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Tenth-Order Lepton Anomalous Magnetic Moment – Sixth-Order Vertices Containing Vacuum-Polarization Subdiagrams

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Abstract

This paper reports the values of contributions to the electron $g-2$ from 300 Feynman diagrams of the gauge-invariant Set III(a) and 450 Feynman diagrams of the gauge-invariant Set III(b). The evaluation is carried out in two versions. Version A is to start from the sixth-order magnetic anomaly $M_6$ obtained in the previous work. The mass-independent contributions of Set III(a) and Set III(b) are 2.1275 (2) and 3.3271 (6) in units of $(\alpha/\pi)^5$, respectively. Version B is based on the recently-developed automatic code generation scheme. This method yields 2.1271 (3) and 3.3271 (8) in units of $(\alpha/\pi)^5$, respectively. They are in excellent agreement with the results of the first method within the uncertainties of numerical integration. Combining these results as statistically independent we obtain the best values, 2.1273 (2), and 3.3271 (5) times $(\alpha/\pi)^5$, for the mass-independent contributions of the Set III(a) and Set III(b), respectively. We have also evaluated mass-dependent contributions of diagrams containing muon and/or tau-particle loop. Including them the total contribution of Set III(a) is 2.1349 (2) and that of Set III(b) is 3.3299 (5) in units of $(\alpha/\pi)^5$. The total contributions to the muon $g-2$ of various leptonic vacuum-polarization loops of Set III(a) and Set III(b) are 112.418 (32) and 15.407 (5) in units of $(\alpha/\pi)^5$, respectively.

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I. INTRODUCTION

The anomalous magnetic moment $g-2$ of the electron has played the central role in testing the validity of quantum electrodynamics (QED) as well as the Standard Model. The latest measurement of $a_e \equiv (g - 2)/2$ by the Harvard group has reached the precision of $0.24 \times 10^{-9}$ [1, 2]:

$$a_e^{(HV08)} = 1 159 652 180.73 \times 10^{-12} \ [0.24\text{ppb}].$$

(1)

At present the theoretical prediction consists of QED corrections of up to the eighth order [3–5], and hadronic corrections [6–12] and electro-weak corrections [13–15] scaled down from their contributions to the muon $g-2$. To compare the theory with the measurement (1), we also need the value of the fine structure constant $\alpha$ determined by a method independent of $g-2$. The best value of such an $\alpha$ has been obtained recently from the measurement of $h/m_{\text{Rb}}$, the ratio of the Planck constant and the mass of Rb atom, combined with the very precisely known Rydberg constant and $m_{\text{Rb}}/m_e$: [16]

$$\alpha^{-1}(\text{Rb10}) = 137.035 \ 999 \ 037 \ (91) \ [0.66\text{ppb}].$$

(2)

With this $\alpha$ the theoretical prediction of $a_e$ becomes

$$a_e^{(\text{theory})} = 1 159 652 \ 181.13 \ (0.11)(0.37)(0.77) \times 10^{-12},$$

(3)

where the first, second, and third uncertainties come from the calculated eighth-order QED term, the tenth-order estimate, and the fine structure constant (2), respectively. The theory (3) is thus in good agreement with the experiment (1):

$$a_e^{(HV08)} - a_e^{(\text{theory})} = -0.40 \ (0.88) \times 10^{-12},$$

(4)

proving that QED (Standard Model) is in good shape even at this very high precision.

An alternative test of QED is to compare $\alpha(\text{Rb10})$ with the value of $\alpha$ determined from the experiment and theory of $g-2$:

$$\alpha^{-1}(a_e^{08}) = 137.035 \ 999 \ 085 \ (12)(37)(33) \ [0.37\text{ppb}],$$

(5)

where the first, second, and third uncertainties come from the eighth-order QED term, the tenth-order estimate, and the measurement of $a_e^{(HV08)}$, respectively. Although the
uncertainty of \( \alpha^{-1}(a_e,08) \) in (5) is a factor 2 smaller than \( \alpha(Rb10) \), it is not a firm factor since it depends on the estimate of the tenth-order term, which is only a crude guess [17]. For a more stringent test of QED, it is obviously necessary to calculate the actual value of the tenth-order term. In anticipation of this challenge we launched a systematic program several years ago to evaluate the complete tenth-order term [18–20].

The 10th-order QED contribution to the anomalous magnetic moment of an electron can be written as

\[
a_{e}^{(10)} = \left( \frac{\alpha}{\pi} \right)^5 \left[ A_1^{(10)} + A_2^{(10)}(m_e/m_\mu) + A_2^{(10)}(m_e/m_\tau) + A_3^{(10)}(m_e/m_\mu, m_e/m_\tau) \right],
\]

where

\[
m_e/m_\mu = 4.836\,331\,71 (12) \times 10^{-3}
\]

and

\[
m_e/m_\tau = 2.875\,64 (47) \times 10^{-4}
\] [17]. In the rest of this article the factor \( \left( \frac{2}{\pi} \right)^5 \) will be suppressed for simplicity.

The contribution to the mass-independent term \( A_1^{(10)} \) can be classified into six gauge-invariant sets, further divided into 32 gauge-invariant subsets depending on the nature of closed lepton loop subdiagrams. Thus far, numerical results of 27 gauge-invariant subsets, which consist of 3106 vertex diagrams, have been published [18, 21–26]. Five of these 27 subsets were also calculated analytically [27, 28]. Our calculation is in good agreement with the analytic results.

In this paper we report the evaluation of the tenth-order lepton \( g-2 \) from two gauge-invariant subsets called Set III(a) and Set III(b). These diagrams are built from the magnetic moment contribution \( M_6 \) (shown in Fig. 1) which consists of 50 proper sixth-order vertices of three-photon-exchange type, namely diagrams without closed lepton loops (and called \( q \)-type. See Ref. [4] for the definition of \( q \)-type.), by insertion of various lepton vacuum-polarization loops.

FIG. 1: The sixth-order \( q \)-type diagrams. The solid line represents the electron in a constant magnetic field. The time reversal diagrams of \( M_{6D} \) and \( M_{6G} \) are omitted for simplicity.
Set III(a). Diagrams obtained by inserting two second-order vacuum-polarization function \( \Pi_2 \)'s in \( M_6 \). The number of vertex diagrams contributing to \( A_1^{(10)} \) is 300.

Set III(b). Diagrams obtained by inserting the fourth-order vacuum-polarization function \( \Pi_4 \) in \( M_6 \), where \( \Pi_4 \) is the sum of three fourth-order vacuum polarization loops. The number of vertex diagrams contributing to \( A_1^{(10)} \) is 450.

Another set (Set III(c) of Fig. 2) consists of diagrams obtained by inserting a light-by-light scattering subdiagram \( \Lambda_4 \) in \( M_6 \). The total number of these diagrams contributing to \( A_1^{(10)} \) is 390. Since it has a structure different from those of Sets III(a) and III(b), it will be treated in a separate paper.

Evaluation of Set III(a) and Set III(b) is carried out in two ways. Version A is to start from the FORTRAN code of the sixth-order anomalous magnetic moment \( M_6 \), which was obtained in previous works [29] and known to give the result identical with the analytic result [30]. It is thus easy to establish the validity of these FORTRAN codes for Sets III(a) and III(b).

We also evaluate these sets by an alternative method, Version B, using FORTRAN codes generated from scratch by the recently developed automatic code generation scheme [4, 19]. This approach deals with the UV renormalization as well as IR subtraction terms as integral parts of automation. In carrying out this automation scheme, we found it useful to construct IR subtraction terms in a different manner from that of Version A [19]. Thus, Version B provides an independent confirmation of Version A. At the same time it helps to verify the automated code generation scheme, which is developed primarily to deal with the vastly more difficult problem of Set V, which consists of 6354 vertex diagrams with pure radiative correction.

As is well-known, the insertion of vacuum-polarization loop such as \( \Pi^{(2)} \) and \( \Pi^{(4)} \) in an internal photon line of momentum \( q \) can be expressed as a superposition of massive vector
FIG. 3: Typical tenth-order diagrams of Set III(a) obtained by insertion of two second-order vacuum-polarization loops $\Pi_2$ in lepton diagrams of the three-photon-exchange type. The subset $\text{III}(a_d)$ consists of diagrams in which $\Pi_2$ are inserted in different photon lines, while the subset $\text{III}(a_s)$ consists of diagrams in which $\Pi_2$ are inserted in the same photon line. There are 150 diagrams in each subset.

The Set III(a) also contains diagrams in which two vacuum-polarization loops are inserted in the same photon line, which will be denoted as Set III$(a_s)$. For these diagrams a slight extension of Eq. (7) is required. When two vacuum polarization loops are inserted in a photon line of momentum $q$, the result, omitting integrations for simplicity, is given by the left-hand-side of the following equation, which can be rewritten in the form on the right-hand-side:

$$\frac{1}{q^2 - \sigma_a} \frac{1}{q^2 - \sigma_b} \equiv \frac{\sigma_a}{\sigma_a - \sigma_b} \frac{1}{q^2 - \sigma_a} - \frac{\sigma_b}{\sigma_a - \sigma_b} \frac{1}{q^2 - \sigma_b}. \tag{8}$$

Note that the right-hand-side is a linear combination of propagators of mass-square $\sigma_a$ and $\sigma_b$ with coefficients $\sigma_a/(\sigma_a - \sigma_b)$ and $-\sigma_b/(\sigma_a - \sigma_b)$ . This enables us to write the Feynman-parametric integrals for the diagrams in Set III$(a_s)$ by a simple extension of $M_6$ integrals.

Eq. (8) can be readily extended to the case in which three or more vacuum-polarization loops are inserted in the same photon line.
These adaptations require a slight modification of the numerator function $V$, which, for $M_6$, is given by

$$V_0 = \sum_{i=1}^{5} z_i(1 - A_i)m_e^2,$$

$$V = V_0 + (z_a + z_b + z_c)\lambda^2,$$  (9)

where $z_i (i = 1, \cdots, 5)$, and $z_j (j = a, b, c)$ are Feynman parameters assigned to the fermion propagators and the photon propagators, respectively. $m_e$ and $\lambda$ are masses of the electron and photon, respectively. $A_i (i = 1, \cdots, 5)$ are scalar currents flowing in the fermion line $i$ (see the exact definition of $A_i$ in Ref. [29]). $A_i$ is expressed by the Feynman parameters and its expression depends on the structure of a diagram. But, the expression of $V$ in terms of $A_i$ is identical for all diagrams of $M_6$.

