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Abstract

We study supercurrent conservation for the four-dimensional Wess-Zumino model formulated on the lattice. The formulation is one that has been discussed several times, and uses Ginsparg-Wilson fermions of the overlap (Neuberger) variety, together with an auxiliary fermion (plus superpartners), such that a lattice version of $U(1)_R$ symmetry is exactly preserved in the limit of vanishing bare mass. We show that the almost naive supercurrent is conserved at one loop. By contrast we find that this is not true for Wilson fermions and a canonical scalar action. We provide nonperturbative evidence for the nonconservation of the supercurrent in Monte Carlo simulations.

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I. INTRODUCTION

The formulation of supersymmetric field theories on a spacetime lattice is of interest because nonperturbative dynamics play an important role in the theory of supersymmetry breaking and its transmission to the visible sector of particle physics. Theories such as super-QCD and $\mathcal{N} = 4$ super-Yang-Mills are of particular interest, but it seems wise to refine methods using simpler toy models such as the Wess-Zumino model, given the difficulties with supersymmetry on the lattice. Hence, we continue our investigations of a lattice formulation that was studied by several groups a few years ago [1–6]. What we are developing here is a methodology for analyzing the extent to which supersymmetry is a feature of the low energy effective theory. Since the lattice formulation explicitly breaks supersymmetry, this symmetry must be accidental. In fact, it will arise from a fine-tuning of bare lattice parameters, corresponding to the ultraviolet definition of the theory. In order to identify the supersymmetric point in that parameter space, we must detect the conservation of the supercurrent. That is a nontrivial task since the naive supercurrent will mix with other operators, due to the explicit violation of the symmetry by the discretization.

The problems that we face are not by any means unique to the Wess-Zumino model. Four-dimensional supersymmetric models on the lattice[1] generically require fine-tuning of counterterms. This is to be contrasted with lower dimensional theories where lattice symmetries can eliminate the need for such fine-tuning; see [10] for further details. The one known four-dimensional exception is pure $\mathcal{N} = 1$ super-Yang-Mills using Ginsparg-Wilson fermions; the domain wall variety has been the subject of past [13] and recent [9, 14–18] simulations. Clearly we would like to go beyond pure $\mathcal{N} = 1$ super-Yang-Mills, and in fact all other models contain scalar fields—which are the source of many difficulties due to unwanted renormalizations that cannot be forbidden by symmetries. Recently it was proposed [19] that an acceptable amount of fine-tuning could be efficiently performed using a multicanonical Monte Carlo [20] simulation together with Ferrenberg-Swendsen reweighting [21–23] in a large class of theories; see also [9]. In any such program, it is necessary to study the divergence of the supercurrent and its renormalization, such as we are doing here for the Wess-Zumino model.

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[1] For reviews with extensive references see [7–12].

A. Summary of our previous work

The theory that we study is the four-dimensional Wess-Zumino model, formulated on the lattice with overlap (Neuberger) fermions [24], as well as numerous auxiliary fields. The goal of the formulation is to impose the Majorana condition and simultaneously preserve the chiral $U(1)_R$ symmetry [1–6] that is present in the continuum in the massless limit. As was shown in our recent work [25], preserving this symmetry significantly limits the number of counterterms that must be fine-tuned in order to obtain the supersymmetric continuum limit. In addition to overlap fermions, the lattice formulation has auxiliary fermions (plus superpartner fields) that couple to the overlap fermions through the Yukawa coupling, as in [26]. It is possible to integrate out the auxiliary fermions (and superpartner fields), and when one does this a nonanalytic Dirac operator results for the surviving fermionic field. Thus, as has been discussed originally in [1], and at greater length in [2, 5] the action is singular once auxiliary fields are integrated out. However, as we described in [25], there is a sensible resolution of this singularity by taking the theory to "live" inside a finite box, with antiperiodic boundary conditions in the time direction for the fermions. The singularity of the Dirac operator that this resolves is related to nonpropagating modes in the infinite volume limit; the fact that these are nonpropagating was shown in [6]. However, singularities in the Dirac operator raise the spectre of possible nonlocalities in the continuum limit, as was found in gauge theories with the SLAC derivative [27]. In [25] we measured the degree of localization of the Dirac operator following the approach of [28]. We found that while there is localization, it is less pronounced than the exponential localization of the overlap operator.

The divergences that need to be cancelled in order to renormalize the lattice theory at one-loop turn out to be strictly wave function renormalization. The wave function renormalization of the fermion and the physical scalar match at one loop in the continuum limit of the lattice expressions; but, the auxiliary scalar has a mismatched wave function counterterm. These findings appeared previously in [1]; thus our work [25] was a confirmation of those results.

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B. Plan of this paper

In Section II we define the Wess-Zumino model both in the continuum and on the lattice. We discuss the $U(1)_R$ symmetry of the lattice theory, as well as supersymmetry transformations that are a symmetry in the limit of the free theory. Finally we discuss the fine-tuning action that must be used to obtain a supersymmetric continuum limit. In Section III we describe the supercurrent and an almost naive transcription of it to the lattice. We also briefly touch on the form of the renormalized supercurrent in terms of bare lattice operators. In Section IV we perform a one-loop perturbative analysis of the four-divergence of the supercurrent. We find that in lattice perturbation theory the supercurrent is conserved. We conclude this section by mentioning two-loop diagrams where we expect that the asymmetric self-energy of the auxiliary field (at one-loop) will be important, leading to nonconservation of the lattice supercurrent. In Section V we describe the results of our Monte Carlo simulations, where we have measured the four-divergence of the supercurrent nonperturbatively. We find that the violation of supersymmetry is consistent with contributions beginning at the two-loop order. In Section VI we give our concluding remarks.

