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Cosmological observables, IR growth of fluctuations, and scale-dependent anisotropies

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Abstract

We simplify and extend semiclassical methods in inflationary cosmology that capture leading IR corrections to correlators. Such IR effects can be absorbed into a coordinate change when examining sufficiently local observables, but not when comparing observations at large separation in scales, such as seen by a late-time observer. The analysis is facilitated by definition of a scale-dependent metric and physical momentum. These assist definition of “IR-safe” observables seen by a post-inflationary observer, which are contrasted to those based on the local geometry of the reheating surface. For such observables, the observer’s horizon provides an effective IR cutoff. IR growth of fluctuations contributes to enhanced statistical inhomogeneities/anisotropies at short scales, observation of which by a present day observer might be sought in 21 cm measurements. Such IR corrections are argued to become large for a very late-time observer.

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1 Introduction

While inflation has successfully resolved a number of conceptual questions, and nicely matches existing cosmological data, it has raised some deeper puzzles. Notable among these is the problem of large infrared (IR) effects. The simplest of these arises from a very basic mechanism (see, *e.g.*, [1]): inflation produces fluctuations of a light field outside the horizon; these fluctuations then “freeze” in amplitude, and the cosmological expansion causes them to accumulate at long wavelength. This produces IR divergences in correlation functions, and is suggestive of a strongly fluctuating structure on the longest scales.

The spectrum of light fields depends on the model considered, but includes the inflaton, and certainly the modes of the graviton. Indeed, one guise of this effect is self-reproducing inflation, in which inflaton fluctuations produce a large-scale spacetime structure with regions of very different effective cosmological constant. An important question is how to define sensible observables in such a situation, particularly in light of the IR growth. Specifically, the massless tensor graviton modes are always present, and are described by correlators that exhibit both IR divergences and IR growth at long times. An important question regards whether these have observable consequence.

Part of the problem is how to formulate gauge-invariant observables, which is an outstanding issue in quantum gravity; diffeomorphism invariance implies that local observables familiar from field theory are not gauge invariant. This indicates that observables must take a more nonlocal form. On the practical level, one would like to understand formulation of such observables describing our actual observations; on a more formal level one would like to find, *e.g.*, gauge-invariant observables that reduce to the local observables of field theory in certain approximations [2].

In fact, it seems useful to have language to distinguish these two notions. If we imagine that there is a yet-unknown complete quantum-mechanical description of cosmology, then one of its basic features should be a set of gauge-invariant quantum-mechanical observables; we will refer to these as *q-observables*. These, however, may or may not be observable by us given our limitations as Earth-bound observers in a particular era of cosmology. Thus there is a much more restricted set of in-practice *observables*. Of course, an important question is what *q-observables* are actually observable. One possible way to think of this is as requiring a *portal*,¹ which is a mechanism for the *q-observable* to imprint its information in physics visible to us.

Returning to metric fluctuations, then, there are two questions: first, what are gauge-invariant *q-observables*, for example sensitive to the large IR fluctuations, and second, are there effects of these which are actually observable. This paper will advance positive arguments on both fronts, and in the process propose an answer to the question of how to formulate “IR-safe” observables free of IR divergences.

Consider a perturbation of an inflating cosmological metric, with flat spatial sections and scale factor $a(t)$,

$$ds^2 = -dt^2 + a^2(t)(e^\Gamma)_{ij}dx^i dx^j , \tag{1.1}$$

¹Analogous to a similar notion in describing hidden-sector physics.

where Γ_{ij} parameterizes the perturbation, and can have both scale and traceless tensor parts. Such perturbations redshift to long wavelengths, and are at least locally unobservable. Indeed, in the long-wavelength limit, Γ_{ij} is a constant, and can be removed by a change of coordinates:

$$\tilde{x}_i = (e^{\Gamma/2})_{ij} x_j . \quad (1.2)$$

this is seen if one attempts to formulate observables in terms of local scalar quantities, such as the curvature; the curvature due to the perturbation vanishes in the limit of large redshift.

However, the perturbation is nonetheless still present in the geometry investigated at longer length scales, and one question is how to describe this presence in terms of suitable q-observables. We will present aspects of this story in future work [3], and instead focus on the more practical question – can such strong fluctuations manifest themselves in quantities we actually observe?

