Mars seasonal polar caps as a test of the equivalence principle

David Parry Rubincam

Phys. Rev. D **84**, 042001 — Published 4 August 2011

DOI: 10.1103/PhysRevD.84.042001

MARS SEASONAL POLAR CAPS AS A TEST OF THE EQUIVALENCE PRINCIPLE

David Parry Rubincam

Planetary Geodynamics Laboratory, NASA Goddard Space Flight Center, Greenbelt, MD 20771, USA

Abstract

The seasonal polar caps of Mars can be used to test the equivalence principle in general relativity. The north and south caps, which are composed of carbon dioxide, wax and wane with the seasons. If the ratio of the inertial (passive) to gravitational (active) masses of the caps differs from the same ratio for the rest of Mars, then the equivalence principle fails, Newton's third law fails, and the caps will pull Mars one way and then the other with a force aligned with the planet's spin axis. This leads to a secular change in Mars's along-track position in its orbit about the Sun, and to a secular change in the orbit's semimajor axis. The caps are a poor Eötvös test of the equivalence principle, being 4 orders-of-magnitude weaker than laboratory tests and 7 orders-of-magnitude weaker than that found by lunar laser ranging; the reason is the small mass of the caps compared to Mars as a whole. The principal virtue of using Mars is that the caps contain carbon, an element not normally considered in such experiments. The Earth with its seasonal snow cover can also be used for a similar test.

I. INTRODUCTION

Mars has an atmosphere composed primarily of carbon dioxide. Each Martian year a significant fraction of the CO_2 atmosphere freezes out in the form of polar caps [1, 2]. These caps, which wax and wane with the seasons, can be used to test the equivalence principle in general relativity.

The basic idea is the following. Assume for the moment Mars has only one polar cap. If the equivalence principle fails, then the gravitational mass of the cap attracts the rest of Mars with a gravitational force which differs in magnitude from Mars attracting the cap's inertial mass. It follows that Newton's third law fails and the planet will self-accelerate. The absence of a measurable self-acceleration indicates the equivalence principle holds, at least within the limits of error. The spirit of this astronomical test is thus similar to that of Bartlett and Van Buren [3], who used the heterogeneity of the lunar crust and the Moon's lack of observable self-acceleration to make a stringent test of the equivalence principle.

The terminology here follows that of Turyshev [4], who uses the terms "gravitational mass" and "inertial mass". The terms "active mass" and "passive mass" are often used instead.

The qualitative details of the polar cap test are shown in the schematic diagram in Figure 1. Mars orbits around the Sun. For the purposes of illustration, Mars is shown to have an obliquity (axial tilt) of 90°. The Sun heats up Mars's north pole when it is over that pole (bottom of the figure). The northern polar cap shrinks over time while the southern cap grows. This creates a net gain of CO_2 in the southern hemisphere (right) and a net self-acceleration (if there is one) along the spin axis, as shown by the thick arrow.

The arrow opposes the motion, as shown in the figure, if the effect is such that the southern cap pulls on Mars more than Mars pulls on the cap. When the Sun shines on Mars's south polar cap (top), that cap shrinks and the northern cap grows, and once again the self-acceleration opposes the motion (left) for the sign of the effect adopted in the figure. The end result is a net negative along-track acceleration when averaged over one revolution of Mars about the Sun; the semimajor axis of the orbit shrinks with time.

The thick arrows are shown in Figure 1 as opposing the motion. However, because the sign of the self-acceleration is not known *a priori* without a theory, the arrows could just as easily point the other way. In this case, the orbit would expand secularly with time. The lag between the maximum insolation and the minimum size of the cap is taken to be 90° in Figure 1 for ease of illustration; the lag for the real Mars is less than 90° (see below), but not zero.