When one vacuum-polarization function is inserted in a photon line, we must replace the mass square $\lambda^2$ of the photon in Eq. (9) by $p(t)$:

$$\lambda^2 \rightarrow p(t) \equiv \frac{4m_{vp}^2}{1 - t^2},$$  (10)

where $m_{vp}$ is the rest mass of the fermion forming the vacuum-polarization loop and the interval $4m_{vp}^2 \leq \sigma < \infty$ of Eq. (7) is mapped onto $(0 < t < 1)$ for the sake of convenience.

When two vacuum-polarization functions are inserted in the same photon line $a$, it follows from Eq. (8) that the denominators must be modified as follows:

$$\frac{1}{V} \rightarrow \frac{V_0}{V_1 V_2},$$

$$\frac{1}{V^2} \rightarrow \frac{V_0^2 - z_0^2p_1(t_1)p_2(t_2)}{(V_1 V_2)^2},$$

$$\frac{1}{V^3} \rightarrow \frac{V_0^3 - 3V_0z_0^2p_1(t_1)p_2(t_2) - z_0^3p_1(t_1)p_2(t_2)(p_1(t_1) + p_2(t_2))}{(V_1 V_2)^3},$$  (11)

where

$$V_i \equiv V_0 + z_0 p_i(t_i) + (z_b + z_c)\lambda^2, \quad i = 1, 2,$$  (12)

for the first or second vacuum-polarization functions.

Throughout this article we use the exact renormalized forms of $\Pi_2$ and $\Pi_4$ instead of intermediately renormalized forms to take advantage of the known analytic forms of their spectral functions [31].
II. SET III(a)

Diagrams belonging to the Set III(a) are generated by inserting two second-order vacuum-polarization loops $\Pi_2$ in the photon lines of $M_6$. Using an identity derived from the Ward-Takahashi identity [18] and time-reversal invariance and summing up all possible insertions of the photon spectral function reduce the number of independent integrals from 300 to 16. For programming purpose it is convenient to treat Set III(a$_d$) and Set III(a$_s$) separately.

A. Set III(a$_d$)

Let $M_{6\alpha,P2P2}$ be the magnetic moment projection of the Set III(a$_d$) generated from a self-energy diagram $M_{6\alpha}$ ($\alpha = A$ through $H$) by insertion of two electron vacuum-polarization loops $\Pi_2$ in different photon lines (see Fig. 3). The subscript $P2P2$ implies that two second-order vacuum polarization function $P2$’s are inserted in different photon lines of the proper diagram $M_{6\alpha}$. To be precise $M_{6\alpha,P2P2}$ should be written as $M_{6\alpha,P2P2}^{(l_1l_2l_3)}$, where the first superscript $l_1$ refers to the open lepton line and $l_2$ and $l_3$ refer to closed lepton loops. When $l_1$, $l_2$, and $l_3$ are identical so that $M_{6\alpha,P2P2}$ is mass-independent, we omit the superscripts for simplicity. Distinction by superscript becomes necessary in Sec. II A 3 where mass-dependent terms are treated.

1. Electron $g-2$: Version A

In Version A the renormalized contribution of the diagrams of Set III(a$_d$) can be written as [32]

$$a_e^{(10)}[\text{Set III(a$_d$)}] = \sum_{\alpha=A}^{H} a_{6\alpha,P2P2},$$

with

$$a_{6\alpha,P2P2} = \Delta M_{6\alpha,P2P2} + \text{residual renormalization terms},$$

where $\Delta M_{6\alpha,P2P2}$ is the UV- and IR-finite part of $M_{6\alpha,P2P2}$ after all divergences are removed by intermediate renormalization by $K_S$ and $I_R$ operations. See Ref. [32] for definitions of $K$-operation and $I$-operation.

When summed over all diagrams of Set III(a$_d$), the UV- and IR-divergent pieces cancel
out and the total contribution to \( a^{(10)} \) can be written as a sum of finite pieces [18]:

\[
a^{(10)}_e[\text{Set III}(a_d) : \text{Ver. } A] = \sum_{\alpha=A}^{H} \Delta M_{6\alpha,P2P2} \\
- 3\Delta B_{2,P2}\Delta M_{4,P2} - 3\Delta B_2\Delta M_{4,P2P2} \\
+ \Delta \delta m_{4,P2}(M_{2^*P2}[I] - M_{2^*P2}) + \Delta \delta m_{4,P2P2}(M_{2^*}[I] - M_{2^*}) \\
- [\Delta B_{4,P2} + 2\Delta L_{4,P2} - 4\Delta B_2\Delta B_{2,P2}]M_{2,P2} \\
- [\Delta B_{4,P2P2} + 2\Delta L_{4,P2P2} - 2(\Delta B_{2,P2})^2]M_{2},
\]

(15)

where the number of vertex diagrams represented by \( \Delta M_{6\alpha,P2P2} \) is 15 for \( \alpha = A, B, C, E, F, H \) and 30 for \( \alpha = D, G \) (see Fig. 1). We use the compactified notations for the magnetic moment, mass renormalization constant, wave-function renormalization constant, and vertex renormalization constants of fourth order [32]:

\[
\Delta M_4 \equiv \Delta M_{4a} + \Delta M_{4b}, \\
\Delta \delta m_4 \equiv \Delta \delta m_{4a} + \Delta \delta m_{4b}, \\
\Delta B_4 \equiv \Delta B_{4a} + \Delta B_{4b}, \\
\Delta L_4 \equiv \sum_{i=1}^{3}(\Delta L_{4a,i} + \Delta L_{4b,i}),
\]

(16)

where \( 4a \) and \( 4b \) refer to fourth-order diagrams with two photons crossed and uncrossed, respectively, and \( i = 1, 2, 3 \) refers to three consecutive lepton lines of the diagram of type \( 4a \) or \( 4b \). \( M_{2^*} \) is the second-order magnetic moment with a two-point vertex insertion. \( M_{2^*}[I] \) is the specific limit of \( M_{2^*} \) related to \( I \)-operation defined in Refs. [32, 33]. The subscript \( P2 \) in Eq. (15) means that a second-order vacuum-polarization function \( \Pi_2 \) is inserted in one of photon lines of the proper diagram in all possible ways. Similarly, \( P2P2 \) means that two \( \Pi_2 \)'s are inserted in two different photon lines in all possible ways.

The numerical values of \( \Delta M_{6\alpha,P2P2} \) are summarized in Table I. Numerical values of auxiliary integrals needed to complete the renormalization are listed in Table II.

Substituting the values listed in Tables I and II into Eq. (15), we obtain

\[
a^{(10)}_e[\text{Set III}(a_d) : \text{Ver. } A] = 0.94192 \quad (7).
\]

(17)
TABLE I: Contributions of the Set III\((a_d)\) diagrams to the electron \(g - 2\) evaluated in Version A. Both closed loops are electron loops. \(n_F\) is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision. First 50 iterations are carried out using \(1 \times 10^8\) sampling points per iteration. We then estimate how many more sampling points are needed to reach the desired precision. We chose \(1 \times 10^9\) sampling points per iteration and iterated 50 more times.

<table>
<thead>
<tr>
<th>Integral</th>
<th>(n_F)</th>
<th>Value (Error) including (n_F)</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta M_{6A,P2P2})</td>
<td>15</td>
<td>(-0.108,564) (3)</td>
<td>(1 \times 10^8, 1 \times 10^9)</td>
<td>50,50</td>
</tr>
<tr>
<td>(\Delta M_{6B,P2P2})</td>
<td>15</td>
<td>(0.107,954) (11)</td>
<td>(1 \times 10^8, 1 \times 10^9)</td>
<td>50,50</td>
</tr>
<tr>
<td>(\Delta M_{6C,P2P2})</td>
<td>15</td>
<td>(0.193,333) (6)</td>
<td>(1 \times 10^8, 1 \times 10^9)</td>
<td>50,50</td>
</tr>
<tr>
<td>(\Delta M_{6D,P2P2})</td>
<td>30</td>
<td>(0.176,456) (20)</td>
<td>(1 \times 10^8, 1 \times 10^9)</td>
<td>50,50</td>
</tr>
<tr>
<td>(\Delta M_{6E,P2P2})</td>
<td>15</td>
<td>(0.142,839) (11)</td>
<td>(1 \times 10^8, 1 \times 10^9)</td>
<td>50,50</td>
</tr>
<tr>
<td>(\Delta M_{6F,P2P2})</td>
<td>15</td>
<td>(0.194,882) (15)</td>
<td>(1 \times 10^8, 1 \times 10^9)</td>
<td>50,50</td>
</tr>
<tr>
<td>(\Delta M_{6G,P2P2})</td>
<td>30</td>
<td>(0.542,183) (34)</td>
<td>(1 \times 10^8, 1 \times 10^9)</td>
<td>50,50</td>
</tr>
<tr>
<td>(\Delta M_{6H,P2P2})</td>
<td>15</td>
<td>(-0.190,026) (43)</td>
<td>(1 \times 10^8, 1 \times 10^9)</td>
<td>50,50</td>
</tr>
</tbody>
</table>

2. Electron \(g - 2\): Version B

The FORTRAN programs of Version B were generated by the automation code GENCODEN with slight modification. Given one-line information specifying a diagram, GENCODEN produces a set of programs for a \(q\)-type diagram of any order of the perturbation theory\[19, 20\]. The insertion of the vacuum polarization function in a photon line is a trivial task requiring modification of just a few lines of the GENCODEN source code. The \(K\)-operation method developed in Ref. \[33\] can be easily automated and incorporated in GENCODEN\[19\] to deal with UV divergence. IR divergence, on the other hand, is somewhat differently treated.

The \(I\)-operation defined in the previous work \[32, 35\] successfully generates the IR subtraction terms for a \(q\)-type diagram of up to the eighth-order of the perturbation theory. Actually, the \(I\)-operation works even for the tenth-order case, except that the automation becomes tremendously complicated. This is why we sought another way to handle the IR divergence. Namely, we deviated from the strict IR power counting, on which the \(I\)-operation
TABLE II: Auxiliary integrals for the Set III(a_4) and Set III(a_8) with \((l_1l_2l_3) = (eee)\), where \((l_1l_2l_3)\) is defined in II A. Six lines in the middle are for Set III (a_4) and bottom four lines are for Set III (a_8). Some integrals are known exactly. Other integrals are obtained by VEGAS integration [34].

