p)\, \mathcal{J}^-(-p) \rangle &= \frac{kb}{2\pi G_3} \frac{p_+}{p_-} \\ \langle \mathcal{T}_{++}(p)\, \mathcal{J}^-(-p) \rangle &= -\frac{c_V c_E}{2\pi G_3} \frac{p_+}{p_-} \end{aligned} \tag{8.15}$$

Using again (8.13) and the vanishing of the correlators of mixed chirality, we derive,

$$\begin{aligned} \langle \mathcal{J}_+(p)\, \mathcal{J}_+(-p) \rangle &= \frac{kb}{4\pi G_3} \frac{p_+}{p_-} \\ \langle \mathcal{T}_{++}(p)\, \mathcal{J}_+(-p) \rangle &= -\frac{c_V c_E}{4\pi G_3} \frac{p_+}{p_-} \end{aligned} \tag{8.16}$$

These results hold to all orders in q_+. The coefficient on the first line coincides with the one computed for the purely magnetic solution (since it is independent of q_+), while the second coefficient is the one announced in (5.18) and (5.19).

8.2 Sourcing only $\mathcal{T}^{--}$

We source $\mathcal{T}^{--}$ by turning on $g^{(0)}_{--}$ while keeping $g^{(0)}_{+-} = g^{(0)}_{++} = g^{(0)}_{ii} = A^{(0)}_{\pm} = 0$. In view of (7.9), the field N_1 is turned on since $s_{--} = s^{[0]}_{--} = 4bg^{(0)}_{--}$. One might now solve successively the fluctuation equations of (6.30) and (6.31) following the scheme of (6.34). Actually there is a simpler and more illuminating way of proceeding using the exact $SL(2,\mathbf{R})$ invariance of the reduced field equations in the far region, given in (2.9). In the far region, the source $g^{(0)}_{--}$ may be turned on by performing the following $SL(2,\mathbf{R})$ transformation Λ on the fields,

$$\Lambda = \begin{pmatrix} 1 & \varepsilon g^{(0)}_{--} e^{ipx} \\ 0 & 1 \end{pmatrix} \tag{8.17}$$

Using the transformation rules of (2.9) on the full fields L, M, N, E, P, V of (6.27) leads to the following exact solution to the fluctuation equations in the far region,

$$\begin{aligned} M_1(r) &= 0 & L_1(r) &= g^{(0)}_{--} M_0(r) \\ E_1(r) &= 0 & N_1(r) &= 2g^{(0)}_{--} L_0(r) \\ V_1(r) &= 0 & P_1(r) &= -g^{(0)}_{--} E_0(r) \end{aligned} \tag{8.18}$$

This far region solution is incompatible with the matching conditions in the overlap region, however, because the non-zero source $s^{[0]}_{--} = 4bg^{(0)}_{--} \neq 0$ implies that $t^{[0]}_{++} \neq 0$ is non-local, in view of (7.11). Therefore it cannot be consistently omitted in our calculations. This shortcoming is easily remedied by adding to M_1 the homogeneous solutions L_0 and L_0^c, with coefficients to be determined by the matching in the overlap region. We will do so below.

Since the above solution has a non-trivial P_1-field, we are inadvertently turning on an unwanted source in the field A_-. From the preceding section, we know how to turn this source off by a shift in q_+ as in (8.3). Since the fluctuation equations are linear, the two solutions may simply be added in the far region. The resulting total solution takes the form,

$$\begin{aligned} L_1(r) &= g^{(0)}_{--} M_0(r) \\ M_1(r) &= \frac{s_{++}}{2b} L_0(r) - 2bt_{++} L_0^c(r) + 2\tau M_0(r) \\ N_1(r) &= 2g^{(0)}_{--} L_0(r) \\ a_+(r) &= \sigma_+ + \tau A_0(r) & E_1 &= \tau E_0 \\ a_-(r) &= \sigma_- + g^{(0)}_{--} A_0(r) & P_1 &= -g^{(0)}_{--} E_0 \end{aligned} \tag{8.19}$$

while we continue to have $V_1 = 0$. This solution holds to all orders in q_+ as may readily be checked by explicit calculation.

8.2.1 Absence of sources and matching the solution in the overlap region

The absence of the sources $g_{++}^{(0)}$ and $A_\pm^{(0)}$, and the presence of the source $g_{--}^{(0)}$ imposes the following relations on the metric coefficients,

$$\begin{aligned} s_{++} &= 2\alpha_c \tau \\ s_{--}^{[0]} &= 4b g_{--}^{[0]} \\ s_{--}^{[n]} &= 0 \qquad\qquad n \geq 1 \end{aligned} \tag{8.20}$$

and the relations

$$\begin{aligned} 0 &= \sigma_+ + \tau \frac{c_V c_E}{kb} \\ 0 &= \sigma_- + g_{--}^{(0)} \frac{c_V c_E}{kb} \end{aligned} \tag{8.21}$$

on the Maxwell coefficients. We solve (8.21) for τ by exploiting the overlap region relation $p_+\sigma_- - p_-\sigma_+ = 0$, then use this result for τ to solve (8.20) for s_{++}, and we find,

$$\tau = \frac{p_+}{p_-} g_{--}^{(0)} \qquad\qquad s_{++} = 2\alpha_c \frac{p_+}{p_-} g_{--}^{(0)} \tag{8.22}$$