II. DEFINITIONS

The Euclidean continuum theory has action

$$
S = -\int d^4x \left\{ \frac{1}{2}\chi^T C M \chi + \phi^*\Box\phi + F^*F + F^*(m^*\phi^* + g^*\phi^{*2}) + F(m\phi + g\phi^2) \right\},
$$
$$
M = \slashed{\partial} + (m + 2g\phi)P_+ + (m^* + 2g^*\phi^*)P_-, \tag{2.1}
$$

invariant under supersymmetry transformations. The lattice action is a special case of the formulations of [1, 2]; we write the lattice action in forms given in [4–6]. For this, a few lattice derivative operators must be introduced.

$$
A = 1 - aD_W, \quad D_W = \frac{1}{2}\gamma_\mu(\partial_\mu^* + \partial_\mu) - \frac{1}{2}a\partial_\mu^*\partial_\mu
$$
$$
D_1 = \frac{1}{2}\gamma_\mu(\partial_\mu^* + \partial_\mu)(A^\dagger A)^{-1/2}
$$
$$
D_2 = \frac{1}{a}\left[1 - \left(1 + \frac{1}{2}a^2\partial_\mu^*\partial_\mu\right)(A^\dagger A)^{-1/2}\right]
$$
$$
D = D_1 + D_2 = \frac{1}{a}\left(1 - A(A^\dagger A)^{-1/2}\right) \tag{2.2}
$$

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where ∂_μ and ∂_μ^* are the forward and backward difference operators respectively. Note that D is the overlap Dirac operator. As shown in [5] the lattice action can be written with a full supermultiplet of auxiliary fields and the kinetic term for the fermion χ involves the overlap Dirac operator D. The use of the other operators D_1 and D_2 in the scalar part of the action is a departure from what one might do naively, and is the reason for favorable renormalization of the action at one-loop. After integrating out the auxiliary fields one passes to the tilded fields in terms of which the lattice action is:

$$S = -a^4 \sum_x \left\{ \frac{1}{2} \tilde{\chi}^T CM\tilde{\chi} - \frac{2}{a} \tilde{\phi}^* D_2 \tilde{\phi} + \tilde{F}^* (1 - \frac{a}{2} D_2)^{-1} \tilde{F} \right.$$
$$\left. + \tilde{F}^* (m^* \tilde{\phi}^* + g^* \tilde{\phi}^{*2}) + \tilde{F}(m\tilde{\phi} + g\tilde{\phi}^2) \right\}. \tag{2.3}$$

The fermion matrix is:

$$M = \slashed{D} + mP_+ + m^* P_- + 2g\tilde{\phi}P_+ + 2g^* \tilde{\phi}^* P_-, \quad \slashed{D} = (1 - \frac{a}{2} D_2)^{-1} D_1 \tag{2.4}$$

The lattice realization of the chiral $U(1)_R$ symmetry in the $m \to 0$ limit takes a particularly simple form on the tilded variables:

$$\delta \tilde{\chi} = i\alpha\gamma_5 \tilde{\chi}, \quad \delta \tilde{\phi} = -2i\alpha\tilde{\phi}, \quad \delta \tilde{F} = 4i\alpha\tilde{F} \tag{2.5}$$

The supersymmetry transformations of the tilded fields are:

$$\delta_\epsilon \tilde{\phi} = \sqrt{2}\epsilon^T CP_+ \tilde{\chi}, \quad \delta_\epsilon \tilde{\phi}^* = \sqrt{2}\epsilon^T CP_- \tilde{\chi},$$
$$\delta_\epsilon \tilde{\chi}_\beta = -\sqrt{2}(P_+(D_1\tilde{\phi} + \tilde{F})\epsilon)_\beta - \sqrt{2}(P_-(D_1\tilde{\phi}^* + \tilde{F}^*)\epsilon)_\beta,$$
$$\delta_\epsilon \tilde{F} = \sqrt{2}\epsilon^T CD_1 P_+ \tilde{\chi}, \quad \delta_\epsilon \tilde{F}^* = \sqrt{2}\epsilon^T CD_1 P_- \tilde{\chi} \tag{2.6}$$

This is not a symmetry of the lattice action for $g \neq 0$, but is a symmetry in the free case. Our perturbative analysis in Section IV will identify the corresponding conserved current.

We can integrate out the auxiliary fields $\tilde{F}, \tilde{F}^*$ to obtain the action

$$S = a^4 \sum_x \left\{ -\frac{1}{2}\tilde{\chi}^T CM\tilde{\chi} + \frac{2}{a}\tilde{\phi}^* D_2 \tilde{\phi} + (m^*\tilde{\phi}^* + g^*\tilde{\phi}^{*2})(1 - \frac{a}{2}D_2)(m\tilde{\phi} + g\tilde{\phi}^2) \right\} \tag{2.7}$$

This is the action that is used in our numerical simulations.

When fine-tuning of the lattice action is performed, we must invoke the most general lattice action consistent with symmetries. Since we perform our simulations at $m \neq 0$, this is just the action with all dimension ≤ 4 operators built out of the physical fields, $\tilde{\phi}$ and $\tilde{\chi}$.

It was given in our previous work [25]. Whereas in the CP invariant supersymmetric theory there are two real parameters, in the most general CP invariant theory we have a total of ten parameters. Holding two fixed, we must tune the other eight to achieve the supersymmetric limit. Conducting a fine-tuning in an eight-dimensional parameter space is a daunting task.