Ref. [4] outlined a story of this nature, understood via a semiclassical picture. To summarize and extend this story, scalar and tensor fluctuations can contribute to observable quantities such as the microwave background fluctuations via their gravitational couplings to shorter-wavelength fluctuations. At short distances compared to the longer mode’s wavelength, its effect can be removed by a transformation such as (1.2) and is thus invisible. However, for the purposes of observations of a more global nature, the fluctuation persists, and alters the spectrum because of mismatches between the local metrics at points separated by long distances, due to the effect of the long wavelength modes. The effect on the spectrum can be understood by noting that the spectrum will be determined in terms of the local proper momentum when the corresponding perturbations leave the horizon scale. By comparing spectra over a sufficiently large hierarchy of scales, the long-wavelength fluctuation can be detected, via its distortion of this spectrum. This also indicates a resolution of the IR divergence problem: we can transform away fluctuations that have wavelengths long as compared to the region we observe. But – a late-time observer sees more and more fluctuating volume, and so can see an IR growth of fluctuations, apparently leading to a breakdown of such a perturbative treatment.

The present paper will further develop and clarify this picture. The next section outlines a specific example framework, single-field slow-roll inflation, although the conclusions extend more broadly. The third section introduces the notion of a scale-dependent metric perturbation and proper momentum, and uses these to give a simple description of the effect on the spectrum of fluctuations, extending [4]. The fourth section describes large-scale effects of metric fluctuations, contrasting their description in terms of the geometry of the reheating surface, to that in terms of the observations of late-time observers such as ourselves. In the process, we also give a prescription that eliminates IR-cutoff-dependent effects due to presently-unobservable longer wavelength fluctuations. But, tensor fluctuations can lead to statistical inhomogeneities/anisotropies that one might try to observe in 21 cm measurements, by comparing scalar fluctuations at different observable locations and scales. We conclude with general comments on relation to renormalization group ideas, and to deeper questions in quantum gravity.

2 Action and perturbation expansion

For a simple example framework, consider single-field, slow-roll inflation, with inflaton field ϕ and potential V :

$$S = \frac{1}{2} \int \sqrt{-g} \left[\frac{R}{8\pi G} - \partial_\mu \phi \partial^\mu \phi - 2V(\phi) \right] . \quad (2.1)$$

R is the Ricci scalar, and we choose units $8\pi G = 1$.

A perturbative description of the coupled metric and matter fluctuations can be derived using the Arnowitt–Deser–Misner [5] (ADM) parameterization of the line element,

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) , \quad (2.2)$$

where N , N^i are the lapse and the shift functions. We further decompose the metric as

$$h_{ij} = a^2(t) e^{2\zeta(x,t)} [e^\gamma]_{ij} , \quad (2.3)$$

where $a(t)$ and $\phi = \phi_0(t)$ parameterize the classical, homogeneous slow-roll solution, and $\gamma_{ii} \equiv \delta^{ij} \gamma_{ij} = 0$. The physical tensor degrees of freedom are contained in $\gamma_{ij}(x, t)$, and a scalar degree of freedom in a combination of ζ and ϕ depending on gauge.

The dynamics are found by expanding (2.1) in ζ , γ_{ij} , and $\phi - \phi_0$; the lapse and shift are determined by their equations of motion, the constraints. The quadratic action determines a mode decomposition for the degrees of freedom. For example, in the gauge $\partial_i \gamma_{ij} = 0$,

$$\gamma_{ij}(x) = \sum_{s=+, \times} \int \frac{d^3 k}{(2\pi)^3} \left[b_{\mathbf{k}}^s \epsilon_{ij}^s(\mathbf{k}) \gamma_k(t) + b_{-\mathbf{k}}^{s\dagger} \epsilon_{ij}^{s*}(-\mathbf{k}) \gamma_k^*(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{d^3 k}{(2\pi)^3} \gamma_{ij}(k; t, x) , \quad (2.4)$$

where we defined the modes $\gamma_{ij}(k; t, x)$ for later use, and $b_{\mathbf{k}}^s$ is an annihilation operator, corresponding to helicity $s = +, \times$, and satisfying

$$\left[b_{\mathbf{k}}^s, b_{\mathbf{k}'}^{s\dagger} \right] = (2\pi)^3 \delta_{ss'} \delta^3(\mathbf{k} - \mathbf{k}') . \quad (2.5)$$