II. EQUATIONS OF MOTION

The body of Mars will be taken to be a sphere. Each seasonal polar cap will be assumed to be a spherical cap centered on the pole, and with a uniform mass density which increases and decreases with time. The surface density of the northern polar cap will be $\sigma_N = m_N/A_N$, where m_N is the mass of seasonal north polar cap, and $A_N = 2\pi R^2 (1 - \cos\theta_N)$ is the cap's area, where θ_N is the colatitude of the edge of the cap. Likewise, the southern seasonal cap has the corresponding quantities $\sigma_S = m_S/A_S$ and θ_S. In the following θ_N and θ_S are each taken to be constant. Thus a cap's edge does not advance or retreat, as observed. The surface densities σ_N and σ_S are constant in latitude or longitude, but vary with time. In other words, at any given time the CO_2 cover in each cap has a constant thickness, but the thickness changes with time. These assumptions will introduce an error whose magnitude is estimated below.

The gravitational surface mass density σ_{Ng} of the northern seasonal cap will exert a force

$$df = \frac{GM_i\sigma_{Ng}dA}{R^2}$$

on the inertial mass M_i of Mars, where G is the universal constant of gravitation, R is Mars's radius, and $dA = R^2 \sin\theta\, d\theta\, d\lambda$ is an element of area, with θ being colatitude and λ being longitude. The force df is directed radially outward from the center of Mars. The total component of force along Mars's spin axis is then

$$f_N \mathbf{p} = G M_i \sigma_{Ng} \int_0^{\theta_N} \int_0^{2\pi} \sin\theta \cos\theta d\lambda d\theta = \frac{G M_i m_{Ng}}{R^2} K(\theta_N) \mathbf{p}$$

where **p** is the unit vector in the positive spin axis direction, m_{Ng} is the gravitational mass of the northern cap, and

$$K(\theta_N) = \frac{1 - \cos 2\theta_N}{4(1 - \cos\theta_N)} \quad . \tag{1}$$

The body of Mars will pull on the northern cap with a force $F_N = G M_g\, m_{Ni}\, K(\theta_N)/R^2$ in the direction opposite to f_N, where m_{Ni} is the inertial mass of the northern cap, and M_g is the gravitational mass of the rest of Mars. The acceleration of Mars will then be approximately

$$a_N \mathbf{p} \approx \left(\frac{f_N - F_N}{M_p} \right) \mathbf{p} = \frac{\Gamma G m_{Ni}}{R^2} K(\theta_N) \mathbf{p}$$

due to the northern cap alone, where

$$\Gamma = \frac{m_{Ng}}{m_{Ni}} - \frac{M_g}{M_i} \quad . \tag{2}$$

Thus the test involves the CO_2 caps and the crust-mantle-core of Mars, whose compositions are quite different. The equivalence principle fails if $\Gamma \neq 0$. Note that the equivalence principle can still fail if $(m_{Ng}/m_{Ni}) \neq 1$ and $(M_g/M_i) \neq 1$, but $(m_{Ng}/m_{Ni}) = (M_g/M_i)$; in such a case $\Gamma = 0$ and the failure will be invisible in this test.

Analogously, the southern cap will give an acceleration a_S

$$a_S \mathbf{p} \approx -\left(\frac{f_S - F_S}{M_p} \right) \mathbf{p} = \frac{\Gamma G m_{Si}}{R^2} K(\theta_S) \mathbf{p} \quad .$$

in the direction opposite to a_N. The net acceleration of Mars along its spin axis will then be

$$a_{NS}\mathbf{p} = (a_N - a_S)\mathbf{p} = \frac{\Gamma G}{R^2}[m_{Ni}K(\theta_N) - m_{Si}K(\theta_S)]\mathbf{p} \tag{3}$$

If the quantity in brackets is nonzero and $\Gamma \neq 0$, then Mars will self-accelerate.

If all of the mass in a cap is concentrated at the pole, then $K = 1$. Observations indicate that each polar cap extends to a maximum of about 35° in latitude from their respective poles [5]. In this case $K(\theta_N) \approx K(\theta_S) = 0.91$, so that (1) is fairly insensitive to cap size. In the following, it will be assumed that $K(\theta_N) = K(\theta_S) = K$, so that (3) becomes

$$a_{NS}\mathbf{p} = (a_N - a_S)\mathbf{p} = \frac{\Gamma G K}{R^2}(m_{Ni} - m_{Si})\mathbf{p} \tag{4}$$

and in the numerical calculation below K is taken to be 0.9.