<table>
<thead>
<tr>
<th>Integral</th>
<th>Value/Error</th>
<th>Integral</th>
<th>Value/Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_2)</td>
<td>0.5</td>
<td>(M_2^\ast)</td>
<td>1.0</td>
</tr>
<tr>
<td>(M_2^\ast[I])</td>
<td>-1.0</td>
<td>(\Delta B_2)</td>
<td>0.75</td>
</tr>
<tr>
<td>(\Delta M_4)</td>
<td>0.030 833 612...</td>
<td>(\Delta \delta m_4)</td>
<td>1.906 340 (21)</td>
</tr>
<tr>
<td>(\Delta L_4)</td>
<td>0.465 024 (17)</td>
<td>(\Delta B_4)</td>
<td>-0.437 094 (21)</td>
</tr>
<tr>
<td>(M_{2,P2})</td>
<td>0.015 687 421...</td>
<td>(M_{2^\ast,P2})</td>
<td>0.044 077 4 (3)</td>
</tr>
<tr>
<td>(M_{2^\ast,P2}[I])</td>
<td>0.010 255 3 (11)</td>
<td>(\Delta M_{4,P2})</td>
<td>-0.106 707 082...</td>
</tr>
<tr>
<td>(\Delta B_{2,P2})</td>
<td>0.063 399 266...</td>
<td>(\Delta \delta m_{4,P2})</td>
<td>0.679 769 (15)</td>
</tr>
<tr>
<td>(\Delta L_{4,P2})</td>
<td>0.200 092 (14)</td>
<td>(\Delta B_{4,P2})</td>
<td>-0.314 320 (10)</td>
</tr>
<tr>
<td>(\Delta M_{4,P2,P2})</td>
<td>-0.026 682 (2)</td>
<td>(\Delta \delta m_{4,P2,P2})</td>
<td>0.105 075 (11)</td>
</tr>
<tr>
<td>(\Delta L_{4,P2,P2})</td>
<td>0.005 481 (8)</td>
<td>(\Delta B_{4,P2,P2})</td>
<td>-0.071 017 (4)</td>
</tr>
<tr>
<td>(M_{2,P2,2})</td>
<td>0.002 558 524...</td>
<td>(M_{2^\ast,P2,2})</td>
<td>0.008 482 (1)</td>
</tr>
<tr>
<td>(M_{2^\ast,P2,2}[I])</td>
<td>0.032 904 (9)</td>
<td>(\Delta M_{4,P2,2})</td>
<td>-0.057 587 8 (9)</td>
</tr>
<tr>
<td>(\Delta B_{2,P2,2})</td>
<td>0.027 902 3 (4)</td>
<td>(\Delta \delta m_{4,P2,2})</td>
<td>0.439 326 (81)</td>
</tr>
<tr>
<td>(\Delta L_{4,P2,2})</td>
<td>0.094 940 (26)</td>
<td>(\Delta B_{4,P2,2})</td>
<td>-0.199 173 (89)</td>
</tr>
</tbody>
</table>

is defined, and took a more diagrammatic approach.

The new scheme to deal with the IR divergence, called \(I/R\)-subtraction, consists of two parts: One is the \(R\)-subtraction that removes the UV-finite part of mass-renormalization term, which is the cause of linear IR divergence. (The UV-divergent part of the mass renormalization is removed by the \(K\)-operation.) Once the mass renormalization is completed, the remaining IR divergence is only logarithmic and is easily subtracted by the second part called \(I\)-subtraction. This \(I\)-subtraction is similar to the previous \(I\)-operation, except that it uses the finite part of a vertex renormalization constant in addition to the logarithmic IR-divergent part as an IR-counter term. The \(I/R\)-subtraction can be readily incorporated in GENCODEN [20].

As far as the sixth-order diagrams are concerned, two methods of IR treatment, \(I\)-
operation or $I/R$-subtraction, work fine making no significant difference. The difference is only finite amount in the amplitude of the magnetic moments, which can be identified analytically. Taking it into account, we obtain the relation of the magnetic moment amplitudes in Version A and Version B as follows:

\[
\begin{align*}
\Delta M_{6A,P2P2}^{(B)} &= \Delta M_{6A,P2P2}^{(A)} - 2\Delta L_{4b,1,P2}M_2 - 2\Delta L_{4b,1,P2P2}M_2, \\
\Delta M_{6B,P2P2}^{(B)} &= \Delta M_{6B,P2P2}^{(A)} - \Delta L_{4b,2,P2}M_2 - \Delta L_{4b,2,P2P2}M_2 - \Delta \delta m_{4b,P2}(M_2 - M_2[I]), \\
\Delta M_{6C,P2P2}^{(B)} &= \Delta M_{6C,P2P2}^{(A)} - \Delta \delta m_{4a,P2}(M_2 - M_2[I]), \\
\Delta M_{6D,P2P2}^{(B)} &= \Delta M_{6D,P2P2}^{(A)} - 2\Delta L_{4a,1,P2}M_2 - 2\Delta L_{4a,1,P2P2}M_2, \\
\Delta M_{6E,P2P2}^{(B)} &= \Delta M_{6E,P2P2}^{(A)} - \Delta L_{4a,2,P2}M_2 - \Delta L_{4a,2,P2P2}M_2, \\
\Delta M_{6F,P2P2}^{(B)} &= \Delta M_{6F,P2P2}^{(A)}, \\
\Delta M_{6G,P2P2}^{(B)} &= \Delta M_{6G,P2P2}^{(A)}, \\
\Delta M_{6H,P2P2}^{(B)} &= \Delta M_{6H,P2P2}^{(A)}. 
\end{align*}
\]  

(18)

Note that the Version B of $\Delta M_{6a,P2P2}$ absorbs not only $\Delta \delta m$ terms but also part of $\Delta L_4$ terms. From Eqs. (15) and (18) we obtain

\[
a_e^{(10)}[\text{Set III}(a_d) : \text{Ver. B}] = \sum_{a=A}^H \Delta M_{6a,P2P2}^{(B)} - 3\Delta B_{2,P2}\Delta M_{4,P2} - 3\Delta B_2\Delta M_{4,P2P2} - [\Delta B_{4,P2} + \Delta L_{4,P2} - 4\Delta B_2\Delta B_{2,P2}]M_{2,P2} - (\Delta B_{4,P2P2} + \Delta L_{4,P2P2} - 2(\Delta B_{2,P2}^2))M_2. 
\]  

(19)

Of course this shift of terms in Eq. (15) does not affect the final result.

This is a trivial change for the Set III. However, in Set V, which consists entirely of $q$-type tenth-order diagrams, residual renormalization terms of $\Delta \delta m$ type give rise to linear IR-divergences which complicate the analysis of the renormalization scheme. Thus there is an advantage in removing the self-mass terms completely, not just their UV-divergent parts.

Substituting the values listed in Tables II and III in Eq. (19), we obtain

\[
a_e^{(10)}[\text{Set III}(a_d) : \text{Ver. B}] = 0.941 \, 81 \quad (12),
\]  

(20)

which is in good agreement with (17).
TABLE III: Version B contributions of the Set III(a_d) diagrams to the electron $g - 2$. The corresponding programs are created by GENCODEN. Both closed loops are electron loops. $n_F$ is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision.

<table>
<thead>
<tr>
<th>Integral</th>
<th>$n_F$</th>
<th>Value (Error) including $n_F$</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta M_{6A,P2P2}$</td>
<td>15</td>
<td>$-0.130 , 613 , (29)$</td>
<td>$1 \times 10^8$</td>
<td>700</td>
</tr>
<tr>
<td>$\Delta M_{6B,P2P2}$</td>
<td>15</td>
<td>$-0.031 , 263 , (26)$</td>
<td>$1 \times 10^8$</td>
<td>700</td>
</tr>
<tr>
<td>$\Delta M_{6C,P2P2}$</td>
<td>15</td>
<td>$0.080 , 981 , (24)$</td>
<td>$1 \times 10^8$</td>
<td>700</td>
</tr>
<tr>
<td>$\Delta M_{6D,P2P2}$</td>
<td>30</td>
<td>$0.170 , 496 , (41)$</td>
<td>$1 \times 10^8$</td>
<td>700</td>
</tr>
<tr>
<td>$\Delta M_{6E,P2P2}$</td>
<td>15</td>
<td>$0.183 , 485 , (29)$</td>
<td>$1 \times 10^8$</td>
<td>700</td>
</tr>
<tr>
<td>$\Delta M_{6F,P2P2}$</td>
<td>15</td>
<td>$0.194 , 756 , (31)$</td>
<td>$1 \times 10^8$</td>
<td>700</td>
</tr>
<tr>
<td>$\Delta M_{6G,P2P2}$</td>
<td>30</td>
<td>$0.541 , 900 , (69)$</td>
<td>$1 \times 10^8$</td>
<td>700</td>
</tr>
<tr>
<td>$\Delta M_{6H,P2P2}$</td>
<td>15</td>
<td>$-0.189 , 816 , (54)$</td>
<td>$1 \times 10^8$</td>
<td>700</td>
</tr>
</tbody>
</table>

3. Mass-dependent terms $A_2$ and $A_3$ of Set III(a_d)

Once FORTRAN programs for mass-independent Set III(a_d) diagrams are obtained, it is straightforward to evaluate contributions of mass-dependent term $A_2^{(10)}(m_e/m_\mu)$, $A_2^{(10)}(m_e/m_\tau)$, and $A_3^{(10)}(m_e/m_\mu, m_e/m_\tau)$. We just have to choose an appropriate fermion mass $m_{vp}$ in Eq. (10). Obviously the residual renormalization terms of Set III(a_d) are slightly more complicated because of insertions of vacuum-polarization loops in several photon lines. Note also that the integrands of these sets may be strongly peaked because of their dependence on $(m_e/m_\mu)^2$ or $(m_e/m_\tau)^2$ which makes them more susceptible to the digit deficiency problem.

Of course we can evaluate them by either Version A or Version B. Since we have established their equivalence, we may choose either one, say Version A.

In the general case $(l_1l_2l_3)$, where $l_2 \neq l_3$, the residual renormalization terms of Set III(a_d)
in Version A have the form

\[
a_e^{(10)}[\text{Set III}(a_d)^{(l_1l_2l_3)}] = \sum_{\alpha=A}^H \Delta M_{6\alpha,P2P2}^{(l_1l_2l_3)}
- 3\Delta B_{2,P2}^{(l_1l_2)} \Delta M_{4,P2}^{(l_1l_3)} - 3\Delta B_{2,P2}^{(l_1l_3)} \Delta M_{4,P2}^{(l_1l_2)} - 3\Delta B_2 \Delta M_{4,P2P2}^{(l_1l_2l_3)}
+ \Delta \delta m_{4,P2}^{(l_1l_2)} (M_{2^*,P2}^{(l_1l_2)}[I] - M_{2^*,P2}^{(l_1l_3)})
+ \Delta \delta m_{4,P2P2}^{(l_1l_2l_3)} (M_{2^*,P2}^{(l_1l_2)}[I] - M_{2^*,P2}^{(l_1l_3)})
- [\Delta B_{4,P2}^{(l_1l_2)} + 2\Delta I_{4,P2}^{(l_1l_2)} - 4\Delta B_2 \Delta B_{2,P2}^{(l_1l_2)}]M_{2,P2}^{(l_1l_3)}
- [\Delta B_{4,P2}^{(l_1l_3)} + 2\Delta I_{4,P2}^{(l_1l_3)} - 4\Delta B_2 \Delta B_{2,P2}^{(l_1l_3)}]M_{2,P2}^{(l_1l_2)}
- [\Delta B_{4,P2P2}^{(l_1l_2l_3)} + 2\Delta I_{4,P2P2}^{(l_1l_2l_3)} - 4\Delta B_2 \Delta B_{2,P2}^{(l_1l_2l_3)}]M_2.
\]

(21)

For instance, for \((l_1l_2l_3) = (eem)\), the first, second, and third symbols refer to the open electron line, electron loop, and muon loop, respectively. Some superscripts are denoted as \((l_1l_2)\) or \((l_1l_3)\) since they have only one internal loop. Superscripts \((l_1)\) on \(\Delta B_2, M_2\), etc., are omitted for simplicity since these terms are mass-independent. Note also that the second and third loops appear interchangeably in the case of Set III(a). Thus Eq. (21) represents the sum of \((eem)\) and \((eme)\).

