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# Fluctuation, dissipation, and thermalization in non-equilibrium $\text{AdS}_5$ black hole geometries

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We give a simple recipe for computing dissipation and fluctuations (commutator and anti-commutator correlation functions) for non-equilibrium black hole geometries. The recipe formulates Hawking radiation as an initial value problem, and is suitable for numerical work. We show how to package the fluctuation and dissipation near the event horizon into correlators on the stretched horizon. These horizon correlators determine the bulk and boundary field theory correlation functions. In addition, the horizon correlators are the components of a horizon effective action which provides a quantum generalization of the membrane paradigm. In equilibrium, the analysis reproduces previous results on the Brownian motion of a heavy quark. Out of equilibrium, Wigner transforms of commutator and anti-commutator correlation functions obey a fluctuation-dissipation relation at high frequency.

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## I. MOTIVATION

One of the motivations for studying high temperature gauge theories at strong coupling is the striking results from the Relativistic Heavy Ion Collider and the Large Hadron Collider [1, 2]. Results on collective flow and the energy loss of energetic probes (in particular heavy quarks [3]) indicate that the nuclear size is sufficiently large that macroscopic quantities such as temperature, pressure and flow velocity, are useful concepts when characterizing heavy ion events. A back of the envelope calculation [4] shows that this would not be possible unless the typical relaxation time is of order a thermal wavelength  $\tau_R \sim \hbar/T$ , placing the QCD plasma in a strong coupling regime. The AdS/CFT correspondence has led to many important insights into the nature of strongly coupled plasmas and energy loss [5]. Of particular relevance to this work is the computation of the heavy quark drag and diffusion coefficient in  $\mathcal{N} = 4$  Super Yang Mills (SYM) at large  $N_c$  and strong coupling [6–8]. Indeed heavy quarks in heavy ion collisions exhibit a strong energy loss and a larger than expected elliptic flow, which is qualitatively consistent with a small diffusion coefficient [3]. The actual interpretation of the RHIC results is more complicated, since radiative energy loss plays a significant (perhaps dominant) role at high momentum where the measurements exist [9]. The current results from the Relativistic Heavy Ion Collider on heavy quarks are confusing and not generally understood.

The diffusion of heavy quarks in AdS/CFT is also interesting from the perspective of black hole physics. Indeed, the primary goal of this paper is to better understand the thermal properties of black holes using the diffusion of heavy quarks in  $\mathcal{N} = 4$  SYM as a constrained theoretical laboratory.

On the field theory side of the correspondence, the diffusion of heavy quarks is the result of a competition between the drag and the noise, which must be precisely balanced if the quark is to reach equilibrium. In particular, over a time scale which is long compared to medium correlations, but short compared to the equilibration time of a heavy quark, the heavy quark is expected to obey a Langevin equation

$$\frac{dp^i}{dt} = -\eta v^i + \xi^i, \quad (1.1)$$

where the drag coefficient  $\eta$  and the random noise  $\xi$  are balanced by the fluctuation-dissipation relation

$$\langle \xi^i(t) \xi^j(t') \rangle = 2T\eta \delta^{ij} \delta(t - t'). \quad (1.2)$$

Previously it was shown how this Brownian equation of motion is reproduced by the correspondence [10, 11]. In AdS/CFT a heavy quark is dual to a long straight string which stretches from the boundary to the horizon. At a classical level the straight string is a solution to the EOM and doesn't move. This is not the dual of a heavy quark in plasma. Fig. 1(a) shows the geometry of black hole AdS together with a long straight string. In our AdS conventions the horizon is at  $r = 1$  while the boundary of AdS is at  $r = \infty$ . The stretched horizon (see below) is at  $r_h = 1 + \epsilon$ . Hawking radiation from the horizon causes the string to flip-flop back and forth stochastically as exhibited in Fig. 1(b). The random tugs from this flip-flopping string give rise to a random force on the boundary quark which is related to the dissipation by the Einstein relation, Eq. (1.2).

The derivation of this result left much to be desired. In Ref. [10] it was simply assumed that the modes are in equilibrium at the Hawking temperature. With this assumption it is not difficult to show that the commutator and anti-commutator of string correlations are related via a bulk version of the Fluctuation Dissipation Theorem (FDT). When this bulk FDT is translated to the boundary theory, the bulk FDT leads to Eq. (1.2). While this derivation is physically reasonable, the calculation provides little guidance to out of equilibrium geometries. In Ref. [11], the bulk FDT was derived following a rather complicated and un-intuitive formalism [12, 13]. This formalism involves analytically continuing modes as is typical in many derivations of Hawking radiation [14–16]. The primary goal of this paper is to provide a much more physical derivation of the bulk FDT based upon propagating near-horizon quantum fluctuations from the distant past up to the bulk.

In Ref. [11] the effects of Hawking radiation were packaged into a horizon effective action. This effective action dictates the dynamics of the fields at  $r = 1 + \epsilon$ , and provides a quantum generalization of membrane paradigm. Although the derivation of the membrane effective action involved a complicated analytic continuation, the final form of the effective action is very natural. The classical part of the effective action can be determined using the classical membrane paradigm, while the quantum part of the effective action is dictated by the classical dissipation, and the fluctuation dissipation relation. Once the horizon effective action is written down, a short exercise shows how the horizon fluctuation dissipation relation leads to the FDT in the bulk and boundary theories. Since the FDT is a direct consequence of the fact that the density matrix is  $\exp(-\beta H)$ , the extent to which this relation holds provides an unequivocal measure of equilibrium in the bulk geometry.

In this paper we will determine the horizon effective action by solving equations of motion with appropriate initial conditions rather than analytically continuing modes. Since Fourier transforms and analytic continuations are never

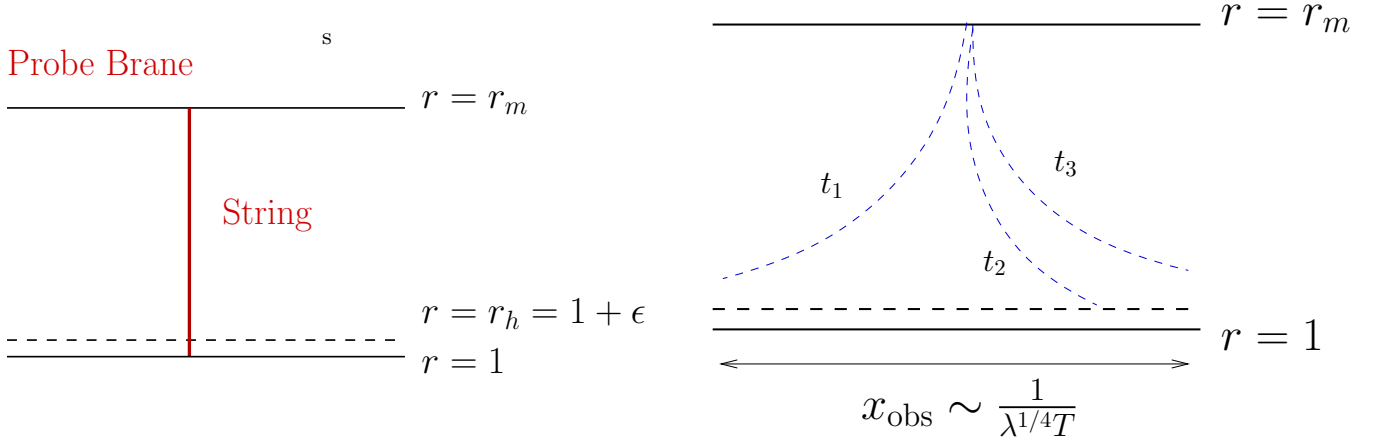


FIG. 1. Two figures from Ref. [11] which motivate this work. (a) A schematic of a classical string in  $\text{AdS}_5$  corresponding to a heavy quark. The horizon is at  $r = 1$  in the coordinates of this work. The stretched horizon is at  $r_h = 1 + \epsilon$  and the endpoint of the string is at the boundary  $r_m$  with  $r_m \gg 1$ . Gravity pulls downward in this figure. (b) Hawking radiation from the horizon induces stochastic motion of the string in the bulk which we show for three subsequent time steps,  $t_1, t_2, t_3$ . The random string configurations give rise to a random force in the boundary theory. The Hawking radiation is encoded in an effective action on the stretched horizon  $r_h = 1 + \epsilon$ . The string fluctuations are small,  $x_{\text{obs}} \sim 1/\lambda^{1/4}T$ .

introduced, it is possible to apply these same techniques to non-equilibrium geometries. We also study the thermalization of a string in a non-equilibrium geometry. Furthermore, while we focus on string fluctuations in this paper, our analysis easily generalizes to other fields such as the graviton.

## II. PRELIMINARIES

In this subsection we will give a brief summary of some of the results of Ref. [11] in order to establish notation. The metric of the black hole AdS space is

$$ds^2 = (\pi T)^2 L^2 \left[ -r^2 f(r) dt^2 + r^2 d\mathbf{x}^2 \right] + \frac{L^2 dr^2}{f(r)r^2}, \quad (2.1)$$

where the horizon is at  $r = 1$  and the boundary is at  $r = \infty$ .  $L$  is the AdS radius,  $f(r) = 1 - 1/r^4$ , and  $T$  is the Hawking temperature.  $r$  is a dimensionless coordinate which measures energy in units of temperature. We will also use Eddington-Finkelstein (EF) coordinates to describe the near horizon dynamics. In this coordinate system the metric is

$$ds^2 = (\pi T)^2 L^2 \left[ -A(r) dv^2 + \frac{2}{\pi T} dr dv + r^2 d\mathbf{x}^2 \right], \quad (2.2)$$

where  $A = r^2 f(r)$  and  $v$  is the EF time

$$v \equiv t + \frac{1}{\pi T} \int \frac{dr}{f r^2}. \quad (2.3)$$

Ingoing lightlike radial geodesics have  $v = \text{const}$ , while outgoing lightlike radial geodesics satisfy  $dr/dv = \pi T A/2$ . From now on we will set the AdS radius to one,  $L = 1$ .

For simplicity, consider fluctuations along an infinitely long straight string (*i.e.* an infinitely massive quark) stretching from the horizon to the boundary. The stationary boundary endpoint is at  $x = 0$ , and small fluctuations around the straight string solution are parameterized by  $x(t, r)$ , where  $x$  denotes displacement of the string in the  $x$  direction. Either  $t$  and  $r$  or  $v$  and  $r$  are taken to be the world sheet parameters,  $t = \sigma^0$  and  $r = \sigma^1$ . The action of these world sheet fluctuations is derived by linearizing the Nambu-Goto action:

$$S = \frac{\sqrt{\lambda}}{2\pi} \int dt dr g_{xx} \left[ -\frac{1}{2} \sqrt{h} h^{\mu\nu} \partial_\mu x \partial_\nu x \right], \quad (2.4)$$

where  $\mu, \nu$  run over  $t, r$  or  $v, r$  depending on the coordinate system. For example, the world sheet metric in EF coordinates is

$$h_{\mu\nu}d\sigma^\mu d\sigma^\nu = (\pi T)^2 \left[ -A(r)dv^2 + \frac{2}{\pi T}drdv \right]. \quad (2.5)$$

We note that the drag coefficient of the heavy quark (see Eq. (1.1)) is related to the coupling between the metric and world sheet fluctuations [6–8]

$$\eta \equiv \frac{\sqrt{\lambda}}{2\pi} g_{xx}(r_h) = \frac{\sqrt{\lambda}}{2\pi} (\pi T)^2. \quad (2.6)$$

The analyses in the following sections make this physical interpretation of  $\eta$  clear.

