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Phys. Rev. D **84**, 034034 — Published 19 August 2011

DOI: 10.1103/PhysRevD.84.034034

SLAC–PUB–14481
June, 2011

Antenna Splitting Functions for Massive Particles

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ABSTRACT

An antenna shower is a parton shower in which the basic move is a color-coherent $2 \to 3$ parton splitting process. In this paper, we give compact forms for the spin-dependent antenna splitting functions involving massive partons of spin 0 and spin $\frac{1}{2}$.

Submitted to *Physical Review* **D**

[1]Work supported by the US Department of Energy, contract DE–AC02–76SF00515.

1 Introduction

The modeling of physics at high energy colliders relies heavily on our understanding of QCD. Quarks and gluons – collectively, partons – that are produced in high energy reactions are observed as jets of hadrons. The structure of each jet is determined by the pattern of radiation of additional partons from the original one produced in the central hard scattering reaction. For this reason, much attention has been given the past few years to the development of methods for creating parton showers, systems of partons created with the distributions predicted by QCD.

The traditional approach to the generation of parton showers is based on splitting off partons through a $1 \to 2$ branching process. This philosophy is incorporated in the widely used event generator programs PYTHIA [1] and HERWIG [2]. The construct of building a shower from $1 \to 2$ branching, often called a 'dipole shower', omits an important aspect of the physics. The longitudinal momentum distribution in the $1 \to 2$ splitting is given by the Altarelli-Parisi splitting functions [3]. In QCD, partons are emitted coherently from the two legs of a color dipole. The emission amplitude is then enhanced inside the dipole and, more importantly, cancels outside the dipole. In the 1980's, Marchesini and Webber argued that this effect could be incorporated into dipole showers by imposing angular ordering of emissions [4]. Thus, HERWIG is built around an angular-ordered parton shower, and PYTHIA, though it uses a different ordering scheme to choose its branchings, vetos emissions that are out of angular ordering.

Alternatively, one might build up a parton shower directly from the color dipoles, using the $2 \to 3$ process of emission of a parton by a dipole as the basic branching process. This construct is called an 'antenna shower'. The scheme was realized in the program ARIADNE, by Andersson, Gustafson, Lönnblad, and Pettersson [5] and, more recently, by the program VINCIA, by Giele, Kosower, and Skands [6]. The approach is of interest both in creating new parton shower codes for the purpose of matrix element-parton shower matching and because of its promise to yield a more accurate treatment of color dynamics in parton showers.

Recently, there has been much interest in the tagging of boosted heavy particles such as the top and Higgs observed as exotic jets [7]. Since tagging methods rely heavily on color flow, it is interesting to have a variety of approaches to the simulation of color flow in parton showers in order to test the robustness of these algorithms.

We have been engaged in providing a well-defined foundation for antenna showers, giving explicit calculations of the splitting functions that generate these showers and generalizing previous work to spin-dependent formulae. In a previous paper, we presented the complete set of spin-dependent antenna splitting functions needed to describe quark and gluon parton showers [8]. In this paper, we continue our study of this approach by presenting the spin-dependent antenna splitting functions for showers with massive particles. In constructing a shower for massless particles, spin-dependence is a convenience, especially for matching with full QCD amplitudes. For massive particles, it is more important to preserve spin information, because the decays of heavy particles such as the top quark are spin-dependent and so the experimental acceptance for the heavy particles varies significantly with their longitudinal polarization.

The formalism presented here has the same strengths and weaknesses as our previous work. We will calculate in the kinematics of final-state showers, using effective operators of definite spin to represent the 2-particle color dipole state before the splitting. We will work in the limit of a large number of colors in QCD for which the concept of a color dipole is strictly defined. Within this approximation, we will derive formulae for splitting functions with any ratio m/Q between the mass of the particle and the mass of the two-particle system. These formulae will necessarily be less simple than those found in [8] for the massless case. We will see, though, that we can make use of spinor product formalism [9] to write these splitting functions relatively compactly. The simplicity of these expressions is connected to their relation to the Maximally Helicity Violating amplitudes of QCD. This point was originally made for the massless case in [10] and is discussed in some detail in [8].

The formalism of QCD antennae was originally developed as a tool for the subtraction of infrared divergences in higher-order QCD calculations. This approach to QCD calculation was pioneered by by Kosower [11,12]. Gehrmann-De Ridder, Gehrmann, Glover, and their students have developed this approach into a sophisticated method applicable to NLO and even NNLO computations [13,14]. Using this formalism, Gehrmann-De Ridder, Gehrmann, and Glover have proposed forms for the spin-summed antenna splitting functions of massless quarks and gluons [15,16]. Our previous paper reviews this latter work and compares the results from our method to theirs. There is no universal form for antenna splitting functions. The behavior of the splitting functions is prescribed in the soft and collinear limits but, away from those limits, different expressions are possible, depending on the framework used in the derivation. The systematic differences between the different proposals are explored in [8].

Following the methods of [15,16], splitting functions for massive, spin summed antennae were constructed in [17–19]. Again, our expressions agree with these in having the correct soft and quasi-collinear behavior but differ away from these limits. The addition of mass greatly complicates both the expressions for the splitting functions and the precise specification of the boundaries of phase space. Because of this, we do not present a detailed comparison to other massive splitting functions here.

The outline of this paper is as follows: In Section 2, we will analyze the case of gluon radiation from an antenna composed of a massive spin-$\frac{1}{2}$ fermion (Q) and a massless spin $\frac{1}{2}$ fermion (q) in a configuration of zero helicity. All of the new complications that arise when we deal with massive particles can be illustrated in this context. We will write expressions for the splitting functions in terms of spinor products of lightlike vectors associated with the massive vectors of the particles before and after the splitting. In Section 3, we will discuss the kinematics of these massive splittings and the evaluation of the the spinor product expressions.

With this introduction, we can go systematically through the various cases of antennae composed of massive and massless particles. In Sections 4 and 5 we will analyze in turn the cases of antennae with spin 0 and spin $\frac{1}{2}$ massive particles recoiling against quarks and gluons in which the antennae emits another quark or gluon. In Section 6, we discuss the analysis of the general case of a pair of massive particles, spin 0 or spin $\frac{1}{2}$, radiating gluons. In Section 7, we discuss antennae that create a pair of massive particles. Section 8 gives some conclusions. We collect the complete set of massive antenna splitting functions derived in this paper in Appendix A.

2 The spin zero fermion-quark antenna

The simplest case of a splitting function with massive particles arises in the system of a massive and a massless fermion created by a spin 0 operator. In this section, we will work out the spin-dependent splitting functions for this case following the prescriptions in [8]. We will then discuss the interpretation of these formulae and their comparison to the standard Altarelli-Parisi splitting functions for a massive quark [20].

In [8], each case of a spin-dependent splitting is associated with a gauge-invariant operator that creates the antenna. For this case, the required operator is

$$\mathcal{O} = \overline{Q}_L q_R \tag{1}$$

where q is an ordinary quark whose mass can be ignored and Q is a massive quark. This operator creates a 2-particle state

$$Q_L \overline{q}_L \tag{2}$$

with total spin zero about the production axis. Antennae with overall opposite helicity or with antiquarks have the same splitting functions, by the P and C invariance of QCD.

In [8], we wrote the basic formula for final-state antennae splitting of massless particles in the following way: Notate the splitting as $AB \to acb$, with

$$(A+B)^2 = s_{AB} = Q^2 \ . \tag{3}$$

Throughout this paper, for any 4-vectors i, j, we will define

$$s_{ij} = (i+j)^2 = m_i^2 + 2i\cdot j + m_j^2 \ . \tag{4}$$

Let z_a, z_b, z_c be the momentum fractions of a, b, and c relative to their maximum value,

$$z_a = \frac{2Q\cdot q}{Q^2} \ , \ \text{etc.} \qquad z_a + z_b + z_c = 2 \ . \tag{5}$$

Then the probability of a splitting is given by

$$\int d\,\mathrm{Prob} = N_c \frac{\alpha_s}{4\pi}(\frac{Q}{2K}) \int dz_a dz_b \, \mathcal{S}(z_a, z_b, z_c) \ . \tag{6}$$

where $N_c = 3$ is the number of colors in QCD and K is the momentum of the partons in the center of mass system of the original 2-particle antenna. In the massless case, $Q/2K = 1$. The distribution $\mathcal{S}$ is the splitting function. In [8], we computed this function as the ratio of 3- to 2- body amplitudes of an appropriate local operator,

$$\mathcal{S} = Q^2 \left| \frac{\mathcal{M}(\mathcal{O}\to acb)}{\mathcal{M}(\mathcal{O}\to AB)} \right|^2 \ . \tag{7}$$

This formula is still correct for the massive particle antennae discussed in this paper. We will discuss the kinematics of these antennae in more detail in Section 3.

In the limit in which c becomes collinear with a or b, the antenna splitting functions reduce to the Altarelli-Parisi functions $P(z)$ that describe $1 \to 2$ splittings. For this limit, the formulae are not as simple in the massive case as they are in the all-massless case. We will present the explicit formulae and check them for the spin zero antenna later in this section.

To compute the amplitudes in (7), we use the spinor product formalism for massive particles of Schwinn and Weinzierl [21]. For a massless particle, the states of definite helicity are well-defined and Lorentz invariant. For a massive particle, the spin states depend on the frame chosen to evaluate them. In the Schwinn-Weinzierl formalism, a massless reference vector q is used to define that frame. The spinors for an outgoing massive fermion of mass m are written

$$\overline{u}_L(p) = \frac{[q(\not p + m)}{[qp^\flat]} \qquad \overline{u}_R(p) = \frac{\langle q(\not p + m)}{\langle qp^\flat\rangle} \ , \tag{8}$$

where the flatted vector $p^\flat$ is defined by

$$p^\flat = p - \frac{m^2}{2q\cdot p} q \tag{9}$$

A particularly useful choice for q is the lightlike vector in the opposite direction from p. Rotating coordinates so that

$$p = (E, 0, 0, p) \qquad \text{with } E^2 = p^2 + m^2 \ , \tag{10}$$

let

$$p^\sharp = \frac{1}{2}(E+p)(1, 0, 0, -1) \ , \tag{11}$$

Then if we set $q = p^\sharp$, the flatted vector is

$$p^\flat = \frac{1}{2}(E+p)(1, 0, 0, 1) \ . \tag{12}$$

This is very convenient. With this choice of q, the spinors defined in (8) are just the usual spinors of definite helicity. Using the basis of Dirac matrices where γ^5 is diagonal, it is easy to see that (8) reduces to

$$\begin{aligned} \overline{u}_L &= \begin{pmatrix} \sqrt{\frac{E-p}{2}} & \sqrt{\frac{E+p}{2}} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \\ \overline{u}_R &= \begin{pmatrix} \sqrt{\frac{E+p}{2}} & \sqrt{\frac{E-p}{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} \ . \end{aligned} \tag{13}$$

Using these conventions, we can easily compute the 2 particle matrix elements of the operator (1). Denote the momenta of the initial-state heavy quark and light antiquark as A and B, respectively. Then

$$\begin{aligned} \mathcal{M}(Q_L\overline{q}_L) &= \frac{[qAB\rangle}{[qA^\flat]} = \langle A^\flat B\rangle \\ \mathcal{M}(Q_R\overline{q}_L) &= \frac{m\langle qB\rangle}{\langle qA^\flat\rangle} \end{aligned} \tag{14}$$

The helicity of the $\overline{q}$ must be L, but the heavy quark created by (1) could be in either spin state. However, with the usual definition of helicity, the production of $Q_R\overline{q}_L$ from a spin 0 operator would be forbidden by angular momentum. Indeed, when we set $q = A^\sharp$,

$$\mathcal{M}(Q_R\overline{q}_L) \sim \langle A^\sharp B\rangle = 0 \ , \tag{15}$$

because $A^\sharp$ is a lightlike vector parallel to B. The only nonzero matrix element is then

$$\mathcal{M}(Q_L\overline{q}_L) = \langle A^\flat B\rangle \ ; \tag{16}$$

this gives the denominator in (6). It is convenient that

$$|\langle A^\flat B\rangle|^2 = Q^2 - m^2 = 2QK \ , \tag{17}$$

with K as in (6).

