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Phys. Rev. D **84**, 046006 — Published 11 August 2011

DOI: [10.1103/PhysRevD.84.046006](https://doi.org/10.1103/PhysRevD.84.046006)

Analytic Non-integrability in String Theory

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Using analytic techniques developed for Hamiltonian dynamical systems we show that a certain classical string configurations in $AdS_5 \times X_5$ with X_5 in a large class of Einstein spaces, is non-integrable. This answers the question of integrability of string on such backgrounds in the negative. We consider a string localized in the center of AdS_5 that winds around two circles in the manifold X_5 .

PACS numbers: 11.25.Tq, 05.45.-a, 11.30.Na

Introduction. – Chaotic motion has been one of the most studied aspects of nonlinear dynamical systems as its application extend to many areas [1]. Although its mathematical roots date back to Poincaré and the three-body problem, it was really during the last part of the XX century when its study flourished largely thanks to new advances in computing power. Naturally, under the shadow of quantum mechanics it is logical to try to understand the quantum properties in systems whose classical limit is chaotic, this area has become known as quantum chaos [2]. In the context of the AdS/CFT correspondence [3], there is a particularly special chance to understand some of these questions as we have a setting in which the classical regime of a theory is dual to the highly quantum regime of another. Understanding classical chaos and the corresponding quantization in the context of string theory provides a new framework with enhanced interpretational opportunities.

The simplest version of the AdS/CFT correspondence [3] states a complete equivalence between strings on $AdS_5 \times S^5$ with Ramond-Ramond fluxes and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) with gauge group $SU(N)$. Chaotic behavior of some classical configurations of strings in the context of the AdS/CFT has been recently established for several interesting string theory backgrounds: ring strings in the Schwarzschild black hole in asymptotically AdS_5 backgrounds [4], strings in the AdS soliton background [5] and in $AdS_5 \times T^{1,1}$ [6] which is a coset but not a maximally symmetric one.

In general the question of integrability is settled through a numerical analysis of the system [1]. Over the last decades an analytical approach has been developed to determine whether a system is integrable. Some powerful results due to Ziglin [7, 8] and further refined by Morales-Ruiz and Ramis [9] turn the question of integrability of some simple systems into an algorithmic process. In this paper we study a large class of systems that appear in string theory. We generalize some of our

previous results for classical strings on $AdS_5 \times T^{1,1}$ [6] to include more general backgrounds of the form $AdS_5 \times X_5$, where X_5 is in a general class of five-dimensional Einstein spaces admitting a $U(1)$ fibration.

Analytic Non-integrability. – The general basis for proving nonintegrability of a system of differential equations $\dot{\vec{x}} = \vec{f}(\vec{x})$ is the analysis of the variational equation around a particular solution $\bar{x} = \bar{x}(t)$ [9, 10]. The variational equation around $\bar{x}(t)$ is a linear system obtained by linearizing the vector field around $\bar{x}(t)$. If the nonlinear system admits some first integrals so does the variational equation. Thus, proving that the variational equation does not admit any first integral within a given class of functions implies that the original nonlinear system is nonintegrable. In particular one works in the analytic setting where inverting the solution $\bar{x}(t)$ one obtains a (noncompact) Riemann surface Γ given by integrating $dt = dw/\dot{\bar{x}}(w)$ with the appropriate limits. Linearizing the system of differential equations around the straight line solution yields the *Normal Variational Equation* (NVE), which is the component of the linearized system which describes the variational normal to the surface Γ .

Given a Hamiltonian system, the main statement of Ziglin's theorems is to relate the existence of a first integral of motion with the monodromy matrices around the straight line solution [7, 8]. The simplest way to compute such monodromies is by changing coordinates to bring the normal variational equation into a known form (hypergeometric, Lamé, Bessel, Heun, etc).