On the other hand in the limit of vanishing bare fermion mass we can impose the $U(1)_R$ symmetry (2.5). This restricts the action to

$$
S = a^4 \sum_x \left\{ -\frac{1}{2} \tilde{\chi}^T C \slashed{D} \tilde{\chi} + \frac{2}{a} \tilde{\phi}^* D_2 \tilde{\phi} + m_2^2 |\tilde{\phi}|^2 + \lambda_1 |\tilde{\phi}|^4 \right.
$$
$$
\left. - \tilde{\chi}^T C (y_1 \tilde{\phi} P_+ + y_1^* \tilde{\phi}^* P_-) \tilde{\chi} \right\} \tag{2.8}
$$

If we hold y_1 fixed (corresponding to some choice of g_R in the long-distance effective theory), then only m_2^2 and λ_1 must be fine-tuned. Conducting a search in a two-dimensional parameter space, with both coming from bosonic terms, is manageable. The difficult part is that we must extrapolate to the massless fermion limit. Another potential problem is that we impose antiperiodic boundary conditions for the fermion in the time direction, but must impose periodic boundary conditions for the scalar in order for the action to be single-valued on the circle in the time direction. This breaks supersymmetry explicitly by boundary conditions. At finite mass this should be an effect that can be made arbitrarily small by taking the large volume limit. However at vanishing mass, it is important that we take $T \gg 1/ma$ as m is sent to zero, where T is the number of sites in the time direction.

What we have seen in [25] is that the one-loop behavior of the theory (2.3) closely follows that of the continuum, so that no new operators are generated at this order. Thus at this level of approximation, a fine-tuning of the general lattice action is not needed. Due to this good one-loop behavior it is of interest to study the original lattice action (2.7) in our simulations, without any fine-tuning. By measuring the degree of supersymmetry breaking through nonconservation of the supercurrent, we gain information about the higher orders and nonperturbative aspects of the lattice theory.

III. SUPERCURRENT, MIXING AND RENORMALIZATION

For a general superpotential $W(\phi)$, the supercurrent is

$$
S^\mu = \sqrt{2} \left[\slashed{\partial} \phi \gamma^\mu P_- \chi + \slashed{\partial} \phi^* \gamma^\mu P_+ \chi + \frac{\partial W}{\partial \phi} \gamma^\mu P_+ \chi + \left(\frac{\partial W}{\partial \phi} \right)^* \gamma^\mu P_- \chi \right] \tag{3.1}
$$

and in our case $\partial W / \partial \phi = m\phi + g\phi^2$. There is also a form with the auxiliary field:

$$S^\mu = \sqrt{2} \left[\slashed{\partial}\phi\gamma^\mu P_-\chi + \slashed{\partial}\phi^*\gamma^\mu P_+\chi - F^*\gamma^\mu P_+\chi - F\gamma^\mu P_-\chi \right] \tag{3.2}$$

Because of the supersymmetry breaking on the lattice, this will mix with other operators in the same symmetry channel. If the lattice action is fine-tuned, then in the long distance effective theory there will be a supercurrent that is conserved in the continuum limit. The way to detect the existence of supersymmetry in this fine-tuning process is to consider linear combinations of bare lattice operators and search for one that has vanishing four-divergence in the supersymmetric limit, modulo contact terms. We briefly describe that approach in Section III B below. Before doing so, we give an almost naive discretization of the continuum supercurrent (3.1) that will turn out to be the conserved supercurrent of the one-loop analysis below.

A. Almost naive lattice supercurrent

We formulate this lattice supercurrent in terms of tilded fields:

$$S_\mu = \sqrt{2} \left[D_1\tilde{\phi}\gamma_\mu P_-\tilde{\chi} + D_1\tilde{\phi}^*\gamma_\mu P_+\tilde{\chi} + \frac{\partial W}{\partial\tilde{\phi}}\gamma_\mu P_+\tilde{\chi} + \left(\frac{\partial W}{\partial\tilde{\phi}}\right)^* \gamma_\mu P_-\tilde{\chi} \right]$$
$$\frac{\partial W}{\partial\tilde{\phi}} = m\tilde{\phi} + g\tilde{\phi}^2 \tag{3.3}$$

It is almost naive because $\slashed{\partial}$ has been replaced by D_1, rather than a more naive prescription such as

$$D_S = \sum_\mu \gamma_\mu \partial_\mu^S, \quad \partial_\mu^S = \frac{1}{2}(\partial_\mu + \partial_\mu^*) \tag{3.4}$$

Note that (3.3) is the form of the supercurrent with the auxiliary fields eliminated. Thus in working with it we should use the Feynman rules corresponding to the action without auxiliary fields. In our perturbative calculations below, that will be the action (2.7). We do not need to use the more general (fine-tuning) action for the $\mathcal{O}(g)$ calculations that we do, since the leading violation of supersymmetry is an $\mathcal{O}(g^2)$ nonsupersymmetric wavefunction renormalization for the auxiliary field, as will be discussed below.

We could also work with a supercurrent containing auxiliary fields:

$$S^\mu = \sqrt{2} \left[D_1\tilde{\phi}\gamma^\mu P_-\tilde{\chi} + D_1\tilde{\phi}^*\gamma^\mu P_+\tilde{\chi} - \tilde{F}^*\gamma^\mu P_+\tilde{\chi} - \tilde{F}\gamma^\mu P_-\tilde{\chi} \right] \tag{3.5}$$

At finite a, this will lead to slightly different results than (3.3), since the lattice equations of motion for the auxiliary field are

$$-\tilde{F}^* = \left(1 - \frac{a}{2} D_2\right)(m\tilde{\phi} + g\tilde{\phi}^2) \tag{3.6}$$

In fact we will also consider the form (3.5) in our perturbative calculations below. It is a convenient choice, since we know crucial facts about the nonsupersymmetric renormalization of the auxiliary field from our previous one-loop analysis of counterterms.