It is straightforward to work out the numerator of (6) for the four possible spin states of the 3-particle system Qgq_L. As in [8], we label the three final-state momenta as (a, c, b), with the emitted particle as c. The results, using a general reference vector q in (8), are

$$\begin{aligned}
\mathcal{M}(Q_L g_L \overline{q}_L) &= -\frac{1}{[qc]}\left\{\frac{\langle ca^\flat\rangle[qQb\rangle}{s_{ac}-m^2} + \frac{[qQa^\flat\rangle}{[bc]}\right\} \\
\mathcal{M}(Q_L g_R \overline{q}_L) &= -\frac{\langle a^\flat b\rangle[cQb\rangle}{\langle bc\rangle(s_{ac}-m^2)} \\
\mathcal{M}(Q_R g_L \overline{q}_L) &= -\frac{m}{[a^\flat c]\langle qa^\flat\rangle}\left\{\frac{\langle cq\rangle[a^\flat Qb\rangle}{s_{ac}-m^2} + \frac{[a^\flat Qq\rangle}{[bc]}\right\} \\
\mathcal{M}(Q_R g_R \overline{q}_L) &= -\frac{m\langle qb\rangle[cQb\rangle}{\langle bc\rangle\langle qa^\flat\rangle(s_{ac}-m^2)}
\end{aligned} \tag{18}$$

We have omitted the overall factor of (gT^a). When we put $q = a^\sharp$, we can recognize the simplification

$$[a^\sharp Q a^\flat\rangle = [a^\flat Q a^\sharp\rangle = 0 \ . \tag{19}$$

This follows from the fact that the 4-vector Q is a linear combination of the two lightlike vectors $a^\flat$ and $a^\sharp$. Now square these expressions and combine with (17) to evaluate (7). This gives

$$\begin{aligned}
\mathcal{S}(Q_L g_L \overline{q}_L) &= \frac{Q}{2K}\left|\frac{\langle a^\flat c\rangle[a^\sharp Qb\rangle}{[a^\sharp c][cac\rangle}\right|^2 \\
\mathcal{S}(Q_L g_R \overline{q}_L) &= \frac{Q}{2K}\left|\frac{\langle a^\flat b\rangle[cQb\rangle}{\langle bc\rangle[cac\rangle}\right|^2 \\
\mathcal{S}(Q_R g_L \overline{q}_L) &= \frac{m^2 Q}{2K}\left|\frac{\langle a^\sharp c\rangle[a^\flat Qb\rangle}{\langle a^\sharp a^\flat\rangle[a^\flat c][cac\rangle}\right|^2 \\
\mathcal{S}(Q_R g_R \overline{q}_L) &= \frac{m^2 Q}{2K}\left|\frac{\langle a^\sharp b\rangle[cQb\rangle}{\langle a^\sharp a^\flat\rangle\langle bc\rangle[cac\rangle}\right|^2
\end{aligned} \tag{20}$$

In the all-massless case, we managed to produce antenna splitting functions that were simple rational functions of the z_a [8]. Here, the antenna splitting functions are more complicated, but not excessively so. The main complications come from the denominators $(s_{ac} - m^2) = [cac\rangle$, which do not factorize simply, and from the multiple lightlike vectors needed to characterize the state of the massive quark. In this case, it is not so difficult to write the splitting functions in terms of 4-vector products:

$$\begin{aligned}
\mathcal{S}(Q_L g_L \overline{q}_L) &= \frac{Q}{K}\frac{s_{a^\flat c}(2a^\sharp\cdot Q\, b\cdot Q - a^\sharp\cdot b\, Q^2)}{s_{a^\sharp c}(s_{ac}-m^2)^2} \\
\mathcal{S}(Q_L g_R \overline{q}_L) &= \frac{Q}{K}\frac{s_{a^\flat b}(2b\cdot Q\, c\cdot Q - b\cdot c\, Q^2)}{s_{bc}(s_{ac}-m^2)^2} \\
\mathcal{S}(Q_R g_L \overline{q}_L) &= \frac{m^2 Q}{K}\frac{s_{a^\sharp c}(2a^\flat\cdot Q\, b\cdot Q - a^\flat\cdot b\, Q^2)}{s_{a^\sharp a^\flat} s_{a^\flat c}(s_{ac}-m^2)^2} \\
\mathcal{S}(Q_R g_R \overline{q}_L) &= \frac{m^2 Q}{K}\frac{s_{a^\sharp b}(2b\cdot Q\, c\cdot Q - b\cdot c\, Q^2)}{s_{a^\sharp a^\flat} s_{bc}(s_{ac}-m^2)^2}
\end{aligned} \tag{21}$$

However, the structure of the expressions is more clearly visible in the form (20).

The expressions (20) contain exact tree-level matrix elements for the transition of the operator $\mathcal{O}$ to a three-particle state. They are correctly used in a parton shower for any values of m/Q and p_T/Q among the final-state particles, as long as the virtuality at the previous and successive branchings of the shower are well separated from Q. In the all-massless case discussed in [8], we made approximations to the splitting functions valid in the soft and collinear limits. It is less obvious here which approximations are appropriate, and, in any case, we did not see how to achieve much further simplification. So we will stop at this point for this set of splitting fuctions and for all of the massive particle splitting functions quoted in this paper.

To evaluate expressions of the type of (20), we find it easiest not to convert the expressions in (20) into 4-vector products or dimensionless scalars built from these but, rather, to directly evaluate the spinor brackets. We will discuss a strategy to evaluate these brackets in the next section.

Finally, we must discuss the collinear limits and the connection to the the Altarelli-Parisi splitting functions. For the spin zero antennae, this connection is easiest to discuss for the limit $c \parallel b$, where only massless particles are involved. We must still take account of the fact that, because b and c recoil against a massive particle, their maximum momentum is limited. To account for this, let

$$\tilde{z}_{b,c} = \frac{z_{b,c}}{(1-m^2/Q^2)} \ . \tag{22}$$

so that $\tilde{z}_b$ and $\tilde{z}_c$ run from 0 to 1 and, in the limit $c \parallel b$, $\tilde{z}_b + \tilde{z}_c = 1$. Then, in this collider limit, $\mathcal{S}$ has the singularity

$$\mathcal{S} \sim \delta_{a,A}\frac{Q^2}{s_{bc}} P_{B\to c}(\tilde{z}_c) \ . \tag{23}$$

The expressions in (20) satisfy this relation. The splitting functions to $Q_R g_{L,R}$ must have no collinear singularity. This follows from the fact that $[a^\flat Q b\rangle$ and $\langle a^\sharp b\rangle$ vanish when b becomes opposite to a. The cases of $Q_L g_{L,R}$ do have singularities proportional to $s_{a^\sharp c}^{-1}$ and s_{bc}^{-1}, with the correct coefficients to match (22).

In the limit $c \parallel a$, where the $1 \to 2$ splitting involves a massive particle, the limit is slightly more complicated. For the splitting of a massive particle, the usual Altarelli-Parisi formula for the collinear splitting is conventionally rewritten as

$$\int d\,\mathrm{Prob} = N_c \frac{\alpha_s}{2\pi} \int dz \int \frac{dp_T^2}{(p_T^2 + z^2 m^2)} P(z, p_T) \ . \tag{24}$$

We divide the usual expressions for $P(z, p_T)$ by 2 so that these functions give the contribution from one of the two antennae that contribute to a collinear singularity. Mass-suppressed terms can contain an additional factor of $(p_T^2 + z^2 m^2)$ in the denominator; this is why we have allowed the Altarelli-Parisi function to depend on p_T. With this formalism, for c becoming parallel to a,

$$\mathcal{S}(z_a, z_b, z_c) \to \frac{Q^2}{s_{ac} - m_A^2} P(\tilde{z}_c, p_T) \tag{25}$$

where $s_{ac} = (a + c)^2$. Here again, the parameter $\tilde{z}_c$ must be scaled to equal 1 at its maximum value, as in (22). For the present case in which the (ac) system recoils against a massless parton, $\tilde{z}_c = z_c$.

To discuss the limits $c \parallel a$, we first need to recall the Altarelli-Parisi functions for splitting of a gluon from a massive fermion. The Altarelli-Parisi functions are defined in the limit of not only collinear but also high energy emission. For a particle of energy E splitting to particles with transverse momentum p_T and finite masses m_i, these functions describe the regime $p_T \sim m_i \ll E$. For a splitting $Q \to gQ$, as we have in this case, the spin-summed splitting function is [20]

$$P(z) = \frac{1 + (1-z)^2}{z} - \frac{m^2}{a \cdot c} \tag{26}$$

This expression becomes clearer when it is written as a set of spin-dependent Altarelli-Parisi functions. In the convention defined by (24),

$$\begin{aligned} P(Q_L \to Q_L g_L) &= \frac{p_T^2}{p_T^2 + z^2 m^2} \frac{1}{z} \\ P(Q_L \to Q_L g_R) &= \frac{p_T^2}{p_T^2 + z^2 m^2} \frac{(1-z)^2}{z} \\ P(Q_L \to Q_R g_L) &= \frac{m^2}{p_T^2 + z^2 m^2} \frac{z^4}{z} \\ P(Q_L \to Q_R g_R) &= 0 \end{aligned} \tag{27}$$

The sum of these terms does reproduce (26). The placement of the factors of z implements the *dead cone* in which soft radiation from a massive particle is suppressed within a cone of size $1/\gamma$, where γ is the boost of the heavy particle [22,23].

We can now compare the $c \parallel a$ limits of our antenna splitting functions to (27). In the collinear limit,

$$s_{ac} - m^2 = \frac{p_T^2 + z^2 m^2}{z(1-z)} \ . \tag{28}$$

Using this formula and the collinear limits of the spinor products, we find that (20) does satisfy (25) with (27), up to corrections of relative order m^2/Q^2. In particular, in the limit $c \parallel a$, $a^\sharp$ becomes collinear with b. Then the vanishing of $\langle a^\sharp b \rangle$ with no compensatory vanishing in the denominator gives the zero in the last line of (27).

The spin-dependent splitting functions in the remaining sections of this paper also satisfy these checks on the collinear limits. For convenience, we list the complete set of mass-dependent, spin-dependent Altarelli-Parisi splitting functions that are needed for these checks in Appendix B.

3 Kinematics of massive antennae

The splitting functions computed in the previous section were written in terms of spinor products of massless vectors associated with the massive 4-vectors of the antenna. One should ask, how are these massless vectors computed? A similar question arises in the context of the formula (6) for the antenna splitting probability. This equation is easily written down as the ratio of a cross section to produce a 3-body final state, integrated over 3-body phase space, to the cross section to produce a 2-body final state, without a radiated parton, integrated over 2-body phase space. In particular, the integral $\int dz_a dz_b$ is an integral over 3-body phase space. One should ask, what is the boundary of the region of integration for these variables, and how does one sample points in the interior of this region?