The polarization tensors ϵ_{ij}^s are chosen to satisfy transversality and tracelessness conditions, along with the completeness relation $\epsilon_{ij}^s(\mathbf{k}) \epsilon_{ij}^{s'}(\mathbf{k}) = 2\delta_{ss'}$. The mode functions $\gamma_k(t)$ depend on the slow-roll potential and solution ϕ_0 .

The two-point function gives an important measure of the gravitational fluctuations; specifically, consider the double trace,

$$\langle \gamma^2(x, x') \rangle = \frac{1}{4} \langle \gamma_{ij}(x) \gamma_{ij}(x') \rangle = \int \frac{d^3 k}{(2\pi)^3} |\gamma_k(t)|^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} . \quad (2.6)$$

The coincident limit $\mathbf{x} = \mathbf{x}'$, which gives the variance [6], is ultraviolet (UV) divergent, but may be regulated by choosing a minimum physical separation $a(t)|\mathbf{x}-\mathbf{x}'| \sim 1/H$, at the Hubble scale $H = \dot{a}/a$, effectively providing a UV cutoff at $k \approx a(t)H$. Eq. (2.6) can also be large in the IR,

for inflation of sufficient duration. For example, in the case of de Sitter space, $V = \Lambda = \text{const.}$, the mode functions are easily written in terms of the conformal time $\eta = \int dt/a(t) = -1/(Ha)$,

$$\gamma_k(\eta) = \frac{H}{\sqrt{k^3}}(1 + ik\eta)e^{-ik\eta} . \quad (2.7)$$

The variance (2.6) then has a logarithmic IR divergence. We can artificially introduce a comoving IR cutoff L , which we might imagine as parameterizing a finite beginning of inflation, in which case

$$\langle \gamma^2(x) \rangle = 2 \left(\frac{H}{2\pi} \right)^2 \int_{1/L}^{a(t)H} \frac{dk}{k} (1 + k^2 \eta^2) \approx 2 \left(\frac{H}{2\pi} \right)^2 \log[a(t)HL] , \quad (2.8)$$

There can be similar growth in slow-roll, and also for the variance of the scalar mode. One current goal will be to improve understanding of the physical implications of this growth, and of the physical framework necessary to eliminate the artificial dependence on the IR cutoff L .

3 Contributions of IR modes: semiclassical relations

We will be interested in the effects of buildup of IR modes on cosmological observables such as the scalar spectrum. These can be calculated in various frameworks and gauges; for our discussion we will fix the time slicing during inflation by working on slices of constant inflaton,

$$\phi = \phi_0(t) . \quad (3.1)$$

This comoving condition leaves gauge freedom corresponding to choice of spatial coordinates. This can be further fixed via the transversality condition, $\partial_i \gamma_{ij} = 0$. One can then show that in this gauge, the constraints are consistent with $N - 1$ and N^i vanishing at first order in the perturbations.

The two point functions for ζ and γ are then computed at gaussian (free-field) level by standard methods, as described above. Our current focus is on one loop corrections to these correlators. These were computed by two methods in [4] for de Sitter space: a direct one-loop calculation, and via the semiclassical relations outlined there. These were found to agree in some detail, and it was argued (with additional supporting evidence) that the latter method also applies to slow-roll. Further elaboration and checks were given in [7–10].

Specifically, in the latter semiclassical method, the correlators are obtained from a simple and intuitive physical picture: when a given mode exits the horizon, the spectrum is determined in terms of the *physical* momentum, computed by treating the longer-wavelength modes as providing a background metric.