The goal of this paper is to show by solving equations of motion that in equilibrium the retarded propagator,

$$iG_{ra}(t_1 r_1 | t_2 r_2) \equiv \theta(t - t') \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle, \quad (2.7)$$

and the symmetrized propagator,

$$G_{rr}(t_1 r_1 | t_2 r_2) \equiv \frac{1}{2} \langle \{ \hat{x}(t_1, r_1), \hat{x}(t_2, r_2) \} \rangle, \quad (2.8)$$

are related by the fluctuation dissipation theorem

$$G_{rr}(\omega, r, \bar{r}) = -(1 + 2n(\omega)) \text{Im} G_{ra}(\omega, r, \bar{r}). \quad (2.9)$$

Here  $n(\omega) = 1/(\exp(\omega/T) - 1)$  is the Bose-Einstein distribution function. This relation is a direct consequence of the fact that the density matrix is  $\exp(-H/T)$  and signifies that the fluctuations are in equilibrium with the black hole at temperature  $T$ .

The advanced propagator is related to the retarded propagator by time reversal,

$$iG_{ar}(t_1 r_1 | t_2 r_2) \equiv iG_{ra}(t_2 r_2 | t_1 r_1), \quad (2.10)$$

while the spectral density is the full commutator

$$iG_{ra-ar}(t_1 r_1 | t_2 r_2) \equiv \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle. \quad (2.11)$$

Simple manipulations show that

$$-2\text{Im} G_{ra}(\omega, r, \bar{r}) = iG_{ra-ar}(\omega, r, \bar{r}), \quad (2.12)$$

and thus the fluctuation dissipation theorem is a relation between the commutator and anti-commutator which signifies equilibrium. Both the commutator and the anti-commutator will be determined by solving equations of motion with appropriate initial conditions.

### III. EQUILIBRIUM STRING FLUCTUATIONS IN $AdS_5$ FROM EQUATIONS OF MOTION

#### A. Equations of motion and boundary conditions

Let us analyze the equations of motion in more detail. Recall that the retarded correlator is a Green function of the equations of motion

$$\frac{\sqrt{\lambda}}{2\pi} \left[ \partial_\mu g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] G_{ra}(t_1 r_1 | t_2 r_2) = \delta(t_1 - t_2) \delta(r_1 - r_2), \quad (3.1)$$

and is required to vanish when  $t_1 r_1$  is outside the future light cone of  $t_2 r_2$ . By contrast, the full commutator (*i.e.* the spectral density) is not a Green function but satisfies the homogeneous equations of motion

$$\frac{\sqrt{\lambda}}{2\pi} \left[ \partial_\mu g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] G_{ra-ra}(t_1 r_1 | t_2 r_2) = 0, \quad (3.2)$$

where the initial conditions are determined by the canonical commutation relations. Similarly, the symmetrized correlation function also satisfies the homogeneous equations of motion

$$\frac{\sqrt{\lambda}}{2\pi} \left[ \partial_\mu g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] G_{rr}(t_1 r_1 | t_2 r_2) = 0, \quad (3.3)$$

but the initial conditions are determined by the density matrix of the quantum system far in the past. The appropriate initial conditions for  $G_{rr}$  and  $G_{ra-ar}$  are discussed more fully in Section III C. Finally all bulk to bulk correlation functions ( $G_{ra}, G_{ra-ar}, G_{rr}$ ) satisfy Dirichlet, or normalizable, boundary conditions for asymptotically large radius, *i.e.*  $G \rightarrow 0$  for  $r_1, r_2 \rightarrow \infty$ .

Since the supergravity equations of motion are essentially coupled oscillators, it is useful to recognize that the retarded propagator for the simple harmonic oscillator is independent of the density matrix. Only symmetrized correlations depend on the density matrix and reveal a thermal state. Since the simple harmonic oscillator clearly illustrates the role of the density matrix, we show how to compute commutator and anti-commutator oscillator correlations using the Keldysh formalism in Appendix A.

## B. Horizon Correlators

The equations of motion propagate initial data in the past to the future. This can be made manifest for the symmetrized correlator by writing down a formal solution to Eq. (3.3) in terms of retarded correlators. Specifically, given  $G_{rr}$  and its time derivatives on some time slice  $t_1 = t_2 = t_0$ , the solution to Eq. (3.3) at later times is given by

$$G_{rr}(1|2) = \left[ \frac{\sqrt{\lambda}}{2\pi} \int dr'_1 g_{xx} \sqrt{h} h^{tt}(r'_1) G_{ra}(1|1') \overleftrightarrow{\partial}_{t'_1} \right] \times \left[ \frac{\sqrt{\lambda}}{2\pi} \int dr'_2 g_{xx} \sqrt{h} h^{tt}(r'_2) G_{ra}(2|2') \overleftrightarrow{\partial}_{t'_2} \right] G_{rr}(1'|2'), \quad (3.4)$$

where  $t'_1$  and  $t'_2$  are set equal to  $t_0$  after differentiating, and  $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$ . This formula expresses the uniqueness of the correlator given its value and time derivatives on a Cauchy surface.<sup>1</sup> The physical interpretation of this solution is easy to understand. The two retarded Green functions appearing in the integrals are convoluted with the separate arguments of the initial data. These retarded Green functions causally propagate the initial data forward in time.

To gain qualitative insight into how initial data is propagated by the retarded Green functions, Figure 2 shows a congruence of outgoing radial null geodesics starting at time  $t = t_0$ . A generic geodesic reaches the boundary in a time  $\Delta t \sim 1/T$ . The information which is propagated along such trajectories reflects off the boundary and falls into the black hole with an infall time also of order  $1/T$ . Thus, at times  $\Delta t \gg 1/T$ , the only outgoing geodesics which populate the geometry above the stretched horizon are those which started exponentially close to the horizon at  $t = t_0$ . Moreover, the initial data propagated from this exponentially narrow strip to the above-horizon geometry will be dramatically redshifted. Because of the redshift, the only finite wavelength contributions to the symmetrized correlator will come from the UV part of the initial data near the horizon. This UV part simply comes from coincident point singularities in  $G_{rr}$ , which encode quantum fluctuations in the exponentially narrow strip near the event horizon,  $r = 1$ .

This discussion suggests a strategy for computing the symmetrized correlator at times long after  $t_0$ . First, all initial data above the stretched horizon at  $r_h \equiv 1 + \epsilon$  can be neglected as it gets absorbed by the black hole at times  $t \sim t_0 + 1/T$ . Initial data which lies in the strip between the event horizon and the stretched horizon can be evolved until it reaches the stretched horizon, where it determines an effective horizon correlator. This correlator encodes quantum fluctuations in the past and sources radiation on the stretched horizon that subsequently propagates up to the boundary. This is depicted graphically below in Figure 3. Given the discussion of the preceding paragraph, we expect that the form of the horizon correlator will be independent of details of the initial data specified in the distant past.

<sup>1</sup> The formula manifestly satisfies the equations of motion (3.3) for  $t_1, t_2 > t_0$ . To see that it satisfies the boundary conditions in the limit  $t_1 \rightarrow t_0$ , one must know the time derivatives of  $G_{ra}(1|1')$  for  $t_1 \rightarrow t'_1$ . This derivative can be obtained by using the fact that  $G_{ra}(1|1')$  vanishes for  $t_1 < t'_1$ , and by integrating  $t_1$  across  $t'_1$  with the equations of motion (3.1) to yield the canonical commutation relations

$$\lim_{t_1 \rightarrow t'_1} \frac{\sqrt{\lambda}}{2\pi} g_{xx} \sqrt{h} h^{tt} \partial_{t_1} G(1|1') = \delta(r_1 - r'_1).$$

An analogous formula holds for the derivative with respect to  $t'_1$ .

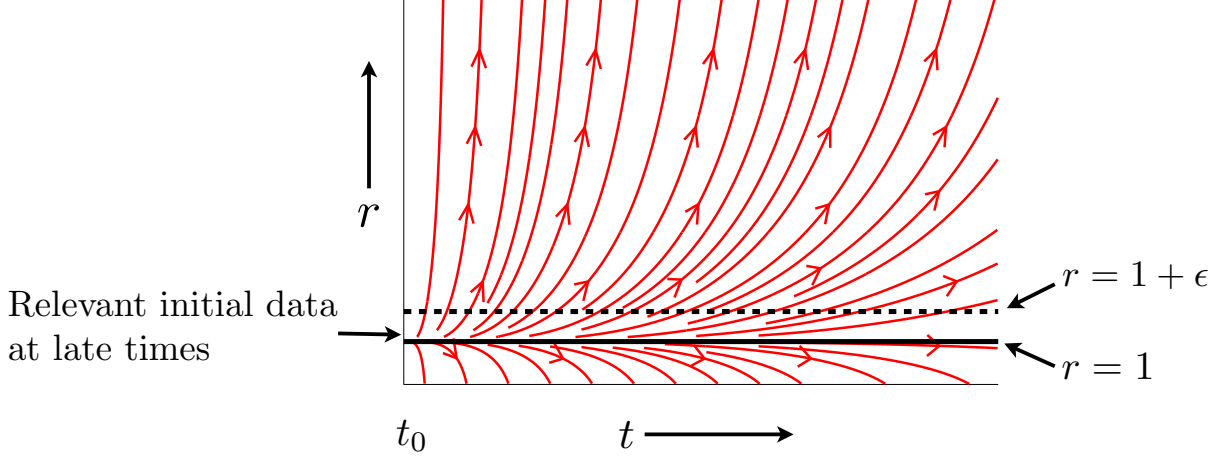


FIG. 2. A congruence of outgoing null radial geodesics starting at time  $t_0$ . Qualitative insight on the propagation of initial data specified on the slice at  $t = t_0$  can be understood from the congruence. Generic geodesics reach the boundary in a time  $\Delta t \sim 1/T$ . Initial data propagated on such trajectories reflects from the boundary and falls into the horizon with an infall time of order  $1/T$ . Geodesics originating exponentially close to the horizon take much longer to escape. Consequently, at late times the above-horizon geometry is filled with geodesics emanating exponentially close to the horizon at  $t = t_0$ . Because of this, the only initial data relevant at late times consists of the initial data exponentially close to the horizon at  $t = t_0$ .