The variation of the action under the lattice version of the supersymmetry transformation was given in [5] and is equal to:[2]

$$\delta S = \sqrt{2} a^4 \sum_x \tilde{\chi}^T C \left[g P_+(2\tilde{\phi} D_1 \tilde{\phi} - D_1 \tilde{\phi}^2) + g^* P_-(2\tilde{\phi}^* D_1 \tilde{\phi}^* - D_1 \tilde{\phi}^{*2})\right] \epsilon \tag{3.7}$$

Note that this is $\mathcal{O}(a)$, since

$$\lim_{a \to 0} \sum_x \tilde{\chi}^T C P_+(2\tilde{\phi} D_1 \tilde{\phi} - D_1 \tilde{\phi}^2) = 0 \tag{3.8}$$

Also note the presence of g in (3.7). In order to see violation of the supersymmetric identity $\partial_\mu S_\mu = 0$ in the continuum limit, diagrams involving the coupling g must be included. Loop diagrams are required, in order to get the divergences that cancel the $\mathcal{O}(a)$ factor coming from (3.7) in the continuum limit.

B. Renormalized supercurrent

Having written down the almost naive supercurrent, which will be the subject of all of our computations in this paper, we next mention the more general case. That is, the form of the renormalized supercurrent, which is expected to be different from the almost naive version. At a given engineering dimension, we write down all operators that have the index structure of $S_{\mu\alpha}(x)$ and dimension less than or equal to $7/2$. We denote these as $\mathcal{O}_{\mu\alpha,j}^{(n/2)}$ where $n/2$ is the engineering dimension; thus n takes odd values $3, 5, 7$, and the index j labels the different operators of dimension $n/2$. A linear combination of these is the long distance effective supercurrent at lattice spacing a:

$$S_{\mu\alpha}(x) = \sum_{n=3,5,7} \sum_j b_j^{(n/2)} a^{(n-7)/2} \mathcal{O}_{\mu\alpha,j}^{(n/2)}(x) \tag{3.9}$$

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[2] The variation of the action under a modified supersymmetry transformation was also given in [4], which reduces to the result (3.7) in the appropriate limits.

Clearly it will be a demanding task to identify this S_μ nonperturbatively. However, there is no other way to properly fine-tune the action. We must find S_μ and the point in parameter space where it is conserved. Before tacking this problem (in future work), we will (in this paper) examine the properties of the almost naive supercurrent.

IV. PERTURBATIVE ANALYSIS

Here we compute $\langle \partial_\mu S_\mu(x) \mathcal{O}(0)\rangle$ for a few different choices of $\mathcal{O}$, with our first choice being $\mathcal{O} = \tilde{\chi}$, which gets an $\mathcal{O}(g)$ contribution at one-loop. As will be seen, this leads naturally to a second choice, $\mathcal{O} = \tilde{\phi}^* \tilde{\chi}$, which gets an $\mathcal{O}(g^0)$ contribution at one-loop.

All other choices correspond to higher loop diagrams. As summarized above, we know that $Z_\phi = Z_\chi \neq Z_F$ at one-loop. These self-energy diagrams can occur on an internal line of $\langle \partial_\mu S_\mu(x) \mathcal{O}(0)\rangle$, giving for example an $\mathcal{O}(g^2)$ contribution when $\mathcal{O} = \tilde{\phi}^* \tilde{\chi}$, at two-loops. Because of the mismatch of the Z factors, we can be confident that the self-energies as a function of loop momentum are also mismatched, so that the cancellations that occur in the continuum theory will not happen. An example diagrams will be presented below. However, the point is that we will need to be able to compute two-loop diagrams in order to begin fine-tuning the action using the conservation of the supercurrent as a probe. This is beyond the scope of the present paper and is left to future work.

In the perturbative analysis, the following lattice propagators are used:

$$S(p) = \sum_x a^4 e^{ip \cdot x} \langle \tilde{\chi}(x) \tilde{\chi}^T(0) C\rangle = \frac{-D_1(p) + (1 - \frac{a}{2}D_2(p))(m^* P_+ + m P_-)}{\frac{2}{a}D_2(p) + (1 - \frac{a}{2}D_2(p))|m|^2}$$

$$G(p) = \sum_x a^4 e^{ip \cdot x} \langle \tilde{\phi}(x) \tilde{\phi}^*(0)\rangle = \frac{-1}{\frac{2}{a}D_2(p) + (1 - \frac{a}{2}D_2(p))|m|^2} \tag{4.1}$$

where

$$D_1(p) = \frac{-ia^{-1} \sum_\mu \gamma_\mu \sin(p_\mu a)}{\sqrt{[1 - 2\sum_\mu \sin^2(p_\mu a/2)]^2 + \sum_\mu \sin^2(p_\mu a)}} \tag{4.2}$$

$$D_2(p) = \frac{1}{a}\left(1 - \frac{1 - 2\sum_\mu \sin^2(p_\mu a/2)}{\sqrt{[1 - 2\sum_\mu \sin^2(p_\mu a/2)]^2 + \sum_\mu \sin^2(p_\mu a)}}\right) \tag{4.3}$$

The fermion propagator will be represented by a solid line and the scalar propagator by a dashed line.
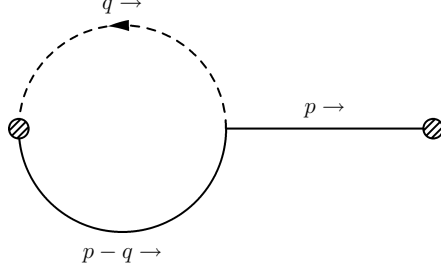
FIG. 1: The Feynman diagrams for the correlation function that we study. In actuality, there are two such diagrams, one with operator $D_1\tilde{\phi}\gamma_\mu P_-\tilde{\chi}$ on the left-hand side, and one with operator $\tilde{\phi}\gamma^\mu P_+\tilde{\chi}$ on the left-hand side.