In order to formulate this condition more precisely, we will define a notion of a *scale-dependent metric fluctuation*, $\Gamma_{ij}(q, t, x)$ at scale q , via the formula

$$\Gamma_{ij}(q, t, x) = \int_{L^{-1}}^q \frac{d^3 k}{(2\pi)^3} [2\zeta(k; t, x)\delta_{ij} + \gamma_{ij}(k; t, x)] \quad (3.2)$$

where $\zeta(k; t, x)$ and $\gamma_{ij}(k; t, x)$ are mode functions with comoving momentum k , as in the expansion (2.4). In particular, if we restrict to the domain $q < a(t)H(t)$, this expression will be constant in time to a very good approximation, as excitations are frozen outside the horizon, but will vary in space over comoving distances $\gtrsim 1/q$. One can then define a *scale-dependent physical momentum* (modulo rescaling by $a(t)$) via

$$\kappa_{q,i}(k, x) = [e^{-\Gamma(q,x)/2}]_{ij} k_j \quad ; \quad \kappa_q^2(k, x) = [e^{-\Gamma(q,x)}]_{ij} k_i k_j \quad , \quad (3.3)$$

as well as a scale-dependent metric, $h_{ij}(q; x, t) = a^2(t)[\exp\{\Gamma(q, x)\}]_{ij}$. These expressions are again constant in x in fixed regions of comoving size $\ll 1/q$, but vary on scales $\gtrsim 1/q$. An extended statement of the proposal of [4] is that leading IR-dependent higher-order effects are incorporated into the spectrum P by writing the *tree-level* two-point function $P_0(k)$ instead as a function of $\kappa_k(k, x)$:

$$P(k, x)d^3k = P_0(\kappa_k(k, x))d^3\kappa_k \quad . \quad (3.4)$$

That is, the physics determines the spectrum in terms of the instantaneous *physical* momentum of the mode when it is exiting the horizon, as expected. The same reasoning applies whether the fluctuation in question is in ζ , the tensor modes γ_{ij} , or in another ‘‘spectator’’ field σ . For example in the former case, we have the definitions²

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle_0 = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') P_{\zeta,0}(k) \quad , \quad P_{\zeta,0}(k) = (2\pi^2) \mathcal{P}_{\zeta,0}(k)/k^3 \quad (3.5)$$

and can parameterize $\mathcal{P}_{\zeta,0}(k) \propto k^{n_s-1}$, with $n_s = 1$ the scale-independent case.

Note that it is simply the Taylor expansion of the right hand side of (3.4) around the *tree-level* two-point function $P_0(k)$ that gave us the order-by-order corrections to the power spectrum in [4], as can be seen from *e.g.* eq. (4.5) or eq. (4.11) of that paper³. (Eq. (4.7) of the present paper contains an expansion of the same form, for the related quantity (4.4).) The prescription (3.4) extends to higher-point functions as well, with examples given in [4].

As a function of k , $P(k, x)$ is sensitive to the IR cutoff, which enters via (3.2). This is what explicit loop calculations show, in de Sitter space [4]. However, rewriting P as a function of $\kappa_k(k, x)$ resums leading IR logs, giving a *candidate* ‘‘IR-safe’’ quantity $P_0(\kappa_k(k, x))d^3\kappa_k$, as is shown by the IR match between the expression (3.4) and the one-loop calculation. This latter aspect has been emphasized in other recent discussions [7, 11, 12]. Ref. [13] has also demonstrated cancellation of IR tensor variance contributions, in a position-space analog of (3.4). We will give an improved IR-safe prescription for actual observers in section 4.

4 Late-time observations and IR growth

An important question is how these corrections contribute to the spectrum of fluctuations seen by a late time observer, such as us; this is also a sharper context to investigate IR safety. We

²In a similar definition for P , the delta function is effectively smeared on scales q .

³At one loop order it was sufficient to calculate the scale-dependent metric fluctuation in the Gaussian approximation, but going to higher orders one will also need to calculate the the scale-dependent metric fluctuation self-consistently to higher order in the loop expansion parameter.

first consider the reheating surface, at $\phi = \phi_r$ which is a constant time slice, $t = t_r$, in our gauge (3.1). Fluctuations with a longer wavelength than the horizon scale at that time will have been generated with the spectrum P in (3.4), and then stay constant outside the horizon. Thus, for $\kappa_k(k, x) < a(t_r)H(t_r)$, these fluctuations will be given by this spectrum.