The effective horizon correlation function can be found by exploiting the composition law obeyed by retarded propagators. Let  $g_{ra}(t_1 r_1 | t_2 r_2)$  denote the retarded propagator in the region  $1 \leq r \leq 1 + \epsilon$  subject to the reflective Dirichlet condition<sup>2</sup> at  $r = 1 + \epsilon$

$$g_{ra}(t_1 r_1 = r_h | t_2 r_2) = 0, \quad g_{ra}(t_1 r_1 | t_2 r_2 = r_h) = 0. \quad (3.5)$$

Appendix B analyzes the Wronskian of the retarded propagators  $G_{ra}$  and  $g_{ra}$ , and determines the following composition rule

$$G_{ra}(1|1') = \int dt_2 G_{ra}(1|2) \left[ \eta \sqrt{h} h^{rr}(r_2) \partial_{r_2} \right]_{r_2=r_h} g_{ra}(2|1'), \quad (3.6)$$

where  $t_1 r_1$  is outside the stretched horizon, while  $t'_1 r'_1$  is inside the stretched horizon. This identity, schematically depicted in Fig. 3, is the mathematical statement of how information is propagated up from near the horizon to the stretched horizon, and then up from the stretched horizon towards the boundary. Substituting this composition law into the solution (3.4) and neglecting contributions to the integrals from  $r \geq 1 + \epsilon$ , we find that above the stretched horizon  $G_{rr}$  takes the form

$$G_{rr}(t_1 r_1 | t_2 r_2) = \int dt'_1 dt'_2 [-G_{ra}(t_1 r_1 | t'_1 r_h)] [-G_{ra}(t_2 r_2 | t'_2 r_h)] G_{rr}^h(t'_1 | t'_2), \quad (3.7)$$

where the “horizon correlator”  $G_{rr}^h$  is determined only by the dynamics between the horizon and the stretched horizon:

$$G_{rr}^h(t_1 | t_2) = \left[ -\eta \sqrt{h} h^{rr}(r_1) \partial_{r_1} \right] \left[ -\eta \sqrt{h} h^{rr}(r_2) \partial_{r_2} \right] g_{rr}(t_1 r_1 | t_2 r_2) \Big|_{r_1, r_2=r_h}. \quad (3.8)$$

Here  $g_{rr}(t_1 r_1 | t_2 r_2)$  is the solution to the homogeneous equations of motion (3.3) with reflective Dirichlet boundary conditions at the stretched horizon, together with the prescribed initial conditions close to the horizon at  $t = t_0$ .<sup>3</sup> We will always denote bulk correlation functions inside the strip with lower case letters.<sup>4</sup>

<sup>2</sup> We note that it is not necessary that  $g_{ra}$  satisfy reflective Dirichlet condition at  $r = r_h$ . This choice is simply a matter of latter convenience.

<sup>3</sup> In particular,  $g_{rr}(1|2)$  is given by the same expression in Eq. (3.4), but with the replacements  $G_{rr}(1|2) \rightarrow g_{rr}(1|2)$ ,  $G_{ra}(1|2) \rightarrow g_{ra}(1|2)$  and with the limits of integration running from  $1 \leq r \leq 1 + \epsilon$ .

<sup>4</sup> This correlator will always be written with its arguments  $g_{rr}(v_1 r_1 | v_2 r_2)$ , and can not be confused with the metric coefficient  $g_{rr}(r, v)$ .

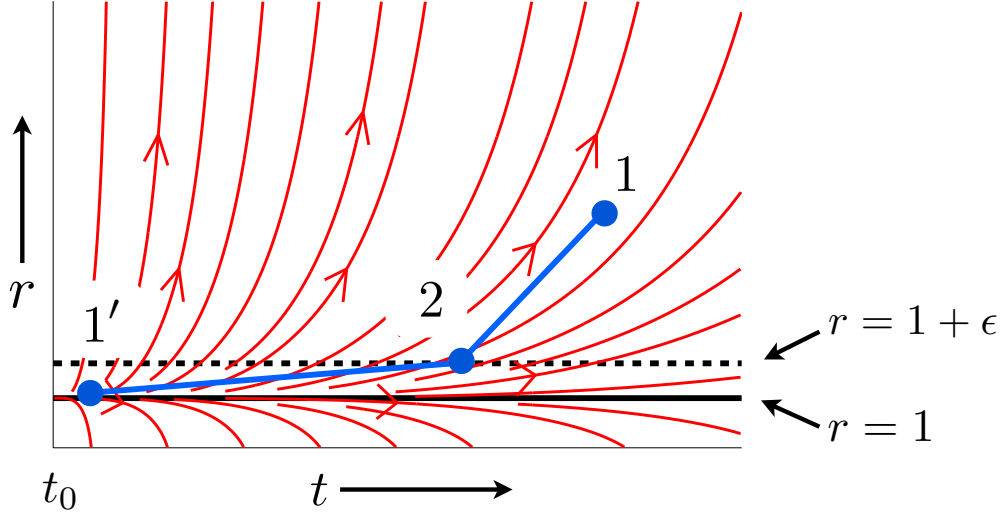


FIG. 3. The composition law for retarded Green functions. The stretched horizon separates the two steps of the evolution. From the perspective of an observer in the exterior, the strip between the horizon at  $r = 1$  and the stretched horizon at  $r = 1 + \epsilon$  produces a simple horizon effective correlator, which acts like a source of radiation from the stretched horizon.

While the symmetrized correlator measures the degree of occupation of microstates, the spectral density  $iG_{ra-ar}(\omega)$  measures the density of available states. Since the spectral density is also a solution to the homogeneous equations of motion, the discussion of the previous paragraph can be repeated *mutatis, mutandis* yielding

$$iG_{ra-ar}(t_1 r_1 | t_2 r_2) = \int dt'_1 dt'_2 [-G_{ra}(t_1 r_1 | t'_1 r_h)] [-G_{ra}(t_2 r_2 | t'_2 r_h)] iG_{ra-ar}^h(t'_1 | t'_2), \quad (3.9)$$

where the horizon spectral density  $G_{ra-ar}^h$  is related to the spectral density inside strip  $g_{ra-ar}$  as in Eq. (3.8)

$$G_{ra-ar}^h(t_1 | t_2) = \left[ -\eta \sqrt{h} h^{rr}(r_1) \partial_{r_1} \right] \left[ -\eta \sqrt{h} h^{rr}(r_2) \partial_{r_2} \right] g_{ra-ar}(t_1 r_1 | t_2 r_2) \Big|_{r_1, r_2 = r_h}. \quad (3.10)$$

In this equation  $g_{ra-ar}$  is subject to the same reflective Dirichlet boundary conditions at  $r = 1 + \epsilon$ . We will see that the horizon correlators,  $G_{ra-ar}^h$  and  $G_{rr}^h$ , form the components of a horizon effective action, which we will study more completely in Section IV.

Eq. (3.7) and Eq. (3.9) are the key equations in our study of thermalization. Together they show that emission and absorption of fluctuations in the bulk can be encapsulated into a horizon fluctuation and a horizon resistance. These equations show that we can focus our attention on the dynamics inside a small strip between the horizon and stretched horizon. Thermalization inside the strip will be simple to understand analytically.

### C. Propagating correlations to the stretched horizon

#### 1. The horizon symmetrized correlation

As is evident from Eqs. (3.7) and (3.8), to determine the symmetrized correlator we must first determine  $g_{rr}(t_1 r_1 | t_2 r_2)$  in the strip between the horizon and the stretched horizon. For describing the evolution inside the strip it is useful to use Eddington-Finkelstein coordinates. In these coordinates the equations of motion for  $r < r_h$  are

$$\frac{\partial}{\partial r_1} \left[ 2h^{vr} \frac{\partial}{\partial v_1} + h^{rr}(r_1) \frac{\partial}{\partial r_1} \right] g_{rr}(v_1 r_1 | v_2 r_2) = 0, \quad (3.11)$$

and

$$\frac{\partial}{\partial r_2} \left[ 2h^{vr} \frac{\partial}{\partial v_2} + h^{rr}(r_2) \frac{\partial}{\partial r_2} \right] g_{rr}(v_1 r_1 | v_2 r_2) = 0, \quad (3.12)$$



where the metric coefficients are given in Eq. (2.5). Since we are interested in the ultra-violet irregular solution to Eq. (3.11) and Eq. (3.12), we have neglected the radial derivatives of  $g_{xx}$ , which are small compared to the radial derivatives of  $g_{rr}(v_1 r_1 | v_2 r_2)$ .<sup>5</sup> With this short-distance approximation, the linear operator in Eq. (3.11) becomes a product of two operators,  $\partial_r$  and  $\partial_v + \frac{\pi T}{2} A \partial_r$ . Recall that in Eddington-Finkelstein coordinates, outgoing null radial geodesics satisfy  $\frac{dr}{dv} = \frac{\pi T}{2} A$ , whereas infalling null geodesics satisfy  $v = \text{const}$ . Thus, the derivatives act along these geodesics, implying that functions of  $e^{-2\pi T v}(r-1)$  and  $v$  satisfy the equations of motion in the near horizon limit. The general solution is therefore

$$g_{rr}(v_1 r_1 | v_2 r_2) = f_1(e^{-2\pi T v_1}(r_1 - 1), e^{-2\pi T v_2}(r_2 - 1)) + f_2(e^{-2\pi T v_1}(r_1 - 1), v_2) \\ + f_3(v_1, e^{-2\pi T v_2}(r_2 - 1)) + f_4(v_1, v_2), \quad (3.13)$$

where the  $f_n(x_1, x_2)$  are arbitrary functions. Requiring that the boundary conditions

$$g_{rr}(v_1, r_1 = r_h | v_2 r_2) = 0, \quad \text{and} \quad g_{rr}(v_1 r_1 | v_2, r_2 = r_h) = 0,$$

are satisfied at all times, we find that

$$g_{rr}(v_1 r_1 | v_2 r_2) = + f(e^{-2\pi T v_1}(r_1 - 1), e^{-2\pi T v_2}(r_2 - 1)) \\ - f(e^{-2\pi T v_1}(r_1 - 1), e^{-2\pi T v_2}(r_h - 1)) \\ - f(e^{-2\pi T v_1}(r_h - 1), e^{-2\pi T v_2}(r_2 - 1)) \\ + f(e^{-2\pi T v_1}(r_h - 1), e^{-2\pi T v_2}(r_h - 1)), \quad (3.14)$$

where  $f(x_1, x_2)$  is determined by initial conditions.

The above solution is a linear combination of modes which are outgoing from the horizon and modes which are infalling towards the horizon. The infalling modes are in fact a consequence of the outgoing modes, as the reflective Dirichlet boundary conditions at the stretched horizon turn outgoing modes into infalling modes, which are subsequently absorbed by the black hole. Inspection of Eq. (3.14) shows that if  $f(x_1, x_2)$  is analytic, this reflection and absorption would always lead to  $g_{rr}(v_1 r_1 | v_2 r_2) \rightarrow 0$  at late times. Furthermore, all outgoing modes originate exponentially close to the horizon. This follows from the fact that as  $e^{-2\pi T v}(r-1)$  is constant on outgoing null geodesics, in the limit  $v \rightarrow -\infty$  we must have  $r \rightarrow 1$ . Therefore, for evolution near  $v = 0$ , relevant initial data specified in the distant past will come from an exponentially narrow strip which is exponentially close to the horizon at  $r = 1$ . In this exponentially narrow strip  $g_{rr}(v_1 r_1 | v_2 r_2)$  is not analytic and contains a logarithmic coincident point singularity

$$g_{rr}(v_1 r_1 | v_2 r_2) = -\frac{1}{4\pi\eta} \log |\mu(v_1 - v_2)(r_1 - r_2)| + \text{exponentially small terms}, \quad (3.15)$$

where  $\mu$  is a constant.<sup>6</sup>

By matching Eq. (3.14) on to Eq. (3.15) we conclude that we must have

$$f(x_1, x_2) = -\frac{1}{4\pi\eta} \log |x_1 - x_2| + \text{exponentially small terms}. \quad (3.16)$$

Substituting Eq. (3.16) into Eq. (3.14), we find that long after initial conditions were specified and up to exponentially small corrections, we have

$$g_{rr}(v_1 r_1 | v_2 r_2) = -\frac{1}{4\pi\eta} \log |e^{-2\pi T \Delta v}(r_1 - 1) - (r_2 - 1)| + \frac{1}{4\pi\eta} \log |e^{-2\pi T \Delta v}(r_1 - 1) - (r_h - 1)| \\ + \frac{1}{4\pi\eta} \log |e^{-2\pi T \Delta v}(r_h - 1) - (r_2 - 1)| - \frac{1}{4\pi\eta} \log |e^{-2\pi T \Delta v}(r_h - 1) - (r_h - 1)|. \quad (3.17)$$

In writing Eq. (3.17) we have pulled out a common factor of  $e^{-2\pi T v_2}$  which cancels between the different terms due to the reflective boundary conditions.

