Covariant formulation of the post-1-Newtonian approximation to general relativity

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Covariant formulation of the post-1-Newtonian approximation to General Relativity

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We derive a coordinate-independent formulation of the post-1-Newtonian approximation to general relativity. This formulation is a generalization of the Newton-Cartan geometric formulation of Newtonian gravity. It involves several fields and a connection, but no spacetime metric at the fundamental level. We show that the usual coordinate-dependent equations of post-Newtonian gravity are recovered when one specializes to asymptotically flat spacetimes and to appropriate classes of coordinates.

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I. INTRODUCTION AND SUMMARY

A. Background and Motivation

The weak field, slow motion approximation to general relativity, also called the post-Newtonian approximation, consists of expanding in the small parameters v^2/c^2 and Φ/c^2, where v is a typical velocity of the system under consideration, Φ is the Newtonian potential, and c is the speed of light. At the leading order Newton's theory is recovered, and higher order corrections are called post-1-Newtonian corrections, post-2-Newtonian corrections and so on. This aproximation scheme is very useful in astrophysical applications and is very well developed. Reviews can be found in Ref. [1] and in the book by Will [2], and a historical review can be found in Ref. [3].

There are two types of of equations that arise in post-Newtonian theory. The first are continuum equations of motion, for example for gravity coupled to a perfect fluid, for which one obtains generalizations of the equations of Newtonian hydrodynamics. Such continuum equations have been used extensively in numerical simulations (although fully relativistic simulations are now the state of the art [4]). The second type of equations are "point particle equations", which describe the motions of bodies whose sizes are small compared to their separations. Such point particle equations (and extensions to include spins) can be derived from the underlying continuum theory by a variety of methods. Currently the equations of motion for two point particles are known up to post-3.5-Newtonian order, see, for example Ref. [5] and references therein. In this paper we shall be concerned only with continuum equations.

Over the years, the foundations of the Newtonian and post-Newtonian approximations have been studied in detail and with considerable mathematical rigor by a number of researchers. Futamase and Schutz [6] have shown that the various orders of post-Newtonian approximation are asymptotic approximations to fully relativisitic solutions. Frittelli and Reula [7] showed that, given a solution of the equations of Newtonian gravity, there exists a one parameter family of exact relativistic solutions which for a finite amount of time are close to the Newtonian solution. Rendall [8] gave a mathematically rigorous derivation of both the Newtonian and post-Newtonian approximations from a precise set of axioms.

However, there remains one aspect of post-Newtonian theory which has not been fully explored: there is as yet no covariant version of the theory. Usually, in order to find the equations of post-Newtonian theory, one first introduces specializations of the coordinate system (or gauge), and then Einstein's equations are expanded order by order. The gauge specializations are chosen to simplify the resulting equations. For example, at post-1-Newtonian order, the harmonic gauge condition and the so-called standard post-Newtonian gauge condition [9] are often used. In each of these gauges the post-Newtonian equations take a different form. The situation is similar to knowing the laws of electromagnetism only in a handful of gauges (e.g. the Lorentz and the Coulomb gauge), without knowing the underlying gauge independent equations.

This lack of covariance of post-Newtonian theory as currently formulated has several disadvantages:

- If one is attempting to compare two different calculations, it is often helpful to identify and compute gauge-invariant quantities. Generally, such quantities are easier to identify starting from a covariant formulation of the theory.

- In attempting to developing intuition from the post-Newtonian equations, it can be difficult to sort out which aspects of the equations contain the actual gauge-independent physics and which aspects are gauge dependent. For example, there is a well-known analogy between the equations of post-1-Newtonian theory in certain gauges and those of electromagnetism. In this case the analogy with electrostatics and magnetostatics is physical, but the additional aspects of the analogy concerning magnetodynamics are gauge.[1]

- At post-1-Newtonian order, there is a well-developed framework for celestial mechanics that describes N interacting, deformable bodies [9–12]. This framework is quite complicated, involving separate coordinate systems for each body as well as a global coordinate system, together with a set of fields associated with each of the coordinate systems. A covariant formulation might simplify some aspects of this framework.

B. Covariant post-Newtonian theory

The purpose of this paper is to derive a coordinate independent formulation of post-1-Newtonian theory. We will derive a fully covariant set of equations, involving a number of tensor fields and a connection, which reduce to the standard post-Newtonian equations in specific coordinate systems for asymptotically flat spacetimes. In the case of Newton's theory, such a geometric formulation has already been found by Cartan and others [13–17], and is called Newton-Cartan theory. Building on earlier work of Dautcourt [18, 19], our derivation will reproduce Newton-Cartan theory at leading order, and at the next order will give a covariant version of post-1-Newtonian theory which we call "post-Newton-Cartan theory".

We will actually derive two different versions of post-Newton-Cartan theory. The first version, which we call perturbative post-Newton-Cartan theory, allows one to compute the leading order corrections to a solution of Newton-Cartan theory. It is a unique theory, in which post-Newtonian corrections to Newtonian quantities are treated as independent variables to be solved for. The second version, which we call combined post-Newton-Cartan theory, combines some of the Newtonian and post-Newtonian variables together. It is more economical and convenient to use than the first version, because it has fewer variables and fewer equations. Solutions of this theory will be accurate to post-1-Newtonian order, but will in addition contain post-2-Newtonian, post-3-Newtonian etc. pieces. The combined theory is not unique; different choices could be made to define different theories whose solutions differ at post-2-Newtonian and higher orders. We note that standard, coordinate-specific post-Newtonian theory also comes in "perturbative" and "combined" versions [6].

C. Results

We now turn to a description of our results. We start by reviewing Newton-Cartan theory, then we describe the two versions of post-Newton-Cartan theory.

TABLE I: Fields and equations of Newton-Cartan theory.

Field	Description
h^{ab}	spatial metric, $(0,+,+,+)$
t_a	time one-form
D_a	symmetric connection
$\mathcal{T}^{ab}$	matter stress-energy tensor
Equation	**Description**
$h^{ab}t_b = 0$	Orthogonality
$D_a h^{bc} = 0$	Compatibility with connection
$D_a t_b = 0$	Compatibility with connection
$h^{e[a}R_{e(bc)}{}^{d]} = 0$	Trautman condition
$R_{ab} = 4\pi t_a t_b t_c t_d \mathcal{T}^{cd}$	Field equation
$D_a \mathcal{T}^{ab} = 0$	Stress-energy conservation
$R_{abc}{}^d = R_{abc}{}^d(D_e)$	definition of Riemann

1. *Newton-Cartan theory*

The variables of Newton-Cartan theory are a one-form t_a, a symmetric contravariant tensor field h^{ab} with signature $(0,+,+,+)$, and a torsion-free connection D_a. Matter is described by a symmetric contravariant stress-energy tensor $\mathcal{T}^{ab}$. The fields t_a and h^{ab} are non-dynamical, background fields, while D_a and $\mathcal{T}^{ab}$ are dynamical.

The equations of the theory are the orthogonality condition

$$h^{ab}t_b = 0, \tag{1.1}$$

the compatibility of the fields h^{ab} and t_a with the connection,

$$D_a h^{bc} = 0, \tag{1.2a}$$
$$D_a t_b = 0, \tag{1.2b}$$

and the Trautman condition

$$h^{f[a}R_{f(bc)}{}^{d]} = 0, \tag{1.3}$$

where $R_{abc}{}^d$ is the Riemann tensor associated with the connection D_a. In addition we have the field equation

$$R_{ab} = 4\pi t_a t_b t_c t_d \mathcal{T}^{cd}, \tag{1.4}$$

where $R_{ab} = R_{acb}{}^c$, and the stress-energy conservation equation

$$D_a \mathcal{T}^{ab} = 0. \tag{1.5}$$

These fields and equations are summarized in Table I. In Sec. III A below we review the derivation of this Newton-Cartan theory from general relativity, and in Sec. IV B we review how Newton-Cartan theory reduces to standard Newtonian gravity in appropriate circumstances and in appropriate coordinate systems.

[1] If the analogy with electromagnetism were complete there would be radiation in the post-1-Newtonian theory.

TABLE II: Additional fields and equations of perturbative post-Newton-Cartan theory.

Field	Description
k^{ab}	contravariant metric perturbation
p_{ab}	covariant metric perturbation
$\Delta^{c}{}_{ab}$	connection perturbation
$\mathcal{S}^{ab}$	matter stress-energy perturbation
Equation	**Description**
$h^{ab}p_{bc} - k^{ab}t_b t_c = \delta^a_c$	Orthogonality
$D_a k^{bc} + \Delta^{b}{}_{ad}h^{dc} + \Delta^{c}{}_{ad}h^{bd} = 0$	Compatibility with connection
$D_a p_{bc} + \Delta^{d}{}_{ab}t_d t_c + \Delta^{d}{}_{ac}t_b t_d = 0$	Compatibility with connection
$k^{e[a}R_{e(bc)}{}^{d]} - h^{e[a}D_e\Delta^{d]}{}_{bc} + h^{e[a}D_{(b}\Delta^{d]}{}_{c)e} = 0$	Trautman condition
$-D_a\Delta^{b}{}_{bc} + D_b\Delta^{b}{}_{ac} = 4\pi\left[t_a t_b t_c t_d \mathcal{S}^{cd} - 4t_c t_{(a}p_{b)d}\mathcal{T}^{cd} + p_{cd}\mathcal{T}^{cd}t_a t_b + t_c t_d \mathcal{T}^{cd}p_{ab}\right]$	Field equation
$D_a\mathcal{S}^{ab} + \Delta^{a}{}_{ac}\mathcal{T}^{cb} + \Delta^{b}{}_{ac}\mathcal{T}^{ac} = 0$	Stress-energy conservation

TABLE III: Fields and equations of combined post-Newton-Cartan theory.