If \(l_2\) and \(l_3\) represent identical particles, duplicate terms of Eq. (21) must be dropped to avoid double counting.

Substituting the values listed in Tables IV and XI into Eq. (21), we obtain

\[
a_e^{(10)}[\text{Set III}(a_d)^{(eem)}] = 0.003 166 (4).
\]

(22)

The contributions of diagrams of \((eet)\), \((emm)\), etc., can be calculate by just changing the mass parameters in FORTRAN programs. The residual renormalization can be carried out using Eq. (21) paying attention to whether \(l_2 = l_3\) or not. We present the final results without giving details:

\[
a_e^{(10)}[\text{Set III}(a_d)^{(eet)}] = 0.000 031 55 (8),
\]

(23)

\[
a_e^{(10)}[\text{Set III}(a_d)^{(emm)}] = 0.000 080 45 (11),
\]

(24)

\[
a_e^{(10)}[\text{Set III}(a_d)^{(emt)}] = 0.000 005 56 (2).
\]

(25)

These results have been confirmed by comparison with the results of Version B.

The trend of mass dependence of these results indicates clearly that \((eet)\) case will be an order of magnitude smaller than (25). Thus it may be ignored at present.
TABLE IV: Mass-dependent contributions of the Set III(\(a_d\)) diagrams to the electron \(g-2\) evaluated in Version A. One closed loop is electron loop and the other is muon loop. Each integral is the sum of \((eem)\) and \((eme)\) types. \(n_F\) is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision.

<table>
<thead>
<tr>
<th>Integral</th>
<th>(n_F)</th>
<th>Value (Error) including (n_F)</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta M_{6A,P2P2}^{(eem)})</td>
<td>30</td>
<td>(-0.000324 (1))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M_{6B,P2P2}^{(eem)})</td>
<td>30</td>
<td>(0.000357 (2))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M_{6C,P2P2}^{(eem)})</td>
<td>30</td>
<td>(0.000496 (1))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M_{6D,P2P2}^{(eem)})</td>
<td>60</td>
<td>(0.000533 (3))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M_{6E,P2P2}^{(eem)})</td>
<td>30</td>
<td>(0.000316 (2))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M_{6F,P2P2}^{(eem)})</td>
<td>30</td>
<td>(0.000659 (3))</td>
<td>(1 \times 10^7)</td>
<td>40</td>
</tr>
<tr>
<td>(\Delta M_{6G,P2P2}^{(eem)})</td>
<td>60</td>
<td>(0.001657 (12))</td>
<td>(1 \times 10^7)</td>
<td>60</td>
</tr>
<tr>
<td>(\Delta M_{6H,P2P2}^{(eem)})</td>
<td>30</td>
<td>(-0.000313 (12))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
</tbody>
</table>

4. Muon \(g-2\). Set III(\(a_d\))

The leading contribution to the muon \(g-2\) comes from the \((mee)\) case where both loops consist of electrons, and \(m\) stands for the open muon line. Results of numerical evaluation in Version A are listed in Table V. From this Table and Table VI we obtain

\[
a_{\mu}^{(10)}[\text{Set III}(a_d)^{(mee)}] = 42.4604 (188). \tag{26}
\]

Next largest contribution comes from \((mme)\). We list only the result:

\[
a_{\mu}^{(10)}[\text{Set III}(a_d)^{(mme)}] = 11.4169 (22). \tag{27}
\]

We also obtained

\[
a_{\mu}^{(10)}[\text{Set III}(a_d)^{(met})] = 0.42197 (19), \tag{28}
\]
\[
a_{\mu}^{(10)}[\text{Set III}(a_d)^{(mnt})] = 0.11071 (1), \tag{29}
\]
\[
a_{\mu}^{(10)}[\text{Set III}(a_d)^{(mtt})] = 0.00850 (10). \tag{30}
\]

These results are in good agreement with the results of Version B.
Both closed loops are electron loops. $n_F$ is the number of Feynman diagrams represented by the integral. All integrals are evaluated in real(10) arithmetic built in gfortran to take advantage of the extended-precision format of a processor. This reduces possible digit-deficiency problem [36] substantially and runs much faster than software-implemented real(16) arithmetic.

**TABLE V:** Contributions of the Set III(a) diagrams to the muon $g-2$ evaluated in Version A. 

<table>
<thead>
<tr>
<th>Integral</th>
<th>$n_F$</th>
<th>Value (Error) including $n_F$</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta M^{(mee)}_{6A,P2P2}$</td>
<td>15</td>
<td>$-35.558 \pm 8 (48)$</td>
<td>$1 \times 10^7, 4 \times 10^7$</td>
<td>20, 180</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{6B,P2P2}$</td>
<td>15</td>
<td>$44.427 \pm 6 (68)$</td>
<td>$1 \times 10^7, 4 \times 10^7$</td>
<td>20, 180</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{6C,P2P2}$</td>
<td>15</td>
<td>$19.208 \pm 7 (63)$</td>
<td>$1 \times 10^7, 4 \times 10^7$</td>
<td>20, 180</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{6D,P2P2}$</td>
<td>30</td>
<td>$28.117 \pm 1 (70)$</td>
<td>$1 \times 10^7, 4 \times 10^7$</td>
<td>20, 280</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{6E,P2P2}$</td>
<td>15</td>
<td>$30.980 \pm 7 (64)$</td>
<td>$1 \times 10^7, 4 \times 10^7$</td>
<td>20, 80</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{6F,P2P2}$</td>
<td>15</td>
<td>$18.790 \pm 9 (60)$</td>
<td>$1 \times 10^7, 4 \times 10^7$</td>
<td>20, 180</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{6g,P2P2}$</td>
<td>30</td>
<td>$58.890 \pm 0 (71)$</td>
<td>$1 \times 10^7, 4 \times 10^7$</td>
<td>20, 180</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{6h,P2P2}$</td>
<td>15</td>
<td>$-51.550 \pm 0 (70)$</td>
<td>$1 \times 10^7, 4 \times 10^7$</td>
<td>20, 280</td>
</tr>
</tbody>
</table>

**TABLE VI:** Auxiliary integrals for Set III(a) and Set III(a), where $(l_1l_2l_3) = (mee)$. $M^{(mee)}_{2,P2}$ and $M^{(mee)}_{2,P2:2}$ are known exactly [37–39] and their uncertainties are due to the uncertainty of the measured electron-muon mass ratio only. Other integrals are obtained by VEGAS integration [34].

<table>
<thead>
<tr>
<th>Integral</th>
<th>Value(Error)</th>
<th>Integral</th>
<th>Value(Error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^{(mee)}_{2,P2}$</td>
<td>$1.094258 \pm 3086 (80)$</td>
<td>$M^{(mee)}_{2,P2}$</td>
<td>$2.34975 \pm 29 (29)$</td>
</tr>
<tr>
<td>$M^{(mee)}_{2,P2:2}[I]$</td>
<td>$-2.18321 \pm 16 (16)$</td>
<td>$\Delta B^{(mee)}_{2,P2}$</td>
<td>$1.885733 \pm 16 (16)$</td>
</tr>
<tr>
<td>$\Delta \delta m^{(mee)}_{4,P2}$</td>
<td>$11.15107 \pm 49 (49)$</td>
<td>$\Delta M^{(mee)}_{4,P2}$</td>
<td>$-0.628831 \pm 80 (2)$</td>
</tr>
<tr>
<td>$\Delta L^{(mee)}_{4,P2}$</td>
<td>$3.11986 \pm 66 (66)$</td>
<td>$\Delta B^{(mee)}_{4,P2}$</td>
<td>$-3.42788 \pm 49 (49)$</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{4,P2P2}$</td>
<td>$-1.95937 \pm 30 (30)$</td>
<td>$\Delta \delta n^{(mee)}_{4,P2P2}$</td>
<td>$16.57579 \pm 52 (52)$</td>
</tr>
<tr>
<td>$\Delta L^{(mee)}_{4,P2P2}$</td>
<td>$4.96040 \pm 63 (63)$</td>
<td>$\Delta B^{(mee)}_{4,P2P2}$</td>
<td>$-6.35375 \pm 62 (62)$</td>
</tr>
<tr>
<td>$M^{(mee)}_{2,P2:2}$</td>
<td>$2.718655 \pm 851 (82)$</td>
<td>$M^{(mee)}_{2,P2:2}$</td>
<td>$6.16233 \pm 39 (39)$</td>
</tr>
<tr>
<td>$M^{(mee)}_{2,P2:2:2}[I]$</td>
<td>$-5.10735 \pm 28 (28)$</td>
<td>$\Delta B^{(mee)}_{2,P2:2}$</td>
<td>$5.33035 \pm 12 (12)$</td>
</tr>
<tr>
<td>$\Delta M^{(mee)}_{4,P2:2:2}$</td>
<td>$-3.48452 \pm 83 (83)$</td>
<td>$\Delta \delta n^{(mee)}_{4,P2:2:2}$</td>
<td>$35.7422 \pm 12 (12)$</td>
</tr>
<tr>
<td>$\Delta L^{(mee)}_{4,P2:2:2}$</td>
<td>$10.6215 \pm 13 (13)$</td>
<td>$\Delta B^{(mee)}_{4,P2:2:2}$</td>
<td>$-12.8119 \pm 12 (12)$</td>
</tr>
</tbody>
</table>
B. Set III(a$_s$)

1. Electron $g-2$. Version A

Let $M_{6\alpha,P2:2}$ be the magnetic moment projection of the Set III(a$_s$) generated from self-energy-like diagrams $6\alpha$ ($\alpha = A$ through $H$) by insertion of two $\Pi_2$'s in the same photon line (see Fig. 3). The renormalized contribution due to these diagrams can be written in a way similar to Eq. (13).

