Central charges of para-Liouville and Toda theories from M5-branes

Tatsuma Nishioka and Yuji Tachikawa

Para-Liouville/Toda central charges from M5-branes

Tatsuma Nishioka$^\heartsuit$ and Yuji Tachikawa$^{\clubsuit *}$

$^\heartsuit$ Department of Physics, Princeton University, Princeton, NJ 08544, USA
$^\clubsuit$ School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA

abstract

We propose that N M5-branes, put on $\mathbb{R}^4/\mathbb{Z}_m$ with deformation parameters $\epsilon_{1,2}$, realize two-dimensional theory with $\widehat{\mathrm{SU}}(m)_N$ symmetry and m-th para-W_N symmetry. This includes the standard W_N symmetry for $m = 1$ and super-Viraroro symmetry for $m = N = 2$. We provide a small check of this proposal by calculating the central charge of the 2d theory from the anomaly polynomial of the 6d theory.

---

*on leave from IPMU, the University of Tokyo

Introduction: N M5-branes, put on $\mathbb{R}^4$ with Nekrasov's deformation parameters $\epsilon_{1,2}$, are now believed to realize two-dimensional theory with W_N symmetry [1, 2]. One check of this statement was given in [3] following the observation made in [4]. Namely, the equivariant integral on $\mathbb{R}^4$ of the anomaly polynomial of N M5-branes determined in [5] gives the anomaly polynomial of the 2d theory, from which the central charge of the Toda theory with W_N symmetry can be reproduced. The same analysis can be performed for 6d $\mathcal{N} = (2,0)$ theory of type $G = A, D, E$ whose anomaly polynomial is also known [6, 7]; and it correctly reproduces the central charge of the Toda theory of type G.

In a recent paper [8], it was proposed that two M5-branes on $\mathbb{R}^4/\mathbb{Z}_2$ with deformations $\epsilon_{1,2}$ give rise to a system with the $\widehat{\mathrm{SU}}(2)_2$ symmetry and the super-Virasoro symmetry. As a generalization, we propose that N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ realize a 2d system with a free boson, $\widehat{\mathrm{SU}}(m)_N$, and the m-th para-W_N symmetry[1]. Here $\mathbb{Z}_m$ acts as $(z, w) \mapsto (e^{2\pi i/m}z, e^{-2\pi i/m}w)$ on $(z, w) \in \mathbb{C}^2 \simeq \mathbb{R}^4$. We give a small piece of supporting evidence by calculating the central charge from the 6d anomaly polynomial. We will also speculate what happens if the 6d $\mathcal{N} = (2,0)$ theory of type G is used instead. In the following G stands for one of A_n, D_n or E_n; r_G, h_G and d_G are the rank, the (dual) Coxeter number and the dimension of G, respectively. They satisfy $d_G = r_G(h_G + 1)$.

Para-W symmetry and para-Toda theory: One way to realize the $W(\widehat{G})$ symmetry [10, 11, 12, 13, 14, 15] is to consider the chiral algebra of the coset

$$\widehat{G}_k \times \widehat{G}_m/\widehat{G}_{k+m} , \tag{1}$$

for $m = 1$. The m-th para-$W(\widehat{G})$ symmetry is obtained by taking an arbitrary positive integer m in this coset; in particular, it reduces to the super-Virasoro algebra when $m = N = 2$. Generalization of NSR superstrings using these algebras were explored e.g. in [16, 17].

The m-th para-$W(\widehat{G})$ algebra is the symmetry of the m-th para-Toda model of type G, which has the following action [18]

$$S = S\left(\frac{\widehat{G}_m}{\widehat{\mathrm{U}}(1)^{r_G}}\right) + \int d^2x \left[\partial_\mu \Phi \partial_\mu \Phi + \sum_{i=1}^{r_G} \Psi_i \bar{\Psi}_i \exp\left(\frac{b}{\sqrt{m}}\alpha_i \cdot \Phi\right)\right] . \tag{2}$$

Here, $\widehat{G}_m/\widehat{\mathrm{U}}(1)^{r_G}$ describes the generalized parafermions Ψ_i of type G [19], α_i are simple roots of G, Φ are r_G free bosons with background charge $(b + 1/b)\rho/\sqrt{m}$ with the Weyl

---
[1]See also [9] for a related work.