Field	Description
$\hat{h}^{ab}$	spatial metric, $(0,+,+,+)$
$\hat{t}_a$	time one-form
$\hat{k}^{ab}$	contravariant metric perturbation
$\hat{p}_{ab}$	covariant metric perturbation
$\hat{D}_a$	symmetric connection
$\hat{\mathcal{T}}^{ab}$	matter stress-energy tensor
Equation	**Description**
$\hat{h}^{ab}\hat{t}_b = 0$	Newtonian orthogonality
$\hat{h}^{ab}\hat{p}_{bc} - \hat{k}^{ab}\hat{t}_b\hat{t}_c = \delta^a_c$	post-Newtonian orthogonality
$\hat{D}_a(\hat{h}^{bc} + \hat{k}^{bc}) = 0$	Compatibility with connection
$\hat{D}_a(-\hat{t}_b\hat{t}_c + \hat{p}_{bc}) = 0$	Compatibility with connection
$(\hat{h}^{e[a} + \hat{k}^{e[a})R_{e(bc)}{}^{d]}(\hat{D}_e) = 0$	Trautman condition
$R_{ab}(\hat{D}_c) = 4\pi\left[\hat{t}_a\hat{t}_b\hat{t}_c\hat{t}_d\hat{\mathcal{T}}^{cd} - 4\hat{t}_c\hat{t}_{(a}\hat{p}_{b)d}\hat{\mathcal{T}}^{cd} + \hat{p}_{cd}\hat{\mathcal{T}}^{cd}\hat{t}_a\hat{t}_b + \hat{t}_c\hat{t}_d\hat{\mathcal{T}}^{cd}\hat{p}_{ab}\right]$	Field equation
$\hat{D}_a\hat{\mathcal{T}}^{ab} = 0$	Stress-energy conservation

2. Perturbative post-Newton-Cartan theory

Turn, now to the perturbative post-Newton-Cartan theory. This theory contains the Newtonian fields h^{ab}, t_a, D_a and $\mathcal{T}^{ab}$, and in addition four new fields: a symmetric, contravariant tensor k^{ab}, a symmetric, covariant tensor p_{ab}, a perturbation $\Delta^{c}{}_{ab}$ to the connection, and a perturbation $\mathcal{S}^{ab}$ to the matter stress-energy tensor.

The equations of the theory are the six Newtonian equations (1.1) – (1.5), together with six post-Newtonian equations: (i) the orthogonality condition

$$h^{ab}p_{bc} - k^{ab}t_b t_c = \delta^a_c; \tag{1.6}$$

(ii) the compatibility of k^{ab} with the connection,

$$D_a k^{bc} + \Delta^{b}{}_{ad}h^{dc} + \Delta^{c}{}_{ad}h^{bd} = 0; \tag{1.7}$$

(iii) the compatibility of p_{ab} with the connection,

$$D_a p_{bc} + \Delta^{d}{}_{ab}t_d t_c + \Delta^{d}{}_{ac}t_b t_d = 0; \tag{1.8}$$

(iv) the post-Newtonian Trautman condition

$$k^{e[a}R_{e(bc)}{}^{d]} - h^{e[a}D_e\Delta^{d]}{}_{bc} + h^{e[a}D_{(b}\Delta^{d]}{}_{c)e} = 0; \tag{1.9}$$

(v) the field equation

$$\begin{aligned} -D_a\Delta^{b}{}_{bc} + D_b\Delta^{b}{}_{ac} \;=\;& 4\pi\big[t_a t_b t_c t_d \mathcal{S}^{cd} - 4t_c t_{(a}p_{b)d}\mathcal{T}^{cd} \\ &+ p_{cd}\mathcal{T}^{cd}t_a t_b + t_c t_d \mathcal{T}^{cd}p_{ab}\big]; \end{aligned} \tag{1.10}$$

and (vi) the stress-energy conservation equation

$$D_a\mathcal{S}^{ab} + \Delta^{a}{}_{ac}\mathcal{T}^{cb} + \Delta^{b}{}_{ac}\mathcal{T}^{ac} = 0. \tag{1.11}$$

These fields and equations are summarized in Table II. In Sec. III A below we review the derivation of this perturbative post-Newton-Cartan theory from general relativity, and in Sec. IV C we review how it reduces to standard post-1-Newtonian gravity in appropriate circumstances and in appropriate coordinate systems.

3. Combined post-Newton-Cartan theory

The combined post-Newton-Cartan theory has aspects of both the Newton-Cartan and the perturbative post-Newton-Cartan theories. The independent variables are a one-form $\hat{t}_a$, a symmetric contravariant tensor field $\hat{h}^{ab}$ with signature $(0,+,+,+)$, a symmetric, contravariant tensor $\hat{k}^{ab}$, and a symmetric, covariant tensor $\hat{p}_{ab}$. There is also a torsion-free connection $\hat{D}_a$, which is defined accurate to post-1-Newtonian order. Matter is described by a symmetric contravariant stress-energy tensor $\hat{\mathcal{T}}^{ab}$, which is also defined accurate to post-1-Newtonian order. The fields $\hat{t}_a$ and $\hat{h}^{ab}$ are non-dynamical, background fields, while the remaining fields are dynamical.

The equations of the theory are: (i) the orthogonality conditions

$$\hat{h}^{ab}\hat{t}_b \;=\; 0, \tag{1.12a}$$
$$\hat{h}^{ab}\hat{p}_{bc} - \hat{k}^{ab}\hat{t}_b\hat{t}_c \;=\; \delta^a_c; \tag{1.12b}$$

(ii) the connection compatibility conditions

$$\hat{D}_a(\hat{h}^{bc} + \hat{k}^{bc}) \;=\; 0, \tag{1.13a}$$
$$\hat{D}_a(-\hat{t}_b\hat{t}_c + \hat{p}_{bc}) \;=\; 0; \tag{1.13b}$$

(iii) the Trautman condition

$$(\hat{h}^{e[a} + \hat{k}^{e[a})R_{e(bc)}{}^{d]}(\hat{D}_e) = 0; \tag{1.14}$$

(iv) the field equation

$$R_{ab}(\hat{D}_c) \;=\; 4\pi\big[\hat{t}_a\hat{t}_b\hat{t}_c\hat{t}_d\hat{\mathcal{T}}^{cd} - 4\hat{t}_c\hat{t}_{(a}\hat{p}_{b)d}\hat{\mathcal{T}}^{cd} + \hat{p}_{cd}\hat{\mathcal{T}}^{cd}\hat{t}_a\hat{t}_b + \hat{t}_c\hat{t}_d\hat{\mathcal{T}}^{cd}\hat{p}_{ab}\big]; \tag{1.15}$$

and (v) the stress-energy conservation equation

$$\hat{D}_a\hat{\mathcal{T}}^{ab} = 0. \tag{1.16}$$

These fields and equations are summarized in Table III. The derivation of this combined post-Newton-Cartan theory from the Newton-Cartan and post-Newton-Cartan theories is given in Sec. III B below.

D. Organization of this paper

In Sec. II we list the assumptions underlying our derivation, and discuss the motivation for these assumptions. Section III gives the derivation of the Newton-Cartan theory and both versions of the post-Newton-Cartan theory, from these assumptions. In Sec. IV we show that the equations of perturbative post-Newton-Cartan theory reduce to the standard coordinate-specific equations of post-Newtonian theory, under certain conditions and in certain coordinate systems. We conclude by summarizing our results and their implications in Sec. V.

Some technical and side issues are discussed in the Appendices. Appendix A derives an alternative form of one of our assumptions. Appendix B derives the gauge transformation properties of the fields of the perturbative post-Newton-Cartan theory. Finally appendix C specializes the formalism, as an example, to perfect fluids.

E. Notation

Throughout we will use the metric and sign conventions of Misner, Thorne and Wheeler [17]. Furthermore we will use Penrose's abstract index notation [20], with indices $(a, b, \ldots)$ from the beginning of the Latin alphabet denoting general tensors. When we specialize to particular coordinate systems, we will use indices $(\alpha, \beta, \ldots)$ from the Greek alphabet to denote general components of tensors. We will also use indices $(i, j, k, \ldots)$ from the middle of the Latin alphabet to denote spatial components of tensors, and an index 0 to denote time components, in particular coordinate systems. For example, u^a will denote a four-velocity vector, u^μ will be the components of the four-velocity in some coordinate system, and u^i and u^0 will be the spatial and time components in this coordinate system. We will work with units in which $G = c = 1$, except for some special cases in Sec. II B where factors of c and G are explicitly included. Finally, we will use the conventional definitions of the two order symbols $O()$ and $o()$.

II. FOUNDATIONS AND ASSUMPTIONS

In this section we define and discuss the assumptions that we make in order to derive the Newton-Cartan and post-Newton-Cartan theories from general relativity. At Newtonian order we follow closely the treatments of Dautcourt [18, 19] and of Rendall [8].