The second relation provides s_{++} with a simple expansion in powers of q_+ given by,

$$\begin{aligned} s_{++}^{[0]} &= s_{++}^{[1]} = 0 \\ s_{++}^{[2]} &= 2\alpha_c \frac{p_+}{p_-} g_{--}^{(0)} \\ s_{++}^{[n]} &= 0 \qquad\qquad n \geq 3 \end{aligned} \tag{8.23}$$

Using the remaining equations of (8.20), and the matching conditions for the metric (7.11),

$$\begin{array}{llll} t_{++}^{[0]} = \dfrac{p_+^3}{24bp_-} s_{--}^{[0]} & t_{++}^{[2]} = \dfrac{p_+p_-}{24b} s_{++}^{[2]} & t_{++}^{[1]} = t_{--}^{[1]} = 0 & \\ t_{--}^{[0]} = \dfrac{p_+p_-}{24b} s_{--}^{[0]} & t_{--}^{[2]} = \dfrac{p_-^3}{24bp_+} s_{++}^{[2]} & t_{++}^{[n]} = t_{--}^{[n]} = 0 & n \geq 3 \end{array} \tag{8.24}$$

Substitution of $s_{++}^{[2]}$, given by (8.23) into the expressions for $t_{\pm\pm}^{[2]}$ immediately reveals that these quantities are local, and so of course is $t_{--}^{[0]}$. Retaining only source terms and non-local contributions in the fields gives $V_1 = 0$ and,

$$\begin{aligned}
L_1(r) &= g^{(0)}_{--} M_0(r) \\
M_1(r) &= 2\,\frac{p_+}{p_-} g^{(0)}_{--}\Big(M_0(r) + \frac{\alpha_c}{2b}\, L_0(r)\Big) - \frac{p_+^3}{bp_-} g^{(0)}_{--} L_0^c(r) \\
N_1(r) &= 2g^{(0)}_{--} L_0(r) \\
a_+(r) &= \frac{p_+}{p_-} g^{(0)}_{--} \left(A_0(r) - \frac{c_V c_E}{kb}\right) \\
a_-(r) &= g^{(0)}_{--} \left(A_0(r) - \frac{c_V c_E}{kb}\right)
\end{aligned} \tag{8.25}$$

This solution holds to all orders in q_+, within the approximations of section 5.3. Extracting the leading asymptotics, it is now verified that only the fluctuations L_1 and N_1 have a source, while M_1, and $a_\pm$ do not.

The presence of a linear term in $L_1(r)$ as $r \to \infty$ implies that we have turned on a source to the operator $\mathcal{T}^{+-}$ as well as to $\mathcal{T}^{--}$. More precisely, we have introduced the coupling of $g^{(0)}_{--}$ to the combination

$$\mathcal{T}^{--} - 2\beta \mathcal{T}^{+-} = 4\mathcal{T}_{++} + 8\beta^2 \mathcal{T}_{--} \tag{8.26}$$

From the sub-leading asymptotics in (8.25), we derive the non-local parts of the expectation values. We clearly find $J_- = T_{+-} = T_{--} = T_K = 0$, and these relations imply the vanishing of the following correlators,

$$\begin{aligned}
\Big\langle \mathcal{J}_-(p)\,(\mathcal{T}_{++} + 2\beta^2 \mathcal{T}_{--})(-p)\Big\rangle &= 0 \\
\Big\langle \mathcal{T}_{+-}(p)\,(\mathcal{T}_{++} + 2\beta^2 \mathcal{T}_{--})(-p)\Big\rangle &= 0 \\
\Big\langle \mathcal{T}_{--}(p)\,(\mathcal{T}_{++} + 2\beta^2 \mathcal{T}_{--})(-p)\Big\rangle &= 0 \\
\Big\langle \mathcal{T}_K(p)\,(\mathcal{T}_{++} + 2\beta^2 \mathcal{T}_{--})(-p)\Big\rangle &= 0
\end{aligned} \tag{8.27}$$

These relations are consistent with the relations (5.10) and (5.12) derived on general grounds.

The remaining expectation values are non-vanishing, and given by,

$$\begin{aligned}
g^{(4)}_{++} &= \frac{p_+^3}{b\, c_V\, p_-} g^{(0)}_{--} + \frac{8 c_V c_E^2\, p_+}{kb\, p_-} g^{(0)}_{--} \\
A^{(2)}_+ &= -\frac{4 c_E\, p_+}{p_-} g^{(0)}_{--}
\end{aligned} \tag{8.28}$$

whence we derive the expectation values of the normalized current and stress tensor,

$$\begin{aligned}
J_+ &= -\frac{c_V c_E p_+}{2\pi G_3\, p_-}\, g^{(0)}_{--} \\
T_{++} &= \frac{p_+^3}{16\pi b G_3\, p_-} g^{(0)}_{--} + \frac{c_V^2 c_E^2\, p_+}{2\pi G_3 kb\, p_-} g^{(0)}_{--}
\end{aligned} \tag{8.29}$$