vector ρ. The central charge is given by

$$c = c\left(\frac{\widehat{G}_m}{\widehat{U}(1)^{r_G}}\right) + r_G + \frac{h_G d_G}{m}\left(b + \frac{1}{b}\right)^2 = \frac{m d_G}{m + h_G} + \frac{h_G d_G}{m}\left(b + \frac{1}{b}\right)^2 . \qquad (3)$$

Note that the parafermion Ψ_i has dimension $1 - 1/m$, and the exponential of bosons has dimension $1/m$, so that the interaction terms are marginal. For $m = 1$ this is the usual affine Toda theory, and for $m = N = 2$ this is the $\mathcal{N} = 1$ super-Liouville theory.

N **M5-branes on** $\mathbb{R}^4/\mathbb{Z}_m$: The anomaly polynomial I_8 of 6d $\mathcal{N} = (2,0)$ is given by the general form

$$I_8 = \mathcal{A}\, I_8(1) + \mathcal{B}\, p_2(NW)/24 , \qquad (4)$$

where $I_8(1)$ is the anomaly polynomial of a single M5-brane, NW is the normal bundle to the worldvolume W of the 6d theory, and $\mathcal{A}, \mathcal{B}$ are integers determined by the type of the 6d theory. For N M5-branes, $\mathcal{A} = N$ and $\mathcal{B} = N^3 - N$. When compactified on a four-manifold X_4 with a suitable twist, it was determined in [3] that the resulting 2d theory has the central charge

$$c = \chi(X_4)\mathcal{A} + (P_1(X_4) + 2\chi(X_4))\mathcal{B} . \qquad (5)$$

Here $\chi(X_4)$ and $P_1(X_4)$ are the Euler number and three times the signature of X_4, respectively. We let X_4 be $\mathbb{R}^4/\mathbb{Z}_m$ with the deformations $\epsilon_{1,2}$. Then χ and P_1 are to be taken in the equivariant sense, and are given by[2]

$$\chi(X_4) = m , \qquad P_1(X_4) = \frac{1}{m}\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} - 2m . \qquad (6)$$

Therefore, N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ give rise to a 2d system with the central charge

$$c = Nm + \frac{N^3 - N}{m}\left(b + \frac{1}{b}\right)^2 , \qquad (7)$$

where we used the standard identification $\epsilon_1/\epsilon_2 = b^2$.

According to the general lore, the Hilbert space of the 2d theory comes from the BPS states of the supersymmetric quantum mechanics on the moduli space of instantons on

---

[2]The calculation is done as follows. Let us parameterize $\mathbb{R}^4 \simeq \mathbb{C}^2$ by (z, w), on which two rotations act via $(z, w) \mapsto (e^{\epsilon_1}z, e^{\epsilon_2}w)$. On the blowup of $\mathbb{R}^4/\mathbb{Z}_m$, we have m fixed points of $U(1)^2$ actions, whose local coordinates are given by $(z_j, w_j) = (z^{m-j+1}w^{1-j}, z^{j-m}w^j)$ for $j = 1, \ldots, m$. Let us define $\epsilon_{1,2}(j)$ by the $U(1)^2$ action at the fixed points: $(z_j, w_j) \mapsto (e^{\epsilon_1(j)}z_j, e^{\epsilon_2(j)}w_j)$. Then the topological numbers are given by the fixed point formula: $\chi = \sum_j 1$, and $P_1 = \sum_j (\epsilon_1(j)^2 + \epsilon_2(j)^2)/(\epsilon_1(j)\epsilon_2(j))$.

X_4. In our case, we consider $U(N)$ instantons on $\mathbb{R}^4/\mathbb{Z}_m$, for which it is known that there is an action of a free boson and of $\widehat{SU}(m)_N$; this was found in [20, 21] and string theory interpretation was later given e.g. in [22]. Then, the central charge (7) needs to be subdivided to

$$c = 1 + c\left(\widehat{SU}(m)_N\right) + \left[\frac{m(N^2-1)}{m+N} + \frac{N^3-N}{m}\left(b+\frac{1}{b}\right)^2\right] . \tag{8}$$

The third term is the central charge of the m-th para-Toda theory of type $SU(N)$ we found in Eq. (3).