A. Assumptions

The starting point is to assume a one-parameter family $g_{ab}(\varepsilon), T^{ab}(\varepsilon)$ of exact solutions of Einstein's equations

$$G^{ab}[g_{cd}(\varepsilon)] = 8\pi T^{ab}(\varepsilon), \tag{2.1}$$

where ε is a real parameter with $0 < \varepsilon < \varepsilon_0$ for some ε_0. We assume that the solutions are smooth functions of spacetime and of ε for $\varepsilon > 0$. We do not assume the existence of a solution at $\varepsilon = 0$.

The key assumptions we make are:

1. The contravariant metric $g^{ab}(\varepsilon)$ can be expanded near $\varepsilon = 0$ as

$$g^{ab}(\varepsilon) = h^{ab} + \varepsilon k^{ab} + \varepsilon^2 j^{ab} + \varepsilon^3 l^{ab} + o(\varepsilon^3), \tag{2.2}$$

where h^{ab}, k^{ab}, j^{ab} and l^{ab} are ε-independent symmetric tensor fields on spacetime. Furthermore h^{ab} has signature

$$h^{ab} \sim (0,+,+,+), \tag{2.3}$$

and $k^{ab}t_a t_b$ is everywhere nonzero, where t_a is the direction defined by $h^{ab}t_b = 0$.

2. The matter stress-energy tensor can be expanded near $\varepsilon = 0$ as

$$T^{ab}(\varepsilon) = \varepsilon^2 \mathcal{T}^{ab} + \varepsilon^3 \mathcal{S}^{ab} + O(\varepsilon^4), \tag{2.4}$$

where again $\mathcal{T}^{ab}$ and $\mathcal{S}^{ab}$ are ε-independent tensor fields on spacetime.

3. The connection ∇_a associated with the metric $g_{ab}(\varepsilon)$ has a continuous limit D_a as $\varepsilon \to 0$,

$$\nabla_a = D_a + O(\varepsilon). \tag{2.5}$$

B. Motivation and Discussion

We now discuss the motivation for and properties of these assumptions.

First, we note that since the limiting contravariant metric h^{ab} is degenerate, the limit as $\varepsilon \to 0$ of the covariant metric does not exist. Therefore there is no limiting, background solution of Einstein's equations at $\varepsilon = 0$ in this framework, unlike the situation for standard perturbation theory [8, 18, 19].

Next, the assumptions are explicitly local and covariant. In Sec. III below we will show that the Newton-Cartan and post-Newton-Cartan theories can be derived from them in a local and covariant manner. Suppose now that one demands that the assumptions apply only in a given, finite region of spacetime. Then, the Newton-Cartan and post-Newton-Cartan equations of Tables I and II will be satisfied in that region. However, as is well known, it does *not* follow that the usual equations of Newtonian gravity will be satisfied, since Newton-Cartan theory contains more local degrees of freedom than Newtonian gravity. The physical reason for this will be discussed in Sec. IV B below. In order to obtain Newtonian gravity, it is necessary to assume an asymptotically flat spacetime and to impose the assumptions (2.1) – (2.5) throughout all of spacetime [8, 18, 19]. We will find a similar situation for the post-Newton-Cartan theory in Sec. IV below: it contains more local degrees of freedom than standard post-1-Newtonian general relativity, and reduces to it only when the assumptions (2.1) – (2.5) hold globally in an asymptotically flat spacetime.

Consider now an isolated physical system that is characterized by some mass scale $\mathcal{M}$, lengthscale $\mathcal{L}$, and timescale $\mathcal{T}$. Then, from Newton's constant of gravitation G and the speed of light c one can form two dimensionless parameters:

$$\hat{c} \equiv \frac{c\mathcal{T}}{\mathcal{L}}, \quad \hat{G} \equiv \frac{G\mathcal{M}\mathcal{T}^2}{\mathcal{L}^3}. \tag{2.6}$$

The Newtonian limit of general relativity is the limit $\hat{c} \to \infty$ at fixed $\hat{G}$.

The first assumption, the expansion (2.2) of the contravariant components of the metric, can now be motivated as follows. The spacetime metric should be close to the flat, Minkowski metric in the Newtonian limit. In system-adapted units where $\mathcal{T} = \mathcal{M} = \mathcal{L} = 1$, this is $ds^2 = -\hat{c}^2 dt^2 + dx^2 + dy^2 + dz^2$. Now identifying $\varepsilon = \hat{c}^{-2}$ gives

$$g^{\mu\nu} = \mathrm{diag}(-\varepsilon, 1, 1, 1) = O(1) + O(\varepsilon),$$

which satisfies assumption 1 and has signature $(0,+,+,+)$ at order $O(\varepsilon^0)$. Corrections from the Newtonian potential do not change this conclusion.

The second assumption, the expansion (2.4) of the stress-energy tensor, can be motivated similarly using dimensional analysis. In a general system of units Einstein's equation is

$$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}, \tag{2.7}$$

where from dimensional analysis $T^{tt} \sim \mathcal{M}\mathcal{L}^{-3}$, $T^{ti} \sim \mathcal{M}\mathcal{L}^{-2}\mathcal{T}^{-1}$, and $T^{ij} \sim \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-2}$. In particular all the components of $T^{\alpha\beta}$ are independent of c to leading order. If we now specialize to system-adapted units, all of the components of $T^{\alpha\beta}$ are of order unity, and the right hand side of Einstein's equation is of order $\hat{G}/\hat{c}^4 \propto \varepsilon^2$, since we are considering a limit in which $\hat{G}$ is held fixed[2].

Before discussing the third assumption, it is useful to consider the gauge freedom present in the formalism. It may appear that the formalism so far is explicitly gauge-invariant, since it is covariant under general diffeomorphisms. However, given a one-parameter family of solutions $g_{ab}(\varepsilon), T^{ab}(\varepsilon)$ of Einstein's equations on a manifold M, the gauge freedom consists of a one parameter family of diffeomorphisms $\varphi_\varepsilon : M \to M$, which act on the solutions via

$$g_{ab}(\varepsilon) \to \varphi_{\varepsilon\,*}\, g_{ab}(\varepsilon), \qquad T^{ab}(\varepsilon) \to \varphi_{\varepsilon\,*} T^{ab}(\varepsilon). \tag{2.8}$$

Here $\varphi_{\varepsilon\,*}$ is the pullback mapping on tensor fields that is defined in, for example, Appendix C of Ref. [20]. An important point is that, since we do not require the existence of a solution at $\varepsilon = 0$, there is no reason to require the diffeomorphisms φ_ε to have a well defined limit as $\varepsilon \to 0$. Thus, there are two subclasses of gauge transformations:

- Transformations which we will call *regular*, consisting of smooth one parameter families of diffeomorphisms which have a smooth limit as $\varepsilon \to 0$. Such

[2] If we replace the assumption (2.4) with a power series expansion of T^{ab} that starts with a term proportional to ε^ν with $\nu \neq 2$, we would obtain a non-Newtonian limit of general relativity where $\hat{c} \to \infty$ with $\hat{G}\hat{c}^{2\nu-4}$ held fixed.

families can be parameterized in terms of a fixed, ε-independent diffeomorphism φ_0 and a set of vector fields $\xi^a_{(1)}$, $\xi^a_{(2)}, \ldots$ via the expansion [21]

$$\varphi_\varepsilon = \varphi_0 \circ \mathcal{D}_{\vec{\xi}_{(1)}}(\varepsilon) \circ \mathcal{D}_{\vec{\xi}_{(2)}}(\varepsilon^2) \circ \ldots, \tag{2.9}$$

where for any vector field $\vec{\xi}$, $\mathcal{D}_{\vec{\xi}}(\varepsilon)$ is the diffeomorphism given by moving any point ε units along an integral curve of $\vec{\xi}$.

- Transformations which we will call *irregular*, consisting of smooth one parameter families φ_ε of diffeomorphisms which have do not have a smooth limit as $\varepsilon \to 0$.

Our assumptions are explicitly covariant under regular gauge transformations, as will be discussed in more detail in Appendix B below. However, they are not covariant under irregular gauge transformations. For example, consider the prototypical Newtonian-order metric that satisfies our assumptions (2.1) – (2.5), namely

$$\begin{aligned} ds^2 \;&=\; -\frac{1}{\varepsilon}\left[1 + 2\varepsilon\Phi(t,x^i) + O(\varepsilon^2)\right] dt^2 \\ &\quad + \left[\delta_{ij} + O(\varepsilon)\right] dx^i dx^j, \end{aligned} \tag{2.10}$$

where Φ is the Newtonian potential. Under the irregular gauge transformation $t \to \sqrt{\varepsilon}t$, $x^i \to x^i$, this metric is transformed into

$$\begin{aligned} ds^2 \;&=\; -\left[1 + 2\varepsilon\Phi(\sqrt{\varepsilon}t,x^i) + O(\varepsilon^2)\right] dt^2 \\ &\quad + \left[\delta_{ij} + O(\varepsilon)\right] dx^i dx^j, \end{aligned} \tag{2.11}$$

which does not satisfy our assumptions (2.1) – (2.5). Thus, our assumptions do entail a certain limited amount of gauge specialization[3], even though they are covariant under transformations of the form (2.9).