These relations hold to all orders in q_+. We readily deduce the correlators in momentum space, and find,

$$\begin{aligned}
\Big\langle \mathcal{J}_+(p)\,(\mathcal{T}_{++}+2\beta^2\mathcal{T}_{--})(-p)\Big\rangle &= -\frac{c_V c_E p_+}{4\pi G_3\, p_-} \\
\Big\langle \mathcal{T}_{++}(p)\,(\mathcal{T}_{++}+2\beta^2\mathcal{T}_{--})(-p)\Big\rangle &= \frac{p_+^3}{32\pi b\, G_3\, p_-}+\frac{c_V^2 c_E^2\, p_+}{4\pi G_3 k b\, p_-}
\end{aligned} \tag{8.30}$$

Using the vanishing of the corresponding mixed chirality correlators from (5.10) and (5.12), we obtain our final result,

$$\begin{aligned}
\langle \mathcal{J}_+(p)\mathcal{T}_{++}(-p)\rangle &= -\frac{c_V c_E p_+}{4\pi G_3\, p_-} \\
\langle \mathcal{T}_{++}(p)\mathcal{T}_{++}(-p)\rangle &= \frac{p_+^3}{32\pi b\, G_3\, p_-}+\frac{c_V^2 c_E^2\, p_+}{4\pi G_3 k b\, p_-}
\end{aligned} \tag{8.31}$$

The result on the first line agrees with the first line of (8.15): this gives a check that a correlator which may be evaluated in two different ways does indeed come out uniquely. The first term on the second line reproduces the purely magnetic result on the second line of (5.1), while the second term gives the result announced in (5.4) and (5.19).

9 Discussion

The purpose of this work was to compute long-distance correlation functions in the charged magnetic background solution as a means of probing the low-energy dynamics of the dual system of fermions at finite density and magnetic field. The results obtained shed light on some properties of this theory, but various questions remain.

In particular, the existence of a quantum critical point in this system is now well established, both numerically and analytically, but we lack a good understanding of what is driving this transition. Since we are working with a fully top-down construction in which the dual gauge theories are completely specified (e.g., $\mathcal{N}$=4 SYM), we can in principle hope to obtain complementary descriptions of the transition mechanism within both gauge theory and gravity. It would be very instructive to have such an understanding of a finite density quantum phase transition from these two points of view. Note that in contrast to models of quantum criticality involving probe fermions [30, 31, 32, 33], here the gauge theory fermions are expected to be important players in the dynamics of the phase transition, rather than spectators.

The results obtained here for the correlators of the stress tensor and U(1) current provide new information about the nature of the critical point. At least as probed by these operators, the low energy theory appears to retain the character of a 1+1 dimensional CFT, albeit with twisted Virasoro generators and renormalized propagation speed. This behavior seems consistent with a general 1+1 dimensional Luttinger liquid description. On the other hand, what is puzzling is that this gives no hint as to why the low temperature thermodynamics exhibits the nontrivial specific heat behavior displayed in equation (1.2) of the Introduction; such behavior is not characteristic of Luttinger liquid behavior, but rather indicates the breakdown of such a description.

To appreciate this point, it is useful to compare the situation found here to one in which the low energy effective theory is Lorentz invariant. In a Lorentz and scale invariant theory with a traceless stress tensor, it is simple to show that there is a direct relation between the stress tensor correlator and the specific heat exponent; the power law obeyed by the former determines the latter. For instance, a $1/x^4$ falloff of the stress tensor correlator in 1+1-dimensions fixes the specific heat to be linear in temperature. In our case, we also have a 1+1-dimensional effective theory with a stress tensor correlator exhibiting $1/x^4$ falloff, but the specific heat is not, in general, linear in the temperature. This can be traced back to the lack of Lorentz invariance; specifically, due to the fact that the stress tensor cannot be made to be both symmetric and traceless (see [34] for a recent discussion of this point). The conclusion is that in a non-Lorentz invariant theory one cannot necessarily expect to see nontrivial specific heat behavior mirroring itself in the behavior of the stress tensor correlator, which is consistent with our computations.

The question thus remains as to what other observable might exhibit qualitative features sensitive to the degrees of freedom that are going critical at the quantum phase transition. One natural possibility is to consider correlation functions of charged fields in the bulk. We expect that these can be computed using the same methods employed in this paper. Ideally, such correlators will exhibit behavior at the critical point that can be used to elucidate the mechanism driving the transition in the dual gauge theory. We leave these investigations for future work.

A Scaling laws for low temperature thermodynamics

In this appendix we review the k-dependent behavior of the low temperature thermodynamics around the quantum critical point. Recalling that we can, without loss of generality, assume $k \geq 0$, there turn out to be three distinct regions : $0 \leq k < 1/2$; $1/2 \leq k \leq 3/4$; and $k \geq 3/4$.

We recall that for our system the quantum critical point arises at a particular value of the dimensionless magnetic field, $\hat{B} = \hat{B}_c$, obeying the following properties. For $\hat{B} < \hat{B}_c$ the entropy density remains finite at zero temperature; for $\hat{B} > \hat{B}_c$ the entropy density goes to zero linearly with temperature; and for $\hat{B} = \hat{B}_c$ the entropy density vanishes as a nontrivial power law, $\hat{s} \sim T^\alpha$. The values of $\hat{B}_c$ and α are k-dependent quantities in general.

For $k < 1/2$ there is no quantum critical point, as the zero temperature entropy density is found to be nonzero for any value of $\hat{B}$. This was established numerically in [10]. The special role of $k = 1/2$ can be seen by examining the near horizon solution (2.15). This solution clearly breaks down at $k = 1/2$, and for $k < 1/2$ has the wrong large r behavior of g_{++} to match onto an asymptotically AdS_5 geometry. This near horizon geometry thus plays no role for $k < 1/2$, and the low temperature behavior is instead governed by a finite entropy density solution whose form has not been determined analytically as yet.