The northern and southern cap masses vary with time. The principal terms are

$$m_{Np} = c_{N1} \sin (L - \delta_{N1}) + c_{N2} \sin (2L - \delta_{N2}) \tag{5}$$

$$m_{Sp} = c_{S1} \sin (L - \delta_{S1}) + c_{S2} \sin (2L - \delta_{S2}) \tag{6}$$

where $L = \omega + f$ is the areocentric longitude of the Sun measured in a Mars-fixed frame, with ω being the L of perigee and f being the true anomaly of Mars's orbit about the Sun. Because Mars's orbit is fairly eccentric with an eccentricity $e = 0.09$, f does not increase uniformly with time. However, the mean anomaly M does increase almost uniformly with time. To average equations over time to get the secular effects on Mars's orbit, it proves convenient to express L in terms of M. To first order in e [6]

$$\sin f \cong \sin M \tag{7}$$

$$\cos f \cong -e + \cos M \tag{8}$$

$$\sin 2f \cong -2e \sin M + \sin 2M \tag{9}$$

$$\cos 2f \cong -2e \cos M + \cos 2M \tag{10}$$

Using these in the expression for m_{Np} in (5) yields

$$\begin{aligned} m_{Ni} = &-e\, c_{N1} (\cos \delta_{N1} \sin \omega - \sin \delta_{N1} \cos \omega) \\ &+ c_{N1} (\cos \delta_{N1} \cos \omega + \sin \delta_{N1} \sin \omega) \sin M \\ &+ c_{N1} (\cos \delta_{N1} \sin \omega - \sin \delta_{N1} \cos \omega) \cos M \\ &- 2e\, c_{N2} (\cos \delta_{N2} \cos 2\omega + \sin \delta_{N2} \sin 2\omega) \sin M \\ &- 2e\, c_{N2} (\cos \delta_{N2} \sin 2\omega - \sin \delta_{N2} \cos 2\omega) \cos M \\ &+ c_{N2} (\cos \delta_{N2} \cos 2\omega + \sin \delta_{N2} \sin 2\omega) \sin 2M \\ &+ c_{N2} (\cos \delta_{N2} \sin 2\omega - \sin \delta_{N2} \cos 2\omega) \cos 2M \end{aligned} \tag{11}$$

with an analogous expression for m_{Si}.

The evolution of Mars's orbital semimajor axis a with time t is given by

$$\frac{da}{dt} = \frac{2}{nr(1-e^2)^{1/2}}[a(1-e^2)S + (er \sin f)U] \tag{12}$$

where $r = a(1 - e^2)/(1+ e \cos f)$ is the Mars-Sun distance, n = Mars's mean motion about the Sun, U is Mars's acceleration directed radially outward from the Sun, and S is the acceleration in the orbital plane and is perpendicular to U, so that S is nearly the along-track acceleration for modest orbital eccentricities [7]. Using (7) – (10) in (12) yields

$$\frac{da}{dt} \approx \frac{2}{n}(S + eS \cos M + eU \sin M) \tag{13}$$

to first order in e.

Let $\mathbf{r} = \cos L\ \mathbf{x} + \sin L\ \mathbf{y}$ be the unit vector in the direction of U, where unit vector $\mathbf{x}$ points to the right in the plane of the orbit as shown in Fig. 1, and $\mathbf{y}$ points to the top of the figure. The unit vector $\mathbf{t} = -\sin L\ \mathbf{x} + \cos L\ \mathbf{y}$ points in the direction of S. Unlike what is shown in Fig. 1, Mars's spin axis $\mathbf{p} = +\sin \Theta\, \mathbf{y} + \cos \Theta\, \mathbf{z}$ lies out of the orbital plane, where $\Theta = 25°$ is the present obliquity of Mars and $\mathbf{z}$ is the unit vector normal to the orbital plane. Then $S = (\mathbf{p} \bullet \mathbf{t})\, a_{NS}$ and $U = (\mathbf{p} \bullet \mathbf{t})\, a_{NS}$, where

$$(\mathbf{p} \bullet \mathbf{t}) = \sin \Theta \cos L \approx \sin \Theta\, (-e \cos \omega - \sin \omega \sin M + \cos \omega \cos M) \tag{14}$$