We interpret this calculation as a check to the proposal that N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ with deformations $\epsilon_{1,2}$ give rise to a 2d system with actions of a free boson, $\widehat{SU}(m)_N$ and m-th para-W_N algebra with central charge (3). This statement reduces to the now-standard relations in [1, 2] when $m = 1$, and to the proposal in [8] when $m = N = 2$.

Speculation concerning 6d theory of general type on $\mathbb{R}^4/\mathbb{Z}_m$: First let us consider what happens if we start from 6d $\mathcal{N} = (2,0)$ theory of type A_{N-1}, instead of N M5-branes. One needs to decouple the center-of-mass mode, which changes $\mathcal{A}, \mathcal{B}$ in Eqs. (4), (5) to $\mathcal{A} = N - 1$ and $\mathcal{B} = N^3 - N$. The resulting 2d theory has the central charge of the form

$$c = c\left(\frac{\widehat{SU}(m)_N}{\widehat{U}(1)^{m-1}}\right) + \left[c\left(\frac{\widehat{SU}(N)_m}{\widehat{U}(1)^{N-1}}\right) + (N-1) + \frac{N^3-N}{m}\left(b+\frac{1}{b}\right)^2\right] . \tag{9}$$

We see two cosets realizing generalized parafermions, known also as $\mathrm{RCFT}[A_{m-1}, A_{N-1}]$ and $\mathrm{RCFT}[A_{N-1}, A_{m-1}]$ in the terminology of [23], respectively. In general, $\mathrm{RCFT}[\Gamma, G]$ for $\Gamma, G = A, D, E$ is a rational CFT with central charge

$$c\left(\mathrm{RCFT}[\Gamma, G]\right) = \frac{h_\Gamma r_\Gamma r_G}{h_\Gamma + h_G} . \tag{10}$$

Note that $\mathrm{RCFT}[G, A_{m-1}]$ is the generalized parafermion of type G, but that $\mathrm{RCFT}[\Gamma, G]$ is not yet constructed for general pair of G, Γ. We have

$$c\left(\mathrm{RCFT}[\Gamma, G]\right) + c\left(\mathrm{RCFT}[G, \Gamma]\right) = r_\Gamma r_G , \tag{11}$$

and it is believed they comprise a 'level-rank-dual' pair of RCFTs.

The 6d theory of type G has $\mathcal{A} = r_G$ and $\mathcal{B} = d_G h_G$ in Eqs. (4), (5) [6, 7]. Using $d_G = r_G(h_G + 1)$, we find that the 6d theory on $\mathbb{R}^4/\mathbb{Z}_m$ has the central charge

$$c = c\left(\mathrm{RCFT}[A_{m-1}, G]\right) + \left[c(\mathrm{RCFT}[G, A_{m-1}]) + r_G + \frac{d_G h_G}{m}\left(b+\frac{1}{b}\right)^2\right] , \tag{12}$$

3

where the second term is the central charge of the m-th para-Toda theory of type G we saw in Eq. (3). This suggests that the resulting 2d theory has the symmetry RCFT$[A_{m-1}, G]$ and the m-th para-$W(\widehat{G})$ symmetry. Note that RCFT$[A_{m-1}, G]$ is not yet constructed when $G \neq A$.

We can further generalize the system by considering instantons of gauge group $G = A, D, E$ on the ALE orbifold of type Γ. Nekrasov's deformation cannot be performed when $\Gamma \neq A$, because the ALE orbifold of type D and E does not have U$(1)^2$ isometry. We can still expect that this construction might naturally give us RCFT$[\Gamma, G]$+RCFT$[G, \Gamma]$. The symmetry under G and Γ can be understood once one realizes that the 6d theory of type G is Type IIB string on the ALE orbifold of type G. Then, the 2d system is Type IIB string on the ALE orbifold of type G times the ALE orbifold of type Γ, which is manifestly symmetric under the exchange of G and Γ.

References

[1] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-Dimensional Gauge Theories," *Lett. Math. Phys.* **91** (2010) 167–197, `arXiv:0906.3219 [hep-th]`.

[2] N. Wyllard, "A_{N-1} Conformal Toda Field Theory Correlation Functions from Conformal $\mathcal{N} = 2$ $SU(N)$ Quiver Gauge Theories," *JHEP* **11** (2009) 002, `arXiv:0907.2189 [hep-th]`.