Morales-Ruiz and Ramis proposed a major improvement on Ziglin's theory by introducing techniques of differential Galois theory [11–13]. The key observation is to change the formulation of integrability from a question of monodromy to a question of the nature of the Galois group of the NVE. Intuitively, if we go back to Kovalevskaya we are interested in understanding whether the KAM tori are resonant or not resonant or, in simpler terms, if their characteristic frequencies in the action-angle formalism are rational or irrational (see the pedagogical introductions provided in [9, 14]). This statement turns out to be dealt with most efficiently in terms

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of the Galois group of the NVE. The key result is now stated as: If the differential Galois group of the NVE is non-virtually Abelian, that is, the identity connected component is a non-Abelian group, then the Hamiltonian system is non-integrable. The calculation of the Galois group is rather intricate but the key simplification comes through the application of Kovacic's algorithm [15]. Kovacic's algorithm implements Picard-Vessiot theory for second order homogeneous linear differential equations with polynomial coefficients giving a constructive answer to the existence of integrability by quadratures. Fortunately Kovacic's algorithm is implemented in most computer algebra software including Maple and Mathematica. It is a little tedious but straightforward to check the algorithm manually. So, once we write down our NVE in a suitable linear form with polynomial coefficients, it becomes a simple task to check their solvability in quadratures. An important property of the Kovacic's algorithm is that it works if and only if the system is integrable, thus a failure of completing the algorithm equates to a proof of non-integrability. This route of declaring systems non-integrable has been successfully applied to various situations, some interesting examples include: general homogeneous potentials [16], cosmological models [17], fluid dynamics [18], generalizations of the Hénon-Heiles system [14] and various others [9].

Wrapped Strings in General $AdS_5 \times X_5$. – The methods of analytic non-integrability can be applied to a large class of spaces in string theory. Let us start by considering a five-dimensional Einstein space X_5 , with $R_{ij} \sim g_{ij}$. Any such Einstein space furnishes a solution to the type IIB supergravity equations known as a Freund-Rubin compactification [19]. The solution takes the form

$$ds^2 = ds^2(AdS_5) + ds^2(X_5), \quad F_5 = (1 + \star)\text{vol}(AdS_5),$$

where vol is the volume five-form and \star is the Hodge dual operator. Of particular interest in string theory is the case when X_5 is Sasaki-Einstein, that is, on top of being Einstein it admits a spinor satisfying $\nabla_{\mu}\epsilon \sim \Gamma_{\mu}\epsilon$.

The configuration that we are interested in exploring is a string sitting at the center of AdS_5 and winding in the *circles* provided by the base space. More explicitly, consider the AdS_5 metric in global coordinates: $ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2$. Then, our solutions is localized at $\rho = 0$. Largely inspired by the Sasaki-Einstein class we consider spaces X_5 that are a $U(1)$ fiber over a four-dimensional manifold. In the case of topologically trivial fibration we are precluded from applying our argument, those manifolds can be considered separately. The general local structure of Sasaki-Einstein metrics is

$$ds_{X_{S-E}}^2 = (d\psi + \frac{i}{2}(K_{,i}dz^i - K_{,\bar{i}}d\bar{z}^{\bar{i}}))^2 + K_{,i\bar{j}}dz^i d\bar{z}^{\bar{j}}, \quad (1)$$

where K is a Kähler potential on the complex base with coordinates z_i with $i = 1, 2$. This is the general structure that will serve as our guiding principle but we will not be limited to it. Roughly our Ansatz for the classical string configuration is $z_i = r_i(\tau)e^{i\alpha_i\sigma}$, where τ and σ are

the worldsheet coordinates of the string. Crucially we have introduced winding of the strings characterized by the constants α_i . The goal is to solve for the functions $r_i(\tau)$.

To apply the tools of analytic non-integrability to the class of solutions above we will: 1) Select a particular solution, that is, define the *straight line solution*. 2) Write the normal variational equation (NVE). 3) Check if the identity component of the differential Galois group of the NVE is Abelian, that is, apply the Kovacic's algorithm to determine if the NVE is integrable by quadrature.

Given this Ansatz above we can now summarize the general results. We prove that the corresponding effective Hamiltonian systems have two degrees of freedom and admit an invariant plane $\Gamma = \{r_2 = \dot{r}_2 = 0\}$ whose normal variational equation around integral curves in Γ we study explicitly.