<sup>5</sup> A posteriori one can verify that the neglected derivatives, *e.g.*  $(\partial_r g_{xx})(\partial_v g_{rr}(v_1 r_1 | v_2 r_2))$ , are small compared to the terms which are kept, *e.g.*  $g_{xx} \partial_r h^{vr} \partial_v g_{rr}(1|2)$ .

<sup>6</sup> The coincident point limit can be found by computing the anti-commutator for a free massless scalar field theory in 1+1d flat space

$$\frac{1}{2} \langle \{\phi(X), \phi(0)\} \rangle = -\frac{1}{4\pi} \log |\mu \eta_{\mu\nu} X^\mu X^\nu|.$$

Eq. (3.15) is found by the principle of equivalence and by comparing the normalization of the string action (2.4) to the canonical normalization.

The fact that  $g_{rr}(v_1 r_1 | v_2 r_2)$  is time translationally invariant at late times is due to the competition between emission and absorption. Quantum fluctuations give rise to the logarithmic coincident point singularity in  $g_{rr}(v_1 r_1 | v_2 r_2)$ , and this singularity acts as a constant source of radiation. Thus, radiation is continually produced, redshifted, reflected off the stretched horizon, and absorbed by the black hole. This dynamical process reaches a steady state which determines  $g_{rr}(v_1 r_1 | v_2 r_2)$  at late times. As we will discuss in Section III D, this steady state is in fact a thermal state.

With  $g_{rr}(v_1 r_1 | v_2 r_2)$  known, we may compute the horizon correlator  $G_{rr}^h(v_1, v_2)$  via Eq. (3.8). The result reads

$$G_{rr}^h(v_1, v_2) = -\frac{\eta}{\pi} \partial_{v_1} \partial_{v_2} \log |1 - e^{-2\pi T(v_1 - v_2)}|. \quad (3.18)$$

## 2. The horizon spectral density

As in the case of  $g_{rr}(t_1 r_1 | t_2 r_2)$ , it is convenient to use EF coordinates  $(v, r)$  to determine  $g_{ra-ar}(t_1 r_1 | t_2 r_2)$ . The spectral density  $g_{ra-ar}(v_1 r_1 | v_2 r_2)$  obeys the same equation of motion as  $g_{rr}(v_1 r_1 | v_2 r_2)$  and has the same reflective boundary conditions at the stretched horizon. Thus, the general solution for  $g_{ra-ar}(v_1 r_1 | v_2 r_2)$  is similar to the corresponding solution for  $g_{rr}(v_1 r_1 | v_2 r_2)$  in Eq. (3.14). Explicitly, we have

$$\begin{aligned} g_{ra-ar}(v_1 r_1 | v_2, r_2) = & + f_{ra-ar}(e^{-2\pi T v_1}(r_1 - 1), e^{-2\pi T v_2}(r_2 - 1)) \\ & - f_{ra-ar}(e^{-2\pi T v_1}(r_1 - 1), e^{-2\pi T v_2}(r_h - 1)) \\ & - f_{ra-ar}(e^{-2\pi T v_1}(r_h - 1), e^{-2\pi T v_2}(r_2 - 1)) \\ & + f_{ra-ar}(e^{-2\pi T v_1}(r_h - 1), e^{-2\pi T v_2}(r_h - 1)), \end{aligned} \quad (3.19)$$

where  $f_{ra-ar}$  is determined by initial conditions.

In contrast to the symmetrized correlator, where the initial conditions are determined by the density matrix in the past, the initial conditions for the spectral density are state independent and are determined by the canonical commutation relations. Thus, it is not necessary to evolve for a long time before reaching a steady state solution.

For  $v_1 r_1$  close to  $v_2 r_2$  flat-space physics determines  $g_{ra-ar}$ <sup>7</sup>

$$g_{ra-ar}(v_1 r_1 | v_2 r_2) \rightarrow -\frac{1}{4\eta} (\text{sign}(v_1 - v_2) - \text{sign}(r_1 - r_2)). \quad (3.20)$$

Comparing Eq. (3.20) to Eq. (3.19), we conclude that for any  $v_1$  and  $v_2$  we must have

$$f_{ra-ar}(x_1, x_2) = \frac{1}{4\eta} \text{sign}(x_1 - x_2). \quad (3.21)$$

The infalling and outgoing modes cancel at spacelike separation due to the sign function. One can verify (with careful algebra) that the  $g_{ra-ar}$  satisfies the canonical commutation relation

$$\eta \sqrt{\hbar} h^{tt}(r_1) \lim_{t_2 \rightarrow t_1} \partial_{t_1} g_{ra-ar}(t_1 r_1 | t_2 r_2) = \delta(r_1 - r_2). \quad (3.22)$$

Substituting  $g_{ra-ar}$  into Eq. (3.10), we determine the horizon spectral density

$$G_{ra-ar}^h(v_1, v_2) = 2\eta \delta'(v_1 - v_2). \quad (3.23)$$

## D. The Bulk Fluctuation Dissipation Theorem

The interpretation of the above results becomes clear in Fourier space. The Fourier transform of the horizon symmetrized correlator is

$$G_{rr}^h(\omega) = \frac{\eta}{\pi} \int_{-\infty}^{\infty} dv e^{i\omega v} \partial_v^2 \log |1 - e^{-2\pi T v}| = \eta \omega (1 + 2n(\omega)), \quad (3.24)$$

<sup>7</sup> The coincident point limit can be found by computing the commutator for a free massless scalar field theory in 1+1d flat space

$$-i \langle [\phi(X), \phi(0)] \rangle = -\frac{1}{2} \theta(-\eta_{\mu\nu} X^\mu X^\nu) \text{sign}(t) = -\frac{1}{4} (\text{sign}(t+z) - \text{sign}(-t+z)).$$

The sign function is the only functional form which allows left-moving and right-moving modes to cancel at spacelike separations. Eq. (3.20) is found by the principle of equivalence and by comparing the normalization of the string action (2.4) to the canonical normalization.

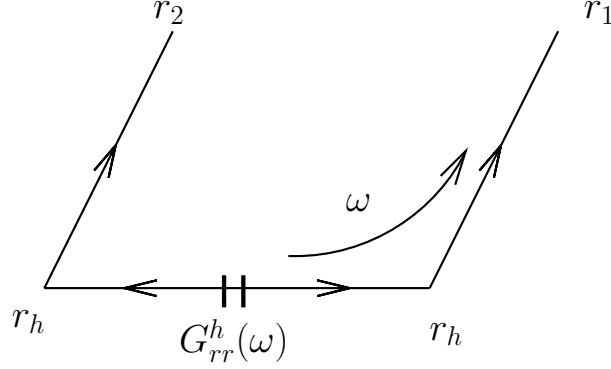


FIG. 4. Feynman graph used for computing the symmetrized correlation function  $G_{rr}(\omega, r_1, r_2)$ , see Eq. (3.27).

where  $n(\omega) = 1/(\exp(\omega/T) - 1)$  is the Bose-Einstein distribution. The Fourier transform of the horizon spectral density is

$$iG_{ra-ar}^h(\omega) = 2\eta\omega. \quad (3.25)$$

Thus, after the decays of transients in the initial data, the horizon correlation functions obey the fluctuation dissipation relation

$$G_{rr}^h(\omega) = iG_{ar-ar}^h(\omega) \left( \frac{1}{2} + n(\omega) \right). \quad (3.26)$$

Now we can see how the bulk will thermalize from these horizon correlations [11]. In Fourier space the expression for  $G_{rr}$  in terms of  $G_{rr}^h$  in Eq. (3.7) becomes

$$G_{rr}(\omega, r_1, r_2) = G_{ra}(\omega, r_1, r_h) G_{ra}(-\omega, r_2, r_h) [G_{rr}^h(\omega)], \quad (3.27)$$

and the spectral density obeys a similar equation

$$iG_{ra-ar}(\omega, r_1, r_2) = G_{ra}(\omega, r_1, r_h) G_{ra}(-\omega, r_2, r_h) [iG_{ra-ar}^h(\omega)]. \quad (3.28)$$

Thus the horizon fluctuation dissipation relation trivially implies the relation in bulk

$$G_{rr}(\omega, r_1, r_2) = iG_{ra-ar}(\omega, r_1, r_2) \left( \frac{1}{2} + n(\omega) \right). \quad (3.29)$$

In Section IV we will show how the convolution in Eq. (3.27) is the result of coupling the horizon effective action to the bulk. Anticipating these results, Fig. 4 shows the Feynman graph corresponding to Eq. (3.27).

Physically, the bulk fluctuation dissipation theorem follows from its horizon counterpart because any fluctuation in the bulk must have crossed the stretched horizon at some point in the past. Previously the form of the horizon correlators was derived either by assuming equilibrium, or by using a complex set of analytic continuations. We see that it is the endpoint of a simple competitive dynamics inside the strip.

#### IV. THE HORIZON EFFECTIVE ACTION

It is enlightening and useful to package the above manipulations into a path integral formalism. The claim is that these steps correspond to integrating out the fields inside the stretched horizon. The horizon correlation functions defined above are the components of the resulting effective action.

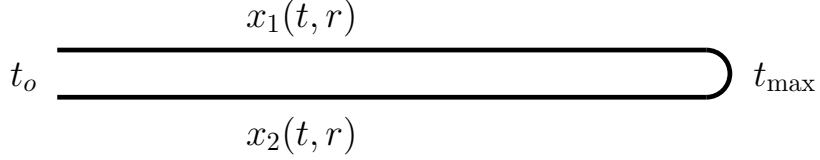


FIG. 5. The Keldysh contour from an initial time  $t_o$  to a final time  $t_{\max}$ .  $x_a(t, r) = x_1(t, r) - x_2(t, r)$  must be zero at  $t_{\max}$ . The initial density matrix  $\rho[x_1(t_o, r), x_2(t_o, r)]$  is specified at time  $t_o$ .