$$(\mathbf{p} \bullet \mathbf{r}) = \sin \Theta \sin L \approx \sin \Theta\, (-e \sin \omega + \cos \omega \sin M + \sin \omega \cos M)\ . \tag{15}$$

by (7) – (10). By (11) and (14), and the acceleration S becomes

$$S \approx (G\Gamma K \sin \Theta / 2R^2)\{-c_{N1} \sin \delta_{N1} + c_{S1} \sin \delta_{S1}$$
$$- 2e[c_{N2} \sin(\omega - \delta_{N2}) - c_{S2} \sin(\omega - \delta_{S2})]\} \tag{16}$$

ignoring the periodic terms and retaining only the secular terms, and remembering that c_{N1}, c_{N2}, c_{S1}, and c_{S2} have units of kilograms. It turns out that by (14) and (15) $eS \cos M + eU \sin M = 0$ to order e in (13), so that

$$\frac{da}{dt} \approx \frac{2S}{n}\ . \tag{17}$$

III. RESULTS AND DISCUSSION

The following numerical values are used for Mars: $R = 3.39 \times 10^6$ m, $e = 0.09$, $n = 1.06 \times 10^{-7}$ s^{-1}, $\Theta = 25°$, and $L = 251°$ [8], while $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}. Also, K is taken to be 0.9, as mentioned above. Smith *et al.* have recovered the coefficients and phase angles in (5) – (6) from the Mars Global Surveyor spacecraft orbiting Mars [2]; which are given in Table 1. These authors find that the cycles vary little from Martian year to Martian year. Using all these numbers in (16) and (17) yields

$$S = 3.5 \times 10^{-9}\ \Gamma \text{ m s}^{-2} \tag{18}$$

and

$$da/dt = 2.1 \times 10^{8}\ \Gamma \text{ m century}^{-1}\ . \tag{19}$$

One source of error in the above estimates is the value for K. This is because the carbon dioxide caps are assumed to have a fixed size, and only thicken or sublime. The thickness of the real caps vary both spatially and with time. However, K is fairly insensitive to size; even if $\theta_N = 50°$, $K(\theta_N)$ goes down to only 0.82. Hence it seems likely that setting $K(\theta_N) = K(\theta_S) = K = 0.9$ produces an error in the treatment of the caps of no more than ~10%.

Another source of error is the neglect of the Mars atmosphere. As the caps wax and wane the atmosphere thins and thickens. The atmosphere "feels" the topography, changing the CO_2 mass distribution and essentially creating another surface mass layer in addition to the caps. The atmospheric effect is about 10% that of the caps [9] and will be ignored here.

It remains to estimate Γ. Krasinsky and Brumberg find that the the solar system, as measured by the Astronomical Unit, may be expanding by 15 ± 4 m century^{-1}, for which they find no satisfactory explanation [10]. Attributing all of the expansion to Mars's polar caps in yields a value of $|\Gamma| \leq 7.1 \times 10^{-8}$ by (19).

The Earth's changing snow cover will also generate a polar cap-type effect, which can complicate solar system tests, since the Earth is often used as one leg of a test. The Earth's seasonal snow cover has a maximum mass of about 10^{15} kg [11], which is about the same as the Martian CO_2 caps. The radius of the Earth is almost twice as great as Mars's in (4), and the Earth's mean motion about the Sun is also almost twice as great as Mars's in (17). If the value for Γ for the Earth's H_2O − (crust-mantle-core) system is comparable to the value of Γ for Mars's CO_2 − (crust-mantle-core) system, then da/dt for Earth is ~15% that of Mars, and a closer look at the Earth's polar cap effect appears to be warranted. The Earth will not be as simple as Mars, because of the distribution of the snow on the irregularly-shaped continents, and the fact that sea level will go up and down as the snow cover changes.

Other solar system tests of the equivalence principle are also not considered (e.g., [12]). Given these complications and the absence of a more detailed study, it seems reasonable to increase the limit on Γ by a factor of 10 to

$$|\Gamma| \leq 7.1 \times 10^{-7} \tag{20}$$

This limit is 4 orders-of-magnitude greater than that given by laboratory experiments, and 7 orders-of-magnitude greater than that given by the Moon (see, e.g., [4]), so that Mars is a poor Eötvös experiment by comparison.