$T^{p,q}$ – These 5-manifolds are not necessarily Sasaki-Einstein, however, some of them are Einstein which allow for consistent string backgrounds. More importantly, some of these spaces provide exact conformal sigma models and are thus exact string backgrounds in all orders in α' [20]. However, they are never maximally symmetric and the integrability discussed for $AdS_5 \times S^5$ is not applicable. In this section we provide a unified treatment of this class for generic values of p and q . The metric has the form

$$ds^2 = a^2(d\psi + p \cos \theta_1 d\phi_1 + q \cos \theta_2 d\phi_2)^2 + b^2(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + c^2(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2).$$

The classical string configuration we are interested is: $\theta_1 = \theta_1(\tau)$, $\theta_2 = \theta_2(\tau)$, $\psi = \psi(\tau)$, $t = t(\tau)$, $\phi_1 = \alpha_1\sigma$, $\phi_2 = \alpha_2\sigma$, where α_i are constants quantifying how the string winds along the ϕ_i directions. Recall that t is from AdS_5 . The Polyakov Lagrangian is

$$\mathcal{L} = -\frac{1}{2\pi\alpha'} \left[\dot{t}^2 - b^2\dot{\theta}_1^2 - c^2\dot{\theta}_2^2 - a^2\dot{\psi}^2 + \alpha_1^2(b^2 - a^2p^2)\sin^2 \theta_1 + \alpha_2^2(c^2 - a^2q^2)\sin^2 \theta_2 + 2\alpha_1\alpha_2pq a^2 \cos \theta_1 \cos \theta_2 \right]. \quad (2)$$

There are several conserved quantities, the corresponding nontrivial equations are

$$\ddot{\theta}_1 + \frac{\alpha_1}{b^2} \sin \theta_1 \left[\alpha_1(b^2 - a^2p^2) \cos \theta_1 - a^2\alpha_2pq \cos \theta_2 \right] = 0,$$

$$\ddot{\theta}_2 + \frac{\alpha_2}{c^2} \sin \theta_2 \left[\alpha_2(c^2 - a^2q^2) \cos \theta_2 - a^2\alpha_1pq \cos \theta_1 \right] = 0.$$

There is immediately some insight into the role of the fibration structure. Note that the topological winding in the space which is described by p and q intertwines with the wrapping of the strings α_1 and α_2 . The effective numbers that appears in the interaction part of the equations are $\alpha_1 p$ and $\alpha_2 q$. For example, from the point of view of the interactions terms, taking $p = 0$ or $q = 0$ is equivalent to taking one of the $\alpha_i = 0$ which leads to

an integrable system of two non-interacting gravitational pendula.

We take the straight line solution to be: $\theta_2 = \dot{\theta}_2 = 0$. The equation for θ_1 becomes

$$\ddot{\theta}_1 + \frac{\alpha_1}{b^2} \left[\alpha_1 (b^2 - a^2 p^2) \cos \theta_1 - a^2 \alpha_2 p q \right] \sin \theta_1 = 0. \quad (3)$$

Let us denote the solution to this equation $\bar{\theta}_1$, it can be given explicitly but we will not need the precise form. This solution also defines the Riemann surface Γ introduced before. The NVE is obtained by considering small fluctuations in θ_2 around the above solutions and takes the form:

$$\ddot{\eta} + \frac{\alpha_2}{c^2} \left[\alpha_2 (c^2 - a^2 q^2) - \alpha_1 p q \cos \bar{\theta}_1 \right] \eta = 0. \quad (4)$$