[3] L. F. Alday, F. Benini, and Y. Tachikawa, "Liouville/Toda Central Charges from M5-Branes," *Phys. Rev. Lett.* **105** (2010) 141601, `arXiv:0909.4776 [hep-th]`.

[4] G. Bonelli and A. Tanzini, "Hitchin Systems, $\mathcal{N} = 2$ Gauge Theories and W-Gravity," *Phys. Lett.* **B691** (2010) 111–115, `arXiv:0909.4031 [hep-th]`.

[5] J. A. Harvey, R. Minasian, and G. W. Moore, "Non-Abelian Tensor-Multiplet Anomalies," *JHEP* **09** (1998) 004, `arXiv:hep-th/9808060`.

[6] K. A. Intriligator, "Anomaly Matching and a Hopf-Wess-Zumino Term in 6D, $N = (2,0)$ Field Theories," *Nucl. Phys.* **B581** (2000) 257–273, `arXiv:hep-th/0001205`.

[7] P. Yi, "Anomaly of (2,0) Theories," *Phys. Rev.* **D64** (2001) 106006, `arXiv:hep-th/0106165`.

[8] V. Belavin and B. Feigin, "Super Liouville Conformal Blocks from $\mathcal{N} = 2\ SU(2)$ Quiver Gauge Theories," `arXiv:1105.5800 [hep-th]`.

[9] T. Kimura, "Matrix Model from $\mathcal{N} = 2$ Orbifold Partition Function," `arXiv:1105.6091 [hep-th]`.

[10] P. Christe and F. Ravanini, "$G_N \times G_L/G_{N+L}$ Conformal Field Theories and Their Modular Invariant Partition Functions," *Int. J. Mod. Phys.* **A4** (1989) 897.

[11] P. Bowcock and P. Goddard, "Coset Constructions and Extended Conformal Algebras," *Nucl. Phys.* **B305** (1988) 685.

[12] S. Nam, "Perturbed $c > 1$ Conformal Field Theories and Generalized Toda Field Theories," *Phys. Lett.* **B243** (1990) 231–236.

[13] C. Ahn, D. Bernard, and A. LeClair, "Fractional Supersymmetries in Perturbed Coset CFTs and Integrable Soliton Theory," *Nucl. Phys.* **B346** (1990) 409–439.

[14] D. Bernard and A. LeClair, "The Fractional Supersymmetric Sine-Gordon Models," *Phys. Lett.* **B247** (1990) 309–316.

[15] D. Nemeschansky and N. P. Warner, "Topological Matter, Integrable Models and Fusion Rings," *Nucl. Phys.* **B380** (1992) 241–264, `arXiv:hep-th/9110055`.

[16] P. C. Argyres, A. LeClair, and S. H. H. Tye, "On the Possibility of Fractional Superstrings," *Phys. Lett.* **B253** (1991) 306–312.

[17] H. Irie, "Fractional Supersymmetric Liouville Theory and the Multi-Cut Matrix Models," *Nucl. Phys.* **B819** (2009) 351–374, `arXiv:0902.1676 [hep-th]`.

[18] A. LeClair, D. Nemeschansky, and N. P. Warner, "S Matrices for Perturbed $\mathcal{N} = 2$ Superconformal Field Theory from Quantum Groups," *Nucl. Phys.* **B390** (1993) 653–680, `arXiv:hep-th/9206041`.

[19] D. Gepner, "New Conformal Field Theories Associated with Lie Algebras and Their Partition Functions," *Nucl. Phys.* **B290** (1987) 10.

[20] H. Nakajima, "Instantons on ALE Spaces, Quiver Varieties and Kac-Moody Algebras," *Duke Math. Journal* **76** (1994) 365.

[21] H. Nakajima, "Heisenberg Algebra and Hilbert Schemes of Points on Projective Surfaces," *Annals of Mathematics* **145** (1997) 379, `alg-geom/9507012`.

[22] R. Dijkgraaf, L. Hollands, P. Sułkowski, and C. Vafa, "Supersymmetric Gauge Theories, Intersecting Branes and Free Fermions," *JHEP* **02** (2008) 106, `arXiv:0709.4446 [hep-th]`.

[23] S. Cecotti, A. Neitzke, and C. Vafa, "R-Twisting and 4D/2D Correspondences," `arXiv:1006.3435 [hep-th]`.