4.1 Geometry(ies) of the reheating surface vs. late time observers

One possible approach, also advocated in [11–14], is to describe the fluctuations, geometrical or otherwise, in terms of the geometry of the reheating surface. This geometry is determined by the spatial metric(s) $h_{ij}(q, x, t_r)$. This will not, in general, be flat, but can be made “as flat as possible” at scale q by a coordinate transformation of the form (*c.f.* (1.2))

$$\tilde{x}_i^q = [e^{\Gamma(q,x)/2}]_{ij} x_j . \quad (4.1)$$

For example, if one takes $q = a(t_r)H(t_r)$, incorporating all fluctuations outside the horizon,⁴ and with the common assumption that fluctuations inside the horizon are unimportant, this puts the metric in a locally flat form at the (arbitrarily-chosen) origin of coordinates. However, this will then introduce gradient terms in the metric of the form $\partial_i \Gamma_{jl}(q, x) x^l$, which become important at sufficiently long distances. One has put the fluctuation spectrum (3.4) in an apparently simpler form, since the conjugate variable to \tilde{x}_i^q is (3.3), up to the same kind of gradient terms. Thus, in the vicinity of the given scale, the spectrum can be written as the uncorrected spectrum $P_0(\kappa_q)$.

In comparing the spectra at significantly different scales on the reheating surface, there will however be mismatches depending on which choice of scale-dependent metric is used in the definition (4.1). Moreover, there have been suggestions [11–13] that gauge-invariant observables are naturally formulated in terms of correlators at fixed proper distance on the reheating surface.⁵ For distances much smaller than the inverse $1/q$ of the scale used in defining the scale-dependent metric (3.2), its geodesics are those of the unperturbed metric, corresponding to the fact that eq. (4.1) is effectively just a constant transformation. But, at longer distances than $1/q$, the fluctuations modify the geodesics, so the scale used to define the metric is relevant. Further examination of this and q -observables characterizing metric fluctuations in the geometry of the reheating surface will be given in [3].

However, a late-time physical observer does not directly measure the local geometry of the reheating surface.⁶ In fact, at the time of reheating, the perturbations on scales relevant for observations today are far outside the horizon, and when comparing correlation functions

⁴For improved higher-order accuracy, this q can be replaced by that determined by $\kappa_q(q, x) = a(t_r)H(t_r)$.

⁵One can in particular write the spectrum P as a function of the average physical momentum on the reheating surface, $\langle \kappa_{q_r}(k, x) \rangle$, computed at the reheating horizon scale $q_r = a_r H_r$. Written in terms of this variable, one finds a different apparent shift in the spectrum, of the form $(n_s - 1)(\langle \zeta^2 \rangle_r - \langle \zeta^2 \rangle_{h.c.}) P_0 / 2$, where h.c. denotes horizon crossing; this is like that found in [11].

⁶Indeed, if we made observations in terms of proper momentum at reheating, our observations would depend on the details of reheating, while the point of working in terms of conserved correlations at horizon exit is that CMB predictions do not depend on such microphysics.

as observed today, corresponding to different causally-disconnected regions on the reheating surface, one has to take into account that the local physical momenta in different regions are different, via (3.3), due to intermediate wavelength fluctuations present within the observable universe at the reheating time.

To a late time observer like us, it is the physical momentum *today* that matters, and today the intermediate wavelength fluctuations, which were background for shorter-wavelength fluctuations at the time of reheating, have entered the horizon and can no longer be counted as background. Thus, for an observer today, observing a largest mode entering the horizon, which exited the horizon 60 e-folds before the end of inflation, the relevant physical momentum coincides with the physical momentum when that mode exited the horizon. This defines a notion of a “small-box” and the corresponding comoving momentum scale. For a later observer, a larger box is relevant.