For massless antenna, the answers to these questions are straightforward. For antenna with both radiators in the final state (FF antennae in the notation of [8]), the complete phase space region is the triangle

$$0 < z_a, z_b < 1 \qquad z_a + z_b > 1 \tag{29}$$

and the region well described by the radiation process $AB \to acb$, with c soft, is the smaller region where

$$0 < z_c < z_a < 1 \quad \text{and} \quad 0 < z_c < z_b < 1 \tag{30}$$

To create an additional radiated particle in a state with N massless particles, we choose a color-connected pair of particles AB, boost so that A and B are of equal length and back-to-back, choose (z_a, z_b) as a random point in the region (30), replace the 2-particle system AB by the chosen 3-particle system acb, and, finally, reverse the boost to bring acb back into the original frame. The corresponding phase space regions and algorithms for antennae including initial-state particles are described in [8]. In this paper, however, we will only discuss final-state showers.

We believe that these 4-vector configurations for massless particles provide a good starting point for constructing 4-vector configurations that include massive particles. Given a point $\{\ell_i\}$ in the phase space of of N massless particles, one can obtain a point $\{k_i\}$ in the phase space of N massive particles by rescaling

$$\vec{k}_i = \lambda \vec{\ell}_i \tag{31}$$

where λ obeys

$$\sum_i \hat{E}_i = E_{\rm CM} \ , \quad \text{with} \quad \hat{E}_i = (|\lambda \vec{\ell}_i|^2 + m_i^2)^{1/2} \ . \tag{32}$$

Conversely, every point of the massive phase space can be constructed uniquely in this way. The scale factor λ is close to unity unless one of the massive particles is nonrelativistic. The relation of the phase space measures for the massive and massless variables is [24]

$$d\Pi_N(k) = d\Pi_N(\ell) \cdot \lambda^{2N-4} \prod_i \frac{|\lambda \vec{\ell}_i|}{E_i} \, \frac{\sum_i |\lambda \vec{\ell}_i|}{\sum_i |\lambda \vec{\ell}_i|^2 / E_i} \ . \tag{33}$$

We will refer to the massless vectors $\{\ell_i\}$ as the *backbone* of the massive configuration.

We now have a strategy for the constructing the N particle phase space of a parton shower that involves massive particles. Starting with a system of 2 massless particles, construct a shower of massless vectors according to the procedure described above. In each antenna, let the momentum fractions of the (massless) final particles a, b be w_a, w_b. Rescale within the antenna by λ and use the massless vectors and this value of λ to compute the splitting probabilities. For example, for the splitting described in the previous section with particle a massive, the equation for λ is

$$E_a + \lambda(|\vec{\ell}_b| + |\vec{\ell}_c|) = Q \ . \tag{34}$$

The splitting probability is given by

$$\int d\,\mathrm{Prob} = N_c \frac{\alpha_s}{4\pi} (\frac{Q}{2K}) \int dw_a dw_b \cdot \lambda^2 \cdot (\frac{\lambda w_a}{E_a}) \frac{\lambda}{|\lambda w_a|^2 Q / 2E_a + \lambda(w_b + w_c)} \ \mathcal{S} \ . \tag{35}$$

To evaluate the splitting function $\mathcal{S}$ we need the flatted and sharped vectors $a^\flat$ and $a^\sharp$. The first of these is given by

$$a^\flat = \frac{1}{2}(E_a/\lambda|\vec{\ell}_a| + 1)\,\lambda\ell_a\ , \tag{36}$$

and $a^\sharp$ is the massless vector of the same length pointing in the opposite direction. Once the configuration is chosen, the three new massless vectors are boosted back to the frame of the shower, and we are ready to generate the next antenna. When the shower is completed, the entire backbone must be rescaled to put the final massive particles on shell. In this prescription, the recoil due to emissions is done locally in each antenna to the extent that the particles are relativistic, but the recoil for nonrelativistic massive particles is distributed over the whole shower.

There is one more complication that should be discussed. For a massless particle, the spin state is determined by the helicity in a way that is independent of frame. For a massive particle, a change of frame can rotate the spin. The helicity is preserved by rotations and by boosts along the direction of motion. Other boosts, at an angle to the direction of motion, change the spin orientation. In the massive particle shower described here, we ignore this effect. In any event, it is unimportant when the massive particles are relativistic, and this accounts for most of the radiation from these particles.

4 Antennae with a massive spin 0 particle

We are now ready to put together a catalogue of the antenna splitting functions that describe the emission of quarks and gluons in the showering of massive particles. We begin with the case of a spin 0 massive particle S recoiling against a quark or a gluon.

In the quark case, the antenna is described by an operator

$$\mathcal{O} = S^\dagger\,\langle 2|\,q_R \tag{37}$$

where $\langle 2|$ is a spin-$\frac{1}{2}$ spurion that controls the quark polarization. Here and in the rest of the paper, we will analyze a subset of the various discrete choices from which the rest can be derived using the P and C symmetries of QCD. Here, for example, the two cases

$$S\overline{q}_L \to Sg_L\overline{q}_L \text{ and } S\overline{q}_L \to Sg_R\overline{q}_L \tag{38}$$

considered below suffice to provide all of the possble spin-dependent splitting functions for $S\overline{q} \to Sg\overline{q}$ and $Sq \to Sgq$.

The 2-particle matrix element of the operator (37) is

$$\mathcal{M}(\mathcal{O} \to S\overline{q}_L) = \langle 2B\rangle \ . \tag{39}$$

Then, for the 2-particle antenna $S\overline{q}_L$ with S moving the $\hat{3}$ direction, 2 should be a massless fermion moving parallel to S. In the following, we will set $2 = A^\flat$. This choice follows the methods used in [8]. In that paper, the polarization vectors associated with operators $\mathcal{O}$ with nonzero spin are built from massless vectors 1 and 2, chosen in the directions of B and A, respectively. With this choice, the denominator of the expression (7) for the splitting function is again evaluated as (17).

The 3-particle matrix elements of (37) are

$$\begin{aligned}\mathcal{M}(\mathcal{O} \to Sg_L\overline{q}_L) &= \frac{\langle A^\flat(b+c)ac\rangle}{[cac\rangle[bc]} \\ \mathcal{M}(\mathcal{O} \to Sg_R\overline{q}_L) &= -\frac{\langle A^\flat b\rangle\langle bac]}{[cac\rangle\langle bc\rangle}\end{aligned} \tag{40}$$

Here again, we strip off the factors of g and color matrices. The final results are surprisingly compact.

For an antenna containing a massive scalar and gluon, we need to find an operator that defines an antenna whose initial state includes a gluon of a definite polarization. For the antenna with a left-handed gluon, we may choose [8]

$$\mathcal{O} = \frac{i}{\sqrt{2}} S^\dagger \langle 2|\,\overline{\sigma}\cdot F\,|2\rangle \tag{41}$$

where

$$\overline{\sigma}\cdot F = \frac{1}{2}\overline{\sigma}^m\sigma^n F_{mn} \ . \tag{42}$$

This operator projects onto anti-self-dual gauge fields or left-handed physical gluons. The corresponding operator $\sigma \cdot F$ can be used to define the antenna with an initial right-handed gluon. The two-particle matrix elements of (41) are

$$\mathcal{M}(\mathcal{O} \to Sg_L) = \langle 2B\rangle^2 \qquad \mathcal{M}(\mathcal{O} \to Sg_R) = 0 \ . \tag{43}$$

The zero for a g_R is just as one should have expected. As above, we set $2 = A^\flat$.

There are two types of 3-particle matrix elements of (41). First, the antenna can radiate a gluon. The corresponding matrix elements are

$$\mathcal{M}(\mathcal{O} \to Sg_Lg_L) = \frac{1}{[bc]}\Big[\frac{\langle A^\flat b\rangle^2[bac\rangle}{[cac\rangle} + 2\langle A^\flat c\rangle\langle A^\flat b\rangle + \frac{\langle A^\flat c\rangle^2[cab\rangle}{[bab\rangle}\Big]$$

$$\mathcal{M}(\mathcal{O} \to S g_R g_L) = -\frac{\langle A^\flat b\rangle^2 \langle bac]}{[cac\rangle\langle bc\rangle}$$
$$\mathcal{M}(\mathcal{O} \to S g_L g_R) = -\frac{\langle A^\flat c\rangle^2 \langle cab]}{[bab\rangle\langle bc\rangle}$$
$$\mathcal{M}(\mathcal{O} \to S g_R g_R) = 0 \ , \qquad (44)$$

following the pattern established in (40). Second, the gluon may split into a quark-antiquark pair. For this, we need the matrix elements

$$\mathcal{M}(\mathcal{O} \to S \overline{q}_R q_L) = -\frac{\langle A^\flat b\rangle^2}{\langle bc\rangle}$$
$$\mathcal{M}(\mathcal{O} \to S \overline{q}_L q_R) = \frac{\langle A^\flat c\rangle^2}{\langle bc\rangle} \ . \qquad (45)$$

The splitting functions derived from these matrix elements using (7) are listed systematically in Appendix A.

5 Antennae with a massive spin $\frac{1}{2}$ particle

In the same way, we can construct operators that correspond to the initial states of antennae involving a massive Dirac fermion Q with a quark or gluon. The massive fermion can have helicity $\pm\frac{1}{2}$. Because the Q is massive, an initial left-handed Q can flip over after radiation to a right-handed Q, or vice versa. We have seen this already in the special case considered in Section 2. In this section, we will recall the results from Section 2 and compare them to those of the other three possible antennae of this type.

The antennae with an initial state containing F and a quark can be arranged in a state with total spin about the axis of motion $|J^3|$ equal to 0 or 1. The spin 0 case was considered in Section 2. The appropriate operator $\mathcal{O}$ is

$$\mathcal{O} = \overline{Q} q_R \ . \qquad (46)$$

The matrix elements of this operator between two-particle $F\overline{q}$ states are

$$\mathcal{M}(\mathcal{O} \to Q_L \overline{q}_L) = \langle A^\flat B\rangle \qquad \mathcal{M}(\mathcal{O} \to Q_L \overline{q}_R) = 0 \qquad (47)$$

in our convention that $A^\sharp$ should be used as the reference vector for Q. The three-particle matrix elements are then readily computed. If we use $a^\sharp$ from the beginnning as the reference vector for Q, (18) gives

$$\begin{aligned}\mathcal{M}(Q_L g_L \overline{q}_L) &= -\frac{\langle c a^\flat\rangle [a^\sharp Q b\rangle}{[a^\sharp c][cac\rangle} \\ \mathcal{M}(Q_L g_R \overline{q}_L) &= -\frac{\langle a^\flat b\rangle [cQb\rangle}{\langle bc\rangle [cac\rangle} \\ \mathcal{M}(Q_R g_L \overline{q}_L) &= -m\frac{\langle c a^\sharp\rangle [a^\flat Q b\rangle}{[a^\flat c]\langle a^\sharp a^\flat\rangle [cac\rangle} \\ \mathcal{M}(Q_R g_R \overline{q}_L) &= -m\frac{\langle a^\sharp b\rangle [cQb\rangle}{\langle bc\rangle\langle a^\sharp a^\flat\rangle [cac\rangle} \ . \end{aligned} \tag{48}$$

The antenna splitting function can be constructed from these elements in the manner described in Section 2.