We now turn to a discussion of the third assumption, the expansion (2.5) of the connection ∇_a. First, we remark that assumptions 1 and 2 are insufficient to characterize the Newtonian limit. For example, consider a one parameter family of static vacuum spacetimes, of the form

$$ds^2 = -e^{2\alpha(x^k,\varepsilon)} dt^2 + h_{ij}(x^k,\varepsilon) dx^i dx^j, \tag{2.12}$$

which is smooth in ε near $\varepsilon = 0$. This one-parameter family of metrics, when written in terms of the coordinates $\bar{t} = \sqrt{\varepsilon}t$ and x^j, satisfies our assumptions 1 and 2, but is not of the type associated with the Newtonian limit. In particular components of the Riemann curvature tensor will diverge in this example as $\varepsilon \to 0$. Therefore some additional assumption like assumption 3 is necessary.[4] Our assumption 3 is actually slightly stronger than is necessary: we show in Appendix A that, whenever assumptions 1 and 2 hold in a local region, and the Riemann tensor $R_{abc}{}^d(\varepsilon)$ is finite as $\varepsilon \to 0$, then there exists a (possibly irregular) gauge transformation of the form (2.8) such that the transformed one parameter family of solutions satisfies assumptions 1, 2 and 3.

Finally we note that, if one wanted to go to higher post-Newtonian orders, the assumptions used here to obtain a covariant approximation scheme would need to be modified. First, to account for dissipative, radiative effects, one would need to introduce half-integer powers of ε in the expansions. These would first arise at order $O(\varepsilon^{5/2})$ in g_{ab} and order $O(\varepsilon^{7/2})$ in g^{ab}. Alternatively one could retain integer powers but make the replacement $\varepsilon \to \varepsilon^2$ throughout. Second, it is well known that solutions of the post-1-Newtonian field equations are not good approximations to exact solutions at distances $\gtrsim 1/\sqrt{\varepsilon}$. That is, although they work well in the near zone they break down in the local wave zone [22]. In order to find solutions which are good approximations everywhere one has to match post-Newtonian solutions onto radiation zone post-Minkowskian solutions; see for example Blanchet [1]. However, the corresponding corrections to the near zone gravitational fields and to the dynamics of the bodies arises at post-2.5-Newtonian order, and will therefore not be important for this paper.

III. DERIVATION OF NEWTON-CARTAN AND POST-NEWTON-CARTAN THEORIES FROM GENERAL RELATIVITY

In this section we derive the equations (1.1) – (1.5) of Newton-Cartan theory, (1.6) – (1.11) of perturbative post-Newton-Cartan theory, and (1.12a) – (1.16) of combined post-Newton-Cartan theory, from the assumptions discussed in Sec. II above. The derivation will be local and covariant[5].

We first note that it follows from assumption 1, together with an adjustment of the normalization of the

[3] Because of the limited gauge dependence of our assumptions, it is not *a priori* obvious that the fields D_a, t_a, h^{ab}, k^{ab}, etc. that characterize the Newton-Cartan and post-Newton-Cartan theories are physically unique. More precisely, suppose that we start with a one parameter family of solutions $g_{ab}(\varepsilon), T^{ab}(\varepsilon)$ which satisfies our assumptions, and is thus characterized by a set of limiting fields D_a, t_a, h^{ab}, k^{ab} etc. Now make a general (possibly irregular) gauge transformation, to obtain a new one parameter family $\bar{g}_{ab}(\varepsilon), \bar{T}^{ab}(\varepsilon)$ of solutions. If this new family also satisfies our assumptions, then it will be characterized by a new set of fields $\bar{D}_a$, $\bar{t}_a$, $\bar{h}^{ab}$, $\bar{k}^{ab}$. We conjecture that in this case there must exist a *regular* gauge transformation of the form (2.9) relating the two sets of fields (in the manner described in Appendix B), so that the the fields are unique in a physical sense.

[4] Dautcourt [19] shows that assumption 3 follows from assumptions 1 and 2 when one makes additional assumptions about the global properties of the spacetime including asymptotic flatness. However, we will not follow this route here, since we want to obtain a purely local derivation of the Newton-Cartan and post-Newton-Cartan theories.

[5] A similar approach to deriving a covariant post-Newtonian theory was undertaken by L. Gunnarsen (unpublished) at the University of Chicago in the 1980s.

one-form t_a if necessary, that the covariant components of the metric can be expanded as

$$g_{ab}(\varepsilon) = -\frac{1}{\varepsilon}t_a t_b + p_{ab} + \varepsilon q_{ab} + o(\varepsilon), \tag{3.1}$$

where t_a, p_{ab} and q_{ab} are ε-independent tensor fields on spacetime. To see this, choose a basis of vector fields $e^a_{\hat{\alpha}}$ for $\hat{\alpha} = 0, 1, 2, 3$ for which $h^{ab} = \delta^{\hat{i}\hat{j}} e^a_{\hat{i}} e^b_{\hat{j}}$. Then by assumption we have $k^{\hat{0}\hat{0}} \neq 0$, and in fact $k^{\hat{0}\hat{0}}$ must be negative in order for $g_{ab}(\varepsilon)$ to have signature $(-,+,+,+)$ for small ε. Now expanding and inverting on this basis the expression (2.2) for the contravariant metric yields an expression of the form (3.1).

Next, in any coordinate system we can compute the coefficients $\Gamma^\alpha_{\beta\gamma}(\varepsilon)$ of the connection by using the expansions (3.1) and (2.2) of the covariant and contravariant metrics. This yields an expression of the form

$$\Gamma^\alpha_{\beta\gamma}(\varepsilon) = O(\varepsilon^{-1}) + O(\varepsilon^0) + O(\varepsilon) + o(\varepsilon), \tag{3.2}$$

where the first three terms can be computed explicitly from the fields appearing in the metric expansions. Now, from assumption 3 it follows that the first term in (3.2) vanishes, and the second term gives the coefficients of the Newtonian connection D_a defined in Eq. (2.5). Therefore we can write, for any one-form w_a,

$$\nabla_a(\varepsilon) w_b = D_a w_b - \varepsilon \Delta^c{}_{ab} w_c + o(\varepsilon), \tag{3.3}$$

where $\Delta^c{}_{ab}$ is an ε-independent tensor field which is symmetric in a and b. This quantity is the post-Newtonian perturbation to the connection. The ε in brackets on the left hand side of Eq. (3.3) indicates the dependence of the derivative operator on ε, not an application of the derivative operator to ε.

A. Newton-Cartan theory and perturbative post-Newton-Cartan theory

1. *Orthogonality conditions*

We start with the definition of the contravariant metric,

$$g^{ab}(\varepsilon) g_{bc}(\varepsilon) = \delta^a_c, \tag{3.4}$$

and insert the expansions (3.1) and (2.2) of the covariant and contravariant metrics. At order $O(\varepsilon^{-1})$ this yields the Newton-Cartan orthogonality condition (1.1), and at order $O(\varepsilon^0)$ it yields the corresponding post-Newton-Cartan condition (1.6).

2. *Compatibility conditions*

Next, we insert the expansions (3.1) and (2.2) of the covariant and contravariant metrics and the expansion (3.3) of the connection into the equations

$$\nabla_a(\varepsilon) g_{bc}(\varepsilon) = 0, \quad \nabla_a(\varepsilon) g^{bc}(\varepsilon) = 0. \tag{3.5}$$

At leading order, this yields the Newton-Cartan compatibility conditions (1.2a) and (1.2b), and at subleading order one obtains the post-Newton-Cartan compatibility conditions (1.7) and (1.8).

3. *Trautman conditions*

We next compute the Riemann tensor using the expansion (3.3) of the connection. This yields

$$\begin{aligned} R_{abc}{}^d[\nabla_e(\varepsilon)] \;=\; & R_{abc}{}^d[D_e] + \varepsilon\left[D_b \Delta^d{}_{ac} - D_a \Delta^d{}_{bc}\right] \\ & + o(\varepsilon). \end{aligned} \tag{3.6}$$

Here the first term on the right hand side is the Riemann tensor of the connection D_a, which we will denote henceforth simply as $R_{abc}{}^d$. From the symmetries of the Riemann tensor it follows that

$$g^{f[a}(\varepsilon) R_{f(bc)}{}^{d]}[\nabla_e(\varepsilon)] = 0, \tag{3.7}$$

which is called the Trautman condition [16] [6]. Inserting the expansion (2.2) of the contravariant metric and (3.6) of the Riemann tensor, and expanding order by order in ε, gives the Newton-Cartan Trautman condition (1.3) at leading order, and the corresponding post-Newton-Cartan condition (1.9) at subleading order.

4. *Stress energy conservation*

Next, we insert the expansions (3.3) and (2.4) of the connection and stress energy tensor into the conservation equation

$$\nabla_a(\varepsilon) T^{ab}(\varepsilon) = 0. \tag{3.8}$$

At leading order this yields the Newtonian stress energy conservation equation (1.5), and at subleading order the post-Newtonian equation (1.11).

5. *Field equation*

Finally, we write the Einstein field equation (2.1) in the form

$$R_{acb}{}^c[\nabla_e(\varepsilon)] \;=\; 4\pi\left[2 g_{ac}(\varepsilon) g_{bd}(\varepsilon) - g_{ab}(\varepsilon) g_{cd}(\varepsilon)\right] T^{cd}(\varepsilon). \tag{3.9}$$

Inserting the expansions (3.6), (3.1) and (2.4) of the Riemann tensor, covariant metric and stress-energy tensor yields at leading order the Newton-Cartan field equation (1.4), and at subleading order the post-Newton-Cartan field equation (1.10).

[6] For many calculations it is advantageous to rewrite the Trautman condition as $g^{ea} R_{ebc}{}^d - g^{ed} R_{ecb}{}^a = 0$.

B. Combined post-Newton-Cartan theory

In this section we derive the equations (1.12a) – (1.16) of combined post-Newton-Cartan theory from the Newton-Cartan and perturbative post-Newton-Cartan theories.

