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Matching factors for $\Delta S = 1$ four-quark operators in RI/SMOM schemes

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The non-perturbative renormalization of four-quark operators plays a significant role in lattice studies of flavor physics. For this purpose, we define regularization-independent symmetric momentum-subtraction (RI/SMOM) schemes for $\Delta S = 1$ flavor-changing four-quark operators and provide one-loop matching factors to the $\overline{\text{MS}}$ scheme in naive dimensional regularization. The mixing of two-quark operators is discussed in terms of two different classes of schemes. We provide a compact expression for the finite one-loop amplitudes which allows for a straightforward definition of further RI/SMOM schemes.

I. INTRODUCTION

The study of physical processes which change the strangeness by one unit ($\Delta S = 1$), such as the decay of a kaon into two pions, is important for the understanding of CP violation within the Standard Model (SM) and its possible extensions. Such processes can be used to measure the parameter of direct CP violation ϵ'/ϵ, to study the $\Delta I = 1/2$ rule, and to calculate long-distance contributions to $K_0 - \overline{K}_0$ mixing and the parameter of indirect CP violation ϵ [1–5]. The resulting constraints for the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements allow for a precise test of the SM. The weak interaction which mediates these processes with change in the strangeness can be described by local four-fermion operators at low energy scales, where the character of the vector boson interaction is essentially point like, see Refs. [6–10] and Refs. [11–13] for reviews. Matrix elements which describe, e.g., two-pion decays of kaons can then be computed with the help of lattice simulations.

In order to perform the renormalization of relevant operators in the lattice computation one can adopt a renormalization scheme which is independent of the regulator. Such a scheme can then be implemented in both non-perturbative lattice calculations and continuum perturbation theory. This allows for a conversion of lattice results to the modified minimal subtraction ($\overline{\text{MS}}$) scheme which is not directly applicable in lattice simulations. In Ref. [14] the non-perturbative renormalization (NPR) technique and regularization independent (RI) momentum-subtraction schemes were defined for this purpose.

In the context of light up, down, and strange quark-mass determinations quark bilinear operators need to be studied for the NPR procedure. The required matching factors which convert the quark masses and fields from the RI schemes to the $\overline{\text{MS}}$ scheme are known up to three-loop order [14–17] in perturbative Quantum Chromodynamics (QCD). The renormalization constants in these regularization independent momentum-subtraction (RI/MOM) schemes are determined at an exceptional momentum point of the considered amplitude. In the case of the quark bilinear operators the exceptional momentum configuration is distinguished by the fact that no momentum leaves the operator. However, a lattice simulation with an exceptional momentum configuration for the renormalization constants is more disposed to effects of chiral symmetry breaking [18]. Furthermore, in the RI/MOM scheme unwanted infrared effects exist and the matching factors show a poor convergence behavior. For these reasons a non-exceptional momentum configuration was proposed in Ref. [18] and the framework and concepts of new RI/SMOM schemes with a symmetric subtraction point were worked out in Ref. [19]. The symmetric subtraction point is characterized by the fact that a momentum leaves the inserted operator.

The matching factors for the conversion of quark masses from the schemes with a symmetric subtraction point to the $\overline{\text{MS}}$ scheme were computed to two-loop order [19–22] by considering the amputated Green's functions with insertion of the scalar or pseudo-scalar operator. These schemes exhibit a better infrared behavior, and also the coefficients of the perturbative expansion of the matching factors are smaller. Therefore their use led to a significant reduction of the systematic uncertainties in the light quark mass determinations following this approach [23] compared to previous studies, see Ref. [24].

For the insertion of any multi-quark operator into an amputated Green's function the fermion field for each external leg needs to be renormalized. In Ref. [19] two schemes, the RI/SMOM and RI/SMOM$_{\gamma_\mu}$ scheme, were suggested for the renormalization with a symmetric subtraction point. It was also shown that the former is equivalent to the RI$'$/MOM scheme and is thus known to three-loop order [14, 16, 17], whereas the latter is known to two-loop order [19–22]. The corresponding calculation requires the computation of amputated Green's functions with insertion of the vector or axial-vector operator and the utilization of Ward-Takahashi-identities.

We would like to mention that also the tensor operator has applications in lattice simulations, see, e.g., Ref. [25] and a scheme with a symmetric subtraction point has been introduced in Ref. [19]. Its matching factor and anomalous dimension is known to two-loop order [19, 21, 22]. Also moments of twist-2 operators used in deep inelastic scattering have been studied in a RI/SMOM scheme in Refs. [26, 27].