The spin 1 case can be treated in the same way. As described in [8] and at the beginning of Section 4, we introduce lightlike vectors 1 and 2 in the direction of B and A, respectively. Then an appropriate operator to define this antenna is

$$\mathcal{O} = \overline{Q}\, 1\rangle [2\, q_L \ . \tag{49}$$

The two-particle matrix elements of this operator are

$$\mathcal{M}(\mathcal{O} \to Q_L \overline{q}_R) = \langle A^\flat 1\rangle\langle 2B\rangle \ , \qquad \mathcal{M}(\mathcal{O} \to Q_R \overline{q}_R) = 0 \ , \tag{50}$$

Thus, this operator does correctly represent the initial situation. We will set $2 = A^\flat$ and $1 = B$ in the following expressions.

The splitting function for the antenna to radiate a gluon is computed from the three-particle matrix elements of this operator to $Fg\overline{q}$ final states. These are

$$\begin{aligned}\mathcal{M}(Q_L g_R \overline{q}_R) &= -\frac{\langle a^\flat B\rangle [A^\flat (b+c) a c]}{\langle cac]\langle bc\rangle} \\ \mathcal{M}(Q_L g_L \overline{q}_R) &= -\frac{[A^\flat b]}{\langle cac][bc][a^\sharp a^\flat]}\Big\{ [a^\sharp a c\rangle [bQB\rangle + m^2 \langle cB\rangle [a^\sharp b] \Big\} \\ \mathcal{M}(Q_R g_R \overline{q}_R) &= \frac{m\langle a^\sharp B\rangle [A^\flat (b+c) a c]}{\langle a^\sharp a^\flat\rangle \langle cac]\langle bc\rangle} \\ \mathcal{M}(Q_R g_L \overline{q}_R) &= \frac{m[A^\flat b]}{\langle a^\sharp a^\flat\rangle\langle cac]\langle bc\rangle}\Big\{ \langle a^\sharp B\rangle \langle cab] + \langle a^\sharp c\rangle\langle Bcb] \Big\} \ . \end{aligned} \tag{51}$$

The splitting functions derived from these formulae and those in (48) are catalogued in Appendix A.

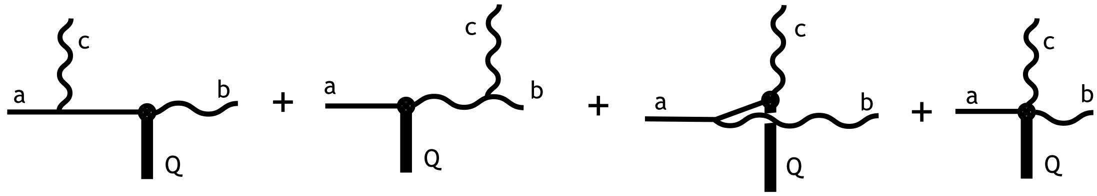

Figure 1: Feynman diagrams for the computation of the $Fg \to Fgg$ splitting functions [8].

For the antennae with Q and a gluon, we again use the operator $\overline{\sigma} \cdot F$ to define the initial state as containing a gluon of definite left-handed polarization. There are two cases, with total spin $\frac{1}{2}$ and $\frac{3}{2}$. For the spin $\frac{1}{2}$ case, the appropriate operator is

$$\mathcal{O} = -\frac{i}{\sqrt{2}}\overline{Q}\,\overline{\sigma} \cdot F\,|2\rangle \ . \tag{52}$$

The dominant two-particle matrix element of this operator is

$$\mathcal{M}(\mathcal{O} \to Q_L \overline{g}_L) = \langle A^\flat B\rangle\langle 2B\rangle \ . \tag{53}$$

If we recall that the vector 2 is identified with $A^\flat$, we see that this puts the initial Q and g into just the correct orientation. The matrix elements to $Q_L g_R$, $Q_R g_L$, and $Q_R g_R$ all vanish if 1 is taken parallel to B.

The splitting functions for the radiation of a gluon from this antenna are given by the matrix elements of (53) to Qgg final states. As in [16] and in [8], these matrix elements are given by the computation of the set of diagrams shown in Fig. 1. The last diagram in the figure comes from the two-gluon vertex of the operator $\overline{\sigma} \cdot F$. The third diagram is required to make the computation gauge-invariant. Its origin is most easily seen by thinking of the Q as a color octet. Then this diagram is obviously an essential contribution to the radiation from the Qg dipole.

With this observation, we find for the three particle matrix elements of (53)

$$\begin{aligned}
\mathcal{M}(Q_L g_L g_L) &= \frac{1}{[bc][a^\sharp a^\flat]}\Big\{\frac{\langle A^\flat b\rangle}{\langle cac]}(Q^2[a^\sharp ac\rangle - m^2[a^\sharp Qc\rangle) + \frac{\langle A^\flat c\rangle}{\langle bab]}(Q^2[a^\sharp ab\rangle - m^2[a^\sharp Qb\rangle)\Big\}\\
\mathcal{M}(Q_L g_R g_L) &= \frac{\langle a^\flat b\rangle\langle A^\flat b\rangle\langle bac]}{\langle cac]\langle bc\rangle}\\
\mathcal{M}(Q_L g_L g_R) &= \frac{\langle a^\flat c\rangle\langle A^\flat c\rangle\langle cab]}{\langle bab]\langle bc\rangle}\\
\mathcal{M}(Q_L g_R g_R) &= 0
\end{aligned}$$

$$
\begin{aligned}
\mathcal{M}(Q_R g_L g_L) &= \frac{m}{\langle a^\sharp a^\flat\rangle [bc]}\left\{\frac{\langle A^\flat b\rangle}{\langle cac]}(\langle a^\sharp a Q c\rangle - Q^2\langle a^\sharp c\rangle) + \frac{\langle A^\flat c\rangle}{\langle bab]}(\langle a^\sharp a Q b\rangle - Q^2\langle a^\sharp b\rangle)\right\}\\
\mathcal{M}(Q_R g_R g_L) &= \frac{m\langle a^\sharp b\rangle\langle A^\flat b\rangle\langle bac]}{\langle a^\sharp a^\flat\rangle\langle cac]\langle bc\rangle}\\
\mathcal{M}(Q_R g_L g_R) &= \frac{m\langle a^\sharp c\rangle\langle A^\flat c\rangle\langle cab]}{\langle a^\sharp a^\flat\rangle\langle bab]\langle bc\rangle}\\
\mathcal{M}(Q_R g_R g_R) &= 0\ .
\end{aligned}
\tag{54}
$$

The case of a Qg antennae in the spin $\frac{3}{2}$ state is treated similarly. The operator that defines the initial state is

$$
\mathcal{O} = -\frac{i}{\sqrt{2}}\overline{Q}\,1]\,\langle 2|\,\overline{\sigma}\cdot F\,|2\rangle\ . \tag{55}
$$

The two-particle matrix elements of this operator are

$$
\mathcal{M}(\mathcal{O}\to Q_R\overline{g}_L) = \langle 1A^\flat\rangle\langle 2B\rangle^2 \tag{56}
$$

and all other matrix elements are equal to zero for the choice of 1 parallel to B. We will set $2 = A^\flat$ and $1 = B$ in the expressions that follow.

The three-particle matrix elements of (55) to Qgg final states are

$$
\begin{aligned}
\mathcal{M}(Q_R g_L \overline{g}_L) &= -\frac{[a^\flat B]}{[bc]}\left\{\frac{\langle A^\flat b\rangle\langle A^\flat(b+c)ac\rangle}{\langle cac]} + \frac{\langle A^\flat c\rangle\langle A^\flat(b+c)ab\rangle}{\langle bab]}\right\}\\
\mathcal{M}(Q_R g_R \overline{g}_L) &= -\frac{\langle A^\flat b\rangle^2}{\langle cac]\langle bc\rangle}\left\{[a^\flat c]\langle bQB] + m^2\frac{\langle a^\sharp b\rangle}{\langle a^\sharp a^\flat\rangle}[cB]\right\}\\
\mathcal{M}(Q_R g_L \overline{g}_R) &= -\frac{\langle A^\flat c\rangle^2}{\langle bab]\langle bc\rangle}\left\{[a^\flat b]\langle cQB] + m^2\frac{\langle a^\sharp c\rangle}{\langle a^\sharp a^\flat\rangle}[bB]\right\}\\
\mathcal{M}(Q_R g_R \overline{g}_R) &= 0\\
\mathcal{M}(Q_L g_L \overline{g}_L) &= -\frac{m[a^\sharp B]}{\langle a^\sharp a^\flat\rangle[bc]}\left\{\frac{\langle A^\flat b\rangle\langle A^\flat(b+c)ac\rangle}{\langle cac]} + \frac{\langle A^\flat c\rangle\langle A^\flat(b+c)ab\rangle}{\langle bab]}\right\}\\
\mathcal{M}(Q_L g_R \overline{g}_L) &= -\frac{m\langle A^\flat b\rangle^2}{[a^\sharp a^\flat]}\frac{([a^\sharp B]\langle bac] + [a^\sharp c]\langle bcB])}{\langle cac]\langle bc\rangle}\\
\mathcal{M}(Q_L g_L \overline{g}_R) &= -\frac{m\langle A^\flat c\rangle^2}{[a^\sharp a^\flat]}\frac{([a^\sharp B]\langle cab] + [a^\sharp b]\langle cbB])}{\langle bab]\langle bc\rangle}\\
\mathcal{M}(Q_L g_R \overline{g}_R) &= 0\ .
\end{aligned}
\tag{57}
$$

The splitting functions for $Qg \to Qgg$ that are derived from these expressions and those in (54) are catalogued in Appendix A.

The Qg antennae can also radiate by gluon splitting to a pair of quarks. For the spin $\frac{1}{2}$ case, the relevant matrix elements are

$$\begin{aligned}
\mathcal{M}(Q_L\overline{q}_R q_L) &= \frac{\langle a^\flat b\rangle\langle A^\flat b\rangle}{\langle bc\rangle} \\
\mathcal{M}(Q_L\overline{q}_L q_R) &= -\frac{\langle a^\flat c\rangle\langle A^\flat c\rangle}{\langle bc\rangle} \\
\mathcal{M}(Q_R\overline{q}_R q_L) &= \frac{m\langle a^\sharp b\rangle\langle A^\flat b\rangle}{\langle a^\sharp a^\flat\rangle\langle bc\rangle} \\
\mathcal{M}(Q_R\overline{q}_L q_R) &= -\frac{m\langle a^\sharp c\rangle\langle A^\flat c\rangle}{\langle a^\sharp a^\flat\rangle\langle bc\rangle}
\end{aligned} \tag{58}$$

For the spin $\frac{3}{2}$ case, the matrix elements are

$$\begin{aligned}
\mathcal{M}(Q_R\overline{q}_R q_L) &= \frac{[a^\flat B]\langle A^\flat b\rangle^2}{\langle bc\rangle} \\
\mathcal{M}(Q_R\overline{q}_L q_R) &= -\frac{[a^\flat B]\langle A^\flat c\rangle^2}{\langle bc\rangle} \\
\mathcal{M}(Q_L\overline{q}_R q_L) &= \frac{m[a^\sharp B]\langle A^\flat b\rangle^2}{[a^\sharp a^\flat]\langle bc\rangle} \\
\mathcal{M}(Q_L\overline{q}_L q_R) &= -\frac{m[a^\sharp B]\langle A^\flat c\rangle^2}{[a^\sharp a^\flat]\langle bc\rangle}
\end{aligned} \tag{59}$$

The splitting functions for $Qg \to Qgg$ that are derived from these expressions are catalogued in Appendix A.