When summed over all the diagrams of Set III(a$_s$), the UV- and IR-divergent pieces cancel out and the total contribution to $a^{(10)}$ can be written in Version A as a sum of finite pieces (which is similar to Eq. (5.39) of Ref. [32]):

$$a_e^{(10)}[\text{Set III(a$_s$)} : \text{Ver. A}] = \sum_{\alpha=A}^{H} \Delta M_{6\alpha,P2:2}$$

$$- 3\Delta B_{2,P2:2} \Delta M_4 - 3\Delta B_2 \Delta M_{4,P2:2}$$
$$+ \Delta \delta m_4(M_{2*,P2:2}[I] - M_{2*,P2:2}) + \Delta \delta m_4,P2:2(M_2,[I] - M_2)$$
$$- [\Delta B_4 + 2\Delta L_4 - 2(\Delta B_2)^2]M_{2,P2:2}$$
$$- [\Delta B_{4,P2:2} + 2\Delta L_{4,P2:2} - 4\Delta B_2 \Delta B_{2,P2:2}]M_2.$$  \hspace{1cm} (31)

The numerical values of $\Delta M_{6\alpha,P2:2}$ are summarized in Table VII. Numerical values of auxiliary integrals needed to complete the renormalization are listed in Table II.

Substituting the values listed in Tables II and VII into Eq. (31), we obtain

$$a_e^{(10)}[\text{Set III(a$_s$)} : \text{Ver. A}] = 1.185\, 56\, (20).$$  \hspace{1cm} (32)

2. Electron $g-2$. Version B

For the reason discussed in Sec. IIA2 we obtain in Version B a formula for $a_e^{(10)}[\text{Set III(a$_s$)}]$ which is different from (31):

$$a_e^{(10)}[\text{Set III(a$_s$)} : \text{Ver. B}] = \sum_{\alpha=A}^{H} \Delta M_{6\alpha,P2:2}^{(B)}$$

$$- 3\Delta B_{2,P2:2} \Delta M_4 - 3\Delta B_2 \Delta M_{4,P2:2}$$
$$- [\Delta B_4 + \Delta L_4 - 2(\Delta B_2)^2]M_{2,P2:2}$$
$$- (\Delta B_{4,P2:2} + \Delta L_{4,P2:2} - 4\Delta B_{2,P2:2} \Delta B_2)M_2.$$  \hspace{1cm} (33)
TABLE VII: Version A contributions of Set III($a_s$) diagrams to the electron $g-2$. $n_F$ is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision.

<table>
<thead>
<tr>
<th>Integral</th>
<th>$n_F$</th>
<th>Value (Error) including $n_F$</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta M_{6A,P2:2}$</td>
<td>15</td>
<td>$-0.204 682 (54)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50,50</td>
</tr>
<tr>
<td>$\Delta M_{6B,P2:2}$</td>
<td>15</td>
<td>$0.413 110 (56)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50,50</td>
</tr>
<tr>
<td>$\Delta M_{6C,P2:2}$</td>
<td>15</td>
<td>$0.458 938 (53)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50,50</td>
</tr>
<tr>
<td>$\Delta M_{6D,P2:2}$</td>
<td>30</td>
<td>$0.281 276 (54)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50,50</td>
</tr>
<tr>
<td>$\Delta M_{6E,P2:2}$</td>
<td>15</td>
<td>$0.220 637 (23)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50,50</td>
</tr>
<tr>
<td>$\Delta M_{6F,P2:2}$</td>
<td>15</td>
<td>$0.317 657 (48)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50,50</td>
</tr>
<tr>
<td>$\Delta M_{6G,P2:2}$</td>
<td>30</td>
<td>$0.765 073 (87)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50,50</td>
</tr>
<tr>
<td>$\Delta M_{6H,P2:2}$</td>
<td>15</td>
<td>$-0.409 439 (98)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50,50</td>
</tr>
</tbody>
</table>

where

\[
\begin{align*}
\Delta M^{(B)}_{6A,P2:2} & = \Delta M^{(A)}_{6A,P2:2} - 2\Delta L_{4b,1} M_{2,P2:2} - 2\Delta L_{4b,1,P2:2} M_2, \\
\Delta M^{(B)}_{6B,P2:2} & = \Delta M^{(A)}_{6B,P2:2} - \Delta L_{4b,2} M_{2,P2:2} - \Delta L_{4b,2,P2:2} M_2 \\
& \quad - \Delta \delta m_{4b} (M_{2,P2:2} - M_{2,P2:2} [I]) - \Delta \delta m_{4b,P2:2} (M_{2,P2:2} - M_{2,P2:2} [I]), \\
\Delta M^{(B)}_{6C,P2:2} & = \Delta M^{(A)}_{6C,P2:2} - \Delta \delta m_{4a} (M_{2,P2:2} - M_{2,P2:2} [I]) \\
& \quad - \Delta \delta m_{4a,P2:2} (M_{2,P2:2} - M_{2,P2:2} [I]), \\
\Delta M^{(B)}_{6D,P2:2} & = \Delta M^{(A)}_{6D,P2:2} - 2\Delta L_{4a,1} M_{2,P2:2} - 2\Delta L_{4a,1,P2:2} M_2, \\
\Delta M^{(B)}_{6E,P2:2} & = \Delta M^{(A)}_{6E,P2:2} - \Delta L_{4a,2} M_{2,P2:2} - \Delta L_{4a,2,P2:2} M_2, \\
\Delta M^{(B)}_{6F,P2:2} & = \Delta M^{(A)}_{6F,P2:2}, \\
\Delta M^{(B)}_{6H,P2:2} & = \Delta M^{(A)}_{6H,P2:2}.
\end{align*}
\]

From Tables II and VIII we obtain

\[
a^{(10)}_e [\text{Set III} (a_s) : \text{Ver. B} ] = 1.185 26 (24),
\]

in good agreement with (32).
TABLE VIII: Version B contributions of Set III(aₚ) diagrams to the electron g−2. Programs are created by GENCODEN. n_F is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision.

<table>
<thead>
<tr>
<th>Integral</th>
<th>n_F</th>
<th>Value (Error) including n_F</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta M_{6A,P2:2})</td>
<td>15</td>
<td>-0.280 605 (31)</td>
<td>(1 \times 10^8)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6B,P2:2})</td>
<td>15</td>
<td>-0.068 830 (70)</td>
<td>(1 \times 10^8)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6C,P2:2})</td>
<td>15</td>
<td>0.070 002 (67)</td>
<td>(1 \times 10^8)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6D,P2:2})</td>
<td>30</td>
<td>0.269 937 (106)</td>
<td>(1 \times 10^8)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6E,P2:2})</td>
<td>15</td>
<td>0.297 943 (60)</td>
<td>(1 \times 10^8)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6F,P2:2})</td>
<td>15</td>
<td>0.317 432 (69)</td>
<td>(1 \times 10^8)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6G,P2:2})</td>
<td>30</td>
<td>0.764 711 (127)</td>
<td>(1 \times 10^8)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6H,P2:2})</td>
<td>15</td>
<td>-0.409 110 (103)</td>
<td>(1 \times 10^8)</td>
<td>600</td>
</tr>
</tbody>
</table>

3. Mass-dependent terms A₂ and A₃ of Set III(aₚ)

For the Set III(aₚ) we have (in Version A)

\[
a^{(10)}_e[\text{Set III}(aₚ)^{(l₁l₂l₃)}] = \sum_{α=A}^{H} \Delta M_{6α,P2:2}^{(l₁l₂l₃)} - 3\Delta B_{2,P2:2}^{(l₁l₂l₃)} \Delta M_{4} - 3\Delta B_{2,P2:2}^{(l₁l₂l₃)} M_{4,P2:2}^{(l₁l₂l₃)} + \Delta \delta m_{4,P2:2}^{(l₁l₂l₃)} (\frac{M_{2,P2:2}^{(l₁l₂l₃)} - M_{2,P2:2}^{(l₁l₂l₃)}}{2}) + \Delta \delta m_{4,P2:2}^{(l₁l₂l₃)} (M_{2,P2:2}^{(l₁l₂l₃)} - M_{2,P2:2}^{(l₁l₂l₃)}) - [\Delta B_{4} + 2\Delta L_{4} - 2(\Delta B_{2})^2] M_{2,P2:2}^{(l₁l₂l₃)} - [\Delta B_{4,P2:2}^{(l₁l₂l₃)} + 2\Delta L_{4,P2:2}^{(l₁l₂l₃)} - 4\Delta B_{2} M_{2,P2:2}^{(l₁l₂l₃)}] M_{2}.
\]

Substituting the values listed in Tables IX and XI into Eq. (36), we obtain

\[
a^{(10)}_e[\text{Set III}(aₚ)^{(eem)}] = 0.004 12 (10).
\]

We also obtained

\[
a^{(10)}_e[\text{Set III}(aₚ)^{(emm)}] = 0.000 144 7 (9),
\]

\[
a^{(10)}_e[\text{Set III}(aₚ)^{(eet)}] = 0.000 045 95 (23),
\]

\[
a^{(10)}_e[\text{Set III}(aₚ)^{(emt)}] = 0.000 009 88 (8).
\]
TABLE IX: Contributions of diagrams of Set III(aₕ) containing one electron vacuum-polarization loop and one muon vacuum-polarization loop. n_F is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision.

<table>
<thead>
<tr>
<th>Integral</th>
<th>n_F</th>
<th>Value (Error) including n_F</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>∆M_{6A,P2,P2}^{(eeem)}</td>
<td>30</td>
<td>0.000 541 ( 4)</td>
<td>1 x 10^7</td>
<td>20</td>
</tr>
<tr>
<td>∆M_{6B,P2,P2}^{(eeem)}</td>
<td>30</td>
<td>0.001 434 (17)</td>
<td>1 x 10^7</td>
<td>20</td>
</tr>
<tr>
<td>∆M_{6C,P2,P2}^{(eeem)}</td>
<td>30</td>
<td>0.001 334 (26)</td>
<td>1 x 10^7</td>
<td>20</td>
</tr>
<tr>
<td>∆M_{6D,P2,P2}^{(eeem)}</td>
<td>60</td>
<td>0.000 747 (26)</td>
<td>1 x 10^7</td>
<td>20</td>
</tr>
<tr>
<td>∆M_{6E,P2,P2}^{(eeem)}</td>
<td>30</td>
<td>0.000 494 (10)</td>
<td>1 x 10^7</td>
<td>20</td>
</tr>
<tr>
<td>∆M_{6F,P2,P2}^{(eeem)}</td>
<td>30</td>
<td>0.001 233 (16)</td>
<td>1 x 10^7</td>
<td>20</td>
</tr>
<tr>
<td>∆M_{6G,P2,P2}^{(eeem)}</td>
<td>60</td>
<td>0.002 397 (78)</td>
<td>1 x 10^7</td>
<td>20</td>
</tr>
<tr>
<td>∆M_{6H,P2,P2}^{(eeem)}</td>
<td>30</td>
<td>−0.000 995 (43)</td>
<td>1 x 10^7</td>
<td>20</td>
</tr>
</tbody>
</table>

TABLE X: Contributions of the Set III(aₕ) diagrams to the muon g − 2 evaluated in Version A. Both closed loops are electron loops. n_F is the number of Feynman diagrams represented by the integral. All integrals are evaluated using real(10) arithmetic built in gfortran.