### A. The Keldysh Formalism

In any quantum statistical system, correlations are computed by tracing the density matrix  $\rho$  over the Heisenberg operators. This is the entire purpose of the Keldysh contour [17]. The generating function of string correlation functions in bulk is

$$Z[\mathcal{F}_1, \mathcal{F}_2] = \text{Tr} \left[ \rho \int_{x_1(t_o, r)}^{x_2(t_o, r)} [\mathbb{D}x_1] [\mathbb{D}x_2] e^{iS_1 - iS_2} e^{i \int dt dr \mathcal{F}_1(t, r) x_1(t, r)} e^{-i \int dt' dr' \mathcal{F}_2(t', r') x_2(t', r')} \right], \quad (4.1)$$

where  $[\mathbb{D}x_1]$  indicates a bulk path integral, *i.e.*

$$[\mathbb{D}x_1] = \prod_{t, r} dx_1(t, r). \quad (4.2)$$

The path integral is defined along the Schwinger-Keldysh contour shown in Fig. 5, where the “1” type path integral is the amplitude of the process, while the “2” type path integral is the conjugate amplitude of the process. The trace is over the initial density matrix  $\rho[x_1(t_o, r), x_2(t_o, r)]$  which determines the initial values  $x_1(t_o, r)$  and  $x_2(t_o, r)$  for the subsequent path integral.  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are sources, which in this case are simply the external forces applied to the string. Variation of generating function with respect to  $\mathcal{F}_1$  and  $\mathcal{F}_2$  yields time ordered, anti-time ordered, and Wightman correlation functions [11].

Instead of using the 12 variables, we will rely on a rewritten version of the Keldysh formalism known as the *ra* formalism which is dramatically simpler. We define retarded (*r*) and advanced (*a*) fields and sources

$$x_r = \frac{x_1 + x_2}{2}, \quad x_a = x_1 - x_2, \quad \mathcal{F}_r = \frac{\mathcal{F}_1 + \mathcal{F}_2}{2}, \quad \mathcal{F}_a = \mathcal{F}_1 - \mathcal{F}_2, \quad (4.3)$$

Since  $x_a$  encodes the differences between the amplitude and conjugate amplitudes,  $x_a$  is a small parameter in the classical limit [18]. In terms of *r* and *a* fields the action becomes

$$S_1 - S_2 = \frac{\sqrt{\lambda}}{2\pi} \int dt dr g_{xx} \left[ -\sqrt{h} h^{\mu\nu} \partial_\mu x_r \partial_\nu x_a \right]. \quad (4.4)$$

The two point functions in the “*ra*” formalism are familiar and explain the notation of Section II

$$iG_{ra}(t_1 r_1 | t_2 r_2) = \langle x_r(t_1, r_1) x_a(t_2, r_2) \rangle = \theta(t_1 - t_2) \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle, \quad (4.5)$$

$$G_{rr}(t_1 r_1 | t_2 r_2) = \langle x_r(t_1, r_1) x_r(t_2, r_2) \rangle = \frac{1}{2} \langle \{ \hat{x}(t_1, r_1), \hat{x}(t_2, r_2) \} \rangle. \quad (4.6)$$

The causal structure of quantum field theory is rendered transparent in *ra* formalism. Since at  $t_{\max}$  along the contour,  $x_1(t_{\max}) = x_2(t_{\max})$ , the path integral must be solved with the boundary condition,  $x_a(t) \rightarrow 0$  for  $t \rightarrow \infty$ . Thus, whenever an “*a*” type field is evaluated at a later time than all other field insertions the correlator vanishes. For example,  $G_{aa}(t, t') = 0$  since an *a* field is always evaluated last. Similarly the retarded correlator,  $G_{ra}(t, t')$ , vanishes whenever the *a* field is evaluated at a later time than the *r* field.  $G_{ra}$  determines (minus) the retarded linear response to a classical force. We will exhibit the retarded correlator with an arrow to indicate the direction of time recorded by this propagator

$$\longrightarrow = iG_{ra}(\omega, r, \bar{r}) .$$

While the retarded correlators reflect the response to a classical force, symmetrized correlators encode the fluctuations in the system. Since the symmetrized correlation function does not represent causal response to a classical force, but

rather a time dependent correlation which arose from a specified initial condition, we will notate this correlation (as in Ref. [19]) with

$$\longleftrightarrow \text{H} \longleftrightarrow = G_{rr}(\omega, r, \bar{r}) \quad . \quad (4.7)$$

Higher point correlation functions involving  $r$  and  $a$  indices have a similarly simple interpretation [17].

### B. The effective action from the path integral

To obtain the horizon effective action we integrate out all field fluctuations inside the stretched horizon at  $r_h = 1 + \epsilon$ . The path integral in Eq. (4.1) becomes

$$Z[\mathcal{F}_1, \mathcal{F}_2] = \int_{r > r_h} [\mathbb{D}x_1 \mathbb{D}x_2] \mathbb{D}x^h e^{iS_1 - iS_2 + iS_{\text{eff}}^h[x^h]} e^{i \int \mathcal{F}_1 x_1} e^{-i \int \mathcal{F}_2 x_2} , \quad (4.8)$$

where the horizon effective action is the path integral

$$e^{iS_{\text{eff}}[x_s^h]} \equiv \text{Tr} \left[ \rho \int_{x_1(t_0, r), x_2(t_0, r)}^{r < r_h} \mathbb{D}x_s \delta [x_s^h(t) - x_s(r_h, t)] e^{iS_1 - iS_2} \right] , \quad (4.9)$$

with fixed boundary values on the stretched horizon  $x_s^h(t)$ . Here the “s” sub-label denotes the Schwinger-Keldysh index. The fixed boundary values couple the field fluctuations with  $r > 1 + \epsilon$  to field fluctuations with  $r < 1 + \epsilon$ . The density matrix in the distant past  $\rho[x_1(t_0, r), x_2(t_0, r)]$  is traced over, and this integration is dominated by initial data extremely close to the event horizon. This trace determines how vacuum fluctuations in the past influence the future dynamics. It should be understood that Eq. (4.8) (where the initial density matrix is retained only inside the stretched horizon) is an approximation that is valid after most of the initial data has fallen into the hole as discussed in Section III.

To quadratic order the effective action can be expanded as:

$$iS_{\text{eff}}[x^h] = - \int dt d\bar{t} x_a^h(t) [iG_{ra}(t, \bar{t})] x_r^h(\bar{t}) - \frac{1}{2} \int dt d\bar{t} x_a^h(t) [G_{rr}^h(t, \bar{t})] x_a^h(\bar{t}) , \quad (4.10)$$

where we can determine the horizon correlators via differentiation, e.g.

$$G_{rr}^h(t, \bar{t}) = \frac{1}{i^2} \frac{i \delta S_{\text{eff}}}{\delta x_a^h(t) \delta x_a^h(\bar{t})} \Big|_{x_r^h, x_a^h=0} , \quad (4.11)$$

$$iG_{ra}^h(t, \bar{t}) = \frac{1}{i^2} \frac{i \delta S_{\text{eff}}}{\delta x_a^h(t) \delta x_r^h(\bar{t})} \Big|_{x_r^h, x_a^h=0} . \quad (4.12)$$

A short calculation<sup>8</sup> shows that every derivative  $-i\delta/\delta x_a$  or  $-i\delta/\delta x_r$  brings down a factor of  $-\eta\sqrt{h}h^{rr}\partial_r x_r$  or  $-\eta\sqrt{h}h^{rr}\partial_r x_a$  respectively. Thus

$$G_{rr}^h(t, \bar{t}) = \lim_{r, \bar{r} \rightarrow r_h} \left[ -\eta\sqrt{h}h^{rr}(r)\partial_r \right] \left[ -\eta\sqrt{h}h^{rr}(\bar{r})\partial_{\bar{r}} \right] g_{rr}(t_1 r_1 | t_2 r_2) , \quad (4.13)$$

$$G_{ra}^h(t, \bar{t}) = \lim_{r, \bar{r} \rightarrow r_h} \left[ -\eta\sqrt{h}h^{rr}(r)\partial_r \right] \left[ -\eta\sqrt{h}h^{rr}(\bar{r})\partial_{\bar{r}} \right] g_{ra}(t_1 r_1 | t_2 r_2) , \quad (4.14)$$

where the lower case correlation functions are defined from the path integral for  $r < r_h$  with vanishing boundary conditions at  $r = r_h$

$$\langle \dots \rangle_h = \frac{1}{Z} \text{Tr} \left[ \rho \int_{x_1(t_0, r), x_2(t_0, r)}^{r < r_h} \mathbb{D}x_s \delta [x_s(t, r_h)] e^{iS_1 - iS_2} \dots \right] . \quad (4.15)$$

<sup>8</sup> This is because the only dependence on  $x_s^h$  comes through the boundary terms of the kinetic term, e.g.

$$- \int dt dr \eta\sqrt{h}h^{rr} \partial_r x_r \partial_r x_a = -\eta\sqrt{h}h^{rr} x_a \partial_r x_r \Big|_{r=r_h} + \int dt dr x_a \partial_r (\eta\sqrt{h}h^{rr} \partial_r x_r) .$$

These relations are familiar from the context of the AdS/CFT correspondence where one takes radial derivatives of the fields as  $r \rightarrow \infty$  [20–22]. The functions  $g_{rr}(t_1 r_1 | t_2 r_2)$  and  $g_{ra}(t_1 r_1 | t_2 r_2)$  (or equivalently  $g_{ra-ar} = g_{ra} - g_{ar}$ ) are the same as in section III and their relation to  $G_{rr}^h$  and  $G_{ra-ar}^h$  is identical to that derived in Eq. (3.8). Further the reflective boundary conditions imposed on  $g_{rr}$  and  $g_{ra}$  appear naturally in the path integral formalism. We conclude that the horizon correlators  $G_{rr}^h$  and  $G_{ra-ar}^h$  defined and computed in the previous section are precisely the components of a well-defined effective action. Given a procedure for regularizing ultraviolet divergences in gravity, this effective action could be computed to any desired order in perturbation theory following the methods of Ref. [23].

### C. Summary of the horizon effective action

The effective action in equilibrium was obtained previously by Son and Teaney [11] by analytically continuing modes across the horizon. Its form is simple,

$$iS_{\text{eff}}^h = - \int \frac{d\omega}{2\pi} x_a^h(-\omega) [iG_{ra}^h(\omega)] x_r^h(\omega) - \frac{1}{2} \int \frac{d\omega}{2\pi} x_a^h(-\omega) [G_{rr}^h(\omega)] x_a^h(\omega), \quad (4.16)$$

where the retarded horizon correlation function is simply

$$G_{ra}^h = -i\omega\eta, \quad (4.17)$$

and the symmetrized part of the horizon action obeys a fluctuation dissipation relation

$$G_{rr}^h = -(1 + 2n(\omega)) \text{Im} G_{ra}^h(\omega). \quad (4.18)$$

Fourier transforming back to time, the horizon action reads

$$iS_{\text{eff}}^h = -i \int dt x_a^h(t) \eta \partial_t x_r^h(t) - \frac{1}{2} \int dt dt' x_a^h(t) \left[ -\frac{\eta}{\pi} \partial_t \partial_{t'} \log |1 - e^{-2\pi T(t-t')}| \right] x_a^h(t'). \quad (4.19)$$

This action compactly summarizes all the correlations that appear through quadratic order in  $x_a$  and agrees with the results of Section III D.