4.2 Late-time observation

We therefore focus on the spectrum seen by a late-time observer. This is due to an essentially familiar effect: at recombination, the fluctuations in ζ are imprinted in the observed $\Delta T/T$ of what are now microwave photons. This follows from the well-developed treatment of the Sachs-Wolfe effect on large angular scales and the acoustic oscillations on smaller scales, which can be conducted precisely by following the metric (2.2) into the matter-dominated regime. The metric perturbations governed by (3.4) then reenter the horizon at times t_p when $\kappa_k(k) = a(t_p)H(t_p)$, and begin oscillating and decaying. However, the leading effect of all this physics is simply the conversion of $\zeta(x, t_r)$, given in the *comoving* coordinates, into $\Delta T(\theta, \phi)/T$. Of course $\Delta T(\theta, \phi)/T$ can be written gauge invariantly and one is allowed to compute it any coordinates one likes. However, in a standard treatment, like the calculation of the second order Sachs-Wolfe effect in [15], one is simply calculating $\Delta T(\theta, \phi)/T$ in terms of the primordial curvature fluctuations in comoving coordinates. Thus, in this approach any new primordial second order effects are compared to the primordial linear spectrum in comoving coordinates.

The physics we are looking at is a small second order effect on the primordial spectrum of comoving curvature perturbations exhibiting mode-mode coupling of long wavelength modes with short wavelength modes. It can easily be summarized in the following way. Today we observe the last-scattering region, which consists of many causally separate patches. When we observe a correlation function of relatively short-wavelength fluctuations in many different places on the last-scattering surface, the background on which this correlation function is evaluated will change from place to place due to long wavelength fluctuations on the last-scattering surface. Of course only wavelengths larger than the wavelength of the correlation function we are evaluating, but smaller than the observed universe (i.e. the observed part of the last-scattering surface) will lead to any additional anisotropy. Wavelengths much larger than the observed universe today will only have an effect which can be absorbed in a coordinate transform (4.1).

Thus, the spectrum of the needed $\zeta(x, t_r)$ is for the most part given by a function of the form (3.4), with k conjugate to comoving coordinate x . But, there is the important subtlety that the metric perturbation $\Gamma_{ij}(q \sim a(t_0)H(t_0))$, where t_0 is the observation time, has not yet

decayed. It is moreover apparently IR cutoff sensitive, or divergent. However, at small scales, its effect can be eliminated by a transformation of the form (4.1), with

$$q_0 = a(t_0)H(t_0) , \quad (4.2)$$

(which again can be improved as in footnote three). Specifically, at short distances as compared to $H^{-1}(t_0)$, the gradient terms described above have effect parametrically small in distance. The transformation (4.1) simply corresponds to scaling out metric fluctuations frozen at longer scales, which are constant on the observer’s Hubble scale.

The net effect is that we work in physical coordinates $\tilde{x}_{0,i} \equiv \tilde{x}_i^{q_0}$ in which the observer’s horizon-scale metric appears flat. And, we correspondingly rewrite the fluctuation spectrum (3.4) in terms of the conjugate physical momentum, $p_{0,i} = [\exp\{-\Gamma(q_0, x_0)/2\}]_{ij}k_j/a(t_0)$, combining the relations to find⁷

$$\kappa_k(k, x)^2 = a_0^2 [e^{\Gamma(q_0, t_0, x_0) - \Gamma(a_0 p_0, t_r, x)}]_{ij} p_{0,i} p_{0,j} , \quad (4.3)$$

and the resulting spectrum given by (3.4),

$$P(p_0)d^3 p_0 = P_0 \left[a_0 \left([e^{\Gamma(q_0, t_0, x_0) - \Gamma(a_0 p_0, t_r, x)}]_{ij} p_{0,i} p_{0,j} \right)^{1/2} \right] d^3 \kappa_k , \quad (4.4)$$

which is now a function of the observed physical momentum p_0 and the comoving momentum q_0 corresponding the size of the observers horizon. (Again, see footnote three.)

Notice that this has accomplished something very important. As a function of k , the spectrum depends on the IR cutoff L , and diverges as it is taken to infinity. But, by working in terms of the observer’s physical momentum p_0 , where longer-wavelength fluctuations are “scaled out,” the observed spectrum instead depends on the quantity

$$\Gamma_{0,ij}(a_0 p_0, t_0, x) = \Gamma_{ij}(a_0 p_0, t_r, x) - \Gamma_{ij}(q_0, t_0, x_0) \quad (4.5)$$

which is *IR safe* – IR cutoff dependence is eliminated, and the observer’s horizon size $1/q_0$ instead functions as an IR cutoff.