Suppose we have a solution of the Newton-Cartan and perturbative post-Newton-Cartan theories, consisting of the fields t_a, p_{ab}, h^{ab}, k^{ab}, D_a, Δ^a_{bc}, $\mathcal{T}^{ab}$ and $\mathcal{S}^{ab}$. Such solutions posess a scaling symmetry corresponding to a change in units of time. Specifically, it is easy to check that for any real number λ there is a mapping of solutions to solutions given by rescaling the fields by $t_a \to e^{\lambda} t_a$, $k^{ab} \to e^{-2\lambda} k^{ab}$, $\mathcal{T}^{ab} \to e^{-4\lambda}\mathcal{T}^{ab}$, $\mathcal{S}^{ab} \to e^{-6\lambda}\mathcal{S}^{ab}$, $\Delta^a_{bc} \to e^{-2\lambda}\Delta^a_{bc}$, with the other fields being left unchanged. If we now consider the expansions (2.2), (3.1), (3.3) and (2.4) of the contravariant metric, covariant metric, connection and stress-energy tensor, truncated to post-1-Newtonian order, we can apply this rescaling to effectively set $\varepsilon = 1$ in these expansions. Specifically, for any $\varepsilon > 0$, we define the hatted fields by

$$\hat{t}_a = \frac{1}{\sqrt{\varepsilon}} t_a, \tag{3.10a}$$

$$\hat{h}^{ab} = h^{ab}, \tag{3.10b}$$

$$\hat{p}_{ab} = p_{ab}, \tag{3.10c}$$

$$\hat{k}^{ab} = \varepsilon k^{ab}, \tag{3.10d}$$

$$\hat{D}_a w_b = D_a w_b - \varepsilon \Delta^c_{ab} w_c, \tag{3.10e}$$

$$\hat{\mathcal{T}}^{ab} = \varepsilon^2 \mathcal{T}^{ab} + \varepsilon^3 \mathcal{S}^{ab}, \tag{3.10f}$$

for any one-form w_a. These will be the fundamental variables of the combined theory.

Next, we note that the orthogonality conditions (1.1) and (1.6) are preserved under rescaling, which yields the orthogonality conditions (1.12a) and (1.12b) for the hatted variables.

Next, using the definitions (3.10) of the hatted fields and the connection-compatibility conditions (1.2), (1.7) and (1.8) we obtain

$$\hat{D}_a(\hat{h}^{bc} + \hat{k}^{bc}) = -\varepsilon^2(\Delta^b{}_{ad}k^{dc} - \Delta^c{}_{ad}k^{bd}), \tag{3.11a}$$

$$\hat{D}_a(-\hat{t}_b\hat{t}_c + \hat{p}_{bc}) = \varepsilon(\Delta^d_{ab}p_{dc} + \Delta^d_{ac}p_{bd}). \tag{3.11b}$$

The right hand sides of both of these equations are of post-2-Newtonian order. Therefore we can drop the right hand sides to obtain the compatibility conditions (1.13); this modification affects the theory only at post-2-Newtonian and higher orders, and not at Newtonian or post-1-Newtonian order.

Finally, we can use exactly analogous arguments to derive the Trautman condition (1.14), field equation (1.15) and stress-energy conservation equation (1.16) of the combined post-Newton-Cartan theory from the corresponding equations of the Newton-Cartan and perturbative post-Newton-Cartan theories.

IV. DERIVATION OF STANDARD, COORDINATE-SPECIFIC POST-NEWTONIAN THEORY FROM PERTURBATIVE POST-NEWTON-CARTAN THEORY

A. Change of viewpoint

So far, we have shown that the Newton-Cartan and post-Newton-Cartan theories can be derived from general relativity together with the three assumptions discussed in Sec. II.

We now make a change of viewpoint, and consider these theories as independent theories in their own right, independent of general relativity. In other words, we forget about the spacetime metric, and instead regard the fields t_a, h^{ab}, D_a of the Newton-Cartan theory, and p_{ab}, k^{ab} and Δ^c_{ab} of the post-Newton-Cartan theory, as fundamental. It is well known that the usual coordinate-dependent formulation of Newtonian gravity can be derived from the resulting Newton-Cartan theory, under the assumption of asymptotic flatness. In this section we will show that, similarly, the usual coordinate-dependent formulations of post-1-Newtonian theory can be derived from the post-Newton-Cartan theory, in suitably chosen coordinate systems, again under the assumption of asymptotic flatness.

B. Derivation of Newtonian theory from Newton-Cartan theory

We start by reviewing the well-known derivation at Newtonian order [19]. We assume that the Newton-Cartan equations (1.1) – (1.5) are valid throughout all of spacetime, and that the Riemann tensor of the connection D_a goes to zero at spatial infinity (asymptotic flatness).

From the metric compatibility condition (1.2b), it follows that there exists a function t on spacetime for which $t_a = D_a t$. We will call this function the time function. We now introduce a coordinate system $x^\alpha = (x^0, x^j)$ with $x^0 = t$. The orthogonality condition (1.1) and Eq. (1.2b) then immediately lead to

$$t_\mu = t_{,\mu} = \delta^0_\mu \tag{4.1}$$

and

$$h^{\mu 0} = 0. \tag{4.2}$$

Next, the compatibility condition (1.2b) gives $\partial_\mu t_\nu - \Gamma^\lambda_{\mu\nu} t_\lambda = 0$, where $\Gamma^\lambda_{\mu\nu}$ are the coefficients of the connection D_a, which yields

$$\Gamma^0_{\mu\nu} = 0 \tag{4.3}$$

from Eq. (4.1). Similarly the compatibility condition (1.2a) yields

$$\partial_\mu h^{\alpha\beta} + \Gamma^\alpha_{\mu\nu} h^{\nu\beta} + \Gamma^\beta_{\mu\nu} h^{\alpha\nu} = 0. \tag{4.4}$$

If we now define a spatial covariant metric h_{kj} by

$$h^{ik}h_{kj} = \delta^i_j, \tag{4.5}$$

then Eq. (4.4) results in

$$\Gamma^i_{lm} = \frac{1}{2}h^{ik}\left(h_{kl,m} + h_{km,l} - h_{lm,k}\right), \tag{4.6}$$

$$\Gamma^i_{0l} = \frac{1}{2}h^{ik}\left(h_{kl,0} - \epsilon_{klm}B_m\right), \tag{4.7}$$

and

$$\Gamma^i_{00} = h^{ik}\Phi_k. \tag{4.8}$$

Here the quantities Φ_k and B_m are still undetermined.

Next, the field Eqs. (1.4) imply

$$R_{i0} = 0, \tag{4.9a}$$

$$R_{ij} = 0, \tag{4.9b}$$

where $R_{\alpha\beta} = R_{\alpha\beta}[D_a]$ is the Ricci tensor computed from the Newtonian connection D_a. Computing the spatial components of this Ricci tensor explicitly, and simplifying using the condition (4.3), gives

$$R_{ij} = \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^l_{ik}\Gamma^k_{jl}. \tag{4.10}$$

Combining this with Eq. (4.6) now gives $R_{ij} = {}^{(3)}R_{ij}[h_{kl}]$, where the right hand side is the three-dimensional Ricci tensor computed from the metric h_{kl} at fixed t. Therefore, from Eq. (4.9b), the Ricci tensor of the metric h_{ij} vanishes. Since this metric is three dimensional, it follows that the metric is flat. Hence at each fixed t we can choose the spatial coordinates so that

$$h_{ij} = \delta_{ij}. \tag{4.11}$$

Next, combining the field equation (4.9a) with the connection coefficients (4.6) – (4.8) and simplifying using the coordinate condition (4.11) gives

$$\epsilon_{ijk}\partial_j B_k = 0. \tag{4.12}$$

Also the Trautman condition (1.3) implies that $\mathbf{B}$ is transverse, $\partial_i B_i = 0$. Together with Eq. (4.12) this implies that the field $\mathbf{B}$ satisfies Laplace's equation, $B_{i,jj} = 0$. Next, using the assumption that $R_{\alpha\beta\gamma}{}^{\delta} \to 0$ as $r \to \infty$, we find that the only allowed nontrivial solutions to Laplace's equation are those with $\mathbf{B} =$ constant. These solutions can be eliminated by transforming to a uniformly rotating coordinate system. It follows that, in a suitably adjusted coordinate system, $B_i = 0$. Finally, the Trautman condition (1.3) implies that $\Phi_{i,j} - \Phi_{j,i} = \epsilon_{ijk}\dot{B}_k = 0$, so that $\Phi_i = \partial_i\Phi$ for some function Φ, which will be the Newtonian potential. It follows that

$$\Gamma^i_{00} = \Phi_{,i}, \tag{4.13}$$

while all the other connection coefficients vanish. The field equation (1.4) then reduces to the Poisson equation $\Phi_{,kk} = 4\pi\mathcal{T}^{00}$, and the stress-energy conservation equation (1.5) reduces to $\partial_\alpha\mathcal{T}^{\alpha 0} = 0$, $\partial_\alpha\mathcal{T}^{\alpha i} + \Phi_{,i}\mathcal{T}^{00} = 0$. These are the standard equations of Newtonian gravity.

We note that the Newton-Cartan theory contains more local degrees of freedom that Newtonian theory. In particular, if one assumes that Newton-Cartan theory holds only in a local region of spacetime, or if one assumes it holds everywhere but drops the assumption of asymptotic flatness, then one obtains a theory with an additional transverse vector field $\mathbf{B}$ that satisfies Laplace's equation. This is just the gravitomagnetic field which normally arises at post-1-Newtonian order. Thus, the Newton-Cartan theory admits source-free gravitomagnetic fields at Newtonian order.