6 Antennae of a pair of massive particles

After a pair of massive scalars or fermions are produced, their first emission of a gluon is described by an antenna in which the two massive particles both appear. For a complete description, we need the splitting functions for these antenna as well. These formulae are somewhat more complicated than those derived above, since some of the simplifications that are possible when the particle b is massless no longer apply. There is little additional complexity in the cases in which the two massive particles have different masses, so we will write the formulae for that more general situation.

The case of a pair of scalars is relatively straightforward. The scalar particles themselves are spinless, so there is only one case, described by the spin-0 operator

$$\mathcal{O} = S_1^\dagger S_2 \ . \tag{60}$$

The matrix element of this operator to create the state $S_1\overline{S}_2$ is simply 1. The matrix elements for gluon emission are

$$\begin{aligned}\mathcal{M}(S_1 g_L \overline{S}_2) &= \frac{1}{[a^\flat c]}\left\{\frac{[a^\flat ac\rangle}{[cac\rangle} - \frac{\langle a^\flat bc]}{\langle cbc]}\right\} \\ \mathcal{M}(S_1 g_R \overline{S}_2) &= -\frac{1}{\langle a^\flat c\rangle}\left\{\frac{\langle a^\flat ac]}{\langle cac]} - \frac{\langle a^\flat bc]}{\langle cbc]}\right\} \ .\end{aligned} \tag{61}$$

Each expression can be brought down to one term using the Schouten identity

$$\langle caf]\langle dbg] - \langle daf]\langle cbg] = -\langle cd\rangle[fabg] \ . \tag{62}$$

This identity is valid when a and b are massive vectors, possibly with different masses; c, d, f, and g must be massless. To prove the identity, write a as a linear combination of $a^\flat$ and $a^\sharp$. Using (62),

$$\begin{aligned}\mathcal{M}(S_1 g_L \overline{S}_2) &= -\frac{\langle cabc\rangle}{\langle cac][cbc\rangle} \\ \mathcal{M}(S_1 g_L \overline{S}_2) &= \frac{[cabc]}{[cac\rangle\langle cbc]} \ .\end{aligned} \tag{63}$$

The splitting functions are readily assembled from these expressions.

For the antenna of a massive fermion and a massive scalar, the general case is described by the spin $\frac{1}{2}$ operator

$$\mathcal{O} = \overline{Q}_1\, 1\rangle S_2 \ . \tag{64}$$

The two-body matrix elements of this operator are

$$\mathcal{M}(Q_{1L}\overline{S}_2) = \langle A^\flat 1\rangle \tag{65}$$

and zero for Q_{1R}. If we take $1 = B^\flat$ following the prescriptions above,

$$|\langle A^\flat 1\rangle|^2 = (E_1 + K)(E_2 + K) \ , \tag{66}$$

where E_1, E_2, and K are the two energies and the momentum in the antenna center of mass frame.

The matrix elements for the operator (64) to create $Qg\overline{S}$ states is given by the expression

$$\mathcal{M} = -\frac{gT^a}{\sqrt{2}}\overline{u}(a)\Big[\frac{\not\epsilon(c)(\not a + \not c + m)}{[cac\rangle} 1\rangle - 1\rangle \frac{2b\cdot\epsilon(c)}{[cbc\rangle}\Big] \ , \tag{67}$$

where $\epsilon(c)$ is the polarization vector of the gluon. A convenient way to treat this is to manipulate

$$\not\epsilon(c)(\not a + \not c + m) = 2a\cdot\epsilon(c) + \not\epsilon(c)\not c \tag{68}$$

plus a term proportional to $(\not a - m)$ that gives zero when applied to $\overline{u}(a)$. The first term in (68) combines with the last term in (67) to give an amplitude proportional that of the scalar-scalar case, (61) or (63) above. The term with $\not\epsilon(c)$ vanishes for g_R and gives a simple but nonzero term for g_L. The final results for the two amplitudes, after dropping the factor of (gT^a), are

$$\begin{aligned}
\mathcal{M}(Q_L g_R \overline{S}) &= \frac{\langle a^\flat 1\rangle\langle cabc\rangle}{\langle cac][cbc\rangle} \\
\mathcal{M}(Q_R g_R \overline{S}) &= \frac{m_1\langle a^\sharp 1\rangle\langle cabc\rangle}{\langle a^\sharp a^\flat\rangle\langle cac][cbc\rangle} \\
\mathcal{M}(Q_L g_L \overline{S}) &= -\frac{\langle a^\flat 1\rangle[cabc]}{[cac\rangle\langle cbc]} + \frac{\langle a^\flat c\rangle\langle c1\rangle}{[cac\rangle} \\
\mathcal{M}(Q_R g_L \overline{S}) &= -\frac{m_1}{\langle a^\sharp a^\flat\rangle}\Big[\frac{\langle a^\sharp 1\rangle[cabc]}{[cac\rangle\langle cbc]} - \frac{\langle a^\sharp c\rangle\langle c1\rangle}{[cac\rangle}\Big] \ .
\end{aligned} \tag{69}$$

Here m_1 is the mass of the fermion Q_1. The formulae apply for any values of the masses of the fermion and scalar, as long as the 4-vectors a and b are properly on mass shell.

The decomposition of the gluon coupling to a massive fermion given in (68) is equivalent the representation of this coupling by the second-order Dirac equation, in which the fermion is replaced by a field with a scalar-type coupling and a magnetic moment coupling. The single-gluon magnetic moment coupling has a chiral structure and vanishes for specific combinations of the fermion and gluon spin. This second-order Dirac formalism is discussed in more detail in [25].

For massive fermions, there are two cases, corresponding to total spin 0 and 1 along the antenna axis. For the spin 0 case, we could use the operator $\overline{Q}_L Q_L$ to create the antenna, similarly to the choices in Sections 2 and 5. However, in the case in which both fermions are massive, that operator creates both $Q_L\overline{Q}_L$ and $Q_R\overline{Q}_R$ states. We will avoid that problem here by taking the operator that creates an initial state of $Q_L\overline{Q}_L$ to be

$$\mathcal{O} = \overline{Q}_1\, 1\rangle\langle 2\, Q_2 \tag{70}$$

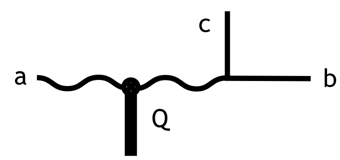

Figure 2: The single Feynman diagram for the computation of the $gg \to gF\overline{F}$ splitting function [8].

The two-body matrix elements of this operator are

$$\mathcal{M}(Q_{1L}\overline{Q}_{2L}) = \langle A^\flat 1\rangle\langle 2B^\flat\rangle \ , \tag{71}$$

and zero for the other three helicity states. Similarly, for the spin 1 case, we will use the operator

$$\mathcal{O} = \overline{Q}_1\, 1\rangle[2\, Q_2 \ . \tag{72}$$

to create an initial state of $Q_L\overline{Q}_R$. The two-body matrix elements of this operator are

$$\mathcal{M}(Q_{1L}\overline{Q}_{2R}) = \langle A^\flat 1\rangle[2B^\flat] \ , \tag{73}$$

and zero for the other three helicity states. The $Qg\overline{Q}$ matrix elements of these operators are easily computed using the methods presented earlier in this section. The results for the splitting functions are tabulated in Appendix A.

7 Antennae with massive particle production

There is one more situation that we must consider. At very high energies, massive particles can be produced by gluon splitting. At the LHC, for example, parton-parton scattering can give quark-gluon and gluon-gluon collisions with center of mass energies well above 1 TeV. Final state gluon antennae in these collsions can produce pairs of top quarks. The pair production amplitudes are relatively simple, since each requires only one Feynman diagram, as shown in Fig. 2 for the $gg \to g\bar{t}t$ case. The final pair of heavy particles must have equal mass and equal spin. However, there are a large number of cases to enumerate. The massive scalar or fermion pair can be formed from a spin $\frac{1}{2}$ or a spin $\frac{3}{2}$ qg antenna or from a spin 0 or spin 2 gg antenna.

For scalar pair production, the formalism is actually quite simple. The spin $\frac{1}{2}$ and spin $\frac{3}{2}$ qg antennae can be represented by the operators

$$\mathcal{O}_{1/2} = -\frac{i}{\sqrt{2}}\overline{q}\,\overline{\sigma}\cdot F\,|2\rangle$$

$$\mathcal{O}_{3/2} = -\frac{i}{\sqrt{2}}\overline{q}\, 1]\, \langle 2|\, \overline{\sigma}\cdot F\, |2\rangle \ . \tag{74}$$

If the gluon splits to a pair of scalars, both cases involve the operator $\overline{\sigma}\cdot F$ dotted with the $gS\overline{S}$ vertex. This product is

$$\frac{1}{2}[(\not{b}+\not{c})\gamma^\mu - \gamma^\mu(\not{b}+\not{c})](b-c)_\mu \equiv [b,c] \tag{75}$$

so that the relevant three-particle matrix elements are

$$\mathcal{M}(q_L\overline{S}S) = -\frac{\langle a[b,c]2\rangle}{s_{bc}} \tag{76}$$

for the spin $\frac{1}{2}$ case and

$$\mathcal{M}(q_R\overline{S}S) = \frac{[a1]\langle 2[b,c]2\rangle}{s_{bc}} \tag{77}$$

for the spin $\frac{3}{2}$ case.

Similarly, the spin 0 and spin 2 gg antennae, corresponding to the g_Lg_L and g_Rg_L initial states, can be represented by the operators

$$\begin{aligned} \mathcal{O}_0 &= \frac{1}{2}\,\mathrm{tr}[(\overline{\sigma}\cdot F)^2] \\ \mathcal{O}_2 &= [1|\, \sigma\cdot F\, |1]\, \langle 2|\, \overline{\sigma}\cdot F\, |2\rangle \ . \end{aligned} \tag{78}$$

The manipulation (75) again gives a simple form for the three-particle matrix elements. The corresponding splitting functions are given in Appendix A.

For the case of massive fermion pair production, this formalism is necessarily more complex. With the choice of helicity states that we have used throughout this paper, the vertex to create a pair of massive fermions is a Dirac matrix. For the case of a final-state $\overline{Q}_RQ_L$, for example, this matrix has the form

$$V = c^\flat]\langle b^\flat - \frac{m^2}{\langle c^\sharp c^\flat\rangle[b^\flat b^\sharp]} b^\sharp]\langle c^\sharp \tag{79}$$

Then the matrix element of $\overline{\sigma}\cdot F$ contains the structure

$$\langle R_1\, [(b+c), V]\, R_2\rangle \tag{80}$$

with a commutator bracketed between reference vectors R_1 and R_2. However, the frame-dependent choice of the vectors $b^\sharp$, $c^\sharp$ makes it difficult to simplify this expression further. It is true that $(b+c) = Q-a$, where a is now massless. In some cases, we have $R_1 = a$, in which case the a term cancels. In other cases, we have $R_1 = 2 = A^\flat$, so that the a term vanishes if a is collinear with A. We list the full expressions for these splitting functions in Appendix A.

8 Conclusion

In this paper, we have provided new materials for the construction of parton showers that include massive spin 0 and spin $\frac{1}{2}$ particles. We hope that this formalism we have presented will be useful in describing the QCD dynamics of the top quark and other heavy particles at LHC.