In view of these successes and advantages, the

RI/SMOM definition has been extended to $\Delta S = 2$ flavor-changing four-quark operators in Ref. [28], where the one-loop QCD corrections to different matching factors have been computed, and also the anomalous dimensions were provided. These results were then used for the determination of the B_K parameter which is needed to parametrize the hadronic matrix element for the theoretical description of $K_0 - \overline{K}_0$ mixing. For the case of an exceptional subtraction point the matching factors were determined at next-to-leading order in Refs. [29, 30] based on Ref. [31]. Similarly the matching for $\Delta S = 1$ flavor-changing four-quark operators with an exceptional subtraction point was determined in Refs. [30, 32]. The purpose of this paper is to introduce RI/SMOM schemes with a non-exceptional subtraction point for $\Delta S = 1$ flavor-changing four-quark operators as well as to provide the corresponding matching factors for the conversion from these RI/SMOM schemes to the $\overline{\text{MS}}$ scheme in naive dimensional regularization (NDR). To this end we first present the framework needed to properly take into account the mixing with two-quark operators which was not needed in the $\Delta S = 2$ case. We then study the insertion of the $\Delta S = 1$ operators into amputated Green's functions in perturbative QCD at one-loop order to determine the renormalization constants.

The outline of this work is as follows. In Sec. II we define the set of $\Delta S = 1$ four-quark operators used in this work. In Sec. III we discuss some generalities of the renormalization of the $\Delta S = 1$ operators in the $\overline{\text{MS}}$[NDR] scheme as well as in a general RI scheme and introduce our notation. In Sec. IV we provide a classification of projectors used to define the RI schemes and present our results for the finite one-loop amplitude as well as conversion factors from different RI/SMOM schemes to the $\overline{\text{MS}}$[NDR] scheme. Finally we close with a summary and conclusions in Sec. V.

II. THE BASES OF $\Delta S = 1$ OPERATORS

In this section we define operator bases of the effective $\Delta S = 1$ Hamiltonian of electroweak interactions, where we closely follow the notation of Ref. [33]. We work in an effective three-flavor theory including the up, down, and strange quark. This effective theory is valid for energies below the charm quark mass. The effective Hamiltonian reads

$$\mathcal{H}_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} \sum_i C_i^x(\mu) O_i^x(\mu) \tag{1}$$

with Fermi coupling constant G_F and renormalization scale μ. The symbols $C_i(\mu)$ denote Wilson coefficients and $O_i(\mu)$ are four-quark operators, which we will discuss in terms of "physical" and "chiral" operator bases in different schemes labeled x. In the case of the physical operator bases the effective Hamiltonian is expressed in terms of ten operators which are grouped into current-current operators, QCD penguin operators, and electroweak penguin operators. The physical operator bases are classified by the physical origin of their respective operators, whereas the chiral operator basis is classified by irreducible representations of the $\text{SU}(3)_L \otimes \text{SU}(3)_R$ chiral symmetry.

A. The physical bases

Let us start with the traditional physical operator basis of Refs. [10, 34–36]. The current-current operators are defined by

$$\begin{aligned} Q_1 &= (\bar{s}_a u_b)_{V-A} (\bar{u}_b d_a)_{V-A} \,, \\ Q_2 &= (\bar{s}_a u_a)_{V-A} (\bar{u}_b d_b)_{V-A} \,, \end{aligned} \tag{2}$$

the QCD penguin operators are defined by

$$\begin{aligned} Q_3 &= (\bar{s}_a d_a)_{V-A} \sum_{q=u,d,s} (\bar{q}_b q_b)_{V-A} \,, \\ Q_4 &= (\bar{s}_a d_b)_{V-A} \sum_{q=u,d,s} (\bar{q}_b q_a)_{V-A} \,, \\ Q_5 &= (\bar{s}_a d_a)_{V-A} \sum_{q=u,d,s} (\bar{q}_b q_b)_{V+A} \,, \\ Q_6 &= (\bar{s}_a d_b)_{V-A} \sum_{q=u,d,s} (\bar{q}_b q_a)_{V+A} \,, \end{aligned} \tag{3}$$

the electroweak penguin operators are defined by

$$\begin{aligned} Q_7 &= \frac{3}{2} (\bar{s}_a d_a)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_b q_b)_{V+A} \,, \\ Q_8 &= \frac{3}{2} (\bar{s}_a d_b)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_b q_a)_{V+A} \end{aligned} \tag{4}$$