Another method that has been used in the literature to exclude these extra degrees of freedom is to assume that [17, 23, 24]

$$h^{ec}R_{abc}{}^{d} = 0, \tag{4.14}$$

where $R_{abc}{}^{d}$ is the Riemann tensor of the connection D_a. Augmenting the equations of Newton-Cartan theory with this assumption yields a covariant theory which is equivalent, locally, to Newtonian gravity. However, the assumption (4.14) cannot be derived from General Relativity in a local manner; its validity requires the use of global information. For this reason we do not use the assumption (4.14) in this paper.

C. Derivation of post-Newtonian theory from perturbative post-Newton-Cartan theory

We now extend the above derivation to post-1-Newtonian order. We continue to use the adapted coordinate system derived above, and we assume that the post-Newton-Cartan equations are valid throughout all of spacetime.

The spatial components of the orthogonality condition (1.6) imply that $h^{ik}p_{kj} = \delta^i_j$. Hence the covariant spatial metric h_{kj} defined in Eq. (4.5) and the spatial components of p_{ab} coincide, and in our adapted coordinate system we have

$$p_{ij} = h_{ij} = \delta_{ij}. \tag{4.15}$$

The remaining components of the orthogonality condition (1.6) yield

$$k^{00} = -1, \tag{4.16}$$

and

$$k^{i0} = h^{ik}p_{k0} = p_{i0}. \tag{4.17}$$

In order to find expressions for p_{i0} and p_{00} we have to consider the metric compatibility condition (1.8), which yields

$$2\Delta^0{}_{\mu 0} = -p_{00,\mu} + 2\Delta^\lambda_{\mu 0}p_{\lambda 0}. \tag{4.18}$$

It follows that

$$\Delta^0{}_{00} = -\frac{1}{2}p_{00,0} + \Phi_{,l}p_{l0} \tag{4.19}$$

and

$$\Delta^0{}_{j0} = -\frac{1}{2}p_{00,j}. \tag{4.20}$$

The condition (1.8) also yields

$$\Delta^0{}_{\mu j} = -p_{0j,\mu} + \Delta^\lambda_{\mu j}p_{\lambda 0} + \Delta^\lambda_{\mu 0}p_{j\lambda}, \tag{4.21}$$

which implies

$$\Delta^0{}_{ij} = -p_{0j,i} \tag{4.22}$$

and

$$\Delta^0{}_{0j} = \Phi_{,j} - p_{j0,0}. \tag{4.23}$$

Since the connection is symmetric, Eq. (4.22) yields $p_{0i,j} - p_{0j,i} = 0$, and thus $p_{0i} = g_{,i}$, for some function g. Now under infinitesimal gauge transformations p_{0i} transforms as, from Eq. (B4b),

$$p_{0i} \to p_{0i} - \xi^0{}_{,i}, \tag{4.24}$$

where ξ^0 is an arbitrary function. Therefore by taking $\xi^0 = g$ we can specialize the gauge to enforce

$$p_{0i} = 0. \tag{4.25}$$

Next, the symmetry of the connection applied to Eqs. (4.20) and (4.23) yields together with Eq. (4.25) that $p_{00,j} = -2\Phi_{,j}$, which implies $p_{00} = -2\Phi + \chi(x^0)$, where $\chi(x^0)$ is a function which depends only on x^0. Now note that the gauge condition (4.25) does not completely fix the gauge; from Eq. (4.24) gauge transformations with $\xi^0{}_{,i} = 0$ leave the condition (4.25) invariant. Since p_{00} transforms like $p_{00} \to p_{00} - 2\xi^0{}_{,0}$ under the infinitesimal gauge transformations of Eq. (B4b), we can further specialize the gauge to enforce

$$p_{00} = -2\Phi. \tag{4.26}$$

Simplifying Eqs. (4.19), (4.20) and (4.22) using the gauge conditions Eqs. (4.25) and (4.26) now yields

$$\Delta^0{}_{00} = \Phi_{,0}, \quad \Delta^0{}_{i0} = \Phi_{,i}, \quad \Delta^0{}_{ij} = 0. \tag{4.27}$$

Next we determine the remaining components of $\Delta^c{}_{ab}$. From the compatibility condition (1.7) we get

$$\Delta^i{}_{\alpha l}h^{lj} + \Delta^j{}_{\alpha l}h^{li} = -k^{ij}{}_{,\alpha} - \Gamma^i_{\alpha\lambda}k^{\lambda j} - \Gamma^j_{\alpha\lambda}k^{i\lambda}. \tag{4.28}$$

This leads to

$$\Delta^i{}_{kj} = -\frac{1}{2}k^{ij}{}_{,k} + W_{ijk} \tag{4.29}$$

and

$$\Delta^i{}_{0j} = -\frac{1}{2}k^{ij}{}_{,0} + V_{ij}, \tag{4.30}$$

where $W_{ijk} = -W_{jik}$ and $V_{ij} = -V_{ji}$ are undetermined. Since the connection is symmetric it follows from Eq. (4.29) that

$$W_{ijk} = W_{ikj} + \frac{1}{2}\left(k^{ij}{}_{,k} - k^{ik}{}_{,j}\right). \tag{4.31}$$

Together with $W_{ijk} = -W_{jik}$ this implies that

$$\Delta^i{}_{jk} = \frac{1}{2}\left(k^{jk}{}_{,i} - k^{ij}{}_{,k} - k^{ik}{}_{,j}\right). \tag{4.32}$$

Thus, from the metric compatibility conditions (1.8) and (1.7) we have been able to determine all the components of $\Delta^\lambda{}_{\mu\nu}$, except for $\Delta^i{}_{0j}$ and $\Delta^i{}_{00}$. To determine those components we use the Trautman condition (1.9), which can be rewritten as

$$\begin{aligned} k^{de}R_{ecb}{}^a - k^{ae}R_{ebc}{}^d &= \left(D_b\Delta^d{}_{ec} - D_e\Delta^d{}_{bc}\right)h^{ae} \\ &+ \left(D_e\Delta^a{}_{bc} - D_c\Delta^a{}_{be}\right)h^{de}. \end{aligned} \tag{4.33}$$

Specializing to $a = n$, $b = 0$, $c = k$ and $d = m$ gives

$$D_0\Delta^m{}_{nk} - D_n\Delta^m{}_{0k} + D_m\Delta^n{}_{0k} - D_c\Delta^n{}_{0m} = 0, \tag{4.34}$$

which using Eqs. (4.32) and (4.30) simplifies to

$$V_{nk,m} - V_{mk,n} - V_{nm,k} = 0. \tag{4.35}$$

Since V_{nm} is antisymmetric the solution of (4.35) is

$$V_{mn} = \frac{1}{2}\left(\gamma_{m,n} - \gamma_{n,m}\right), \tag{4.36}$$

where γ_m is some undetermined vector field.

Next, considering the $a \to n$, $b = c \to 0$, $d \to m$ components of Eq. (4.33), we find

$$\begin{aligned} D_0\Delta^m{}_{n0} - D_n\Delta^m{}_{00} + D_m\Delta^n{}_{00} - D_0\Delta^n{}_{0m} \\ = \frac{1}{2}\left(k^{ml}p_{00,nl} - k^{nl}p_{00,ml}\right). \end{aligned} \tag{4.37}$$

If we define

$$A_m \equiv \Delta^m{}_{00} + \frac{1}{2}p_{00,l}k^{ml} - \gamma_{m,0}, \tag{4.38}$$

then Eq. (4.37) becomes

$$A_{m,n} - A_{n,m} = 0, \tag{4.39}$$

so we can write

$$A_m = -\frac{1}{2}\gamma_{,m}, \tag{4.40}$$

for some function γ. Combining Eqs. (4.38) and (4.40) now yields

$$\Delta^i{}_{00} = -\frac{1}{2}\gamma_{,i} + \gamma_{i,0} - \frac{1}{2}p_{00,l}k^{li}, \tag{4.41}$$

while Eqs. (4.36) and (4.30) lead to

$$\Delta^{i}{}_{0j} = \frac{1}{2}\left(\gamma_{i,j} - \gamma_{j,i}\right) - \frac{1}{2}k^{ij}{}_{,0}. \qquad (4.42)$$

We now define a metric $\hat{g}_{ab}(\varepsilon)$, for each $\varepsilon > 0$, by the formula

$$\hat{g}_{ab}(\varepsilon) = -\frac{1}{\varepsilon}t_a t_b + p_{ab} + \varepsilon q_{ab}, \qquad (4.43)$$

cf. the metric expansion (3.1) above. Here the tensor q_{ab} is defined by

$$\begin{aligned} q_{00} &= \gamma, && (4.44a)\\ q_{0i} &= \gamma_i, && (4.44b)\\ q_{ij} &= -k^{ij}. && (4.44c) \end{aligned}$$

We compute the inverse of this metric, using the values of the components of the fields t_a, p_{ab} and q_{ab} given in Eqs. (4.1), (4.15), (4.25), (4.26) and (4.44). The result is of the form [cf. Eq. (2.2) above]

$$\hat{g}^{ab}(\varepsilon) = h^{ab} + \varepsilon k^{ab} + \varepsilon^2 j^{ab} + O(\varepsilon^3). \qquad (4.45)$$

Here the components of the fields h^{ab} and k^{ab} (except for k^{ij}) are those given by Eqs. (4.2), (4.11), (4.16), (4.17) and (4.25). This result is guaranteed because we have imposed the the orthogonality conditions (1.1) and (1.6). The fact that the spatial components of the coefficient of ε in Eq. (4.45) are k^{ij} follows from the choice (4.44c) of spatial components of q_{ab}. The explicit form of the tensor j^{ab} which appears in Eq. (4.45) will not be needed in what follows.