Next consider the range $1/2 \leq k \leq 3/4$. The low temperature thermodynamics in this region is found to be controlled by the near horizon solution (2.15). This is a scale invariant geometry, and as discussed in section 4.1, the thermodynamic behavior associated with this scaling is:

$$\hat{s} \sim \hat{T}^\alpha \qquad\qquad \alpha = \frac{1-k}{k} \tag{A.1}$$

In Table 1 we compare this prediction against numerical data, and the agreement is seen to be excellent. As we have noted, this scaling behavior only holds in the range $1/2 \leq k \leq 3/4$,

k	α	$(1-k)/k$	$\Delta\alpha$
0.55	0.8182	0.8182	0.0003
0.60	0.6667	0.6667	0.0003
0.65	0.5389	0.5385	0.0003
0.70	0.4319	0.4286	0.0026
0.749	0.3606	0.3351	0.0032

Table 1: Numerical results for the exponent α, its statistical standard deviation $\Delta\alpha$, and comparison with the analytically predicted values $(1-k)/k$, in the scaling relation $\hat{s} \sim \hat{T}^\alpha$ for the range $1/2 < k < 3/4$.

and we now discuss why this is the case. In order for a given near horizon, zero temperature solution, such as (2.15), to control the low temperature thermodynamics, it must be the case that there exists a finite temperature solution that shares the same asymptotics. Only in this case will the scaling behavior of the zero temperature solution manifest itself at finite temperature. What happens for $k > 3/4$ is that at finite temperature the function V appearing in the general metric Ansatz (2.5) turns out to grow as one moves out from the horizon. This is incompatible with the asymptotics of (2.15) in which V is constant. The fact that this behavior sets in at $k = 3/4$ can be seen numerically, but a simple analytical explanation is presently lacking.

In the range $k > 3/4$ the low temperature thermodynamics is not associated with the scaling behavior of a near horizon solution, but rather emerges from the more involved asymptotic matching analysis performed in [11]. This analysis, which confirms the earlier numerical results presented in [10], leads to the scaling behavior $\hat{s} \sim \hat{T}^{1/3}$. Note that this agrees with the value of α quoted in (A.1) for $k = 3/4$.

B Perturbation theory and WKB for scalar correlators

In this appendix, we derive a systematic perturbation theory in powers of small ξ for the scalar correlators of section 3, and develop a WKB approximation for large ξ.

B.1 Perturbative expansion in power of ξ

For $1/2 < k$ the z^{4-4k} term in equation (3.22) is suppressed compared to the z^2 term for large z. For small z, this situation is reversed, but the ν^2 term then dominates over both z^2 and z^{4-4k}, as long as $k < 1$. Thus, in the interval $1/2 < k < 1$, one expects a convergent perturbative expansion in the z^{4-4k} term to hold. The starting point is equation (3.22), which we prepare in a form adapted to carrying out perturbation theory in powers of ξ,

$$x^2\varphi'' + x\varphi' - (x^2 + \nu^2)\varphi = \xi^2\, x^{4-4k}\varphi \tag{B.1}$$

The solutions for $\xi = 0$ are the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$. Regularity of the $\xi = 0$ solution φ_0 near the horizon as $x \to \infty$ requires $\varphi_0(x) = K_\nu(x)$ up to an overall normalization. The Green function corresponding to the differential operator on the left hand side of (B.1) with these large x asymptotics is given by,

$$G(x, y) = \theta(x - y)K_\nu(x)I_\nu(y) + \theta(y - x)K_\nu(y)I_\nu(x) \tag{B.2}$$

Using the Wronskian relation $xK_\nu I'_\nu - xI_\nu K'_\nu = 1$, it is readily checked that,

$$\left(x^2\partial_x^2 + x\partial_x - x^2 - \nu^2\right) G(x, y) = -y\delta(x - y) \tag{B.3}$$

The differential equation (B.1), together with the horizon asymptotics may now be transformed into the following integral equation for φ,

$$\varphi(x) = K_\nu(x) - \xi^2 \int_0^\infty dy\, G(x,y)\, y^{3-4k}\, \varphi(y) \tag{B.4}$$

Next, we show that the perturbative solution to (B.4) is point-wise convergent in the interval $1/2 < k < 1$. The perturbative solution is obtained by iteration,

$$\varphi(x) = \sum_{n=0}^\infty (-\xi^2)^n \varphi_n(x) \qquad\qquad \varphi_{n+1}(x) = \int_0^\infty dy\, G(x,y)\, y^{3-4k}\, \varphi_n(y) \tag{B.5}$$

where $\varphi_0(x) = K_\nu(x)$, and the Green function G was given in (B.2). To investigate the convergence properties of the expansion in powers of ξ, it will be useful to have the asymptotic behavior of the Bessel functions. For $x \to 0$, we have,

$$I_\nu(x) \sim \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} \qquad\qquad K_\nu(x) \sim \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu} \Gamma(\nu) \tag{B.6}$$

while for $x \to \infty$, we have,

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \qquad\qquad K_\nu(x) \sim \frac{e^{-x}}{\sqrt{2x/\pi}} \tag{B.7}$$

Subleading terms will not be needed here. We begin by considering the first iteration,[11]

$$\varphi_1(x) = K_\nu(x) \int_0^x dy\, y^{3-4k}\, I_\nu(y)\, K_\nu(y) + I_\nu(x) \int_x^\infty dy\, y^{3-4k}\, K_\nu(y)^2 \tag{B.8}$$

Clearly, for all $1/2 < k < 1$, both integrals are point-wise convergent for all finite x.