with $e_u = 2/3$, $e_d = e_s = -1/3$, and

$$\begin{aligned} Q_9 &= \frac{3}{2} (\bar{s}_a d_a)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_b q_b)_{V-A} \,, \\ Q_{10} &= \frac{3}{2} (\bar{s}_a d_b)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_b q_a)_{V-A} \,, \end{aligned} \tag{5}$$

where $(\bar{q}q)_{V\pm A}$ refers to the spinor structure $\bar{q}\gamma_\mu(1\pm\gamma_5)q$, a and b are color indices, and u, d, and s are the fields of the up, down, and strange quarks. This basis of operators $\{O_i\} = \{Q_1, \ldots, Q_{10}\}$ is referred to as "basis I" in the following. Alternatively we can Fierz transform Q_1 and Q_2 to

$$\begin{aligned} \tilde{Q}_1 &= (\bar{s}_a d_a)_{V-A} (\bar{u}_b u_b)_{V-A} \,, \\ \tilde{Q}_2 &= (\bar{s}_a d_b)_{V-A} (\bar{u}_b u_a)_{V-A} \,. \end{aligned} \tag{6}$$

The basis of operators $\{O_i\} = \{\tilde{Q}_1, \tilde{Q}_2, Q_3, \ldots, Q_{10}\}$ is called "basis II" in the following. Operators with color contractions as in $\tilde{Q}_1$ are called color diagonal, operators with color contractions as in $\tilde{Q}_2$ are called color mixed. It will also be useful to define the Fierz transformation of Q_3, i.e.,

$$\tilde{Q}_3 = \sum_{q=u,d,s} (\bar{s}_a q_b)_{V-A} (\bar{q}_b d_a)_{V-A} \,. \tag{7}$$

In an explicit four-dimensional regularization scheme, such as lattice regularization, Q_i and $\tilde{Q}_i$ can be used interchangeably. In dimensional regularization, however, the contribution of evanescent operators such as

$$E_{1i} = Q_i - \tilde{Q}_i \tag{8}$$

has to be included. The evanescent operators vanish at tree level if the regulator is removed, see, e.g., Refs. [35, 37–39].

B. The chiral basis

The operators $Q_1, \ldots, Q_{10}$ are not linearly independent, i.e., one can eliminate three operators by expressing them as linear combinations of the remaining ones. In a regularization which breaks Fierz transformations also evanescent operators enter these relations. The reduced operator basis of linearly independent operators can then be classified according to irreducible representations of SU(3) and SU(2) flavor symmetries [6, 8], and the resulting operator basis will be referred to as the "chiral basis" in the following. The linear independence of its elements will become important later for the nonperturbative definition of RI schemes with the help of projectors.

We proceed along the lines of Ref. [2] amending their discussion by the contributions of evanescent operators since we work in dimensional regularization. We first eliminate the operators Q_4, Q_9, and Q_{10} using

$$\begin{aligned} Q_4 &= Q_2 + Q_3 - Q_1 - E_{12} - E_{13} \,, \\ Q_9 &= \frac{3}{2} Q_1 - \frac{1}{2} Q_3 - \frac{3}{2} E_{11} \,, \\ Q_{10} &= \frac{1}{2}(Q_1 - Q_3) + Q_2 + \frac{1}{2} E_{13} - E_{12}. \end{aligned} \tag{9}$$

The remaining seven operators can then be recombined according to irreducible representations of the chiral flavor-symmetry group $\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$. The details of this decomposition including evanescent operators are given in App. A. The chiral operator basis is thus given by

$$\begin{aligned} &(27,1) \qquad Q'_1 = 3Q_1 + 2Q_2 - Q_3 - 3E_{11} \,, \\ &(8,1) \qquad Q'_2 = \frac{1}{5}(2Q_1 - 2Q_2 + Q_3) - \frac{2}{5} E_{11} \,, \\ &(8,1) \qquad Q'_3 = \frac{1}{5}(-3Q_1 + 3Q_2 + Q_3) + \frac{3}{5} E_{11} \,, \\ &(8,1) \qquad Q'_{5,6} = Q_{5,6} \,, \\ &(8,8) \qquad Q'_{7,8} = Q_{7,8} \,, \end{aligned} \tag{10}$$

where (L, R) denotes the respective irreducible representation of $\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$. Instead of expressing Eqs. (9) and (10) in terms of the operators of basis I and evanescent operators we could have also eliminated the latter by introducing the operators $\tilde{Q}_1$, $\tilde{Q}_2$ of basis II and $\tilde{Q}_3$ with the help of Eq. (8).