The horizon effective action is useful. For instance, since the bulk action in Eq. (4.4) has no  $rr$  and no  $aa$  components, the first Feynman graph which contributes to the correlation  $\langle x_r(v_1, r_1) x_r(v_2, r_2) \rangle_h$  is shown in Fig. 4 and is written in full in Eq. (3.27), with the lines explained in Section IV A. Thus, perturbation theory with the bulk and horizon actions transparently produces the convolution formulas given in Eq. (3.7) and Eq. (3.9).

The retarded horizon propagator in the action,  $G_{ra}^h$ , reflects the resistance on the horizon,  $-\eta\dot{x}$ , and is valid at all frequencies [11]. Indeed, the classical dissipation encoded by the  $ra$  part of the action can be derived simply from the classical membrane paradigm. Variation of the effective action gives the horizon force

$$\mathcal{F}_r^h = \frac{\delta S_{\text{eff}}^h}{\delta x_a} = -\eta \partial_t x_r, \quad (4.20)$$

where the force is  $\mathcal{F}_r^h = -\eta\sqrt{\hbar}h^{rr}\partial_r x_r$  as required by the membrane paradigm [24]. In the zero frequency limit the force  $\mathcal{F}_r$  is independent of radius [6, 25], implying that horizon drag coefficient  $\eta$  is the same as boundary drag coefficient in the Langevin equation [6].

## V. NON-EQUILIBRIUM CORRELATORS

### A. Setup

In this section we wish to show how to generalize the non-equilibrium horizon effective action. Interesting non-equilibrium geometries to consider can be found in [26–28]. However, for definiteness we will consider the non-equilibrium geometry discussed in Ref. [26]. In this work, an excited state in the boundary quantum field theory was created by briefly turning on a time-dependent  $4d$  gravitational field, which was taken to be translationally invariant. The gravitational field did work on the quantum system, producing an excited state which subsequently thermalized. In the dual  $5d$  gravitational system, turning on a  $4d$  gravitational field corresponds to deforming the  $4d$  boundary of the  $5d$  geometry. Before the deformation was turned on, the  $5d$  geometry was taken to be  $AdS_5$ , which is dual to

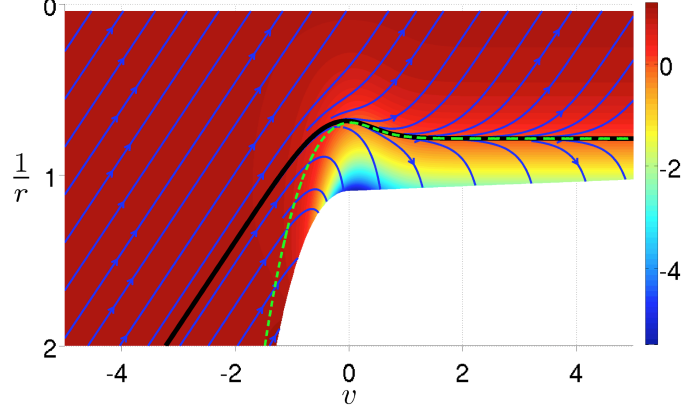


FIG. 6. Figure from Ref. [26]. The congruence of outgoing radial null geodesics. The surface coloring displays  $A(v, r)/r^2$ . The excised region is beyond the apparent horizon, which is shown by the dashed green line. The geodesic shown as a solid black line is the event horizon; it separates geodesics which escape to the boundary from those which cannot escape.

the vacuum state. The deformation in the boundary geometry produced gravitational radiation which fell into the bulk. This infalling radiation resulted in the process of gravitational collapse, changing the initial  $AdS_5$  geometry to one which had a black hole, and the relaxation of the black hole to equilibrium encoded the thermalization of the expectation value of the stress tensor in the dual quantum theory.

Translation invariance allows the  $5d$  metric to be written

$$ds^2 = A dv^2 + 2 dr dv + \Sigma^2 \left( e^B d\mathbf{x}_\perp^2 + e^{-2B} dx_\parallel^2 \right), \quad (5.1)$$

where all coefficients  $A, \Sigma, B$  are functions of radius  $r$  and time  $v$ . The metric coefficient  $A(v, r)$  together with the outgoing radial geodesics calculated in this geometry are shown in Fig. 6 and is reproduced from [26]. Outgoing light like geodesics satisfy  $dr/dv = A/2$ . We will determine the non-equilibrium string correlators in this transient geometry.

A salient feature of the outgoing geodesics is their ultimate bifurcation at the event horizon,  $r_o(v)$ , which is shown by a thick black line in Fig. 6. This bifurcation is reminiscent of Fig. 2. It is convenient to switch coordinates to a system of coordinates where this black line is flat

$$\rho = r - r_o(v). \quad (5.2)$$

Note  $r_o(v)$  defines a light like outgoing radial geodesic, so that the metric is

$$ds^2 = -(A - A_o) dv^2 + 2 d\rho dv + \Sigma^2 \left( e^B d\mathbf{x}_\perp^2 + e^{-2B} dx_\parallel^2 \right), \quad (5.3)$$

where  $A_o(v) = A(r_o(v), v)$ . In the near horizon geometry, where  $r \simeq r_o(v)$  we can approximate

$$A(r, v) - A_o(v) \simeq \left. \frac{\partial A(r, v)}{\partial r} \right|_{r=r_h(v)} \rho, \quad (5.4)$$

where here and below we will define the stretched horizon at  $\rho_h \equiv \epsilon$ , *i.e.*  $r_h(v) \equiv r_o(v) + \epsilon$ . For future convenience we define the “effective temperature”<sup>9</sup>

$$2\pi T_{\text{eff}}(v) \equiv \frac{1}{2} \left. \frac{\partial A(r, v)}{\partial r} \right|_{r=r_h(v)}. \quad (5.5)$$

The action of the string fluctuations in the  $x$  direction is the same as Eq. (2.4), but the metric coefficients depend on time and radius. With the goal of determining the horizon effective action it is useful to define a non-equilibrium drag coefficient

$$\eta(v) \equiv \frac{\sqrt{\lambda}}{2\pi} g_{xx}(r_h(v), v). \quad (5.6)$$

<sup>9</sup> We note that  $T_{\text{eff}}$  should not be interpreted as a temperature at all times. In particular, due to the teleological nature of event horizons,  $T_{\text{eff}}$  is non-zero even before the boundary geometry has changed – *i.e.* when the dual boundary quantum theory is still in the vacuum state. It is only at late times when the black hole starts to thermalize that  $T_{\text{eff}}$  can be interpreted as a temperature.

## B. Calculation

The computational procedure for computing correlators in the non-equilibrium case is remarkably similar to the equilibrium case discussed in Sections III and IV. For both  $G_{rr}$  and  $G_{ra-ar}$  one can write down a solution to the equations of motion in terms of retarded Green functions convoluted with initial data, as in Eq. (3.4). Furthermore, for initial data specified suitably far in the past, the relevant initial data for evolution near  $v = 0$  (when the boundary geometry is changing) will come from a narrow strip near the horizon.<sup>10</sup> One can then repeat the analysis of Section IIIB and conclude that  $G_{rr}$  and  $G_{ra-ar}$  are determined by horizon correlators  $G_{rr}^h$  and  $G_{ra-ar}^h$ , as in Eqs. (3.7) and (3.9).

In the non-equilibrium case  $G_{rr}^h$  is given by

$$G_{rr}^h(v_1, v_2) = \left[ -\eta(v_1) \sqrt{h} h^{\rho\rho}(\rho_1) \partial_{\rho_1} \right] \left[ -\eta(v_2) \sqrt{h} h^{\rho\rho}(\rho_2) \partial_{\rho_2} \right] g_{rr}(v_1 \rho_1 | v_2 \rho_2) \quad (5.7)$$

where  $g_{rr}(v_1 \rho_1 | v_2 \rho_2)$  is a solution of the homogeneous equations, but is confined to the strip,  $0 < \rho \leq \epsilon$ . Furthermore  $g_{rr}(v_1 \rho_1 | v_2 \rho_2)$  should satisfy the boundary conditions  $g_{rr}(v_1, \rho_1 = \epsilon | v_2, \rho_2) = g_{rr}(v_1 \rho_1 | v_2 \rho_2 = \epsilon) = 0$ .

The equations of motion for  $g_{rr}(v_1 \rho_1 | v_2 \rho_2)$  are

$$\left[ \frac{\partial}{\partial v_1} g_{xx} \sqrt{h} h^{\rho\nu} \frac{\partial}{\partial \rho_1} + \frac{\partial}{\partial \rho_1} g_{xx} \sqrt{h} h^{\rho\nu} \frac{\partial}{\partial v_1} + \frac{\partial}{\partial \rho_1} g_{xx} \sqrt{h} h^{\rho\rho} \frac{\partial}{\partial \rho_1} \right] g_{rr}(v_1 \rho_1 | v_2 \rho_2) = 0, \quad (5.8)$$

where all the metric coefficients depend on  $v$  and  $r$ . Without approximation, we have

$$\begin{aligned} \frac{\partial}{\partial \rho_1} \left[ 2g_{xx} \sqrt{h} h^{\rho\nu} \frac{\partial}{\partial v_1} + \frac{\partial(g_{xx} \sqrt{h} h^{\rho\nu})}{\partial v_1} + g_{xx} \sqrt{h} h^{\rho\rho} \frac{\partial}{\partial \rho_1} \right] g_{rr}(v_1 \rho_1 | v_2 \rho_2) \\ - \left[ \frac{\partial(g_{xx} \sqrt{h} h^{\rho\nu})}{\partial \rho_1} \frac{\partial}{\partial v_1} + \frac{\partial(g_{xx} \sqrt{h} h^{\rho\nu})}{\partial \rho_1 \partial v_1} \right] g_{rr}(v_1 \rho_1 | v_2 \rho_2) = 0. \end{aligned} \quad (5.9)$$

The above equation of motion contains both first and second derivatives. However, as we are interested in solutions which are irregular near the horizon, all terms with single derivative operators acting on  $g_{rr}(v_1 \rho_1 | v_2 \rho_2)$  can be neglected. This approximation leads to

$$\frac{\partial}{\partial \rho_1} \left[ 2\eta \sqrt{h} h^{\rho\nu} \frac{\partial}{\partial v_1} + \frac{\partial(\eta \sqrt{h} h^{\rho\nu})}{\partial v_1} + \eta \sqrt{h} h^{\rho\rho} \frac{\partial}{\partial \rho_1} \right] g_{rr}(v_1 \rho_1 | v_2 \rho_2) = 0, \quad (5.10)$$

and an analogous equation for  $v_2$  and  $\rho_2$ . Inspecting the above equation, we see that  $\sqrt{\eta(v_1)\eta(v_2)} g_{rr}(v_1 \rho_1 | v_2 \rho_2)$  is annihilated by the operator  $\partial_\rho [\partial_v + \frac{1}{2} A \partial_\rho]$ . As in Section IIIC1, the consequence of this is that any function which is constant on null radial geodesics satisfies the equations of motion. Near the horizon this translates to  $\sqrt{\eta(v_1)\eta(v_2)} g_{rr}(v_1 \rho_1 | v_2 \rho_2)$  being a function of  $v$  and

$$\rho_{\text{out}}(\rho, v) = \rho e^{-\int_{v_o}^v 2\pi T_{\text{eff}}(v') dv'}. \quad (5.11)$$

As in Eq. (3.14), the general solution to Eq. (5.10) which satisfies the requisite boundary conditions at the stretched horizon reads

$$\begin{aligned} \sqrt{\eta(v_1)\eta(v_2)} g_{rr}(v_1 \rho_1 | v_2 \rho_2) = & f(\rho_{\text{out}}(\rho_1, v_1), \rho_{\text{out}}(\rho_2, v_2)) - f(\rho_{\text{out}}(\rho_h, v_1), \rho_{\text{out}}(\rho_2, v_2)) \\ & - f(\rho_{\text{out}}(\rho_1, v_1), \rho_{\text{out}}(\rho_h, v_2)) + f(\rho_{\text{out}}(\rho_h, v_1), \rho_{\text{out}}(\rho_h, v_2)), \end{aligned} \quad (5.12)$$

where as in the equilibrium case,  $f(x_1, x_2)$  is determined by the initial conditions specified in the distant past.