To investigate higher order terms in the expansion (B.5), we need the asymptotic behavior of φ_1, which may be obtained from the asymptotics in (B.6) and (B.7). For small x, the first integral in (B.8) behaves as x^{4-4k}, while the second integral behaves as follows,

$$\int_x^\infty dy\, y^{3-4k}\, K_\nu(y)^2 \quad \sim \quad \begin{cases} x^{-2\nu+4-4k} & \text{if} \quad -2\nu+4 < 4k \\ x^0 & \text{if} \quad -2\nu+4 > 4k \end{cases} \tag{B.9}$$

Thus, as $x \to 0$, the behavior may be captured as follows,

$$\varphi_1(x) \sim x^\varepsilon K_\nu(x) \qquad \varepsilon = \mathrm{Min}(2\nu, 4-4k) \tag{B.10}$$

[11]The case $k > 1$ can be acommodated as well by relaxing the lower integration boundary to a non-zero positive value (such as ∞), and absorbing an infinite term proportional to $K_\nu(x)$ into $\varphi_0(x)$.

which, for $k < 1$, is softer than the leading order term $K_\nu(x)$. For large x, the integrals behave as,

$$\begin{aligned} \int_0^x dy\, y^{3-4k}\, I_\nu(y)\, K_\nu(y) \quad &\sim \quad \begin{cases} x^{3-4k} & \text{if} \quad k < 3/4 \\ x^0 & \text{if} \quad k > 3/4 \end{cases} \\ \int_x^\infty dy\, y^{3-4k}\, K_\nu(y)^2 \quad &\sim \quad x^{2-4k} e^{-2x} \end{aligned} \tag{B.11}$$

For $k > 1/2$, and as $x \to \infty$, the leading behavior arises entirely from the first integral. For $k > 3/4$, the asymptotic behavior is precisely that of the leading order term,

$$\varphi_1(x) \sim K_\nu(x) \tag{B.12}$$

Thus, the perturbative expansion in this region of k is uniformly convergent. For $k < 3/4$, an extra power arises, and we have,

$$\varphi_1(x) \sim x^{3-4k} K_\nu(x) \tag{B.13}$$

the behavior of which is harder than the lowest order $K_\nu(x)$ term. Still, the contribution to each order is point-wise convergent, and we easily derive the following general leading asymptotic formula for $1/2 < k < 3/4$,

$$\varphi_n(x) \sim \frac{1}{n!} \left(\frac{x^{3-4k}}{2(3-4k)} \right)^n K_\nu(x) \tag{B.14}$$

This leading order asymptotic behavior may be resummed, and we find,

$$\varphi(x) \sim \exp\left(-\frac{\xi^2 x^{3-4k}}{2(3-4k)} \right) K_\nu(x) \tag{B.15}$$

a formula which demonstrates that the perturbation theory is well-defined.

B.2 WKB expansion

A key observation throughout is that the powers of $p_\pm$ occurring in the expansion parameters are independent of ν. This suggests that, for certain regimes, we should be able to take ν^2 large and use WKB. To this end, we use equation (3.20) as our starting point since it clearly exhibits both free parameters p^2 and p_-^2. We now make the following change of variables,

$$\ln z = x + \ln \lambda \qquad\qquad \tilde{\phi}(z) = \varphi(x) \tag{B.16}$$

where λ is a constant. (The change of variables of (B.16) is not to be confused with the one of (3.21).) The scalar equation (3.20) is transformed into the standard Schrödinger form,

$$-\frac{d^2\varphi}{dx^2} + \left(\frac{p^2 \lambda^2}{b^3}\, e^{2x} + \frac{p_-^2\, a_M \lambda^{4-4k}}{36}\, e^{(4-4k)x} \right) \varphi = -\nu^2 \varphi \tag{B.17}$$

Although p^2 and p_-^2 appeared as two independent free parameters in (3.20), only the combination ξ of (3.23) is intrinsic, while p merely sets a scale for the coordinate z, or shifts an origin for the coordinate x. The addition of the constant $\ln \lambda$ to x precisely allows for such a shift. We will choose λ such that it sets the coefficients of both potential terms in (B.17) equal to one another, thereby shifting the natural origin to $x = 0$. The corresponding condition on λ is as follows,

$$\lambda^{4k-2} = \frac{b^3 a_M p_-}{36 p_+} \tag{B.18}$$

With this value of λ, the common factor of the potential terms becomes,

$$\frac{p^2 \lambda^2}{b^3} = \frac{1}{\hbar^2} \qquad\qquad \hbar^{4k-2} = \frac{1}{\xi} = \frac{(p_+)^{2-2k}}{\xi_0 \, (p_-)^{2k}} \tag{B.19}$$

and the Schrödinger equation of (B.17) takes the form,

$$-\hbar^2 \frac{d^2\varphi}{dx^2} + \left(e^{2x} + e^{(4-4k)x}\right) \varphi = -\hbar^2 \nu^2 \varphi \tag{B.20}$$