A.   First choice: $\mathcal{O} = \tilde{\chi}$

It is convenient to work instead with the Fourier transform:

$$\sum_x e^{ip\cdot x}\langle \partial_\mu^S S_\mu(x)\tilde{\chi}(0)\rangle \tag{4.4}$$

The corresponding diagram at one-loop is Fig. 1. Note that this includes the point $x = 0$ so that we have to worry about contact terms. These would involve the supersymmetric variation of $\tilde{\chi}$:

$$\langle \Delta\tilde{\chi}(x)\rangle\delta_{x,0} \propto \langle P_+(D_1\tilde{\phi} + \tilde{F})(x) + P_-(D_1\tilde{\phi}^* + \tilde{F}^*)\rangle\delta_{x,0} \tag{4.5}$$

At one-loop we found in our previous work $\langle\tilde{F}(x)\rangle = \langle\tilde{\phi}(x)\rangle = 0$. So we do not have to worry about contact terms at one-loop, due to the absence of tadpoles.

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It is a straightforward calculation to obtain the four correlation functions that we need:

$$\sum_x e^{ip\cdot x}\langle \partial_\mu^S(D_1\tilde{\phi}\gamma_\mu P_-\tilde{\chi})_\alpha(x)\tilde{\chi}_\beta(0)\rangle$$

$$= -g^*\frac{1}{a}(e^{-ip_\mu a} - e^{ip_\mu a})\int_B \frac{d^4q}{(2\pi)^4}(D_1(q)\gamma_\mu P_-S(p-q)P_-S(p)C)_{\alpha\beta}G(q) \qquad (4.6)$$

$$\sum_x e^{ip\cdot x}\langle \partial_\mu^S(D_1\tilde{\phi}^*\gamma_\mu P_+\tilde{\chi})_\alpha(x)\tilde{\chi}_\beta(0)\rangle$$

$$= -g\frac{1}{a}(e^{-ip_\mu a} - e^{ip_\mu a})\int_B \frac{d^4q}{(2\pi)^4}(D_1(q)\gamma_\mu P_+S(p-q)P_+S(p)C)_{\alpha\beta}G(q) \qquad (4.7)$$

$$\sum_x e^{ip\cdot x}\langle \partial_\mu^S(\tilde{\phi}\gamma_\mu P_+\tilde{\chi})_\alpha(x)\tilde{\chi}_\beta(0)\rangle$$

$$= -g^*\frac{1}{a}(e^{-ip_\mu a} - e^{ip_\mu a})\int_B \frac{d^4q}{(2\pi)^4}(\gamma_\mu P_+S(p-q)P_-S(p)C)_{\alpha\beta}G(q) \qquad (4.8)$$

$$\sum_x e^{ip\cdot x}\langle \partial_\mu^S(\tilde{\phi}^*\gamma_\mu P_-\tilde{\chi})_\alpha(x)\tilde{\chi}_\beta(0)\rangle$$

$$= -g\frac{1}{a}(e^{-ip_\mu a} - e^{ip_\mu a})\int_B \frac{d^4q}{(2\pi)^4}(\gamma_\mu P_-S(p-q)P_+S(p)C)_{\alpha\beta}G(q) \qquad (4.9)$$

Here "B" indicates integration over the first Brillouin zone, $B = [-\pi/a, \pi/a]^4$. Notice that these correlation functions have certain common factors. On the left,

$$\frac{1}{2a}(e^{-ip_\mu a} - e^{ip_\mu a}) \qquad (4.10)$$

and on the right

$$S(p)C \qquad (4.11)$$

Thus when we sum them, we can factor out these bits and the remainder has to be the thing that cancels. We now check whether the four-divergence of the almost naive supercurrent vanishes. We have computed the above integrals numerically for various values of a, having in mind the $a \to 0$ extrapolation.

If the naive supercurrent is to work, then with $p = (p_0, 0, 0, 0)$

$$\mathfrak{S}(p_0)_{-,\alpha\beta} = b_1^{(7/2)}\mathfrak{I}(p_0)_{1-,\alpha\beta} + b_1^{(5/2)}a^{-1}\mathfrak{I}(p_0)_{2-,\alpha\beta}$$

$$\mathfrak{I}(p_0)_{1-,\alpha\beta} = \int_B \frac{d^4q}{(2\pi)^4}(D_1(q)\gamma_0 P_-S(p-q)P_-)_{\alpha\beta}G(q)$$

$$\mathfrak{I}(p_0)_{2-,\alpha\beta} = \int_B \frac{d^4q}{(2\pi)^4}(\gamma_0 P_+S(p-q)P_-)_{\alpha\beta}G(q) \qquad (4.12)$$

the quantity $\mathfrak{S}(p_0)_{-,\alpha\beta}$ has to vanish in the continuum limit. It is not hard to show that a naive continuum limit of this expression does vanish, provided

$$b_1^{(7/2)} = \sqrt{2}, \quad b_1^{(5/2)} = \sqrt{2}ma \qquad (4.13)$$

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a	p_0	$(2\pi)^4 \Im(p_0)_{1-,00}$	$(2\pi)^4 \Im(p_0)_{2-,00}$
0.1	0.1	2.0375(2)	-2.0323(2)
0.1	0.2	4.0700(4)	-4.0595(4)
0.01	0.1	4.2796(4)	-4.2789(4)
0.01	0.2	8.5544(9)	-8.5529(9)
0.001	0.1	6.5529(7)	-6.5513(7)
0.001	0.2	13.1009(13)	-13.0979(13)

TABLE I: Values of the integrals at various values of p_0 and a; the units are chosen such that the numerical value of m is equal to one.

i.e., for the proper coefficients of the almost naive supercurrent. If $\mathfrak{S}(p_0)_{-,\alpha\beta}$ vanishes in the one-loop continuum limit, it would also follow that

$$\mathfrak{S}(p_0)_{+,\alpha\beta} = \int_B \frac{d^4q}{(2\pi)^4} \left\{ b_1^{(7/2)*}(D_1(q)\gamma_0 P_+ S(p-q)P_+)_{\alpha\beta}G(q) \right.$$
$$\left. + b_1^{(5/2)*}a^{-1}(\gamma_0 P_- S(p-q)P_+)_{\alpha\beta}G(q) \right\} \tag{4.14}$$

would vanish as well.