This is not to say that there are no IR large effects. For sufficiently large hierarchy between the scale p_0 of the fluctuation being observed, and the horizon scale $H(t_0)$ for the observer, the integral (4.5) can make a large contribution via (4.4). Specifically, looking at the tensor contribution, the variance, given in de Sitter space by eq. (2.8), signals such large contributions. In this simple case (which approximates slow-roll), we see that this variance grows linearly as

$$\langle \gamma^2 \rangle \approx 2 \left(\frac{H}{2\pi} \right)^2 \log[H(t_0)/p_0] . \quad (4.6)$$

⁷This expression is given modulo commutators. These could, *e.g.* be handled by giving a “scale-ordered” expression, analogous to a path-ordered exponential in other contexts. Indeed, that is one way to give a formal solution to the differential equation (5.1) for the evolution in scale.

Thus, while this is small today, it becomes of order one for a late time observer who is able to see $1/H^2 \sim S$ e-folds of inflation.⁸ A more precise criterion is given in [4], which examines the size of corrections in simple scenarios. This appears to confirm the proposal of [4], that large IR effects can be resummed/absorbed for the purposes of describing sufficiently local observables, but not for purposes of describing sufficiently global observables. This also extends the discussion of [16,17], in which IR effects were argued within the δN approach to be small in the “small box” (defined by absorbing the long wavelength field fluctuations smoothed with a top-hat window function in the local background), but not in a “large box.” However, in the δN approach the IR divergences in the field fluctuations at horizon crossing are not automatically included, and some of the IR divergences captured by the present approach were not included in earlier treatments (see [4] for more discussion of this point).

The spectral distortion due to (4.5) can be examined by expanding (4.4) in $\Gamma_{0,ij}$,

$$P_\zeta(p_0, x) = \left\{ P_{\zeta_0} + \left[-\Gamma_{0,ij}(k, x) + \frac{1}{2}\Gamma_{0,il}(k, x)\Gamma_{0,lj}(k, x) \right] k_i k_j \frac{\partial P_{\zeta_0}}{\partial k^2} + \frac{1}{2}[\Gamma_{0,ij}(k, x)k_i k_j]^2 \left(\frac{\partial}{\partial k^2} \right)^2 P_{\zeta_0} + \dots \right\} \Big|_{k=a_0 p_0} a_0^3 e^{-Tr\Gamma_0(a_0 p_0, x)/2} . \quad (4.7)$$

(See eqs. (4.5) and (4.11) of [4]; other formulas there can likewise be rewritten in terms of p_0 .) The expectation value of the first-order term vanishes at leading order in perturbation theory. However, by comparing different regions of the sky, the observer can in principle see differential isometries. In this way long wavelength modes induces a *statistical inhomogeneity/anisotropy* for correlation functions of shorter wavelength modes. The statistical inhomogeneity arises in part from the fact that if we are restricted to individual smaller patches of the sky, then they are affected by longer wavelength scalar modes, which we can actually measure. The tensor modes, on the other hand, contribute an additional inhomogenous statistical anisotropy. The size of the effects of the scalar and tensor modes are proportional to the typical fluctuation in $\Gamma_{0,ij}$, determined by the accumulated variance (4.6), and thus is largest for the smallest-scale fluctuations. For an approximate scale invariant spectrum, the effect of the scalar modes is slow-roll suppressed, and the effect of the tensor modes dominates. If we are measuring a correlation on a comoving scale $a_0 p_0$, the size of the effect of long wavelength tensor modes can be estimated to be of order the square-root of the variance of long wavelength tensor modes within the horizon, *i.e.*, from (4.6), $(\langle \gamma^2 \rangle_{obs})^{1/2} \approx 2 \times 10^{-5}$, where as example values we assumed a tensor-to-scalar ratio of order $r \sim 0.1$, the scalar amplitude as measured by WMAP7 $A_s = 2.46 \times 10^{-9}$ [18], and $p_0 = 10^3 H_0$. Planck is only expected to be able to probe statistical anisotropy down to a precision around 2%, but it has been predicted that it is possible to see statistical anisotropy all the way down to the 10^{-7} level by using 21 cm emissions to probe the “dark ages,” thereby improving statistics [19]. However, these statements apply to