Since the objects we are after will be independent of ν, we take ν large. Specifically, for $E = -\hbar^2 \nu^2 \gg 1$, this equation admits a perfectly fine WKB expansion, provided $\hbar \ll 1$, or $\xi \gg 1$. This range of parameters includes, for example, the long distance limit in which $p_+ \to 0$ at fixed p_-. The WKB method involves a turning point at $x = x_*$, given by,

$$e^{2x_*} + e^{(4-4k)x_*} = E \tag{B.21}$$

For $x > x_*$, we have an exponentially decaying branch, while for $x < x_*$, two oscillating exponentials contribute, whose behavior accounts for the z^ν and $z^{-\nu}$ asymptotic branches. The WKB method provides a systematic expansion in powers of $\hbar$, independently of the value of E. Therefore, our results for the form of the expansion parameter should be valid for all E, and thus all ν, including the physical region $\nu \geq 1$ of our problem.

C Linearized reduced equations in light-cone gauge

The reduced fluctuation equations in light-cone gauge around the full charged magnetic brane solutions were derived using Maple. They are relatively involved, and will not be presented here. In part, they may be solved iteratively. The $--$ component of the Einstein equation may be solved for $\tilde{h}_{rr}$ in terms of $\tilde{h}_K$ by,

$$\tilde{h}_{rr} = -\frac{2}{K_0 L_0^2} \tilde{h}_K \tag{C.1}$$

The $r-$ component of the Einstein equation may be solved for $\tilde{h}_{r+}$ in terms of $\tilde{h}_K$ by,

$$\tilde{h}_{r+} = -\frac{i}{p_-}\left(\frac{L_0 K_0'}{K_0^2}\tilde{h}_K + \frac{2L_0}{K_0}\tilde{h}_K'\right) \tag{C.2}$$

The $-$ component of the Maxwell equation may be solved for $\tilde{a}_+$ in terms of $\tilde{a}_r$ by,

$$\tilde{a}_+ = \frac{iL_0^2}{K_0 p_-}\Big((K_0 L_0 \tilde{a}_r)' + 2kb\tilde{a}_r\Big) \tag{C.3}$$

The $+-$ component of the Einstein equation may be solved for $\tilde{h}_{++}$ in terms of $\tilde{h}_K$ and $\tilde{a}_r$ (the dependences of the other fields having been eliminated using the above relations) by,

$$\begin{aligned}\tilde{h}_{++} \;=\; -\frac{L_0^2}{3K_0^3 p_-^2}\Big(& -6K_0^2 L_0^2 \tilde{h}_K'' - 3K_0 K_0' L_0^2 \tilde{h}_K' - 12K_0^2 L_0 L_0' \tilde{h}_K' \\ & -3L_0^2 K_0 K_0'' \tilde{h}_K + 24\tilde{h}_K + 3L_0^2 (K_0')^2 \tilde{h}_K + 6K_0^2 L_0 L_0'' \tilde{h}_K \\ & -9K_0 K_0' L_0 L_0' \tilde{h}_K + 6K_0^2 (L_0')^2 \tilde{h}_K - 4iK_0^3 L_0 p_- A_0' \tilde{a}_r\Big)\end{aligned} \tag{C.4}$$

This leaves only the r component of the Maxwell equation and the rr component of the Einstein equation for the fields $\tilde{h}_K$ and $\tilde{a}_r$. They satisfy the following pair of coupled second order equations,

$$\begin{aligned}0 \;=\;& \tilde{h}_K'' + \frac{2L_0'}{L_0}\tilde{h}_K' - \frac{K_0'}{K_0}\tilde{h}_K' - \frac{8}{L_0^2 K_0^2}\tilde{h}_K + \frac{4L_0' K_0'}{L_0 K_0}\tilde{h}_K + \frac{2K_0''}{K_0}\tilde{h}_K \\ & + \frac{M_0 p_-^2}{L_0^4}\tilde{h}_K - \frac{2p_+ p_-}{L_0^3}\tilde{h}_K - \frac{8}{L_0^2}\tilde{h}_K - \frac{2(K_0')^2}{K_0^2}\tilde{h}_K + \frac{8iK_0 A_0' p_-}{3L_0}\tilde{a}_r \\ 0 \;=\;& -\frac{1}{2}K_0 L_0^4 \tilde{a}_r'' - 2K_0 L_0^3 L_0' \tilde{a}_r' - \frac{1}{2}L_0^4 K_0' \tilde{a}_r' - \frac{1}{2}K_0 L_0^3 L_0'' \tilde{a}_r - kbL_0^2 L_0' \tilde{a}_r \\ & - K_0 L_0^2 (L_0')^2 \tilde{a}_r + 2kb\frac{L_0^3 K_0'}{K_0}\tilde{a}_r + \frac{L_0^4 (K_0')^2}{2K_0}\tilde{a}_r + \frac{6k^2 L_0^2}{K_0}\tilde{a}_r - \frac{1}{2}K_0 M_0 p_-^2 \tilde{a}_r \\ & + p_+ p_- K_0 L_0 \tilde{a}_r - \frac{3}{2}L_0^3 L_0' K_0' \tilde{a}_r - \frac{1}{2}L_0^4 K_0'' \tilde{a}_r + ip_- L_0 A_0' \tilde{h}_K\end{aligned} \tag{C.5}$$

The remaining equations are automatically obeyed.