III. RENORMALIZATION

In this section we discuss the renormalization of the four-quark operators O_i starting with some generalities concerning operator renormalization at fixed gauge. We then describe the one-loop off-shell renormalization of the operators in the different bases in the $\overline{\mathrm{MS}}$[NDR] scheme in detail and provide a discussion of renormalization in a general RI scheme. A brief discussion of some details concerning the Wilson coefficients of the chiral basis concludes this section.

A. Renormalization at fixed gauge

The renormalization schemes described in this work are defined at a fixed covariant gauge with gauge fixing parameter ξ, where $\xi = 0$ ($\xi = 1$) corresponds to the Landau (Feynman) gauge. The gauge-fixing procedure explicitly breaks the gauge symmetry, and it can be shown that mixing with three classes of operators can occur [35, 40–44]: (i) gauge-invariant operators which do not vanish using the equations of motion, (ii) gauge-invariant operators which vanish using the equations of motion, and (iii) gauge non-invariant operators which are either BRST invariant or vanish using the equations of motion.

In a general renormalization scheme x, a renormalized four-quark operator O_i^x can be written as

$$O_i^x = Z_{ij}^x O_j + b_{ik}^x F_k + c_{il}^x G_l + d_{im}^x N_m \,, \tag{11}$$

where O_j are the bare four-quark operators, F_k are evanescent operators, G_l (N_m) are gauge-invariant (gauge non-invariant) operators involving only two quark fields. The symbols Z_{ij}^x, b_{ik}^x, c_{il}^x, and d_{im}^x denote renormalization constants. The sum over the respective operator basis for O_j, F_k, G_l, and N_m is implied. The operators O_j belong to class (i), the operators G_l belong to class (i) or (ii), and the operators N_m belong to class (iii). Operators of class (ii) and (iii) do not contribute to physical amplitudes [40–43]. The mixing of operators N_m can be avoided by using the background field gauge [35, 36, 45].

B. The $\overline{\mathrm{MS}}$[NDR] scheme

In the following we discuss the off-shell renormalization of the operators O_i in the $\overline{\mathrm{MS}}$ scheme with massless quarks at one-loop order in perturbative QCD. We use naive dimensional regularization (NDR) in $d = 4 - 2\varepsilon$ space-time dimensions with a naive anti-commutation definition of γ_5. For multi-loop calculations the operator basis of Ref. [46] is more convenient and allows for a straightforward treatment of γ_5. Since we restrict ourselves in this work to the one-loop order, we adhere to the traditional bases in order to connect with previous works in this field. The $\overline{\text{MS}}$-renormalization of the four-quark operators Q_i with a focus on on-shell renormalization is discussed, e.g., in Refs. [10, 47] at one loop and in Refs. [35, 36] at two-loop order.

The four-quark operators O_i are of mass-dimension 6, so that in the massless limit mixing can only occur with operators F_k, G_l, and N_m of mass-dimension 6. For off-shell external states the operator

$$G_1 = \frac{4}{ig^2}\bar{s}\gamma_\nu(1-\gamma_5)[D_\mu,[D^\mu,D^\nu]]d \tag{12}$$

mixes under renormalization with the operators O_i at the one-loop level [35, 48]. This operator is of class (i), i.e., it is nonzero in the limit of on-shell external states. In this limit, however, the operator G_1 becomes linearly dependent on the four-quark operators O_i, and one finds [10, 35, 48]

$$G_1 \overset{\text{on-shell}}{\longrightarrow} Q_p \tag{13}$$

with

$$Q_p = Q_4 + Q_6 - \frac{1}{N_c}(Q_3 + Q_5)\,. \tag{14}$$

At one-loop order the renormalized operator in the $\overline{\text{MS}}$ scheme is given by

$$O_i^{\overline{\text{MS}}} = O_i + a_{ij}^{\overline{\text{MS}}} O_j + b_{ik}^{\overline{\text{MS}}} F_k + c_i^{\overline{\text{MS}}} G_1 \tag{15}$$

with

$$a^{\overline{\text{MS}}} = Z^{\overline{\text{MS}}} - \mathbb{1}\,. \tag{16}$$

There are two different types of diagrams that need to be considered: current-current diagrams, where each fermion line involving a quark field of the operator O_i extends to the external quark fields, and penguin diagrams, where one fermion line involving quark fields of the operator O_i begins and ends at the operator. We depict the corresponding one-loop diagrams in Fig. 1 and Fig. 2. The one-loop current-current diagrams determine the mixing of the four-quark operators with themselves (given in $a^{\overline{\text{MS}}}$), while the penguin diagrams determine the mixing of G_1 with the four-quark operators (given in $c^{\overline{\text{MS}}}$).