<table>
<thead>
<tr>
<th>Integral</th>
<th>n_F</th>
<th>Value (Error) including n_F</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>∆M_{6A,P2P2}^{(mee)}</td>
<td>15</td>
<td>−38.157 8 (106)</td>
<td>1 x 10^7, 4 x 10^7</td>
<td>20, 80</td>
</tr>
<tr>
<td>∆M_{6B,P2P2}^{(mee)}</td>
<td>15</td>
<td>51.107 5 (103)</td>
<td>1 x 10^7, 4 x 10^7</td>
<td>20, 180</td>
</tr>
<tr>
<td>∆M_{6C,P2P2}^{(mee)}</td>
<td>15</td>
<td>20.547 7 (113)</td>
<td>1 x 10^7, 4 x 10^7</td>
<td>20, 180</td>
</tr>
<tr>
<td>∆M_{6D,P2P2}^{(mee)}</td>
<td>30</td>
<td>31.469 8 (113)</td>
<td>1 x 10^7, 4 x 10^7</td>
<td>20, 280</td>
</tr>
<tr>
<td>∆M_{6E,P2P2}^{(mee)}</td>
<td>15</td>
<td>32.799 0 (89)</td>
<td>1 x 10^7, 4 x 10^7</td>
<td>20, 80</td>
</tr>
<tr>
<td>∆M_{6F,P2P2}^{(mee)}</td>
<td>15</td>
<td>19.014 4 (96)</td>
<td>1 x 10^7, 4 x 10^7</td>
<td>20, 180</td>
</tr>
<tr>
<td>∆M_{6G,P2P2}^{(mee)}</td>
<td>30</td>
<td>61.519 4 (101)</td>
<td>1 x 10^7, 4 x 10^7</td>
<td>20, 180</td>
</tr>
<tr>
<td>∆M_{6H,P2P2}^{(mee)}</td>
<td>15</td>
<td>−55.142 4 (95)</td>
<td>1 x 10^7, 4 x 10^7</td>
<td>20, 280</td>
</tr>
</tbody>
</table>

These results are in good agreement with those of Version B. The contribution of the (ett) term is negligibly small.
TABLE XI: Auxiliary integrals which depend on the mass ratio $m_\mu/m_e$. Those for Set III(a$_s$) are listed on the left side. Those for Set III(a$_s$) are listed on the top half of the right side. Those for Set III(b) are listed on the bottom half of the right side.

<table>
<thead>
<tr>
<th>Integral</th>
<th>Value/Error</th>
<th>Integral</th>
<th>Value/Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta M_{4,P2}^{(em)}$</td>
<td>-0.000 018 9 (1)</td>
<td>$\Delta M_{4,P2:2}^{(em)}$</td>
<td>-0.000 074 7 (1)</td>
</tr>
<tr>
<td>$\Delta M_{4,P2,P2}^{(eem)}$</td>
<td>-0.000 019 1 (1)</td>
<td>$M_{2*,P2:2}^{(eem)}$</td>
<td>0.000 000 9 (0)</td>
</tr>
<tr>
<td>$M_{2,P2}^{(em)}$</td>
<td>0.015 690 0 (16)</td>
<td>$M_{2*,P2:2}^{(eem)}[I]^{(eem)}$</td>
<td>0.000 069 7 (1)</td>
</tr>
<tr>
<td>$M_{2*,P2}^{(em)}$</td>
<td>0.044 089 4 (101)</td>
<td>$\Delta B_{2,P2:2}^{(eem)}$</td>
<td>0.000 036 2 (1)</td>
</tr>
<tr>
<td>$M_{2*,P2}^{(eem)}[I]^{(em)}$</td>
<td>0.010 274 2 (256)</td>
<td>$\Delta B_{1,P2:2}^{(eem)}$</td>
<td>-0.000 409 7 (48)</td>
</tr>
<tr>
<td>$\Delta B_{2,P2}^{(em)}$</td>
<td>0.000 009 4 (00)</td>
<td>$\Delta L_{4,P2:2}^{(eem)}$</td>
<td>0.000 016 0 (59)</td>
</tr>
<tr>
<td>$\Delta B_{4,P2}^{(em)}$</td>
<td>-0.000 091 5 (4)</td>
<td>$\Delta \delta m_{4,P2:2}^{(em)}$</td>
<td>0.001 261 2 (63)</td>
</tr>
<tr>
<td>$\Delta L_{4,P2}^{(em)}$</td>
<td>0.000 012 7 (6)</td>
<td>$\Delta \delta m_{4,P2:2}^{(em)}$</td>
<td>0.000 052 0 (1)</td>
</tr>
<tr>
<td>$\Delta B_{4,P2,P2}^{(eem)}$</td>
<td>-0.000 129 0 (3)</td>
<td>$\Delta M_{4,P4}^{(em)}$</td>
<td>-0.000 068 2 (7)</td>
</tr>
<tr>
<td>$\Delta L_{4,P2,P2}^{(em)}$</td>
<td>-0.000 113 6 (6)</td>
<td>$M_{2*,P4}^{(eem)}$</td>
<td>0.000 005 9 (0)</td>
</tr>
<tr>
<td>$\Delta \delta m_{4,P2}^{(em)}$</td>
<td>0.000 253 9 (5)</td>
<td>$M_{2*,P4}^{(eem)}[I]^{(em)}$</td>
<td>0.000 052 0 (1)</td>
</tr>
<tr>
<td>$\Delta \delta m_{4,P2,P2}^{(em)}$</td>
<td>0.000 195 1 (3)</td>
<td>$\Delta B_{1,P4}^{(eem)}$</td>
<td>-0.000 322 0 (9)</td>
</tr>
<tr>
<td>$\Delta \delta m_{4,P4}^{(eem)}$</td>
<td>0.000 054 2 (13)</td>
<td>$\Delta L_{4,P4}^{(eem)}$</td>
<td>0.000 054 2 (13)</td>
</tr>
<tr>
<td>$\Delta \delta m_{4,P4}^{(em)}$</td>
<td>0.000 878 6 (15)</td>
<td>$\Delta \delta m_{4,P4}^{(em)}$</td>
<td>0.000 878 6 (15)</td>
</tr>
</tbody>
</table>

4. Muon $g-2$. Set III(a$_s$)

The leading contribution to the muon $g-2$ comes from the case where both loops consist of electrons, namely the (mee) case, where $m$ stands for the muon. Results of numerical evaluation in Version A are listed in Table X. From this Table and Table VI we obtain

$$a^{(10)}_\mu[\text{Set III}(a_s)^{(mee)}] = 43.048 \, 8 \, (194).$$

(41)

Next leading term is

$$a^{(10)}_\mu[\text{Set III}(a_s)^{(mem)}] = 12.190 \, 3 \, (176).$$

(42)

We also have

$$a^{(10)}_\mu[\text{Set III}(a_s)^{(met})] = 0.469 \, 43 \, (39),$$

(43)

$$a^{(10)}_\mu[\text{Set III}(a_s)^{(mmnt)}] = 0.150 \, 11 \, (21),$$

(44)

$$a^{(10)}_\mu[\text{Set III}(a_s)^{(mtt)}] = 0.013 \, 68 \, (3).$$

(45)
FIG. 4: Typical tenth-order diagrams of Set III(b) obtained by insertion of a fourth-order vacuum-polarization loop $\Pi_4$ in lepton diagrams of the three-photon-exchange type. Altogether there are 450 diagrams of this type.

These results are in good agreement with those of Version B.

III. SET III(B)

Diagrams belonging to this set are generated by inserting a proper fourth-order vacuum-polarization loop $\Pi_4$ (consisting of three diagrams) in the photon lines of $M_6$. Time-reversal invariance and use of the photon spectral function $\rho_4$ reduce the number of independent integrals from 450 to 8. These integrals are represented by the “self-energy-like” diagrams of Fig. 1. A typical diagram is shown in Fig. 4.

1. Electron $g-2$: Version A

Let $M_{6\alpha,P_4}$ be the magnetic moment projection of the set of diagrams generated from a self-energy diagram $\alpha$ (=A through H) of Fig. 1 by insertion of $\Pi_4$ and an external vertex. The renormalized contribution due to the Set III(b) diagrams can then be written as

$$a_e^{(10)}[\text{Set III(b)} : \text{Ver. A}] = \sum_{\alpha=A}^{H} a_{6\alpha,P_4},$$

with

$$a_{6\alpha,P_4} = \Delta M_{6\alpha,P_4} + \text{residual renormalization terms},$$

where all divergences, except those within $\Pi_4$, are removed by intermediate renormalization by $K_S$ and $I_R$ operations. (See Ref. [32].)

The numerical values of Set III(b) integrals are summarized in Table XII. Numerical values of auxiliary integrals needed to complete the renormalization are listed in Table XIII.
TABLE XII: *Version A* contributions of Set III(b) diagrams to the electron g–2. \( n_F \) is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision.