When one looks at the specific values of the integrals, (Table I) what one finds is that we should set the coefficients to the naive values of (4.13). (Of course the overall normalization of $\sqrt{2}$ is a matter of convention.) That is, the almost naive supercurrent is conserved at one loop.

This result suggests that at one-loop there is an argument that the quantity we are computing is determined by the free theory and the supercurrent is not renormalized at one-loop.

B.   Second choice: $\mathcal{O} = \tilde{\phi}^*\tilde{\chi}$

That the forgoing claim is true can be seen from the fact that when we amputate the propagator $S(p)$ from the diagram Fig. 1 (i.e., the fermion propagator that is not in the loop), we obtain the diagram of Fig. 2. However, the latter diagram is just the one that we would obtain in the free theory from evaluating $\langle \partial_\mu^S S_\mu(x)(\phi^*\chi)(0)\rangle$. Since in the free theory the variation of the action under supersymmetry is zero, there should be a conserved current.
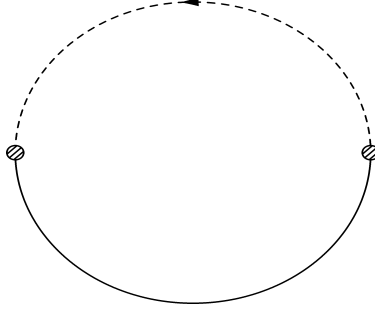
FIG. 2: The simpler correlation function that must vanish in the continuum limit.

What we have just found is that the almost naive lattice supercurrent is that supercurrent to within the accuracy of our numerical evaluation of the integrals.

C.   A more naive discretization

The behavior we have just observed is to be contrasted with what happens in the free theory if we use Wilson fermions and naive scalars so that the action is

$$S = -\sum_x a^4 \{ \frac{1}{2} \chi^T C (D_W + m) \chi + \phi^* \partial_\mu \partial_\mu^* \phi + F^* F + m(F\phi + F^* \phi^*) \} \qquad (4.15)$$

where D_W is defined in Eq. (2.2) and we have specialized to a real mass m. Also, we will use the symmetric difference Dirac operator (3.4) in the discretization of the supercurrent (3.1), $\slashed{\partial} \to D_S$, yielding

$$D_S(p) = -\frac{i}{a} \sum_\mu \gamma_\mu \sin(p_\mu a) \qquad (4.16)$$

in momentum space. Corresponding to (4.15), after integrating out the auxiliary field, there will be fermion propagator

$$S_W(p) = \frac{-D_S(p) + m + \frac{2}{a} \sum_\mu \sin^2(p_\mu a/2)}{\frac{1}{a^2} \sum_\mu \sin^2(p_\mu a) + (m + \frac{2}{a} \sum_\mu \sin^2(p_\mu a/2))^2} \qquad (4.17)$$

and scalar propagator

$$G_N(p) = \frac{-1}{\frac{4}{a^2} \sum_\mu \sin^2(p_\mu a/2) + m^2} \qquad (4.18)$$

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a	p_0	$(2\pi)^4\mathfrak{I}(p_0)_{3-,00}$	$(2\pi)^4\mathfrak{I}(p_0)_{4-,00}$
0.1	0.1	7.772923	-2.076717
0.1	0.2	15.540440	-4.148735
0.01	0.1	65.802(7)	-4.4650(4)
0.01	0.2	131.599(13)	-8.9252(9)
0.001	0.1	623.85116	-6.77184
0.001	0.2	1247.69800	-13.53882

TABLE II: Values of the more naive integrals at various values of p_0 and a; the units are chosen such that the numerical value of m is equal to one.

It is easy to check that the action (4.15) is not invariant under a lattice supersymmetry transformation, such as

$$\delta_\epsilon\phi = \sqrt{2}\epsilon^T C P_+ \chi, \quad \delta_\epsilon\phi^* = \sqrt{2}\epsilon^T C P_- \chi,$$
$$\delta_\epsilon\chi = -\sqrt{2}P_+(D_S\phi + F)\epsilon - \sqrt{2}P_-(D_S\phi^* + F^*)\epsilon,$$
$$\delta_\epsilon F = \sqrt{2}\epsilon^T C D_S P_+ \chi, \quad \delta_\epsilon F^* = \sqrt{2}\epsilon^T C D_S P_- \chi \tag{4.19}$$

in spite of the fact that this is a free theory. I.e., it does not have the behavior (3.7) seen in the discretization that is the main focus of this paper.

The integrals that are the counterparts of (4.12) are:

$$\mathfrak{I}(p_0)_{3-,\alpha\beta} = \int_B \frac{d^4q}{(2\pi)^4}(D_S(q)\gamma_0 P_- S_W(p-q)P_-)_{\alpha\beta}G_N(q)$$
$$\mathfrak{I}(p_0)_{4-,\alpha\beta} = \int_B \frac{d^4q}{(2\pi)^4}(\gamma_0 P_+ S_W(p-q)P_-)_{\alpha\beta}G_N(q) \tag{4.20}$$

It can be seen from Table II that setting (4.13) will not work: there is a significant lattice artifact and the naive supercurrent is not conserved. This shows the benefits of the formulation which is the main focus of this article.