A Catalogue of massive antenna splitting functions

In this appendix, we catalogue the various antenna splitting functions for massless particles derived in this paper. Antenna splitting functions not listed are equal to cases listed below that are related by the P and C symmetries of QCD.

A.1 Splitting functions with one massive scalar

A.1.1 Spin $\frac{1}{2}$ antenna: initial dipole $S\overline{q}_L$

$$\begin{aligned}
\mathcal{S}(Sg_L\overline{q}_L) &= \frac{Q}{2K}\left|\frac{\langle A^\flat(b+c)ac\rangle}{[cac\rangle[bc]}\right|^2 \\
\mathcal{S}(Sg_R\overline{q}_L) &= \frac{Q}{2K}\left|\frac{\langle A^\flat b\rangle\langle bac]}{[cac\rangle\langle bc\rangle}\right|^2
\end{aligned} \tag{81}$$

A.1.2 Spin 1 antenna: initial dipole Sg_L

$$\begin{aligned}
\mathcal{S}(Sg_Lg_L) &= \frac{1}{(2K)^2}\left|\frac{1}{[bc]}\left[\frac{\langle A^\flat b\rangle^2[bac\rangle}{[cac\rangle} + 2\langle A^\flat c\rangle\langle A^\flat b\rangle + \frac{\langle A^\flat c\rangle^2[cab\rangle}{[bab\rangle}\right]\right|^2 \\
\mathcal{S}(Sg_Rg_L) &= \frac{1}{(2K)^2}\left|\frac{\langle A^\flat b\rangle^2\langle bac]}{[cac\rangle\langle bc\rangle}\right|^2 \\
\mathcal{S}(Sg_Lg_R) &= \frac{1}{(2K)^2}\left|\frac{\langle A^\flat c\rangle^2\langle cab]}{[bab\rangle\langle bc\rangle}\right|^2 \\
\mathcal{S}(Sg_Rg_R) &= 0 \\
\mathcal{S}(S\overline{q}_Rq_L) &= \frac{1}{(2K)^2}\frac{[bA^\flat b\rangle^2}{[bcb\rangle}
\end{aligned}$$

$$\mathcal{S}(S\overline{q}_L q_R) = \frac{1}{(2K)^2}\frac{[cA^\flat c\rangle^2}{[bcb\rangle} \tag{82}$$

Note that the last two expressions are already squared and evaluate to values that are real and positive for the case of a final-state antenna. For example,

$$\mathcal{S}(S\overline{q}_R q_L) = \frac{1}{(2K)^2}\frac{(2b\cdot A^\flat)^2}{2b\cdot c} \ .$$

A.2 Splitting functions with one massive fermion

A.2.1 Spin 0 antenna: initial dipole $Q_L\overline{q}_L$

$$\begin{aligned}
\mathcal{S}(Q_L g_L \overline{q}_L) &= \frac{Q}{2K}\left|\frac{\langle a^\flat c\rangle[a^\sharp Qb\rangle}{[a^\sharp c][cac\rangle}\right|^2\\
\mathcal{S}(Q_L g_R \overline{q}_L) &= \frac{Q}{2K}\left|\frac{\langle a^\flat b\rangle[cQb\rangle}{\langle bc\rangle[cac\rangle}\right|^2\\
\mathcal{S}(Q_R g_L \overline{q}_L) &= \frac{m^2 Q}{2K}\left|\frac{\langle a^\sharp c\rangle[a^\flat Qb\rangle}{\langle a^\sharp a^\flat\rangle[a^\flat c][cac\rangle}\right|^2\\
\mathcal{S}(Q_R g_R \overline{q}_L) &= \frac{m^2 Q}{2K}\left|\frac{\langle a^\sharp b\rangle[cQb\rangle}{\langle a^\sharp a^\flat\rangle\langle bc\rangle[cac\rangle}\right|^2
\end{aligned} \tag{83}$$

A.2.2 Spin $\frac{1}{2}$ antenna: initial dipole $Q_L g_L$

$$\begin{aligned}
\mathcal{S}(Q_L g_L g_L) &= \frac{1}{(2K)^2}\left|\frac{\langle A^\flat b\rangle(Q^2[a^\sharp ac\rangle - m^2[a^\sharp Qc\rangle)}{[a^\sharp a^\flat]\langle cac][bc]} + \frac{\langle A^\flat c\rangle(Q^2[a^\sharp ab\rangle - m^2[a^\sharp Qb\rangle)}{[a^\sharp a^\flat]\langle bab][bc]}\right|^2\\
\mathcal{S}(Q_L g_R g_L) &= \frac{1}{(2K)^2}\left|\frac{\langle a^\flat b\rangle\langle A^\flat b\rangle\langle bac]}{\langle cac]\langle bc\rangle}\right|^2\\
\mathcal{S}(Q_L g_L g_R) &= \frac{1}{(2K)^2}\left|\frac{\langle a^\flat c\rangle\langle A^\flat c\rangle\langle cab]}{\langle bab]\langle bc\rangle}\right|^2\\
\mathcal{S}(Q_L g_R g_R) &= 0\\
\mathcal{S}(Q_R g_L g_L) &= \frac{m^2}{(2K)^2}\left|\frac{\langle A^\flat b\rangle(\langle a^\sharp aQc\rangle - Q^2\langle a^\sharp c\rangle)}{\langle a^\sharp a^\flat\rangle\langle cac][bc]} + \frac{\langle A^\flat c\rangle(\langle a^\sharp aQb\rangle - Q^2\langle a^\sharp b\rangle)}{\langle a^\sharp a^\flat\rangle\langle bab][bc]}\right|^2
\end{aligned}$$

$$
\begin{aligned}
\mathcal{S}(Q_R g_R g_L) &= \frac{m^2}{(2K)^2}\left|\frac{\langle a^\sharp b\rangle\langle A^\flat b\rangle\langle bac]}{\langle a^\sharp a^\flat\rangle\langle cac]\langle bc\rangle}\right|^2 \\
\mathcal{S}(Q_R g_L g_R) &= \frac{m^2}{(2K)^2}\left|\frac{\langle a^\sharp c\rangle\langle A^\flat c\rangle\langle cab]}{\langle a^\sharp a^\flat\rangle\langle bab]\langle bc\rangle}\right|^2 \\
\mathcal{S}(Q_R g_R g_R) &= 0 \\
\mathcal{S}(Q_L \overline{q}_R q_L) &= \frac{1}{(2K)^2}\frac{[ba^\flat b\rangle[bA^\flat b\rangle}{[bcb\rangle} \\
\mathcal{S}(Q_L \overline{q}_L q_R) &= \frac{1}{(2K)^2}\frac{[ca^\flat c\rangle[cA^\flat c\rangle}{[bcb\rangle} \\
\mathcal{S}(Q_R \overline{q}_R q_L) &= \frac{m^2}{(2K)^2}\frac{[ba^\sharp b\rangle[bA^\flat b\rangle}{[a^\sharp aa^\sharp\rangle[bcb\rangle} \\
\mathcal{S}(Q_R \overline{q}_L q_R) &= \frac{m^2}{(2K)^2}\frac{[ca^\sharp c\rangle[cA^\flat c\rangle}{[a^\sharp aa^\sharp\rangle[bcb\rangle}
\end{aligned} \tag{84}
$$

As in A.1.2, the last four expressions here are already squared and evaluate to real, positive values.

A.2.3 Spin 1 antenna: initial dipole $Q_L \overline{q}_R$

$$
\begin{aligned}
\mathcal{S}(Q_L g_R \overline{q}_R) &= \frac{1}{(2K)^2}\left|\frac{\langle a^\flat B\rangle[A^\flat(b+c)ac]}{\langle cac]\langle bc\rangle}\right|^2 \\
\mathcal{S}(Q_L g_L \overline{q}_R) &= \frac{1}{(2K)^2}\left|\frac{[A^\flat b]([a^\sharp ac\rangle[bQB\rangle + m^2\langle cB\rangle[a^\sharp b])}{[a^\sharp a^\flat]\langle cac][bc]}\right|^2 \\
\mathcal{S}(Q_R g_R \overline{q}_R) &= \frac{m^2}{(2K)^2}\left|\frac{\langle a^\sharp B\rangle[A^\flat(b+c)ac]}{\langle a^\sharp a^\flat\rangle\langle cac]\langle bc\rangle}\right|^2 \\
\mathcal{S}(Q_R g_L \overline{q}_R) &= \frac{m^2}{(2K)^2}\left|\frac{[A^\flat b](\langle a^\sharp B\rangle\langle cab] + \langle a^\sharp c\rangle\langle Bcb])}{\langle a^\sharp a^\flat\rangle\langle cac]\langle bc\rangle}\right|^2
\end{aligned} \tag{85}
$$

A.2.4 Spin $\frac{3}{2}$ antenna: initial dipole $Q_R g_L$

$$
\mathcal{S}(Q_L g_L g_L) = \frac{m^2 Q^2}{(2QK)^3}\left|\frac{[a^\sharp B]}{[a^\sharp a^\flat][bc]}\left\{\frac{\langle A^\flat b\rangle\langle A^\flat(b+c)ac\rangle}{\langle cac]} + \frac{\langle A^\flat c\rangle\langle A^\flat(b+c)ab\rangle}{\langle bab]}\right\}\right|^2
$$

$$\mathcal{S}(Q_L g_R g_L) = \frac{m^2 Q^2}{(2QK)^3}\left|\frac{\langle A^\flat b\rangle^2([a^\sharp B]\langle bac] + [a^\sharp c]\langle bcB])}{[a^\sharp a^\flat]\langle cac]\langle bc\rangle}\right|^2$$
$$\mathcal{S}(Q_L g_L g_R) = \frac{m^2 Q^2}{(2QK)^3}\left|\frac{\langle A^\flat c\rangle^2([a^\sharp B]\langle cab] + [a^\sharp b]\langle cbB])}{[a^\sharp a^\flat]\langle bab]\langle bc\rangle}\right|^2$$
$$\mathcal{S}(Q_L g_R g_R) = 0$$
$$\mathcal{S}(Q_R g_L g_L) = \frac{Q^2}{(2QK)^3}\left|\frac{[a^\flat B]}{[bc]}\left\{\frac{\langle A^\flat b\rangle\langle A^\flat(b+c)ac\rangle}{\langle cac]} + \frac{\langle A^\flat c\rangle\langle A^\flat(b+c)ab\rangle}{\langle bab]}\right\}\right|^2$$
$$\mathcal{S}(Q_R g_R g_L) = \frac{Q^2}{(2QK)^3}\left|\frac{\langle A^\flat b\rangle^2}{\langle cac]\langle bc\rangle}\left\{[a^\flat c]\langle bQB] + m^2\frac{\langle a^\sharp b\rangle}{\langle a^\sharp a^\flat\rangle}[cB]\right\}\right|^2$$
$$\mathcal{S}(Q_R g_L g_R) = \frac{Q^2}{(2QK)^3}\left|\frac{\langle A^\flat c\rangle^2}{\langle bab]\langle bc\rangle}\left\{[a^\flat b]\langle cQB] + m^2\frac{\langle a^\sharp c\rangle}{\langle a^\sharp a^\flat\rangle}[bB]\right\}\right|^2$$
$$\mathcal{S}(Q_R g_R g_R) = 0$$
$$\mathcal{S}(Q_L \overline{q}_R q_L) = \frac{m^2 Q^2}{(2QK)^3}\frac{[Ba^\sharp B\rangle[bA^\flat b\rangle^2}{[a^\sharp a a^\sharp\rangle[bcb\rangle}$$
$$\mathcal{S}(Q_L \overline{q}_L q_R) = \frac{m^2 Q^2}{(2QK)^3}\frac{[Ba^\sharp B\rangle[cA^\flat c\rangle^2}{[a^\sharp a a^\sharp\rangle[bcb\rangle}$$
$$\mathcal{S}(Q_R \overline{q}_R q_L) = \frac{Q^2}{(2QK)^3}\frac{[Ba^\flat B\rangle[bA^\flat b\rangle^2}{[bcb\rangle}$$
$$\mathcal{S}(Q_R \overline{q}_L q_R) = \frac{Q^2}{(2QK)^3}\frac{[Ba^\flat B\rangle[cA^\flat c\rangle^2}{[bcb\rangle} \tag{86}$$

As in A.1.2, the last four expressions here are already squared and evaluate to real, positive values.