Our goal is to study the NVE. To make the equation amenable to the Kovacic's algorithm we introduce the following substitution: $\cos(\bar{\theta}_1) = z$. In this variable the NVE takes a form similar to Lamé equation

$$f(z)\eta''(z) + \frac{1}{2}f'(z)\eta'(z) + \frac{\alpha_2}{c^2} \left[\alpha_2 (c^2 - a^2 q^2) - \alpha_1 p q z \right] \eta(z) = 0 \quad (5)$$

where, prime now denotes differentiation with respect to z and $f(z) = \frac{\dot{\theta}_1^2 \sin^2(\bar{\theta}_1)}{(6E^2 - \frac{1}{3}(4\alpha_1\alpha_2 z + \alpha_2^2(1-z^2))) (1-z^2)}$. Equation (5) is a second order homogeneous linear differential equation with polynomial coefficients and it is, therefore, ready for the application of Kovacic's algorithm. For generic values of the parameters above the Kovacic's algorithm does not produce a solution meaning the system defined in equations (3) is not integrable.

The case of $T^{1,1}$ is particularly interesting because for this case the supergravity solution is supersymmetric and a lot of attention has been paid to extending configurations of $AdS_5 \times S^5$ to the case of $AdS_5 \times T^{1,1}$ [21–26].

$Y^{p,q}$ – These spaces have played a central role in developments of the AdS/CFT correspondence as they provided an infinite class of dualities. These spaces are Sasaki-Einstein but they are not coset spaces as was the case for the $T^{p,q}$ discussed above. Following the general discussion of Sasaki-Einstein spaces above, we write the metric on these spaces as

$$ds^2 = \frac{1}{9}(d\psi - (1-cy)\cos\theta d\phi + yd\beta)^2 + \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{p(y)}{6}(d\beta + c\cos\theta d\phi)^2,$$

and $p(y) = [a - 3y^2 + 2cy^3]/[3(1-cy)]$. The classical string configuration is described by the Ansatz: $\theta = \theta(\tau)$, $y = y(\tau)$, $\phi = \alpha_1\sigma$, $\beta = \alpha_2\sigma$. The Polyakov

Lagrangian is simply:

$$\mathcal{L} = -\frac{1}{2\pi\alpha'} \left[\dot{t}^2 - \frac{1-cy}{6}\dot{\theta}^2 - \frac{1}{6p(y)}\dot{y}^2 - \frac{1}{9}\dot{\psi}^2 + \frac{1-cy}{6}\alpha_1^2 \sin^2\theta + \frac{p(y)}{6}(\alpha_2 + c\alpha_1 \cos\theta)^2 + \frac{1}{9}(\alpha_2 y - \alpha_1(1-cy)\cos\theta)^2 \right] \quad (6)$$

As in previous cases the equations of motion for t and ψ are integrated immediately leaving only two nontrivial equations for θ and y . The straight line solution can be taken to be $\theta = \dot{\theta} = 0$. Then the equation for y is simplified to

$$\ddot{y} - \frac{p'}{p}\dot{y}^2 + \frac{pp'}{2}(\alpha_2 + c\alpha_1)^2 + \frac{2}{3}p(\alpha_2 + c\alpha_1)(y(\alpha_2 + c\alpha_1) - \alpha_1) = 0.$$

To be able to write the NVE in a form conducive to the application of Kovacic's algorithm we substitute $y_s(t) = y$ and the NVE takes the form

$$(1-cy)\dot{y}^2(t) \frac{d^2\eta}{dy^2} + q(y)\eta(y) \quad (7)$$

$$+ (\ddot{y}(t)(1-cy) - c\dot{y}^2(t)) \frac{d\eta}{dy} = 0, \quad (8)$$

where \dot{y} and \ddot{y} can be written in terms of y as: $\dot{y}^2(t) = 6(E + p(y)V(y, 0)) = 6p(y) \left(\frac{p(y)}{6}(\alpha_2 + c\alpha_1)^2 + \frac{1}{9}(\alpha_2 y - \alpha_1(1-cy)) \right)$ and $q(y) = \alpha_1(1-cy)[5/3\alpha_1 - \frac{c(a-3y^2+2cy^3)(\alpha_2+c\alpha_1)}{(3-3cy)(1-cy)} - 2/3(\alpha_2 + c\alpha_1)y]$. With these identifications we have rewritten the NVE as a homogeneous second order linear differential equation with polynomial coefficients. The Kovacic's algorithm again fails to yield a solution pointing to the fact that the system is generically non-integrable.