D. Beyond one-loop

The result of the one-loop investigation is that there is a conserved supercurrent in the continuum limit, when we work to $\mathcal{O}(g)$. This agrees with the fact that the renormalization at one loop only yields a nonuniform wavefunction renormalization for the auxiliary field:
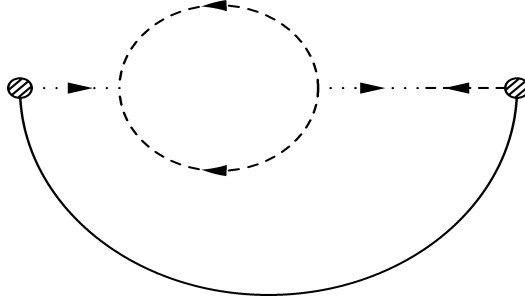
FIG. 3: A diagram where the nonsupersymmetric renormalization of the auxiliary field will play a role. The dashed/dotted line is the mixed $\langle F\phi \rangle$ propagator. The dotted line is the $\langle FF^* \rangle$ propagator.

$Z_F \neq Z_\phi = Z_\chi$ (in the continuum limit). This cannot affect correlation functions at $\mathcal{O}(g)$ since the wavefunction renormalization of the auxiliary field would have to appear on the internal line of a diagram, which necessarily implies a factor of g^2 to occur. For example, the diagram in Fig. 3 would be sensitive to the value of Z_F, and is $\mathcal{O}(g^2)$. Note that for this analysis we have switched to the supercurrent involving the auxiliary field, Eq. (3.2). In the nonperturbative analysis that we discuss next, we find further evidence that the nonconservation of the almost naive supercurrent begins at $\mathcal{O}(g^2)$.

V.   NONPERTURBATIVE ANALYSIS

Here it is convenient to take the spatial transform so that we instead work with

$$a^3 \sum_{\boldsymbol{x}} \partial_\mu^S S_{\mu\alpha}(t, \boldsymbol{x}) = a^3 \sum_{\boldsymbol{x}} \partial_t^S S_{0\alpha}(t, \boldsymbol{x}) = \partial_t^S Q_\alpha(t). \tag{5.1}$$

We evaluate

$$C(t) = \sum_{\mathbf{x}} \langle \partial_\mu^S S_\mu(t, \mathbf{x}) \mathcal{O}(0) \rangle \tag{5.2}$$

Since $S_{\mu\alpha}$ is fermionic, an odd number of $\tilde{\chi}$ fields must appear in nonvanishing correlation functions with $\partial_t Q_\alpha(t)$. The case that we will consider is $\mathcal{O} = \tilde{\chi}^T C$. Because of cluster decomposition, $C(t)$ will fall off with t exponentially, governed by the mass m_{eff} of the lightest state created by $\mathcal{O}$. As we tune the action, m_{eff} and the fall-off with t will change.

Thus we could mistake a decrease in $C(t)$ for an improvement of supersymmetry when it is really an increase in m_{eff}. Similarly, we could think we have worsened supersymmetry when in fact all we did was to decrease m_{eff}. Clearly we need a way to normalize $C(t)$ in order to cancel off this $\exp(-t\, m_{\text{eff}})$ behavior. For this reason we look instead at the ratio

$$R(t) = \frac{|\sum_{\mathbf{x}} \langle \partial_\mu S_{\mu\alpha}(t, \mathbf{x})(\tilde{\chi}^T C)_\beta(0)\rangle|}{|\sum_{\mathbf{x}} \langle \tilde{\chi}_\alpha(t, \mathbf{x})(\tilde{\chi}^T C)_\beta(0)\rangle|} \tag{5.3}$$

and will set $\alpha = \beta = 0$.

Here we use Monte Carlo simulations to nonperturbatively measure the ratio (5.3) using the almost naive supercurrent (3.3). The simulation method is rational hybrid Monte Carlo [29], and the runs were performed on Compute Unified Device Architecture (CUDA) enabled graphics processing units (Nvidia GeForce GTX 285, GTX 480 and Tesla C1060), using code that we developed and tested in our previous work. We have measured the autocorrelation time to be approximately 12 molecular dynamics time units for $ma = 0.1$ bare fermion mass, with a coupling $g = 0.1$. The length of the simulation was 5,000 molecular dynamics time units and we sample at each 5 time units. Errors in the ratio function $R(t)$ are computed by jackknife analysis with data blocked into 5 samples each. The runs are summarized in Table III. As can be seen we also consider the case of the fine-tuning action (2.8) except that we allow for a bare fermion mass m and set $y_1 = g$. Results for $R(t)$ in each case are displayed in Figs. 4 and 5. It can be seen that at large times, where the long distance theory should be obtained, $R(t) \lesssim \mathcal{O}(g^2)$. This is consistent with nonconservation of the almost naive supercurrent beginning at two loops or higher. It can also be seen that the action (2.7) gives a significantly smaller value for $R(t)$ than the action (2.8)—again, a fermion mass term has been added to the latter. This shows how the formulation (2.7) has a particularly small violation of supersymmetry, even at higher orders.