We first consider the off-shell renormalization of operator basis I and II. The current-current contributions in $a^{\overline{\text{MS}}}$ separate in 2×2 blocks given by [10, 35, 36, 47]

$$(a_{1,2}^{\overline{\text{MS}}}) = (a_{3,4}^{\overline{\text{MS}}}) = (a_{9,10}^{\overline{\text{MS}}})$$

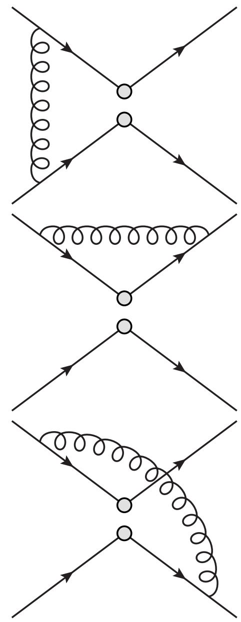

FIG. 1. Current-current contributions at one-loop order.

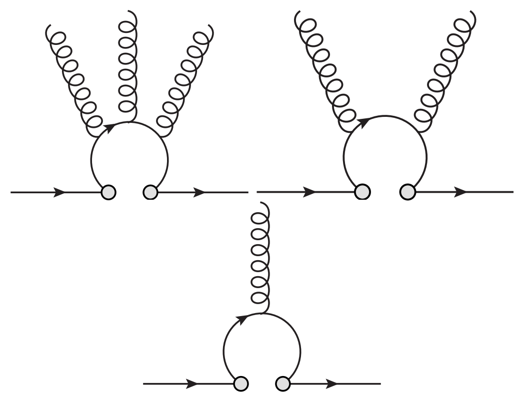

FIG. 2. Penguin-type diagrams with up to three external gluons. Three analog penguin diagrams, where the gluons attach to a closed fermion loop, also need to be considered but are not shown here.

$$= \frac{\alpha_s}{4\pi\epsilon}\begin{pmatrix} -3/N_c & 3 \\ 3 & -3/N_c \end{pmatrix}, \tag{17}$$

and

$$(a_{5,6}^{\overline{\text{MS}}}) = (a_{7,8}^{\overline{\text{MS}}}) = \frac{\alpha_s}{4\pi\epsilon}\begin{pmatrix} 3/N_c & -3 \\ 0 & 3/N_c - 3N_c \end{pmatrix}, \tag{18}$$

where the subscripts i, j of $(a_{i,j}^{\overline{\text{MS}}})$ denote that the matrix acts on the space of operators O_i and O_j. This notation is also used for other block-diagonal matrices in the remainder of this section. The matrix $a^{\overline{\text{MS}}}$ is identical for basis I and II.

The penguin contributions in $c^{\overline{\text{MS}}}$ are given by

$$c_2^{\overline{\text{MS}}} = -c_9^{\overline{\text{MS}}} = \frac{\alpha_s}{4\pi\varepsilon}\frac{1}{3}\,, \qquad c_3^{\overline{\text{MS}}} = \frac{\alpha_s}{4\pi\varepsilon}\frac{2}{3}\,,$$

$$c_4^{\overline{\rm MS}} = c_6^{\overline{\rm MS}} = \frac{\alpha_s}{4\pi\varepsilon}\,, \qquad c_{1,5,7,8,10}^{\overline{\rm MS}} = 0 \tag{19}$$

for basis I and II. For on-shell matrix elements we use Eq. (13) in Eq. (15) and thus reproduce the results for the anomalous dimensions of Refs. [10, 35, 36, 47].