D Gauge change in the overlap region: derivations

In this Appendix, we present the derivation of the solutions in the overlap region to the gauge change equations of (6.23), which we repeat here for convenience,

$$\begin{aligned} h_{MN} e^{ipx} &= \tilde{h}_{MN} e^{ipx} + \nabla_M U_N + \nabla_N U_M \\ a_M e^{ipx} &= \tilde{a}_M e^{ipx} + U^K F_{KM} + \partial_M \Theta \end{aligned} \tag{D.1}$$

Imposing rotation invariance of (6.3) on $\tilde{h}_{MN}$, $\tilde{a}_M$, h_{MN}, and a_M gives $u^1 = u^2 = 0$. We impose the conditions for light-cone gauge of (6.4) on $\tilde{h}_{MN}$ and $\tilde{a}_M$, and for LMN gauge of (6.6) on h_{MN}, and a_M. We assign the expressions for the light-cone gauge solution in the overlap region of (6.16) and (6.18) to the remaining components of $\tilde{h}_{MN}$ and $\tilde{a}_M$. Equations (D.1) for the remaining components of h_{MN} are,

$$\begin{aligned}
h_{++}(r) &= 4ibp_+ru^-(r) + \frac{8ibq}{p_-}\left(c_+r^{2k} + c_-r\right) - \frac{4q^2r^{2k-1}}{2k-1}\,u^r(r) - \frac{4ip_+q^2r^{2k}}{k(2k-1)}\,u^+(r) \\
h_{+-}(r) &= 2bu^r(r) + 2ibp_+ru^+(r) + 2ibp_-ru^-(r) - \frac{2iq^2p_-r^{2k}}{k(2k-1)}\,u^+(r) \\
h_{--}(r) &= 4ibp_-ru^+(r) \\
h_K(r) &= v_+r^\sigma + v_-r^{-1-\sigma}
\end{aligned} \tag{D.2}$$

while those for the gauge field are given by,

$$\begin{aligned}
a_+(r) &= ip_+\theta(r) + (2k-1)\frac{6ibc_+r^k}{p_-} - qr^{k-1}u^r(r) \\
a_-(r) &= ip_-\theta(r)
\end{aligned} \tag{D.3}$$

The equations determining θ and the remaining components of u^M are given by,

$$\begin{aligned}
0 &= u^r(r)' + ip_+u^+(r) + ip_-u^-(r) - \frac{iq^2p_-r^{2k-1}}{bk(2k-1)}\,u^+(r) - v_+r^\sigma - v_-r^{-1-\sigma} \\
0 &= ru^+(r)' + \frac{ip_-u^r(r)}{8b^3r^2} \\
0 &= ru^-(r)' + \frac{ip_+u^r(r)}{8b^3r^2} - \frac{q^2r^{2k}}{bk(2k-1)}\,u^+(r)' - \frac{2i}{p_-}\left(\sigma v_+r^\sigma - (\sigma+1)v_-r^{-1-\sigma}\right) \\
0 &= \theta(r)' + \frac{c_+}{4}r^{k-2} + \frac{c_-}{4}r^{-1-k} + qr^{k-1}\,u^+(r)
\end{aligned} \tag{D.4}$$

From (D.2) and (D.3), it is clear that, in the overlap region, the metric and gauge fluctuation fields h and a are determined completely in terms of $v_\pm$, $c_\pm$, $u^r, u^\pm$, and θ. In turn, $u^r, u^\pm$ and θ are governed by differential equations (D.4) which only involve the data $v_\pm, c_\pm$.

D.1 Form of the LMN gauge solution

Since the overlap region is contained in the far region, it is appropriate to match the solutions in the overlap region to (D.2), (D.3), and (D.4) with the form of the LMN gauge solutions in the far region. The latter are already known from equation (4.3) of [11], and may be parametrized as follows,

$$\begin{aligned} h_{++}(r) &= s_{++}r + t_{++} & a_+(r) &= a^1_+ \, r^{+k} + a^0_+ \\ h_{--}(r) &= s_{--}r + t_{--} & a_-(r) &= a^1_- \, r^{-k} + a^0_- \\ h_{+-}(r) &= s_{+-}r + t_{+-} & h_K(r) &= v_+ r^{\sigma} + v_- r^{-\sigma-1} \end{aligned} \tag{D.5}$$

where the coefficients $v_\pm, s_{\pm\pm}, t_{\pm\pm}, s_{+-}, t_{+-}, a^0_\pm$, and $a^1_\pm$ are independent of r, and the constraint equation implies $s_{+-} = 0$. Terms with different functional dependence on r have been omitted from (D.5), since they may be neglected in the overlap region.

From equations (D.2) and (D.3), we can infer which functional forms of $u^r, u^\pm$, and θ are allowed to contribute to the functional forms of $h_{\pm\pm}, h_{+-}, h_K$ and $a_\pm$ specified in (D.5). The allowed functional forms are found to be as follows,

$$\begin{aligned} u^+(r) \qquad & r^0, \, r^{-1}, \, r^{-2k}, \, r^{-2k+1} \\ u^-(r) \qquad & r^0, \, r^{-1} \\ u^r(r) \qquad & r^0, \, r^1, \, r^{-2k+1}, \, r^{-2k+2} \\ \theta(r) \qquad & r^{+k}, \, r^{-k} \end{aligned} \tag{D.6}$$

From these functional dependences, it is clear that all terms involving r^σ and $r^{-1-\sigma}$ may be dropped from (D.4), since they can never contribute to the relevant functional forms of $u^r, u^\pm, \theta$ in the overlap region given in (D.6).