To determine  $f(x_1, x_2)$  we invoke a similar argument used in Section IIIC1. In particular, assuming initial data for  $g_{rr}(v_1 \rho_1 | v_2 \rho_2)$  was specified in the distant past, the only relevant initial data for evolution near  $v = 0$  comes from

<sup>10</sup> To make this more precise, suppose instead of starting off with an initial geometry which was  $AdS_5$ , the initial geometry consisted of a static black hole geometry at temperature  $T_{\text{initial}} = \delta$ . Assuming initial data is specified at times  $v \ll -1/\delta$  in the past, all relevant initial data for future evolution around  $v = 0$  will come from an exponentially narrow strip which is exponentially close to the horizon. Of course, one can always consider the limit  $\delta \rightarrow 0$  after all calculations are performed.



a very narrow strip which is very close to the event horizon. In this strip the relevant initial data is given by the coincident point limit of the symmetrized correlator as given in Eq. (3.15). Consequently  $f(x_1, x_2)$  must be given by

$$f(x_1, x_2) = -\frac{1}{4\pi} \log |x_1 - x_2|. \quad (5.13)$$

With  $g_{rr}(v_1\rho_1|v_2\rho_2)$  known, Eq. (5.7) yields the symmetrized horizon correlator

$$G_{rr}^h(v_1, v_2) = -\frac{\sqrt{\eta(v_1)\eta(v_2)}}{\pi} \partial_{v_1} \partial_{v_2} \log |1 - e^{-\int_{v_1}^{v_2} 2\pi T_{\text{eff}}(v') dv'}|. \quad (5.14)$$

We will discuss the physical implications of this result in the next section.

The horizon spectral density can be obtained along the same lines as the horizon symmetrized correlator with the subscript replacement  $rr \rightarrow ra - ar$  in Eq. (5.7). Because  $g_{ra-ar}(v_1\rho_1|v_2\rho_2)$  satisfies the same equations of motion and boundary conditions as  $g_{rr}(v_1\rho_1|v_2\rho_2)$ , the general solution for  $g_{ra-ar}(v_1\rho_1|v_2\rho_2)$  takes the same form as Eq. (5.12). However, as in the equilibrium case discussed in Section III C 2, the initial conditions determining  $f(x_1, x_2)$  for the spectral density come purely from the equal time canonical commutation relations and yield

$$f(x_1, x_2) = \frac{1}{4} \text{sign}(x_1 - x_2). \quad (5.15)$$

With  $g_{ra-ar}$  known, the horizon spectral correlator reads

$$G_{ra-ar}^h(v_1, v_2) = 2\sqrt{\eta(v_1)\eta(v_2)}\delta'(v_1 - v_2). \quad (5.16)$$

## VI. SUMMARY

Let us summarize our results. For definiteness we will describe how to compute string fluctuations in the non-equilibrium geometry determined by Chesler and Yaffe [26]. We emphasize, however, that similar formulas can be used for other fields such as gravitons and dilatons. Fig. 6 shows the event horizon  $r_o(v)$  together with the associated bifurcating outgoing geodesics of this geometry. The stretched horizon is located at  $r_h(v) = r_o(v) + \epsilon$

In contrast to a retarded Green function (which is a response to a source), a symmetrized correlation function, or a fluctuation, is a time dependent correlation that arose from a definite initial condition. In the case of Hawking radiation (see Fig. 2), this initial condition is the result of an ultra-violet vacuum fluctuation (or a symmetrized correlation function) which originated close to the event horizon of the developing black hole. This UV fluctuation skims the event horizon following outgoing lightlike geodesics until late times. Then the fluctuation, which is no longer so ultra-violet, leaves the bifurcating horizon and induces stochastic motion in the string. We implemented this picture of Hawking radiation directly.

We showed how the Hawking flux out of equilibrium can be packaged into an effective action on the stretched horizon which can be used to determine the effect of the Hawking radiation on the exterior dynamics. The horizon action through quadratic order is

$$\begin{aligned} iS_{\text{eff}}[x^h] = & -\frac{1}{2} \int dv_1 dv_2 x_a^h(v_1) [iG_{ra-ar}^h(v_1, v_2)] x_r^h(v_2) \\ & - \frac{1}{2} \int dv_1 dv_2 x_a^h(v_1) [G_{rr}^h(v_1, v_2)] x_a^h(v_2), \end{aligned} \quad (6.1)$$

where the horizon correlation functions are the horizon spectral density,

$$G_{ra-ar}^h(v_1, v_2) = 2\sqrt{\eta(v_1)\eta(v_2)}\delta'(v_1 - v_2), \quad (6.2)$$

and the horizon symmetrized correlator,

$$G_{rr}^h(v_1, v_2) = -\frac{\sqrt{\eta(v_1)\eta(v_2)}}{\pi} \partial_{v_1} \partial_{v_2} \log |1 - e^{-\int_{v_1}^{v_2} 2\pi T_{\text{eff}}(v') dv'}|. \quad (6.3)$$

Here  $x^h(v)$  is the location of the string on the stretched horizon as a function of time. The coefficient  $\eta(v) = (\sqrt{\lambda}/2\pi)g_{xx}(v, r_h(v))$  determines the coupling between the world sheet fluctuations and the near horizon geometry, and the effective horizon temperature records the Lyapunov exponent of diverging geodesics along the bifurcating

horizon (see Eq. (5.5) and Eq. (5.11)). More invariantly, it is related to the extrinsic curvature on the stretched horizon. In equilibrium,  $\eta(v)$  is the drag of the heavy quark, and  $T_{\text{eff}}$  is the Hawking temperature. The current analysis can also be extended to a heavy quark moving with velocity  $v$  in a finite temperature background. In this case  $T_{\text{eff}}$  records how geodesics diverge on the string world sheet and differs from  $T$  by a factor of  $\sqrt{\gamma}$ , where  $\gamma = 1/\sqrt{1-v^2}$  [29–33]. This extra factor of  $\sqrt{\gamma}$  influences the velocity dependence of the transverse and longitudinal momentum broadening rates of heavy quarks in strongly coupled plasmas.

The horizon spectral density in Eq. (6.2) is proportional to  $\omega$  and is determined by the canonical commutation relations of the 1+1d effective theory which describes the near horizon dynamics. The horizon symmetrized correlator is determined by the initial density matrix of the effective theory far in the past. When the horizon action is coupled to the bulk, this action generates noise on the stretched horizon which induces the random motion of the quark in the dual field theory [11]. For a stationary black hole, the effective action was determined previously using complicated analytic continuations [11].

The importance of these horizon functions is that they determine the spectral density and symmetrized correlations in the bulk and boundary theories. Indeed, the bulk spectral density (the commutator) and symmetrized correlator (the anti-commutator) are determined by propagating their horizon counterparts away from the stretched horizon

$$iG_{ra-ar}(v_1 r_1 | v_2 r_2) = \int dv'_1 dv'_2 G_{ra}(v_1 r_1 | v'_1 r_h(v'_1)) G_{ra}(v_2 r_2 | v'_2 r_h(v'_2)) iG_{ra-ar}^h(v'_1 | v'_2), \quad (6.4)$$

$$G_{rr}(v_1 r_1 | v_2 r_2) = \int dv'_1 dv'_2 G_{ra}(v_1 r_1 | v'_1 r_h(v'_1)) G_{ra}(v_2 r_2 | v'_2 r_h(v'_2)) G_{rr}^h(v'_1 | v'_2). \quad (6.5)$$

Finally, these bulk correlation functions can be lifted to the boundary to determine the spectral density and symmetrized correlator in the field theory (see Appendix C). When the fluctuations are thermalized, the two correlation functions satisfy the fluctuation dissipation theorem

$$G_{rr}(\omega, r_1, r_2) = \left( \frac{1}{2} + n(\omega) \right) iG_{ra-ar}(\omega, r_1, r_2). \quad (6.6)$$

Thus we can numerically determine the fluctuations and monitor their approach to equilibrium using the formalism of this work. This numerical calculation will be presented in future work.

Even without a complete numerical computation, some preliminary remarks can be made about equilibration in  $AdS_5$ . The Wigner transforms of  $G_{rr}^h(v_1, v_2)$  and  $G_{ra-ar}^h(v_1, v_2)$  obey the fluctuation dissipation theorem at *high* frequency.<sup>11</sup> Specifically, for a typical non-equilibrium time scale  $\tau$ , we have

$$G_{rr}^h(\bar{v}, \omega) \simeq \left( \frac{1}{2} + n(\omega) \right) iG_{ra-ar}^h(\bar{v}, \omega) + O\left( \frac{1}{\tau^2 \omega^2} \right), \quad (6.7)$$

where  $\bar{v} = (v_1 + v_2)/2$ . Thus the string is born into equilibrium at high frequency, and eventually frequencies of order the temperature and below equilibrate. This conclusion seems squarely aligned with the results of Ref. [34] which was limited to operators of high conformal dimension. However, it must be emphasized that the map between the stretched horizon and the boundary is non-trivial, especially for five-dimensional fields where the 3-momentum influences the coupling between the field and the near horizon geometry.

One popular picture of Hawking radiation is based on quantum tunneling (see, for instance, [35]). This picture relates the thermal factor in the emission rate to the change in the black hole entropy, which appears by way of the Euclidean action. It would be interesting to make contact with this picture using the effective action formalism. This would help make the universality of the result fully manifest. It would also be interesting to see if the simple non-equilibrium effective action presented in this paper finds a simple origin in the tunneling picture.

The current derivation of Hawking radiation and correlation functions is similar to the 2PI formalism of non-equilibrium field theory [36], and we hope this will make black hole physics accessible to a wider audience. The derivation uses the unstable nature of the bifurcating event horizon to expand ultraviolet vacuum fluctuations. This exponential sensitivity to initial conditions has been called the transplanckian problem [37], and is characteristic of classically chaotic systems [38]. By exploiting this analogy, we hope that new insight can be found into the transplanckian problem and the Bekenstein entropy. Understanding the Bekenstein-Hawking entropy will require coupling

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<sup>11</sup> The Wigner transform is the Fourier transform with respect to the difference  $v_1 - v_2$ , as a function of the average,  $\bar{v} = (v_1 + v_2)/2$ .

the particle emission to the background metric, leading to a dynamical competition between quantum particles and the classical background field. The particle-field problem has been extensively studied in thermal field theory and the Color Glass Condensate [18, 39]. We hope to pursue these connections in future work.