⁸Here, in describing such a late-time observer, we neglect the effects of late-time inflation that appears to be beginning in the Universe today and interferes with such idealized observations. Alternatively, if the present-day vacuum energy decays, ultimately the relevant information enters the observer’s horizon.

homogeneous anisotropies. It would be interesting to investigate observability of the kind of inhomogeneous anisotropies discussed in this paper, in these futuristic 21 cm measurements. It is also interesting to note that while the present effect of tensor modes is imprinted on scalar modes of all scales, the effect of primordial tensor modes on B -mode polarization in CMB is only on relatively large scales. This suggests that this effect could even be the most sensitive probe of primordial tensor modes in the far future. Indeed, in a related work, Masui and Pen [20] gave arguments for 21 cm observability of the effect (4.7) on the spectrum, with a heuristic re-derivation of the leading effect [4] based on similar considerations, but basing a signal to noise estimate on the approximation of a statistically homogeneous effect.

5 Comments and conclusions

This discussion is expected to generalize in several directions. First, as initially investigated in [4], one can apply a prescription like (4.4) to higher-point functions, using (4.3). Secondly, while we have discussed the essential points in the context of single-field slow-roll inflation, clearly they have broader applicability. For example, to control the statement that fluctuations exiting the horizon are determined in a simple way in terms of a physical momentum defined by the background, really all one needs is an adiabaticity condition that time scales for deviation from de Sitter space are long as compared to the Hubble time $1/H$, here provided by slow-roll. Likewise, the use of a scale-dependent metric and physical momentum have broader possible generality to other inflationary scenarios, and suggest an approach to defining certain IR-safe observables in dS space.

These notions also appear connected to renormalization group methods, of which they have a strong flavor, and which have been applied to inflation in related contexts [21–23]. For example, thinking of q as playing a role like a renormalization scale, one can differentiate to find renormalization group equations. Applied to the scale-dependent momentum, this yields

$$q \frac{\partial}{\partial q} \kappa_{q,i}(x) = -\frac{1}{2} \left(\frac{q}{2\pi} \right)^3 \int d^2 \Omega_q [2\zeta(\mathbf{q}; t, x) \delta_{ij} + \gamma_{ij}(\mathbf{q}; t, x)] \kappa_{q,j}(x) \quad (5.1)$$

with related equations for other quantities. One might think of the right side of (5.1) as giving an analogue of a beta function. Likewise, differentiating the spectrum (4.4) gives

$$q \frac{\partial}{\partial q} P = \left[q \partial_q \Gamma_{ij}(q, x) p_{0,i} p_{0,j} \frac{\partial}{\partial p_0^2} + \frac{1}{2} q \partial_q \text{Tr} \Gamma(q, x) \right] P, \quad (5.2)$$

a “cosmological renormalization group” equation. Note that IR cutoff dependence is also eliminated in these equations, and that similar equations can be written for other correlators.⁹

Finally, growth of the variance as in *e.g.* (2.8) means that for the late time observer seeing $S \sim 1/H^2$ e-folds, the one-loop corrections compete with the lowest-order effects. This suggests

⁹Note that another alternative is to write such equations in terms of the more physical scale given by the variable $\kappa = \kappa_q(q, x)$, instead of q .

a breakdown of this perturbative approach to gravitational fluctuations on these timescales, such as has been argued in [4, 24–26] to occur more generally on time scales $t \sim RS$, with here $R \sim 1/H$, and supports the suggestion [4] that de Sitter space has a sort of instability, arising from accumulation of large-scale fluctuations. Refs. [4, 25] also argue this is parallel to a breakdown of the calculation of the quantum state of a black hole, also on the time scale $t \sim RS$, with R the horizon radius, and S the entropy ([24, 26] argue for a relation via an apparently different proposed mechanism). Thus, there appear to be q-observables sensitive to the perturbative breakdown, describing late-time observation, and we have also argued that there is a portal to present-day observations where one could possibly see a small imprint of statistical anisotropy/inhomogeneity in short wavelength primordial correlation functions in 21 cm measurements. These hint at a possible link between a potentially observable effect, and profound aspects of non-perturbative quantum gravity.

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