To further quantify these statements, we have also computed

$$K = \frac{1}{26} \sum_{t=2}^{14} [R(t) + R(T - t)] \tag{5.4}$$

where $T = 32$; i.e., the average over the plateaus. Errors are again estimated by jackknife procedure, since timeslices will be correlated. For the action (2.7) we obtain

$$K^{(1)} = 0.00118(13) \tag{5.5}$$

whereas for the action (2.8) with a fermion mass term added we obtain
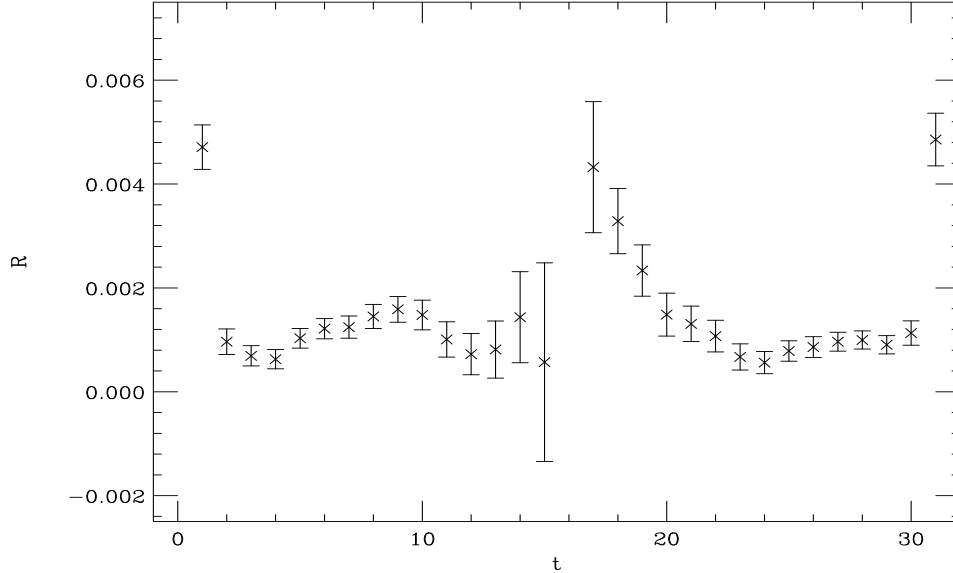
$$K^{(2)} = 0.030(18) \tag{5.6}$$

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FIG. 4: The ratio (5.3) for $ma = 0.1$, $g = 0.1$.

ma	g	$m_2^2 a^2$	λ_1	lattice
0.1	0.1	—	—	$8^3 \times 32$
0.1	0.1	0.01	0.01	$8^3 \times 32$

TABLE III: Parameters of the Monte Carlo simulations that we have performed.

The ratio is therefore

$$K^{(1)}/K^{(2)} = 0.039 \pm 0.024 \tag{5.7}$$

For an unknown reason the statistical errors for the action (2.8) are much larger, which gives our result less significance than we would like. However, even if the actual value for the ratio $K^{(1)}/K^{(2)}$ were 3σ greater than the current estimate, the divergence of the supercurrent is an order of magnitude smaller for the action (2.7) than for the action (2.8).

VI. CONCLUSIONS

We found that at one loop $\langle \partial_\mu^S S_\mu(x) \mathcal{O}(0) \rangle = 0$. This was true without any tuning of the lattice action, and provided the almost naive supercurrent is used. The numerical results were explained by the fact that the one-loop diagram is related to a free theory diagram,
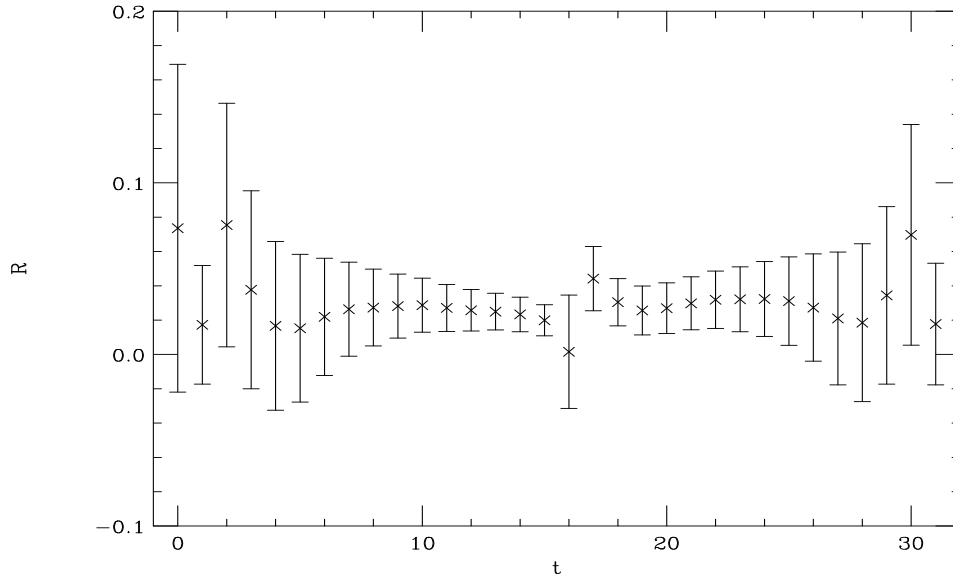
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FIG. 5: The ratio (5.3) for $ma = 0.1$, $g = 0.1$, $m_2^2 a^2 = 0.01$, $\lambda_1 = 0.01$.

and so must vanish. We showed that this result does not hold if a more naive discretization is used. Next we discussed two loop diagrams where we do not expect the cancellations to hold, since they are sensitive to the mismatch in self-energies that was already found in our previous study of one-loop counterterms. We look forward to presenting numerical results for the two loop diagrams in a forthcoming paper. Finally, we provided nonperturbative results with the almost naive supercurrent. It was seen that the nonconservation of the supercurrent is consistent with contributions beginning at two loops. Another direction for future research is the fine-tuning of the action together with the search for the renormalized supercurrent, which will have the more general form (3.9). Investigations in this direction are in progress.

Acknowledgments

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