Furthermore, by scaling a factor $p^2 = p_+ p_-$ out of r, we see that several terms in (D.2), (D.3), and (D.4) exhibit coefficients that contain powers of $\xi \sim (p_+)^{k-1}(p_-)^k$ compared to other terms in the same equation, and are thereby suppressed for $\xi \ll 1$. This is the case for the fourth term of the second equation in (D.2), and the fourth term in the first equation of (D.4), both of which are suppressed by ξ^2 compared to the second terms in each equation. It is also the case for the third term in the third equation of (D.4) which is suppressed by ξ^2 compared to the second term, upon making use of the second equation of (D.4).

Omitting all irrelevant terms leaves simplified equations for the metric fluctuations,

$$\begin{aligned} h_{++}(r) &= 4ibp_+ r u^-(r) + \frac{8ibq}{p_-}\Big(c_+ r^{2k} + c_- r\Big) - \frac{4q^2 r^{2k-1}}{2k-1} u^r(r) - \frac{4ip_+ q^2 r^{2k}}{k(2k-1)} u^+(r) \\ h_{+-}(r) &= 2bu^r(r) + 2ibp_+ r u^+(r) + 2ibp_- r u^-(r) \\ h_{--}(r) &= 4ibp_- r u^+(r) \\ h_K(r) &= v_+ r^{\sigma} + v_- r^{-\sigma-1} \end{aligned} \tag{D.7}$$

and for the gauge field fluctuations,

$$0 = u^r(r)' + ip_+ u^+(r) + ip_- u^-(r)$$

$$
\begin{aligned}
0 &= ru^+(r)' + \frac{ip_- u^r(r)}{8b^3 r^2} \\
0 &= ru^-(r)' + \frac{ip_+ u^r(r)}{8b^3 r^2} \\
0 &= \theta(r)' + \frac{c_+}{4} r^{k-2} + \frac{c_-}{4} r^{-1-k} + q r^{k-1}\, u^+(r)
\end{aligned} \tag{D.8}
$$

Note that the fluctuation h_K decouples from all the others.

D.2 Solutions in LMN gauge for metric fluctuations

To solve the simplified equations of (D.7) and (D.8), we begin by taking the derivative of the first equation in (D.8), and eliminating the derivatives of $u^\pm$ using the second and third equations. This gives the following equation for u^r alone,

$$
0 = u^r(r)'' + \frac{p^2}{4b^3 r^3} u^r(r) \tag{D.9}
$$

The second term may be neglected in the overlap region, since it is suppressed by a factor of $p^2/r \ll 1$. We are left with $u^r(r)'' = 0$. The solutions to this equation and to the second and third equations in (D.8) may be parametrized as follows,

$$
\begin{aligned}
u^r(r) &= u_1^r r + u_0^r \\
u^+(r) &= u_0^+ + \frac{ip_- u_1^r}{8b^3 r} \\
u^-(r) &= u_0^- + \frac{ip_+ u_1^r}{8b^3 r}
\end{aligned} \tag{D.10}
$$

The integration constants $u_0^\pm, u_0^r, u_1^r$ are independent of r. In the result for $u^\pm$ an extra term proportional to u_0^r/r^2 has been omitted, since its functional dependence is not in the list of (D.6). Enforcing also the first equation of (D.8), and omitting a term suppressed by a factor of p^2/r, gives the following non-trivial relation between the integration constants,

$$
u_1^r = -ip_+ u_0^+ - ip_- u_0^- \tag{D.11}
$$

We note that the functional dependences involving r^{-2k}, which were listed in (D.6) for u^+, u^r, do not actually occur. As a result, the last two terms of h_{++} in (D.7) do not contribute. The term in c_+ will also not contribute there because of its functional dependence, and neither will the term in c_- because it is momentum-suppressed in view of (6.17). As a result, the metric fluctuations are indeed given by (D.5) and (6.24), with the coefficients given in (6.25).

D.3 Solution in LMN gauge for gauge field fluctuations

Using the above result for u^+, the solution for θ in the overlap region may be readily computed, and we have,

$$\theta(r) = \theta_0 - \frac{qu_0^+}{k} r^k + \frac{c_-}{4k} r^{-k} \tag{D.12}$$

where θ_0 is a further integration constant. Using this expression in the solutions for $a_\pm$ of (D.3), and retaining only the functional behaviors of the solutions (D.5) suitable in the overlap region, we find,

$$\begin{aligned} a_+^0 &= ip_+\theta_0 & a_+^1 &= (2k-1)\frac{6ibc_+}{p_-} + \frac{qp_+(k-1)}{4bp_-k} s_{--} + \frac{qp_-}{4bp_+} s_{++} \\ a_-^0 &= ip_-\theta_0 & a_-^1 &= \frac{ip_-c_-}{4k} \end{aligned} \tag{D.13}$$

Within our low momentum approximation, we may further set $c_- = 0$.

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