### Comparison with recent literature:

Recently several papers appeared which addressed aspects of this paper as this work was being finalized [40]. First, a paper by Headrik and Ebrahim [40] used the equations of motion to solve for the symmetrized correlation function in  $AdS_3$ -Vaidya spacetimes. Headrik and Ebrahim reported on the “instantaneous thermalization” of  $AdS_3$ . Second, a paper by Balasubramanian *et al* [34] computed the thermalization of operators with high conformal dimension by studying geodesics. Although the current paper is not limited to such operators, the basic conclusion that the field theory thermalizes first at high frequency is consistent with our conclusion about horizon Wigner transforms. However, as emphasized above the map between the stretched horizon and the boundary involves physically important and non-trivial outgoing propagators.

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### Appendix A: Symmetrized correlations and the Keldysh formalism – a lesson from the harmonic oscillator

It is instructive in many respects to compute the symmetrized correlation function of the harmonic oscillator using the Keldysh formalism. The action of the oscillator is

$$iS_1 - iS_2 = i \int_{t_o} dt [m\dot{x}_r\dot{x}_a - m\omega_o^2 x_r x_a] , \quad (A1)$$

and is similar to the string action written in Eq. (4.4). In both systems we see that there are no  $rr$  type propagators in the action itself. This is because symmetrized correlations are the result of a correlation built into the initial state wave functions, *i.e.* the initial density matrix.

In symmetrized type correlation functions the density matrix at an initial time  $t_o$  correlates the initial conditions for subsequent evolution. The density matrix for the harmonic oscillator in the ground state is  $\Psi_o(x_1)\Psi_o^*(x_2)$ , and the symmetrized correlator is

$$G_{rr}(t, \bar{t}) = \int dx_r^o dx_a^o \Psi_o(x_r^o + x_a^o/2) \Psi_o^*(x_r^o - x_a^o/2) \int_{x_r^o, x_a^o} Dx_r Dx_a e^{iS_1 - iS_2} x_r(t) x_r(\bar{t}) . \quad (A2)$$

This is simplified by (i) introducing the Wigner transform,

$$W(x_r^o, p^o) = \int dx_a^o e^{-ip^o x_a^o} \Psi_o(x_r^o + x_a^o/2) \Psi_o^*(x_r^o - x_a^o/2) ,$$

(ii) integrating by parts in the action (with the boundary condition  $x_a \rightarrow 0$  for  $t \rightarrow +\infty$ ),

$$iS_1 - iS_2 = -imx_a^o \dot{x}_r(t_o) - i \int_{t_o} dt x_a [m\ddot{x} + m\omega_o^2 x] ,$$

and finally (iii) integrating over all  $x_a$ , yielding

$$G_{rr}(t, \bar{t}) = \int \frac{dx_r^o dp^o}{2\pi} W(x_r^o, p^o) \delta(p^o - m\dot{x}_r(t_o)) \int_{x_r(t_o)=x_r^o} Dx_r \delta_t [M\ddot{x}_r + m\omega_o x_r] x_r(t) x_r(\bar{t}) . \quad (A3)$$

Here  $\delta_t[f(t)]$  denotes the functional delta function,  $\prod_t \delta(f(t))$ . The meaning of this path integral is that  $G_{rr}(t, \bar{t})$  is found by solving the equation of motion for a specified initial condition,

$$x_r(t) = x_r^o \cos(\omega_o(t - t_o)) + \frac{p^o}{m\omega_o} \sin(\omega_o(t - t_o)) , \quad (A4)$$

and then averaging the square of this solution over the initial conditions specified by the Wigner transform. Performing this average for the ground state wave function of the oscillator reproduces the familiar result

$$G_{rr}(t, \bar{t}) = \frac{1}{2} \langle 0 | \{ \hat{x}(t), \hat{x}(\bar{t}) \} | 0 \rangle = \frac{1}{2m\omega_o} \cos(\omega_o(t - \bar{t})). \quad (\text{A5})$$

The lesson from this analysis is that symmetrized correlation functions invariably arise from correlations in the initial density matrix which are propagated forward by the equations of motion. This dependence on the initial density matrix should be contrasted with retarded propagators which are independent of the wave function of the simple harmonic oscillator, *i.e.*

$$\theta(t - \bar{t}) [\hat{x}(t), \hat{x}(\bar{t})] = \frac{-i\theta(t - \bar{t})}{m\omega_o} \sin(\omega_o(t - \bar{t})), \quad (\text{A6})$$

is a pure number.

## Appendix B: The Green Function Composition Rule

In this appendix we detail the Green function composition rule stated in Eq. (3.6). In this section 1, 2, 3 denote the space-time points, *e.g.*  $1 = (v_1, r_1)$ .

Suppose  $G_{ra}(1|2)$  and  $\hat{G}_{ra}(1|3)$  are retarded Green functions. Then the Wronskian of the two Green functions is

$$W^\mu(2) = \frac{\sqrt{\lambda}}{2\pi} g_{xx} \sqrt{h} h^{\mu\nu}(2) \left[ G_{ra}(1|2) \frac{\overleftrightarrow{\partial}}{\partial 2^\nu} \hat{G}_{ra}(2|3) \right], \quad (\text{B1})$$

where  $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$  and is not intended to act outside of the square braces. Then, a short exercise shows that the divergence is

$$\partial_\mu W^\mu(2) = G_{ra}(1|3) \delta(2, 3) - \hat{G}_{ra}(3|1) \delta(2, 1). \quad (\text{B2})$$

Assuming that 3 is inside the strip and 1 is outside the strip (see Fig. 3) we can integrate over the strip to obtain the retarded Green function:

$$G_{ra}(1|3) = \int_{r < r_h} \frac{\partial}{\partial 2^\mu} W^\mu(2) = \int_2 d\Sigma_\mu \frac{\sqrt{\lambda}}{2\pi} g_{xx} \sqrt{h} h^{\mu\nu}(2) \left[ G_{ra}(1|2) \frac{\overleftrightarrow{\partial}}{\partial 2^\nu} \hat{G}_{ra}(2|3) \right], \quad (\text{B3})$$

where  $d\Sigma_\mu$  is a surface surrounding the strip with outward directed normal, and the integration is over space-time point 2. We next use the near horizon approximation for the leading factors (Eq. (2.6)), and neglect all surface terms except the integral over the stretched horizon. These surface integrals vanish because one of the Green functions vanishes. For instance, on the past surface (where  $v = -\infty$ )  $\hat{G}_{ra}(2|3)$  must vanish since it represents the causal response at point 2 (past infinity) to a source at point 3. Since we have not specified the boundary conditions on the retarded Green function  $\hat{G}_{ra}$ , we are free to specify reflective Dirichlet boundary conditions on the stretched horizon, *i.e.* take  $\hat{G}_{ra}(2|3) = g_{ra}(2|3)$ , as defined in the text. This specification does not interfere with the relevant initial data extremely close to the real event horizon. With this choice Eq. (B3) results in Eq. (3.6) given in the text.

## Appendix C: From bulk to bulk correlators, to boundary to bulk correlators, to CFT correlators

Given a retarded bulk to bulk correlator  $G_{ra}(1|2)$  in the gravitational theory, we will show that the retarded correlator in the CFT is

$$G_{ra}^{CFT}(t_1|t_2) = \lim_{r_1 \rightarrow \text{bnd}} \lim_{r_2 \rightarrow \text{bnd}} \left[ -\frac{\sqrt{\lambda}}{2\pi} g_{xx} \sqrt{h} h^{rr}(1) \partial_{r_1} \right] \left[ -\frac{\sqrt{\lambda}}{2\pi} g_{xx} \sqrt{h} h^{rr}(2) \partial_{r_2} \right] G_{ra}(1|2). \quad (\text{C1})$$

In holography, the CFT correlation function is usually determined by setting the boundary conditions of five dimensional fields as  $r \rightarrow \infty$ . We claim that Eq. (C1) is an equivalent prescription. Indeed, Eq. (C1) arises if the path integral discussion of the horizon effective action in Section IV B is applied to the boundary. Specifically, instead of

integrating out the strip to find an effective action of the stretched horizon, one integrates out the entire bulk to find an effective action of the CFT. Put differently, from the viewpoint of the bulk theory, any small change in a boundary condition must be equivalent to inserting a suitable local operator close to that boundary, and for a Dirichlet condition that operator turns out to be a derivative of the field as in Eq. (C1) (see, for instance, [23]).

To make direct contact with the pioneering work of Policastro, Son, and Starinets [41], we define the boundary to bulk correlator  $f(t_1 r_1 | t_2)$  as a retarded solution which satisfies

$$\lim_{r_1 \rightarrow \text{bnd}} f(t_1 r_1 | t_2) = \delta(t_1 - t_2). \quad (\text{C2})$$

For static geometries in Fourier space this is usually called  $f_\omega(r_1)$ , and the retarded correlator in the CFT is usually

$$G_{ra}^{CFT}(\omega) = + \frac{\sqrt{\lambda}}{2\pi} g_{xx} \sqrt{h} h^{rr} \partial_r f_\omega(r_1). \quad (\text{C3})$$

Now we claim that the boundary to bulk propagator is simply related to the bulk to bulk propagator via

$$f(t_1 r_1 | t_2) = \lim_{r_2 \rightarrow \text{bnd}} \frac{\sqrt{\lambda}}{2\pi} \sqrt{h} h^{rr}(r_2) \partial_{r_2} G_{ra}(t_1 r_1 | t_2 r_2). \quad (\text{C4})$$

To show this we take  $r_1$  fixed and large, and we integrate the equations of motions of the retarded Green function (Eq. (3.1)) with respect to the second argument over the pill-box shown below. The radius of the lower surface is small compared to  $r_1$  (but still large), while the radius of the upper surface is large compared to  $r_1$ .



$$\quad (\text{C5})$$

This yields

$$\begin{aligned} \lim_{r_1 \rightarrow \text{bnd}} \lim_{r_2 \rightarrow \text{bnd}} \frac{\sqrt{\lambda}}{2\pi} g_{xx} \sqrt{h} h^{rr}(r_2) \frac{\partial}{\partial r_2} G(r_1 t_1 | r_2 t_2) \\ - \lim_{r_2 \rightarrow \text{bnd}} \lim_{r_1 \rightarrow \text{bnd}} \frac{\sqrt{\lambda}}{2\pi} g_{xx} \sqrt{h} h^{rr}(r_2) \frac{\partial}{\partial r_2} G(t_1 r_1 | t_2 r_2) = \delta(t_1 - t_2). \end{aligned} \quad (\text{C6})$$

The second term vanishes since  $G_{ra}(t_1 r_1 | t_2 r_2)$  obeys Dirichlet boundary conditions. Thus, the first term in Eq. (C6) obeys the same boundary conditions as the bulk to boundary propagator  $f(t_1 r_2 | t_2)$ , Eq. (C2). Since both functions are retarded and obey the same equations of motion and boundary conditions, they are the same and Eq. (C4) holds. By extension, Eq. (C1) for the CFT correlators from bulk to bulk correlators is equivalent to the usual prescription.

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