<table>
<thead>
<tr>
<th>Integral</th>
<th>( n_F )</th>
<th>Value (Error) including ( n_F )</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta M_{6A,P4} )</td>
<td>15</td>
<td>(-1.275 , 23 \ (8))</td>
<td>( 1 \times 10^8, 1 \times 10^9 )</td>
<td>50,50</td>
</tr>
<tr>
<td>( \Delta M_{6B,P4} )</td>
<td>15</td>
<td>(1.865 , 05 \ (14))</td>
<td>( 1 \times 10^8, 1 \times 10^9 )</td>
<td>50,50</td>
</tr>
<tr>
<td>( \Delta M_{6C,P4} )</td>
<td>15</td>
<td>(1.593 , 72 \ (14))</td>
<td>( 1 \times 10^8, 1 \times 10^9 )</td>
<td>50,50</td>
</tr>
<tr>
<td>( \Delta M_{6D,P4} )</td>
<td>30</td>
<td>(1.166 , 99 \ (14))</td>
<td>( 1 \times 10^8, 1 \times 10^9 )</td>
<td>50,50</td>
</tr>
<tr>
<td>( \Delta M_{6E,P4} )</td>
<td>15</td>
<td>(1.212 , 50 \ (6))</td>
<td>( 1 \times 10^8, 1 \times 10^9 )</td>
<td>50,50</td>
</tr>
<tr>
<td>( \Delta M_{6F,P4} )</td>
<td>15</td>
<td>(1.113 , 25 \ (14))</td>
<td>( 1 \times 10^8, 1 \times 10^9 )</td>
<td>50,50</td>
</tr>
<tr>
<td>( \Delta M_{6G,P4} )</td>
<td>30</td>
<td>(2.948 , 70 \ (24))</td>
<td>( 1 \times 10^8, 1 \times 10^9 )</td>
<td>50,50</td>
</tr>
<tr>
<td>( \Delta M_{6H,P4} )</td>
<td>15</td>
<td>(-2.231 , 76 \ (22))</td>
<td>( 1 \times 10^8, 1 \times 10^9 )</td>
<td>50,50</td>
</tr>
</tbody>
</table>

When summed over all the diagrams of Set III(b), the UV- and IR-divergent pieces cancel out and the total contribution to \( a^{(10)} \) can be written as a sum of finite pieces (which is similar to Eq. (5.39) of Ref. [32]):

\[
a_e^{(10)}[\text{Set III(b) : Ver. } A] = \sum_{\alpha=A}^{H} \Delta M_{6\alpha,P4} \\
- 3\Delta B_{2,P4} \Delta M_{4} - 3\Delta B_{2} \Delta M_{4,P4} \\
+ \Delta \delta m_4(M_{2^*,P4}[I] - M_{2^*,P4}) + \Delta \delta m_{4,P4}(M_{2^*}[I] - M_{2^*}) \\
- [\Delta B_{4} + 2\Delta L_{4} - 2(\Delta B_{2})^2]M_{2,P4} \\
- [\Delta B_{4,P4} + 2\Delta L_{4,P4} - 4\Delta B_{2}\Delta B_{2,P4}]M_{2}. \tag{48}
\]

Terms with suffix \( P_4 \) in Eq. (48) are obtained by insertion of \( \Pi_4 \) in the photon lines of diagrams. Note that \( K \)-operation is not applied to \( \Pi_4 \) so that we have \( M_{2,P4} \), instead of \( \Delta M_{2,P4} \), in Eq. (48).

Substituting the values listed in Tables XII and XIII into Eq. (48), we obtain

\[
a_e^{(10)}[\text{Set III(b) : Ver. } A] = 3.327 \, 14 \ (56). \tag{49}
\]
TABLE XIII: Auxiliary integrals for the Set III(b). Some integrals are known exactly. Other integrals are obtained by VEGAS integration.

<table>
<thead>
<tr>
<th>Integral</th>
<th>Value/Error</th>
<th>Integral</th>
<th>Value/Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_2$</td>
<td>0.5</td>
<td>$M_{2,P4}$</td>
<td>0.052 870 652...</td>
</tr>
<tr>
<td>$M_2^*$</td>
<td>1.0</td>
<td>$M_{2^*,P4}$</td>
<td>0.145 597 (21)</td>
</tr>
<tr>
<td>$M_{2^*[I]}$</td>
<td>-1.0</td>
<td>$M_{2^*,P4[I]}$</td>
<td>-0.016 526 (69)</td>
</tr>
<tr>
<td>$\Delta M_4$</td>
<td>0.030 833 612...</td>
<td>$\Delta M_{4,P4}$</td>
<td>-0.288 997 (12)</td>
</tr>
<tr>
<td>$\Delta \delta m_4$</td>
<td>1.906 340 (21)</td>
<td>$\Delta \delta m_{4,P4}$</td>
<td>1.773 79 (26)</td>
</tr>
<tr>
<td>$\Delta B_2$</td>
<td>0.75</td>
<td>$\Delta B_{2,P4}$</td>
<td>0.183 666 8 (18)</td>
</tr>
<tr>
<td>$\Delta B_4$</td>
<td>-0.437 094 (21)</td>
<td>$\Delta B_{4,P4}$</td>
<td>-0.816 23 (25)</td>
</tr>
<tr>
<td>$\Delta L_4$</td>
<td>0.465 024 (17)</td>
<td>$\Delta L_{4,P4}$</td>
<td>0.559 72 (25)</td>
</tr>
</tbody>
</table>

2. $\text{Electron } g - 2$: $\text{Version } B$

Let us now treat Set III(b) by the method based on the automated code generation scheme. In this approach, the contribution from Set III(b) is expressed as

$$a_e^{(10)}[\text{Set III(b) : Ver. } B] = \sum_{\alpha=A}^{H} \Delta M_{6\alpha,P4}^{(B)}$$

$$- 3\Delta B_2 \Delta M_{4,P4} - 3\Delta B_{2,P4} \Delta M_4$$

$$- (\Delta B_4 + \Delta L_4 - 2(\Delta B_2)^2) M_{2,P4}$$

$$- (\Delta B_{4,P4} + \Delta L_{4,P4} - 4\Delta B_2 \Delta B_{2,P4}) M_2,$$  \hspace{1cm} (50)
\[
\begin{align*}
\Delta M_{6A, P4}^{(B)} &= \Delta M_{6A, P4}^{(A)} - 2\Delta L_{4b,1} M_{2, P4} - 2\Delta L_{4b,1, P4} M_2, \\
\Delta M_{6B, P4}^{(B)} &= \Delta M_{6B, P4}^{(A)} - \Delta L_{4b,2} M_{2, P4} - \Delta L_{4b,2, P4} M_2 \\
& \quad - \Delta \delta m_{4b}(M_{2^* P4} - M_{2^* [I]}), \\
\Delta M_{6C, P4}^{(B)} &= \Delta M_{6C, P4}^{(A)} - \Delta \delta m_{4a}(M_{2^* P4} - M_{2^* [I]}), \\
& \quad - \Delta \delta m_{4a, P4}(M_{2^*} - M_{2^* [I]}), \\
\Delta M_{6D, P4}^{(B)} &= \Delta M_{6D, P4}^{(A)} - 2\Delta L_{4a,1} M_{2, P4} - 2\Delta L_{4a,1, P4} M_2, \\
\Delta M_{6E, P4}^{(B)} &= \Delta M_{6E, P4}^{(A)} - \Delta L_{4b,2} M_{2, P4} - \Delta L_{4b,2, P4} M_2, \\
\Delta M_{6F, P4}^{(B)} &= \Delta M_{6F, P4}^{(A)}, \\
\Delta M_{6G, P4}^{(B)} &= \Delta M_{6G, P4}^{(A)}, \\
\Delta M_{6H, P4}^{(B)} &= \Delta M_{6H, P4}^{(A)}. \\
\end{align*}
\]

Using the code generator we obtained the programs of the magnetic moments \(M_{6\alpha, P4}\), \(\alpha = A, \ldots, H\), and \(M_{4\alpha, P4}\), \(\alpha = A, B\). The programs for the renormalization constants \(L_{4\alpha, P4}, L_{4\alpha, P4}, B_{4\alpha, P4}, B_{4\alpha}, \delta m_{4\alpha, P4}, \delta m_{4\alpha}\) are also automatically generated. Other quantities, \(\Delta B_{2, P4}, M_{2, P4}\) are very simple so that they are calculated by using hand-written programs. The values of \(\Delta B_2\) and \(M_2\) are analytically known.

The results of numerical integration by VEGAS are shown in Table XIV.

Substituting the numbers shown in Tables XIV and XIII into Eq. (50), we obtain

\[
a_e^{(10)}[\text{Set III(b) : Ver. B}] = 3.327 07 (78)
\]

in good agreement with (49), where the uncertainty is from the numerical integration only.
TABLE XIV: Version B contributions of Set III(b) diagrams to the electron $g-2$. Programs are created by GencodeN. $n_F$ is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision.

<table>
<thead>
<tr>
<th>Integral</th>
<th>$n_F$</th>
<th>Value (Error) including $n_F$</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta M_{6A,P4}$</td>
<td>15</td>
<td>$-1.673 94 (34)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50, 50</td>
</tr>
<tr>
<td>$\Delta M_{6B,P4}$</td>
<td>15</td>
<td>$-0.959 45 (24)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50, 50</td>
</tr>
<tr>
<td>$\Delta M_{6C,P4}$</td>
<td>15</td>
<td>$0.427 34 (21)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50, 50</td>
</tr>
<tr>
<td>$\Delta M_{6D,P4}$</td>
<td>30</td>
<td>$1.110 10 (32)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50, 50</td>
</tr>
<tr>
<td>$\Delta M_{6E,P4}$</td>
<td>15</td>
<td>$1.498 03 (18)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50, 50</td>
</tr>
<tr>
<td>$\Delta M_{6F,P4}$</td>
<td>15</td>
<td>$1.113 12 (20)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50, 50</td>
</tr>
<tr>
<td>$\Delta M_{6G,P4}$</td>
<td>30</td>
<td>$2.947 48 (36)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50, 50</td>
</tr>
<tr>
<td>$\Delta M_{6H,P4}$</td>
<td>15</td>
<td>$-2.231 66 (28)$</td>
<td>$1 \times 10^8, 1 \times 10^9$</td>
<td>50, 50</td>
</tr>
</tbody>
</table>

3. Mass-dependent terms $A_2$ of Set III(b)

The residual renormalization scheme (in Version A) for the $(em)$ term is the following:

$$a_e^{(10)}[\text{Set III(b)}^{(em)}] = \sum_{\alpha=A}^H \Delta M_{6\alpha,P4}^{(em)}$$

$$- 3\Delta B_{2,P4}^{(em)} \Delta M_4 - 3\Delta B_2 \Delta M_{4,P4}^{(em)}$$

$$+ \Delta \delta m_4 (M_{2',P4}^{(em)} - M_2^{(em)}) + \Delta \delta m_4^{(em)} (M_{2',[1]} - M_{2'})$$

$$- [\Delta B_4 + 2\Delta L_4 - 2(\Delta B_2)^2] M_{2,P4}^{(em)}$$

$$- [\Delta B_{4,P4}^{(em)} + 2\Delta L_{4,P4}^{(em)} - 4\Delta B_2 \Delta B_{2,P4}^{(em)}] M_2.$$

(53)

Substituting the values listed in Tables XI and XV into Eq. (53), we obtain

$$a_e^{(10)}[\text{Set III(b)}^{(em)}] = 0.002 794 \ (1).$$

(54)

It is easy to obtain the contribution of tau lepton loop instead of the muon loop. We have simply to replace the muon mass by the tau mass in the FORTRAN programs. For instance a crude calculation in Version A yields

$$a_e^{(10)}[\text{Set III(b)}^{(et)}] = 0.000 021 42 \ (1),$$

(55)

which is two orders of magnitude smaller than (54).
Table XV: Contributions of \((em)\) diagrams of Fig. 2 to the Set III(b). \(n_F\) is the number of Feynman diagrams represented by the integral. All integrals are evaluated in double precision.