Next, we compute the coefficients $\hat{\Gamma}^{\alpha}_{\beta\gamma}(\varepsilon)$ of the Levi-Civita connection associated with the metric (4.43), using the expansion (4.45). Suppressing indices, the result is schematically of the form

$$\begin{aligned} \hat{\Gamma} \ \sim \ & \frac{1}{\varepsilon}(ht\partial t) + (h\partial p + kt\partial t) \\ & + \varepsilon(h\partial q + k\partial p + jt\partial t) + O(\varepsilon^2). \qquad (4.46) \end{aligned}$$

The leading order, $O(\varepsilon^{-1})$ term vanishes identically by virtue of Eq. (4.1). We can evaluate the next order, $O(\varepsilon^0)$ term using the the specific values of the components of t_a, p_{ab}, h^{ab} and k^{ab} in our adapted coordinate system, given by Eqs. (4.1), (4.2), (4.11), (4.15), (4.16), (4.17), (4.25) and (4.26). The resulting expressions are just the coefficients $\Gamma^{\alpha}_{\beta\gamma}$ of the Newton-Cartan connection D_a, with the only nonzero component being given by Eq. (4.13). Again, this result is not surprising because we have enforced the compatibility conditions (1.2a), (1.2b), (1.7) and (1.8). Similarly, at the next order, we find that the $O(\varepsilon)$ components in Eq. (4.46) coincide with the components of the post-Newton-Cartan field $\Delta^{\alpha}_{\beta\gamma}$, given by Eqs. (4.27), (4.32), (4.41) and (4.42). [Note that the result is independent of $j^{\alpha\beta}$, by virtue of Eq. (4.1)].

To summarize, we have been able to show that all of our Newton-Cartan and post-Newton-Cartan fields can be derived from the three fields t_a, p_{ab} and q_{ab} that enter into the expansion (4.43) of the metric, using the standard equations of general relativity. Moreover, that metric expansion is of the standard post-Newtonian form; using the specific values of the components of t_a, p_{ab} and q_{ab} given above and writing $\varepsilon = c^{-2}$, Eq. (4.43) takes the form

$$\begin{aligned} ds^2 \ = \ & -c^2\left[1 + \frac{2\Phi}{c^2} - \frac{\gamma}{c^4} + O\left(\frac{1}{c^6}\right)\right]dt^2 \\ & +2\left[\frac{\gamma_i}{c^2} + O\left(\frac{1}{c^4}\right)\right]dx^i dt \\ & +\left[\delta_{ij} + \frac{1}{c^2}q_{ij} + O\left(\frac{1}{c^4}\right)\right]dx^i dx^j. \quad (4.47) \end{aligned}$$

This is the standard starting point for coordinate-specific post-Newtonian theory, involving a post-Newtonian correction to the Newtonian potential Φ, and a gravitomagnetic potential γ_i [7]. It follows that all of the relations of the Newton-Cartan and post-Newton-Cartan theories, when expressed in terms Φ, γ, γ_i and q_{ij}, are either identically satisfied, or reduce to the Einstein equations that one would compute directly from the metric (4.43), i.e. the coordinate-specific post-Newtonian equations. Furthermore we have shown how to obtain the quantities Φ, γ, γ_i and q_{ij}, starting from solutions of the Newton-Cartan and post-Newton-Cartan equations. It follows that the equations of the perturbative post-Newton-Cartan theory are equivalent to those of the standard coordinate-specific post-Newtonian theory.

V. DISCUSSION AND CONCLUSIONS

We have derived a covariant version of the equations of the post-1-Newtonian approximation to general relativity. These equations reduce to the standard coordinate formulation of post-Newtonian theory in asymptotically flat spacetimes in suitable coordinate systems.

Although the covariant formulation is elegant, it does not provide a very compact or efficient representation of the theory. In a general coordinate system the combined post-Newton-Cartan theory involves 74 free functions to describe the geometry, as compared to 4 for standard post-Newtonian theory. The covariant formulation is therefore mostly of formal interest. However it may be useful to connect the different gauge-dependent formulations that are found in the literature. It may also be useful for deriving general properties of post-Newtonian theory. It might also provide insight into the meaning of the parameters of the parameterized post-Newtonian

[7] The spatial tensor q_{ij} does not contain any independent degrees of freedom; using the post-Newtonian field equations and making a gauge specialization gives $q_{ij} = -2\Phi\delta_{ij}$ [9]. This gauge specialization is nearly always adopted in post-Newtonian theory.

(PPN) framework [2], if the analysis of this paper were generalized to the class of theories of gravity encompassed by that framework.

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Appendix A: Limiting behavior of connection derived from other postulates

In this appendix we show that our assumption 3 on the limiting behavior of the connection will hold (up to gauge transformations) whenever the Riemann tensor is bounded as $\varepsilon \to 0$. More precisely, whenever assumptions 1 and 2 hold in a local region, and the Riemann tensor $R_{abc}{}^{d}(\varepsilon)$ is finite as $\varepsilon \to 0$, then we show that there exists a (possibly irregular) gauge transformation of the form (2.8) such that the transformed one parameter family of solutions satisfies assumptions 1, 2 and 3 of Sec. II A.

We start by fixing a coordinate system and computing the connection coefficients using the expansions (2.2) and (3.1) of the contravariant and covariant metrics and the orthogonality condition (1.1). The result is of the form [cf. Eq. (3.2) above]

$$\Gamma^{\alpha}_{\beta\gamma}(\varepsilon) = \varepsilon^{-1}\Gamma^{(-1)\alpha}{}_{\beta\gamma} + \Gamma^{(0)\alpha}{}_{\beta\gamma} + \varepsilon\Gamma^{(1)\alpha}{}_{\beta\gamma} + O(\varepsilon^2), \quad \text{(A1)}$$

with

$$\Gamma^{(-1)\alpha}{}_{\beta\gamma} = -h^{\alpha\lambda}(t_\gamma t_{[\lambda,\beta]} + t_\beta t_{[\lambda,\gamma]}) \quad \text{(A2)}$$

and

$$\begin{aligned}\Gamma^{(0)\alpha}{}_{\beta\gamma} &= -k^{\alpha\lambda}(t_\gamma t_{[\lambda,\beta]} + t_\beta t_{[\lambda,\gamma]} + t_\lambda t_{(\beta,\gamma)}) \\ &\quad + \frac{1}{2}h^{\alpha\lambda}(-p_{\beta\gamma,\lambda} + p_{\beta\lambda,\gamma} + p_{\gamma\lambda,\beta}). \quad \text{(A3)}\end{aligned}$$

Here $\Gamma^{(0)\alpha}{}_{\beta\gamma}$ are the coefficients of the Newtonian connection D_a, which were denoted simply $\Gamma^\alpha_{\beta\gamma}$ in the body of the paper. Also $\Gamma^{(1)\alpha}{}_{\beta\gamma}$ are the coefficients of the post-Newtonian connection perturbation, which were denoted $\Delta^\alpha_{\beta\gamma}$ in the body of the paper. We want to show that $\Gamma^{(-1)\alpha}{}_{\beta\gamma}$ vanishes.

We next compute the expansion of the Einstein tensor, which is of the form

$$\begin{aligned} G^{\alpha\beta}(\varepsilon) \; &= \; \varepsilon^{-2}G^{(-2)\alpha\beta} + \varepsilon^{-1}G^{(-1)\alpha\beta} + G^{(0)\alpha\beta} \\ &\quad + \varepsilon G^{(1)\alpha\beta} + O(\varepsilon^2). \end{aligned} \tag{A4}$$

It follows from assumption 2 of Sec. II A that the first four terms in this expansion all vanish. We find that $G^{(-2)\alpha\beta}$ vanishes identically, while $G^{(-1)\alpha\beta}$ is given by

$$G^{(-1)\alpha\beta} = 2H^\alpha{}_\mu H^\beta{}_\nu h^{\mu\nu} + \frac{1}{2}h^{\alpha\beta}H^\mu{}_\nu H^\nu{}_\mu, \tag{A5}$$

where $H^\alpha{}_\beta = h^{\alpha\gamma}t_{[\gamma,\beta]}$. Setting this expression to zero yields $H^\mu{}_\nu H^\nu{}_\mu = 0$, from which it follows [19] that

$$t_\alpha = f t_{,\alpha} \tag{A6}$$

for some functions f and t. We now specialize the coordinates by choosing $x^0 = t$, so that $h^{0\alpha} = 0$ and $t_\alpha = f\delta^0_\alpha$.