The exceptional case: S^5 – In this section we provide an integrable example where the Kovacic's algorithm should succeed. To expose the Sasaki-Einstein structure of S^5 , it is convenient to write the metric as a $U(1)$ fiber over \mathbb{P}^2 . The round metrics on S^5 may be elegantly expressed in terms of the left-invariant one-forms of $SU(2)$. The left-invariant one-forms can be written as: $\sigma_1 = \frac{1}{2}(\cos(d\psi)d\theta + \sin(\psi)\sin(\theta)d\phi)$, $\sigma_2 = \frac{1}{2}(\sin(\psi)d\theta - \cos(\psi)\sin(\theta)d\phi)$, $\sigma_3 = \frac{1}{2}(d\psi + \cos(\theta)d\phi)$. In terms of these 1-forms, the metrics on \mathbb{P}^2 and S^5 may be written,

$$ds_{\mathbb{P}^2}^2 = d\mu^2 + \sin^2(\mu)(\sigma_1^2 + \sigma_2^2 + \cos^2(\mu)\sigma_3^2), \quad ds_{S^5}^2 = ds_{\mathbb{P}^2}^2 + (d\chi + \sin^2(\mu)\sigma_3)^2 \quad (9)$$

where χ is the local coordinate on the Hopf fibre and $A = \sin^2(\mu)\sigma_3 = \sin^2(\mu)(d\psi + \cos(\theta)d\phi)/2$ is the 1-form potential for the Kähler form on \mathbb{P}^2 .

The classical string configuration is: $\theta = \theta(\tau)$, $\mu = \mu(\tau)$, $\chi = \chi(\tau)$, $\phi = \alpha_1\sigma$, $\psi = \alpha_2\sigma$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2\pi\alpha'} \left[\dot{t}^2 - \dot{\mu}^2 - \frac{1}{4} \sin^2 \mu \dot{\theta}^2 - \dot{\chi}^2 + \frac{1}{4} \sin^2 \mu (\alpha_1^2 \sin^2 \theta + (\alpha_2 + \alpha_1 \cos \theta)^2) \right]. \quad (10)$$

The nontrivial equations of motion are

$$\begin{aligned} \ddot{\mu} + \frac{1}{8} \sin(2\mu) \left[\dot{\theta}^2 - 2\alpha_1\alpha_2 \cos \theta - \alpha_1^2 - \alpha_2^2 \right] &= 0, \\ \ddot{\theta} + 2\dot{\mu}\dot{\theta} \cot(\mu) + \alpha_1\alpha_2 \sin \theta &= 0. \end{aligned} \quad (11)$$

Inspection of the above system shows that we have various natural choices. We discussed the two natural choices of straight line solutions in what follows.

θ straight line – Let us assume $\theta = \dot{\theta} = 0$, then the equation for μ becomes

$$\ddot{\mu} - \frac{1}{8} (\alpha_1 + \alpha_2)^2 \sin(2\mu) = 0. \quad (12)$$

We call the solution of this equation μ_s . The NVE is

$$\ddot{\eta} + 2 \cot(\mu_s) \dot{\mu}_s \dot{\eta} + \alpha_1\alpha_2 \eta = 0. \quad (13)$$

With $\sin(\mu) = z$ the NVE may be written as,

$$r(z) \frac{d^2}{dz^2} \eta(z) + q(z) \frac{d}{dz} \eta(z) + \alpha_1\alpha_2 z^2 \eta(z) = 0 \quad (14)$$

$$(15)$$

with $r(z) = z^2 \left(2E + 1/8 (\alpha_1 + \alpha_2)^2 (1 - 2z^2) \right) (1 - z^2)$ and $q(z) = -1/8z \left(-32E + 48Ez^2 - 2\alpha_1^2 + 9\alpha_1^2 z^2 - 8\alpha_1^2 z^4 - 4\alpha_1\alpha_2 + 18\alpha_1\alpha_2 z^2 - 16\alpha_1\alpha_2 z^4 - 2\alpha_2^2 + 9\alpha_2^2 z^2 - 8\alpha_2^2 z^4 \right)$. This equation is now on the form conducive to Kovacic's algorithm which succeeds and gives a solution. Since the above approach obscures the nature of integrability of $AdS_5 \times S^5$ we consider another example which leaves no doubt about the integrability.