A.3 Splitting functions with two massive scalars

A.3.1 Spin 0 antenna: initial dipole $S_1\overline{S}_2$

$$\mathcal{S}(S_1 g_L \overline{S}_2) = Q^2\left|\frac{\langle cabc\rangle}{\langle cac]\,[cbc\rangle}\right|^2$$
$$\mathcal{S}(S_1 g_R \overline{S}_2) = Q^2\left|\frac{[cabc]}{[cac\rangle\langle cbc]}\right|^2 \tag{87}$$

A.4 Splitting functions with a massive fermion and a massive scalar

A.4.1 Spin $\frac{1}{2}$ antenna: initial dipole $Q_1\overline{S}_2$

$$
\begin{aligned}
\mathcal{S}(Q_{1L}g_L\overline{S}_2) &= \frac{Q^2}{(E_1+K)(E_2+K)}\left|\frac{\langle a^\flat B^\flat\rangle\langle cabc\rangle}{\langle cac][cbc\rangle} - \frac{\langle a^\flat c\rangle\langle cB^\flat\rangle}{\langle cac]}\right|^2 \\
\mathcal{S}(Q_{1R}g_L\overline{S}_2) &= \frac{m_1^2Q^2}{(E_1+K)(E_2+K)}\left|\frac{1}{\langle a^\sharp a^\flat\rangle}\left\{\frac{\langle a^\sharp B^\flat\rangle\langle cabc\rangle}{\langle cac][cbc\rangle} - \frac{\langle a^\sharp c\rangle\langle cB^\flat\rangle}{\langle cac]}\right\}\right|^2 \\
\mathcal{S}(Q_{1L}g_R\overline{S}_2) &= \frac{Q^2}{(E_1+K)(E_2+K)}\left|\frac{\langle a^\flat B^\flat\rangle[cabc]}{[cac\rangle\langle cbc]}\right|^2 . \\
\mathcal{S}(Q_{1L}g_R\overline{S}_2) &= \frac{m_1^2Q^2}{(E_1+K)(E_2+K)}\left|\frac{1}{\langle a^\sharp a^\flat\rangle}\frac{\langle a^\sharp B^\flat\rangle[cabc]}{[cac\rangle\langle cbc]}\right|^2
\end{aligned}
\tag{88}
$$

A.5 Splitting functions with two massive fermions

A.5.1 Spin 0 antenna: initial dipole $Q_{1L}\overline{Q}_{2L}$

$$
\begin{aligned}
\mathcal{S}(Q_{1L}g_L\overline{Q}_{2L}) &= \frac{Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{\langle a^\flat B^\flat\rangle\langle cabc\rangle\langle A^\flat b^\flat\rangle}{\langle cac][cbc\rangle}\right. \\
&\qquad\left. - \frac{\langle a^\flat c\rangle\langle cB^\flat\rangle\langle A^\flat b^\flat\rangle}{\langle cac]} - \frac{\langle a^\flat B^\flat\rangle\langle A^\flat c\rangle\langle cb^\flat\rangle}{[cbc\rangle}\right|^2 \\
\mathcal{S}(Q_{1L}g_L\overline{Q}_{2R}) &= \frac{m_2^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle b^\flat b^\sharp\rangle}\left\{\frac{\langle a^\flat B\rangle^\flat\langle cabc\rangle\langle A^\flat b^\sharp\rangle}{\langle cac][cbc\rangle}\right.\right. \\
&\qquad\left.\left. - \frac{\langle a^\flat c\rangle\langle cB^\flat\rangle\langle A^\flat b^\sharp\rangle}{\langle cac]} - \frac{\langle a^\flat B^\flat\rangle\langle A^\flat c\rangle\langle cb^\sharp\rangle}{[cbc\rangle}\right\}\right|^2 \\
\mathcal{S}(Q_{1R}g_L\overline{Q}_{2L}) &= \frac{m_1^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle a^\sharp a^\flat\rangle}\left\{\frac{\langle a^\sharp B\rangle^\flat\langle cabc\rangle\langle A^\flat b^\flat\rangle}{\langle cac][cbc\rangle}\right.\right. \\
&\qquad\left.\left. - \frac{\langle a^\sharp c\rangle\langle cB^\flat\rangle\langle A^\flat b^\flat\rangle}{\langle cac]} - \frac{\langle a^\sharp B^\flat\rangle\langle A^\flat c\rangle\langle cb^\flat\rangle}{[cbc\rangle}\right\}\right|^2 \\
\mathcal{S}(Q_{1R}g_L\overline{Q}_{2R}) &= \frac{m_1^2m_2^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle a^\sharp a^\flat\rangle\langle b^\sharp b^\flat\rangle}\left\{\frac{\langle a^\sharp B^\flat\rangle\langle cabc\rangle\langle A^\flat b^\sharp\rangle}{\langle cac][cbc\rangle}\right.\right. \\
&\qquad\left.\left. - \frac{\langle a^\sharp c\rangle\langle cB^\flat\rangle\langle A^\flat b^\sharp\rangle}{\langle cac]} - \frac{\langle a^\sharp B^\flat\rangle\langle A^\flat c\rangle\langle cb^\sharp\rangle}{[cbc\rangle}\right\}\right|^2
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}(Q_{1L}g_R\overline{Q}_{2L}) &= \frac{Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{\langle a^\flat B^\flat\rangle[cabc]\langle A^\flat b^\flat\rangle}{\langle cac][cbc\rangle}\right\}\Bigg|^2 \\
\mathcal{S}(Q_{1L}g_R\overline{Q}_{2R}) &= \frac{m_2^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle b^\flat b^\sharp\rangle}\frac{\langle a^\flat B^\flat\rangle[cabc]\langle A^\flat b^\sharp\rangle}{\langle cac][cbc\rangle}\right|^2 \\
\mathcal{S}(Q_{1R}g_R\overline{Q}_{2L}) &= \frac{m_1^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle a^\sharp a^\flat\rangle}\frac{\langle a^\sharp B\rangle^\flat[cabc]\langle A^\flat b^\flat\rangle}{\langle cac][cbc\rangle}\right|^2 \\
\mathcal{S}(Q_{1R}g_R\overline{Q}_{2R}) &= \frac{m_1^2m_2^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle a^\sharp a^\flat\rangle\langle b^\sharp b^\flat\rangle}\frac{\langle a^\sharp B^\flat\rangle[cabc]\langle A^\flat b^\sharp\rangle}{\langle cac][cbc\rangle}\right|^2
\end{aligned}
\tag{89}
$$

A.5.2 Spin 1 antenna: initial dipole $Q_{1L}\overline{Q}_{2R}$

$$
\begin{aligned}
\mathcal{S}(Q_{1L}g_L\overline{Q}_{2L}) &= \frac{m_2^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle b^\flat b^\sharp\rangle}\left\{\frac{\langle a^\flat B^\flat\rangle\langle cabc\rangle[A^\flat b^\sharp]}{\langle cac][cbc\rangle}-\frac{\langle a^\flat c\rangle\langle cB^\flat\rangle[A^\flat b^\sharp]}{\langle cac]}\right\}\right|^2 \\
\mathcal{S}(Q_{1L}g_L\overline{Q}_{2R}) &= \frac{Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{\langle a^\flat B^\flat\rangle\langle cabc\rangle[A^\flat b^\flat]}{\langle cac][cbc\rangle}-\frac{\langle a^\flat c\rangle\langle cB^\flat\rangle[A^\flat b^\flat]}{\langle cac]}\right\}\Bigg|^2 \\
\mathcal{S}(Q_{1R}g_L\overline{Q}_{2L}) &= \frac{m_1^2m_2^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle a^\sharp a^\flat\rangle[b^\flat b^\sharp]}\left\{\frac{\langle a^\sharp B^\flat\rangle\langle cabc\rangle[A^\flat b^\sharp]}{\langle cac][cbc\rangle}-\frac{\langle a^\sharp c\rangle\langle cB^\flat\rangle[A^\flat b^\sharp]}{\langle cac]}\right\}\right|^2 \\
\mathcal{S}(Q_{1R}g_L\overline{Q}_{2R}) &= \frac{m_1^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle a^\sharp a^\flat\rangle}\left\{\frac{\langle a^\sharp B^\flat\rangle\langle cabc\rangle[A^\flat b^\flat]}{\langle cac][cbc\rangle}-\frac{\langle a^\sharp c\rangle\langle cB^\flat\rangle\langle A^\flat b^\flat\rangle}{\langle cac]}\right\}\right|^2 \\
\mathcal{S}(Q_{1L}g_R\overline{Q}_{2L}) &= \frac{m_2^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{[b^\flat b^\sharp]}\left\{\frac{\langle a^\flat B^\flat\rangle[cabc][A^\flat b^\sharp]}{\langle cac][cbc\rangle}-\frac{\langle a^\flat B^\flat\rangle[A^\flat c][cb^\sharp]}{[cbc\rangle}\right\}\right|^2 \\
\mathcal{S}(Q_{1L}g_R\overline{Q}_{2R}) &= \frac{Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{\langle a^\flat B^\flat\rangle[cabc][A^\flat b^\flat]}{\langle cac][cbc\rangle}-\frac{\langle a^\flat B^\flat\rangle[A^\flat c][cb^\flat]}{[cbc\rangle}\right|^2 \\
\mathcal{S}(Q_{1R}g_R\overline{Q}_{2L}) &= \frac{m_1^2m_2^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle a^\sharp a^\flat\rangle[b^\flat b^\sharp]}\left\{\frac{\langle a^\sharp B^\flat\rangle[cabc][A^\flat b^\flat]}{\langle cac][cbc\rangle}-\frac{\langle a^\flat B^\flat\rangle[A^\flat c][cb^\sharp]}{[cbc\rangle}\right\}\right|^2 \\
\mathcal{S}(Q_{1R}g_R\overline{Q}_{2R}) &= \frac{m_1^2Q^2}{((E_1+K)(E_2+K))^2}\left|\frac{1}{\langle a^\sharp a^\flat\rangle}\left\{\frac{\langle a^\sharp B^\flat\rangle[cabc][A^\flat b^\flat]}{\langle cac][cbc\rangle}-\frac{\langle a^\flat B^\flat\rangle[A^\flat c][cb^\flat]}{[cbc\rangle}\right\}\right|^2
\end{aligned}
\tag{90}
$$

A.6 Splitting functions with pair production of scalars

A.6.1 Spin 0 antenna: initial dipole g_Lg_L

$$
\mathcal{S}(g_L\overline{S}S) = \frac{1}{Q^2}\left|\frac{\langle a[b,c]a\rangle}{s_{bc}}\right|^2 \tag{91}
$$