Mars does have the virtue of testing the equivalence principle with a novel combination of elements. The caps are almost entirely CO_2, so that the caps contain carbon. The composition of Mars's body is not entirely certain, but a recent model has the crust and mantle consisting of SiO_2 (44.4% by weight), MgO (30.2%), FeO (17.9%), and Al_2O_3 (3.1%), plus 4.4% other elements; while the core consists of Fe (77.8%), S (14.2%), and Ni (8.0%) [13]. Carbon versus these other heavier elements is not usually used in tests of the equivalence principle.

ACKNOWLEDGMENTS

The paper by David. F. Bartlett and Dave Van Buren and subsequent discussions of it with Bahman Shahid-Saless inspired the present work. I thank an anonymous referee for comments. The support from NASA Headquarters through a SALMON proposal by is gratefully acknowledged.

[1] S. L. Hess, J. A. Ryan, J. E. Tillman, R. M. Henry, and C. B. Leovy, Geophys. Res. Lett. **7**, 197, 1980.

[2] D. E. Smith, M. T. Zuber, M. H. Torrence, P. J. Dunn, G. A. Neumann, F. G. Lemoine, and S. K. Fricke, J. Geophys. Res. **114**, E05002, doi:10.1029/2008JE003267 (2009).

[3] D. F. Barlett and D. Van Buren, Phys. Rev. Lett. **57**, 21 (1986).

[4] S. G. Turyshev, in *Gravitational Physics: Testing Gravity from Submillimeter to Cosmic Scale,* edited by H. A. Morales-Tecotl, L. A. Urena-Lopez, R. Linnares-Romero, and H. H. Garcia-Compean (AIP CP1256, 2010).

[5] P. B. James, H. H. Kieffer, and D. A. Paige, in *Mars*, edited by H. H. Kieffer, B. M. Jakosky, C. W. Snyder, and M. S. Matthews (Univ. Ariz. Pr., 1992).

[6] A. Cayley, Memoirs R. Astron. Soc. **29**, 191, 1861.

[7] V. M. Blanco and S. W. McCuskey, *Basic Physics of the Solar System* (Addison-Wesley, 1961).

[8] H. H. Kieffer, B. M. Jakosky, and C. W. Snyder, in *Mars*, edited by H. H. Kieffer, B. M. Jakosky, C. W. Snyder, and M. S. Matthews (Univ. Ariz. Pr., 1992)

[9] B. F. Chao and D. P. Rubincam, J. Geophys. Res. **95**, 14,755 (1990).

[10] G. A. Krasinsky and V. A. Brumberg, Celes. Mech. & Dyn. Astron. **90**, 267 (2004).

[11] B. F. Chao, W. P. O'Connor, A. T. C. Change, D. K. Hall, and J. L. Foster, J. Geophys. Res. **92**, 9415 (1987).

[12] J. D. Anderson, M. Gross, K. L. Nordtvedt, and S. G. Turyshev, Astrophys. J. **459**, 365 (1996).

[13] H. Wänke, Space Sci. Rev. **56**, 1, (1991).

TABLE I

The coefficients and phase angles for the north and south cap masses from [2].

$c_{N1} = (-1.534 \pm 0.028) \times 10^{15}$ kg

$c_{N2} = (-0.486 \pm 0.029) \times 10^{15}$ kg

$\delta_{N1} = 43.64° \pm 1.07°$

$\delta_{N2} = 47.88° \pm 3.34°$

$c_{S1} = (-3.058 \pm 0.032) \times 10^{15}$ kg

$c_{S2} = (-0.917 \pm 0.033) \times 10^{15}$ kg

$\delta_{S1} = 223.84° \pm 0.61°$

$\delta_{S2} = 52.85° \pm 2.02°$

Figure 1

Schematic of the Mars equivalence principle test. Mars orbits the Sun. Mars's spin axis is shown as lying in the orbital plane for the purposes of illustration. The polar caps (white) wax and wane with insolation. When one cap is larger than the other, there is a self-force (thick arrows) if the equivalence principle fails. The planet self-accelerates, leading to a secular change in the size of the orbit. For the case shown below the orbit shrinks. If the thick arrows pointed the other way, the orbit would expand.