<table>
<thead>
<tr>
<th>Integral</th>
<th>(n_F)</th>
<th>Value (Error) including (n_F)</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta M^{(em)}_{6A,P4})</td>
<td>15</td>
<td>(-0.000\ 389\ (2))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M^{(em)}_{6B,P4})</td>
<td>15</td>
<td>(0.000\ 943\ (6))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M^{(em)}_{6C,P4})</td>
<td>15</td>
<td>(0.000\ 925\ (6))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M^{(em)}_{6D,P4})</td>
<td>30</td>
<td>(0.000\ 592\ (4))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M^{(em)}_{6E,P4})</td>
<td>15</td>
<td>(0.000\ 371\ (4))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
<tr>
<td>(\Delta M^{(em)}_{6F,P4})</td>
<td>15</td>
<td>(0.000\ 773\ (7))</td>
<td>(1 \times 10^7)</td>
<td>40</td>
</tr>
<tr>
<td>(\Delta M^{(em)}_{6G,P4})</td>
<td>30</td>
<td>(0.001\ 626\ (26))</td>
<td>(1 \times 10^7)</td>
<td>60</td>
</tr>
<tr>
<td>(\Delta M^{(em)}_{6H,P4})</td>
<td>15</td>
<td>(-0.000\ 751\ (20))</td>
<td>(1 \times 10^7)</td>
<td>20</td>
</tr>
</tbody>
</table>

4. Muon \(g-2\). Set III(b)

The leading contribution to the muon \(g-2\) comes from the case containing an electron loop, namely the \((me)\) case, where \(m\) stands for the muon. Results of numerical evaluation (Version A) are listed in Table XVI. From this Table and Table XVII we obtain

\[
a^{(10)}_\mu [\text{Set III(b)}^{(me)}] = 11.936\ 7\ (45). \tag{56}
\]

We also obtained (Version A)

\[
a^{(10)}_\mu [\text{Set III(b)}^{(mt)}] = 0.143\ 60\ (1). \tag{57}
\]

These results are confirmed by Version B calculation.

IV. DISCUSSION

As was noted earlier Version A and Version B differ in the treatment of self-energy subtraction and IR divergence. Furthermore, the actual algebraic form of integrands in the first method [29] is quite different from the second one because ”Kirchhoff’s laws” satisfied by the scalar currents [32] were used extensively to make the integrand as compact as possible to save the computing time. Thus the two calculations can be regarded as independent
TABLE XVI: Contributions of \((me)\)-type diagrams of Set III(b) to the muon \(g - 2\). \(n_F\) is the number of Feynman diagrams represented by the integral. All integrals are evaluated using the real(10) arithmetic built in \textit{gfortran}.

<table>
<thead>
<tr>
<th>Integral</th>
<th>(n_F)</th>
<th>Value (Error) including (n_F)</th>
<th>Sampling per iteration</th>
<th>No. of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta M_{6A,P4}^{(me)})</td>
<td>15</td>
<td>(-16.0005) (53)</td>
<td>(1 \times 10^7)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6B,P4}^{(me)})</td>
<td>15</td>
<td>(25.3670) (61)</td>
<td>(1 \times 10^7)</td>
<td>900</td>
</tr>
<tr>
<td>(\Delta M_{6C,P4}^{(me)})</td>
<td>15</td>
<td>(2.8871) (61)</td>
<td>(1 \times 10^7)</td>
<td>900</td>
</tr>
<tr>
<td>(\Delta M_{6D,P4}^{(me)})</td>
<td>30</td>
<td>(12.8166) (79)</td>
<td>(1 \times 10^7)</td>
<td>1000</td>
</tr>
<tr>
<td>(\Delta M_{6E,P4}^{(me)})</td>
<td>15</td>
<td>(13.6292) (44)</td>
<td>(1 \times 10^7)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6F,P4}^{(me)})</td>
<td>15</td>
<td>(6.9195) (59)</td>
<td>(1 \times 10^7)</td>
<td>600</td>
</tr>
<tr>
<td>(\Delta M_{6G,P4}^{(me)})</td>
<td>30</td>
<td>(24.6728) (59)</td>
<td>(1 \times 10^7)</td>
<td>700</td>
</tr>
<tr>
<td>(\Delta M_{6H,P4}^{(me)})</td>
<td>15</td>
<td>(-23.0066) (57)</td>
<td>(1 \times 10^7)</td>
<td>800</td>
</tr>
</tbody>
</table>

TABLE XVII: Auxiliary integrals for the Set III(b)\(^{(me)}\). Some integrals are known exactly. Other integrals are obtained by VEGAS integration.

<table>
<thead>
<tr>
<th>Integral</th>
<th>Value(Err.)</th>
<th>Integral</th>
<th>Value(Err.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_2)</td>
<td>0.5</td>
<td>(M_{2,P4}^{(me)})</td>
<td>1.493 651 (84)</td>
</tr>
<tr>
<td>(M_2^*)</td>
<td>1.0</td>
<td>(M_{2*,P4}^{(me)})</td>
<td>3.122 88 (16)</td>
</tr>
<tr>
<td>(M_2^*[I])</td>
<td>(-1.0)</td>
<td>(M_{2*,P4}^{(me)}[I])</td>
<td>(-2.996 76 (24))</td>
</tr>
<tr>
<td>(\Delta M_4)</td>
<td>0.030 833 612...</td>
<td>(\Delta M_{4,P4}^{(me)})</td>
<td>(-0.438 76 (26))</td>
</tr>
<tr>
<td>(\Delta \delta m_4)</td>
<td>1.906 340 (21)</td>
<td>(\Delta \delta m_{4,P4}^{(me)})</td>
<td>13.651 22 (80)</td>
</tr>
<tr>
<td>(\Delta B_2)</td>
<td>0.75</td>
<td>(\Delta B_{2,P4}^{(me)})</td>
<td>2.439 109 (53)</td>
</tr>
<tr>
<td>(\Delta B_4)</td>
<td>(-0.437 094) (21)</td>
<td>(\Delta B_{3,P4}^{(me)})</td>
<td>(-3.826 32 (71))</td>
</tr>
<tr>
<td>(\Delta L_4)</td>
<td>0.465 024 (17)</td>
<td>(\Delta L_{4,P4}^{(me)})</td>
<td>3.653 31 (42)</td>
</tr>
</tbody>
</table>

of each other and the results agree within their error bars. Thus they may be combined statistically to yield the values listed below:

\[
\alpha_e^{(10)}\[\text{Set III(a)}_d\] = 0.9419 (1),
\]

\[
\alpha_e^{(10)}\[\text{Set III(a)}_s\] = 1.1854 (2),
\]

\[
\alpha_e^{(10)}\[\text{Set III(b)}\] = 3.3271 (5).
\]
The mass-dependent contribution of Set III\( (a_d) \) to the electron \( g-2 \), the sum of (22), (23), (24), and (25), is given by

\[
a^{(10)}_e[\text{Set III}(a_d)\text{(mass-dep})] = 0.00328 (1),
\]
while the mass-dependent contributions of Set III\( (a_s) \) to \( a_e \) is the sum of (37), (38), (39), and (40):

\[
a^{(10)}_e[\text{Set III}(a_s)\text{(mass-dep})] = 0.00432 (10).
\]

The total contribution of Set III\( (a) \) to \( a_e \) is the sum of (58), (59), (61), (62):

\[
a^{(10)}_e[\text{Set III}(a)\text{ (all terms})] = 2.1349 (2).
\]

Similarly, from (54), (55), and (60) we obtain

\[
a^{(10)}_e[\text{Set III}(b)\text{ (all terms})] = 3.3299 (5).
\]

The total contribution of Set III\( (a_d) \) to the muon \( g-2 \), the sum of (26), (27), (28), (29), (30), and (58), is

\[
a^{(10)}_\mu[\text{Set III}(a_d)\text{(all terms})] = 55.360 (19),
\]
while the total contribution of Set III\( (a_s) \) to the muon \( g-2 \), the sum of (41), (42), (43), (44), (45), and (59), is

\[
a^{(10)}_\mu[\text{Set III}(a_s)\text{(all terms})] = 57.058 (26).
\]

The total contribution of Set III\( (a) \) to the muon \( g-2 \), the sum of (65) and (66), is thus

\[
a^{(10)}_\mu[\text{Set III}(a)\text{(all terms})] = 112.418 (32).
\]

The total contribution of Set III\( (b) \) to the muon \( g-2 \), from (56), (57), and (60), is

\[
a^{(10)}_\mu[\text{Set III}(b)\text{(all terms})] = 15.4074 (45).
\]

The contribution of Set III\( (a) \) to the muon \( g-2 \) is very large, which is not unexpected. In particular, the orders of magnitude of contributions from the dominant \( (mee) \) terms of Set III\( (a_d) \) and Set III\( (a_s) \), as well as the \( (me) \) term of Set III\( (b) \), can be estimated crudely since their leading \( \log(m_\mu/m_e) \) term is determined by the renormalization procedure [3, 40]:

\[
\begin{align*}
a^{(10)}_\mu[\text{Set III}(a_d)\text{(mee})] & \sim 3K^2_2a^{(6)}_e\text{(no loop}) \sim 34, \\
a^{(10)}_\mu[\text{Set III}(a_s)\text{(mee})] & \sim 3K^2_2a^{(6)}_e\text{(no loop}) \sim 34, \\
a^{(10)}_\mu[\text{Set III}(b)\text{(me})] & \sim 3K_4a^{(6)}_e\text{(no loop}) \sim 7,
\end{align*}
\]
with

\begin{align*}
K_2 & \sim \frac{2}{3} \ln(m_\mu/m_e) - \ldots, \\
K_{2,2} & \sim K_2^2, \\
K_4 & \sim \frac{1}{2} \ln(m_\mu/m_e) - \ldots,
\end{align*}

(70)

and [30]

\[ a_e^{(6)}(\text{no loop}) = 0.904979 \ldots, \]

(71)

where no loop means diagrams without closed lepton loops of vacuum-polarization type. The factor 3 accounts for the increase in the number of diagrams caused by insertion of vacuum-polarization loops. As is expected from (69) and (70), the values of (26) and (41) are of the same order of magnitude.

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