A.6.2 Spin $\frac{1}{2}$ antenna: initial dipole $q_L g_L$

$$\mathcal{S}(q_L\overline{S}S) = \frac{1}{Q^2}\left|\frac{\langle a[b,c]A\rangle}{s_{bc}}\right|^2 \tag{92}$$

A.6.3 Spin $\frac{3}{2}$ antenna: initial dipole $q_R g_L$

$$\mathcal{S}(q_R\overline{S}S) = \frac{1}{Q^4}\left|\frac{[aB]\langle A[b,c]A\rangle}{s_{bc}}\right|^2 \tag{93}$$

A.6.4 Spin 2 antenna: initial dipole $g_R g_L$

$$\mathcal{S}(g_R\overline{S}S) = \frac{1}{Q^6}\left|\frac{[aB]^2\langle A[b,c]A\rangle}{s_{bc}}\right|^2 \tag{94}$$

A.7 Splitting functions with pair production of fermions

A.7.1 Spin 0 antenna: initial dipole $g_L g_L$

$$\begin{aligned}
\mathcal{S}(g_L\overline{Q}_L Q_L) &= \frac{m^2}{Q^2 s_{bc}^2}\left|\frac{\langle aQb^\sharp]\langle c^\flat a\rangle}{[b^\flat b^\sharp]} + \frac{\langle aQc^\sharp]\langle b^\flat a\rangle}{[c^\flat c^\sharp]}\right|^2 \\
\mathcal{S}(g_L\overline{Q}_L Q_R) &= \frac{1}{Q^2 s_{bc}^2}\left|\langle aQc^\flat]\langle b^\flat a\rangle + \frac{m^2}{[b^\flat b^\sharp]\langle c^\flat c^\sharp\rangle}\langle aQb^\sharp]\langle c^\sharp a\rangle\right|^2 \\
\mathcal{S}(g_L\overline{Q}_R Q_L) &= \frac{1}{Q^2 s_{bc}^2}\left|\langle aQb^\flat]\langle c^\flat a\rangle + \frac{m^2}{\langle b^\flat b^\sharp\rangle[c^\flat c^\sharp]}\langle aQc^\sharp]\langle b^\sharp a\rangle\right|^2 \\
\mathcal{S}(g_L\overline{Q}_R Q_R) &= \frac{m^2}{Q^2 s_{bc}^2}\left|\frac{\langle aQc^\flat]\langle b^\sharp a\rangle}{[b^\flat b^\sharp]} + \frac{\langle aQb^\flat]\langle c^\sharp a\rangle}{[c^\flat c^\sharp]}\right|^2
\end{aligned} \tag{95}$$

A.7.2 Spin $\frac{1}{2}$ antenna: initial dipole $q_L g_L$

$$\mathcal{S}(q_L\overline{Q}_L Q_L) = \frac{m^2}{4Q^2 s_{bc}^2}\left|\frac{\langle aQb^\sharp]\langle c^\flat A\rangle}{[b^\flat b^\sharp]} + \frac{\langle aQc^\sharp]\langle b^\flat A\rangle}{[c^\flat c^\sharp]}\right.$$

$$+\frac{\langle A(Q-a)b^\sharp]\langle c^\flat a\rangle}{[b^\flat b^\sharp]}+\frac{\langle A(Q-a)c^\sharp]\langle b^\flat a\rangle}{[c^\flat c^\sharp]}\Bigg|^2$$

$$\mathcal{S}(q_L\overline{Q}_LQ_R)=\frac{1}{4Q^2s_{bc}^2}\Bigg|\langle aQc^\flat]\langle b^\flat A\rangle+\frac{m^2}{[b^\flat b^\sharp]\langle c^\flat c^\sharp\rangle}\langle aQb^\sharp]\langle c^\sharp A\rangle$$
$$+\langle A(Q-a)c^\flat]\langle b^\flat a\rangle+\frac{m^2}{[b^\flat b^\sharp]\langle c^\flat c^\sharp\rangle}\langle A(Q-a)b^\sharp]\langle c^\sharp a\rangle\Bigg|^2$$

$$\mathcal{S}(q_L\overline{Q}_RQ_L)=\frac{1}{4Q^2s_{bc}^2}\Bigg|\langle aQb^\flat]\langle c^\flat A\rangle+\frac{m^2}{\langle b^\flat b^\sharp\rangle[c^\flat c^\sharp]}\langle aQc^\sharp]\langle b^\sharp A\rangle$$
$$+\langle A(Q-a)b^\flat]\langle c^\flat a\rangle+\frac{m^2}{\langle b^\flat b^\sharp\rangle[c^\flat c^\sharp]}\langle A(Q-a)c^\sharp]\langle b^\sharp a\rangle\Bigg|^2$$

$$\mathcal{S}(q_L\overline{Q}_RQ_R)=\frac{m^2}{4Q^2s_{bc}^2}\Bigg|\frac{\langle aQc^\flat]\langle b^\sharp A\rangle}{[b^\flat b^\sharp]}+\frac{\langle aQb^\flat]\langle c^\sharp A\rangle}{[c^\flat c^\sharp]}$$
$$+\frac{\langle A(Q-a)c^\flat]\langle b^\sharp a\rangle}{[b^\flat b^\sharp]}+\frac{\langle A(Q-a)b^\flat]\langle c^\sharp a\rangle}{[c^\flat c^\sharp]}\Bigg|^2 \tag{96}$$

A.7.3 Spin $\frac{3}{2}$ antenna: initial dipole q_Rg_L

$$\mathcal{S}(q_R\overline{Q}_LQ_L)=\frac{m^2}{Q^4s_{bc}^2}\Bigg|[aB]\Bigg\{\frac{\langle A(Q-a)b^\sharp]\langle c^\flat A\rangle}{[b^\flat b^\sharp]}+\frac{\langle A(Q-a)c^\sharp]\langle b^\flat A\rangle}{[c^\flat c^\sharp]}\Bigg\}\Bigg|^2$$
$$\mathcal{S}(q_R\overline{Q}_LQ_R)=\frac{1}{Q^4s_{bc}^2}\Bigg|[aB]\Bigg\{\langle A(Q-a)c^\flat]\langle b^\flat A\rangle+\frac{m^2}{[b^\flat b^\sharp]\langle c^\flat c^\sharp\rangle}\langle A(Q-a)b^\sharp]\langle c^\sharp A\rangle\Bigg\}\Bigg|^2$$
$$\mathcal{S}(q_R\overline{Q}_RQ_L)=\frac{1}{Q^4s_{bc}^2}\Bigg|[aB]\Bigg\{\langle A(Q-a)b^\flat]\langle c^\flat A\rangle+\frac{m^2}{\langle b^\flat b^\sharp\rangle[c^\flat c^\sharp]}\langle A(Q-a)c^\sharp]\langle b^\sharp A\rangle\Bigg\}\Bigg|^2$$
$$\mathcal{S}(q_R\overline{Q}_RQ_R)=\frac{m^2}{Q^4s_{bc}^2}\Bigg|[aB]\Bigg\{\frac{\langle A(Q-a)c^\flat]\langle b^\sharp A\rangle}{[b^\flat b^\sharp]}+\frac{\langle A(Q-a)b^\flat]\langle c^\sharp A\rangle}{[c^\flat c^\sharp]}\Bigg\}\Bigg|^2 \tag{97}$$

A.7.4 Spin 2 antenna: initial dipole g_Rg_L

$$\mathcal{S}(g_R\overline{Q}_LQ_L)=\frac{m^2}{Q^6s_{bc}^2}\Bigg|[aB]^2\Bigg\{\frac{\langle A(Q-a)b^\sharp]\langle c^\flat A\rangle}{[b^\flat b^\sharp]}+\frac{\langle A(Q-a)c^\sharp]\langle b^\flat A\rangle}{[c^\flat c^\sharp]}\Bigg\}\Bigg|^2$$
$$\mathcal{S}(g_R\overline{Q}_LQ_R)=\frac{1}{Q^6s_{bc}^2}\Bigg|[aB]^2\Bigg\{\langle A(Q-a)c^\flat]\langle b^\flat A\rangle+\frac{m^2}{[b^\flat b^\sharp]\langle c^\flat c^\sharp\rangle}\langle A(Q-a)b^\sharp]\langle c^\sharp A\rangle\Bigg\}\Bigg|^2$$

$$\mathcal{S}(g_R\overline{Q}_R Q_L) = \frac{1}{Q^6 s_{bc}^2}\Big|[aB]^2\Big\{\langle A(Q-a)b^\flat]\langle c^\flat A\rangle + \frac{m^2}{\langle b^\flat b^\sharp\rangle[c^\flat c^\sharp]}\langle A(Q-a)c^\sharp]\langle b^\sharp A\rangle\Big\}\Big|^2$$

$$\mathcal{S}(g_R\overline{Q}_R Q_R) = \frac{m^2}{Q^6 s_{bc}^2}\Big|[aB]^2\Big\{\frac{\langle A(Q-a)c^\flat]\langle b^\sharp A\rangle}{[b^\flat b^\sharp]} + \frac{\langle A(Q-a)b^\flat]\langle c^\sharp A\rangle}{[c^\flat c^\sharp]}\Big\}\Big|^2 \qquad (98)$$

B Spin-dependent Altarelli-Parisi functions for massive particles

In this Appendix, we present the spin-dependent Altarelli-Parisi splitting functions for massless and massive particles. The massless cases were derived in the original paper of Altarelli and Parisi [3]. Spin-summed Altarelli-Parisi functions for the cases with massive particles arise in NLO QCD calculations for supersymmetric particle production. They have been catalogued by Catani, Dittmaier, and Trócsányi in [20]. The spin-dependent functions can be worked out by textbook methods. Here we present these functions in a representation convenient for comparison to the antenna splitting functions derived in this paper. We omit the overall color factor of N_c and divide by 2 so that the splitting accounts the contents of an individual antenna.

Note that, since we work at the leading order in N_c and normalize to a single antenna, there is no difference between the splitting function for a heavy quark or a gluino to radiate a gluon. Thus, there are only two cases, the cases of a heavy scalar S or a heavy quark Q radiating a gluon. The cases of a heavy particle splitting to a heavy particle by radiating a gluon are given by the same expressions with $z \to (1-z)$.

For $S \to gS$,

$$P(S \to Sg_L S) = \frac{p_T^2}{p_T^2 + z^2 m^2}\frac{1-z}{z}$$

$$P(S \to Sg_R S) = \frac{p_T^2}{p_T^2 + z^2 m^2}\frac{1-z}{z} \qquad (99)$$

For $Q \to gQ$,

$$P(Q_L \to Q_L g_L) = \frac{p_T^2}{p_T^2 + z^2 m^2}\frac{1}{z}$$

$$P(Q_L \to Q_L g_R) = \frac{p_T^2}{p_T^2 + z^2 m^2}\frac{(1-z)^2}{z}$$

$$P(Q_L \to Q_R g_L) = \frac{m^2}{p_T^2 + z^2 m^2} \frac{z^4}{z}$$
$$P(Q_L \to Q_R g_R) = 0 \qquad (100)$$

ACKNOWLEDGEMENTS

The authors thank Kassa Betre, Stefan Hoeche, and Jared Kaplan for helpful conversations. This work is supported by the US Department of Energy under contract DE–AC02–76SF00515. A.L. is also supported by an LHC Theory Initiative Travel Award.

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