μ straight line – Let us assume the straight line is now given by $\mu = \pi/2$, $\dot{\mu} = 0$. The equation for θ becomes

$$\ddot{\theta} + \alpha_1\alpha_2 \sin \theta = 0. \quad (16)$$

Let us call the solution to this equation θ_s . Then the NVE is

$$\ddot{\eta} + \frac{1}{4} \left(\dot{\theta}_s^2 - 2\alpha_1\alpha_2 \cos(\theta_s) - \alpha_1^2 - \alpha_2^2 \right) \eta = 0. \quad (17)$$

Note that the equation of motion for θ_s implies

$$\begin{aligned} \ddot{\theta} + \alpha_1\alpha_2 \sin \theta = 0 &\rightarrow \frac{d}{dt} \left(\dot{\theta}_s^2 - 2\alpha_1\alpha_2 \cos \theta_s \right) = 0, \\ &\rightarrow \dot{\theta}_s^2 - 2\alpha_1\alpha_2 \cos \theta_s = C_0 \end{aligned} \quad (18)$$

Thus the NVE equation can be written as a simple harmonic equation

$$\ddot{\eta} + \frac{1}{4} (C_0 - \alpha_1^2 - \alpha_2^2) \eta = 0. \quad (19)$$

We do not require Kovacic's algorithm to tell us that there is an analytic solution for this equation. The power of differential Galois theory also guarantees that the result is really independent of the straight line solution (Riemann surface) that one chooses. We conclude this subsection with the jovial comment that we now know a very precise sense in which *String theory in $AdS_5 \times S^5$ is like a harmonic oscillator*.

Conclusions – In this paper we have shown that certain classical string configurations corresponding to a string winding along two of the angles of a general class of five-dimensional Einstein manifolds X_5 , realized as a nontrivial S^1 fibration over a 4-d base, is non-integrable. The result highlights the limit of integrability within the AdS/CFT correspondence. Integrability has been one of the main areas of study for almost ten years. The paper forces the AdS/CFT to bear and expand with the newly discovered fact that most configurations beyond $AdS_5 \times S^5$ are non-integrable, this requires a new dictionary.

In all the previous examples in the literature homogeneity of the potential played a crucial role in the proof [9, 16, 17]. A mathematical curiosity arises from the fact that traditionally due to the works of Hadamard and later of Anosov, chaos has been associated with the motion of particles in negatively curved spaces through the Jacobi equation. The class of five-dimensional Einstein spaces used here have positive curvature. The main mechanism for non-integrability is provided by winding of the strings which is a property unique to strings and therefore not well understood. More precisely, we found an interesting interplay between topology $c_1 = \int dA$ and dynamics as the Chern class determines the possibility of an interaction term in the dynamical system. As pointed out in the main terms in various cases the interaction, and therefore non-integrability, appears as the product of the Chern number and the winding number of the string.

The direct connection between analytic non-integrability and chaotic behavior is still open. This question has been discussed in the literature and we refer the reader to [9] for further details. For the sake of disclosure we notice that we have not directly proved that the systems discussed here are chaotic. However, together with our previous publication [6], we believe the case for outright chaotic behavior is overwhelmingly strong.

Our work now opens the door to the interpretation of many chaotic quantities in the context of the AdS/CFT correspondence. For example, the meaning of: Lyapunov spectrum, Kolmogorov-Sinai entropy and fractal dimension are but a few quantities expecting its quantum analog in this context.

Acknowledgments. – PB thanks D. Das, S. Das, A. Ghosh and A. Shapere. This work is partially supported by Department of Energy under grant DE-FG02-95ER40899 and the National Science Foundation under grants NSF-PHY-0855614 and PHY-0970069.

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