We now extend this computation to the next order. The orthogonality relation (1.6) implies that

$$k^{00} = -1/f^2, \quad k^{0i} = h^{ij}p_{0j}/f^2, \quad h^{ij}p_{jk} = \delta^i_k. \tag{A7}$$

Using these relations and the expansions (A2) and (A3) gives $G^{(0)00} = G^{(0)0i} = 0$ and

$$G^{(0)ij} = G^{ij}[h_{kl}] - \frac{1}{f}D^iD^jf + \frac{1}{2f}h^{ij}D^kD_kf. \tag{A8}$$

Here the first term denotes the three dimensional Einstein tensor computed from the metric $h_{ij} = p_{ij}$ (the inverse of h^{ij}), and D_i is the covariant derivative associated with that metric. Also the $O(\varepsilon^{-1})$ piece of the Riemann tensor is given by

$$R^{(-1)}{}_{0i0}{}^j = fD_iD^jf, \tag{A9}$$

with the other components being zero. Our assumption on the Riemann tensor forces this quantity to vanish, from which it follows from the vanishing of the expression (A8) for $G^{(0)ij}$ that the Einstein tensor of the metric h_{ij} must be zero. We can therefore specialize the coordinates so that $h_{ij} = h^{ij} = \delta_{ij}$.

It now follows from Eq. (A9) that $f_{,ij} = 0$, so that $f = \alpha(t) + \beta_i(t)x^i$. The leading order expression for the metric is therefore the Rindler metric, and we can apply the standard gauge transformation[8] that takes the Rindler metric to the Minkowski metric. The result of this transformation is to effectively set f to unity, and so from Eq. (A2) it follows that $\Gamma^{(-1)\alpha}{}_{\beta\gamma} = 0$ for the transformed one parameter family of metrics.

[8] In defining this gauge transformation we treat α and β_i as constants; their time dependence affects the final metric only at subleading order. The gauge transformation depends explicitly on ε and is not smooth as $\varepsilon \to 0$.

Appendix B: Gauge freedom in the post-Newtonian fields

In this appendix we derive how the Newtonian and post-Newtonian fields transform under a regular gauge transformation φ_ε of the form (2.9). Such a gauge transformation is parameterized by an ε-independent diffeomorphism φ_0, and by a set of vector fields $\vec{\xi}_{(1)}, \vec{\xi}_{(2)}, \ldots$, one for each order in ε. For simplicity, we will take φ_0 to be the identity mapping, since all quantities will transform trivially under this portion of the overall diffeomorphism. Consider now any tensor field $S(\varepsilon)$ which depends on ε, and has an expansion of the form

$$S(\varepsilon) = S^{(0)} + \varepsilon S^{(1)} + \varepsilon^2 S^{(2)} + O(\varepsilon^3). \tag{B1}$$

Here for brevity we have suppressed any tensor indices on S. We define the transformed expansion coefficients $\bar{S}^{(j)}$ via the expansion

$$\varphi_{\varepsilon\,*}S(\varepsilon) = \bar{S}(\varepsilon) = \bar{S}^{(0)} + \varepsilon\bar{S}^{(1)} + \varepsilon^2\bar{S}^{(2)} + O(\varepsilon^3). \tag{B2}$$

From Eq. (2.9) it now follows that [21]

$$\bar{S}^{(0)} \; = \; S^{(0)}, \tag{B3a}$$

$$\bar{S}^{(1)} \; = \; S^{(1)} + \mathcal{L}_{\vec{\xi}_1}S^{(0)}, \tag{B3b}$$

$$\begin{aligned} \bar{S}^{(2)} \; &= \; S^{(2)} + \mathcal{L}_{\vec{\xi}_2}S^{(0)} + \mathcal{L}_{\vec{\xi}_1}S^{(1)} \\ &\quad + \frac{1}{2}\mathcal{L}_{\vec{\xi}_1}\mathcal{L}_{\vec{\xi}_1}S^{(0)}, \end{aligned} \tag{B3c}$$

where $\mathcal{L}$ is the Lie derivative.

We now apply this formalism to the expansions (2.2), (3.1), (2.4) and (2.5) of the contravariant metric, covariant metric, stress-energy tensor and connection. We use of the compatibility conditions (1.2), denote gauge-transformed quantities with bars, and rewrite Lie derivatives in terms of D_a derivatives. This yields that the Newtonian fields h^{ab}, t_a, D_a and $\mathcal{T}^{ab}$ are invariant, while the post-Newtonian fields transform as

$$\bar{k}^{ab} \; = \; k^{ab} - h^{ac}D_c\xi^b - h^{bc}D_c\xi^a, \tag{B4a}$$

$$\bar{p}_{ab} \; = \; p_{ab} - t_at_cD_b\xi^c - t_bt_cD_a\xi^c, \tag{B4b}$$

$$\bar{\Delta}^c{}_{ab} \; = \; \Delta^c{}_{ab} - 2\xi^dR_{d(ab)}{}^c + D_{(a}D_{b)}\xi^c, \tag{B4c}$$

$$\bar{\mathcal{S}}^{ab} \; = \; \mathcal{S}^{ab} + \xi^cD_c\mathcal{T}^{ab} - 2\mathcal{T}^{c(a}D_c\xi^{b)}. \tag{B4d}$$

Here we have written $\vec{\xi}_{(1)}$ simply as $\vec{\xi}$. One can check that the post-Newton-Cartan equations (1.6)–(1.11) are invariant under these transformations, as they must be.

To obtain the formula (B4c), let $\omega_b(\varepsilon)$ be an arbitrary one form which depends smoothly on ε, with the expansion $\omega_b = \omega_b^{(0)} + \varepsilon\omega_b^{(1)} + O(\varepsilon^2)$. We define the tensor

$S_{ab} = \nabla_a \omega_b$, which has the expansion

$$\begin{aligned} S_{ab}(\varepsilon) &= S^{(0)}_{ab} + \varepsilon S^{(1)}_{ab} + O(\varepsilon^2) \\ &= D_a \omega^{(0)}_b + \varepsilon \left[D_a \omega^{(1)}_b - \Delta^c{}_{ab} \omega^{(0)}_c \right] + O(\varepsilon^2). \end{aligned} \tag{B5}$$

Applying the general transformation rule (B3) now yields

$$S^{(1)}_{ab} \to S^{(1)}_{ab} + \mathcal{L}_\xi S^{(0)}_{ab}, \tag{B6}$$

and

$$\omega^{(1)}_a \to \omega^{(1)}_a + \mathcal{L}_\xi w^{(0)}_a. \tag{B7}$$

Combining Eqs. (B5) – (B7) now yields

$$\begin{aligned} \Delta^c{}_{ab} \omega^{(0)}_c \;\to\; & \Delta^c{}_{ab} \omega^{(0)}_c + \xi^d D_a D_d \omega^{(0)}_b - \xi^d D_d D_a \omega^{(0)}_b \\ & + \omega^{(0)}_d D_a D_b \xi^d \end{aligned} \tag{B8}$$

which results in Eq. (B4c).

Appendix C: perfect fluids

In this appendix we describe as an example how perfect fluids can be described in the covariant formalism. The stress energy tensor is

$$T^{ab} = (\rho + p) u^a u^b + p g^{ab}, \tag{C1}$$

where ρ is the density, p the pressure and u^a the four-velocity. The appropriate form of the expansions of these fields is

$$u^a = \sqrt{\varepsilon} \left[u^a_{\rm n} + \varepsilon u^a_{\rm pn} + O(\varepsilon^2) \right], \tag{C2a}$$

$$\rho = \varepsilon \rho_{\rm n} + \varepsilon^2 \rho_{\rm pn} + O(\varepsilon^3), \tag{C2b}$$

$$p = \varepsilon^2 p_{\rm n} + \varepsilon^3 p_{\rm pn} + O(\varepsilon^3). \tag{C2c}$$

Here the subscripts "n" and "pn" indicate the Newtonian-order and post-Newtonian order pieces. Comparing with the expansion (2.4) of the stress energy tensor yields the formulae

$$\mathcal{T}^{ab} = \rho_{\rm n} u^a_{\rm n} u^b_{\rm n} + p_{\rm n} h^{ab}, \tag{C3a}$$

$$\begin{aligned} \mathcal{S}^{ab} = \; & (\rho_{\rm pn} + p_{\rm n}) u^a_{\rm n} u^b_{\rm n} + 2 \rho_{\rm n} u^{(a}_{\rm n} u^{b)}_{\rm pn} + p_{\rm n} k^{ab} \\ & + p_{\rm pn} h^{ab}. \end{aligned} \tag{C3b}$$

Also the normalization of the four-velocity yields the conditions

$$t_a u^a_{\rm n} = 1, \quad p_{ab} u^a_{\rm n} u^b_{\rm n} = 2 t_a u^a_{\rm pn}. \tag{C4}$$

One can check that inserting these expressions in the stress-energy conservation laws (1.5) and (1.11), using the specific forms of h^{ab}, k^{ab} and Δ^a_{bc} derived in Sec. IV and using the normalization constraints (C4) yields the usual equations of Newtonian and post-Newtonian hydrodynamics.

Alternatively, one can combine the Newtonian and post-Newtonian pieces together, as in the combined post-Newton-Cartan theory derived in Sec. III B. Defining $\hat\rho = \rho_{\rm n} + \varepsilon \rho_{\rm pn}$, $\hat p = p_{\rm n} + \varepsilon p_{\rm pn}$ and $\hat u^a = u^a_{\rm n} + \varepsilon u^a_{\rm pn}$, then the combined stress energy of Sec. III B is

$$\hat{\mathcal{T}}^{ab} = (\hat\rho + \hat p) \hat u^a \hat u^b + \hat p (\hat h^{ab} + \hat k^{ab}), \tag{C5}$$

and the normalization constraint is $\hat u^a \hat u^b (\hat t_a \hat t_b - \hat p_{ab}) = -1$. Again one can check that these expressions lead to the usual post-Newtonian hydrodynamic equations.