The set of evanescent operators $\{F_k\}$ used in Eq. (15) to define the $\overline{\rm MS}$ scheme consists of operators

$$\begin{aligned} E_1 &= (\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L u_b)(\bar u_b \gamma^\mu \gamma^\nu \gamma^{\rho L} d_a) - (16-4\varepsilon) Q_1\,,\\ E_2 &= (\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L u_a)(\bar u_b \gamma^\mu \gamma^\nu \gamma^{\rho L} d_b)\\ &\quad - (16-4\varepsilon) Q_2\,, \end{aligned} \tag{20}$$

and

$$\begin{aligned} \tilde E_1 &= (\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_a)(\bar u_b \gamma^\mu \gamma^\nu \gamma^{\rho L} u_b) - (16-4\varepsilon) \tilde Q_1\,,\\ \tilde E_2 &= (\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_b)(\bar u_b \gamma^\mu \gamma^\nu \gamma^{\rho L} u_a)\\ &\quad - (16-4\varepsilon) \tilde Q_2\,, \end{aligned} \tag{21}$$

as well as

$$\begin{aligned} E_3 &= (\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_a) \sum_{q=u,d,s} (\bar q_b \gamma^\mu \gamma^\nu \gamma^{\rho L} q_b)\\ &\quad - (16-4\varepsilon) Q_3\,,\\ E_4 &= (\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_b) \sum_{q=u,d,s} (\bar q_b \gamma^\mu \gamma^\nu \gamma^{\rho L} q_a)\\ &\quad - (16-4\varepsilon) Q_4\,,\\ E_5 &= (\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_a) \sum_{q=u,d,s} (\bar q_b \gamma^\mu \gamma^\nu \gamma^{\rho R} q_b)\\ &\quad - (4+4\varepsilon) Q_5\,,\\ E_6 &= (\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_b) \sum_{q=u,d,s} (\bar q_b \gamma^\mu \gamma^\nu \gamma^{\rho R} q_a)\\ &\quad - (4+4\varepsilon) Q_6\,, \end{aligned} \tag{22}$$

and

$$\begin{aligned} E_7 &= \frac{3}{2}(\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_a) \sum_{q=u,d,s} e_q (\bar q_b \gamma^\mu \gamma^\nu \gamma^{\rho R} q_b)\\ &\quad - (4+4\varepsilon) Q_7\,,\\ E_8 &= \frac{3}{2}(\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_b) \sum_{q=u,d,s} e_q (\bar q_b \gamma^\mu \gamma^\nu \gamma^{\rho R} q_a)\\ &\quad - (4+4\varepsilon) Q_8\,, \end{aligned} \tag{23}$$

and finally

$$\begin{aligned} E_9 &= \frac{3}{2}(\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_a) \sum_{q=u,d,s} e_q (\bar q_b \gamma^\mu \gamma^\nu \gamma^{\rho L} q_b)\\ &\quad - (16-4\varepsilon) Q_9\,,\\ E_{10} &= \frac{3}{2}(\bar s_a \gamma_\mu \gamma_\nu \gamma_\rho^L d_b) \sum_{q=u,d,s} e_q (\bar q_b \gamma^\mu \gamma^\nu \gamma^{\rho L} q_a)\\ &\quad - (16-4\varepsilon) Q_{10}\,, \end{aligned} \tag{24}$$

where $\gamma_\rho^{L/R} = \gamma_\rho(1 \mp \gamma_5)$, and a and b are color indices. The explicit contributions of order ε are determined by the "Greek" method [49] in accordance with two-loop calculations such as Ref. [30]. For operator basis I we have $\{F_k\} = \{E_1, \ldots, E_{10}\}$. The renormalization coefficients of the evanescent operators in Eq. (15) decompose in 2×2 blocks and are given by

$$\begin{aligned} (b_{1,2}^{\overline{\rm MS}}) &= \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} N_c/4 - 1/(2N_c) & 1/4 \\ 1/2 & -1/(2N_c)\end{pmatrix},\\ (b_{3,4}^{\overline{\rm MS}}) &= (b_{5,6}^{\overline{\rm MS}}) = (b_{7,8}^{\overline{\rm MS}}) = (b_{9,10}^{\overline{\rm MS}})\\ &= \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} -1/(2N_c) & 1/2 \\ 1/4 & N_c/4 - 1/(2N_c)\end{pmatrix} \end{aligned} \tag{25}$$

for basis I. Similarly for basis II we have $\{F_k\} = \{\tilde E_1, \tilde E_2, E_3, \ldots, E_{10}\}$, and one obtains

$$\begin{aligned} (b_{1,2}^{\overline{\rm MS}}) &= (b_{3,4}^{\overline{\rm MS}}) = (b_{5,6}^{\overline{\rm MS}}) = (b_{7,8}^{\overline{\rm MS}}) = (b_{9,10}^{\overline{\rm MS}})\\ &= \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} -1/(2N_c) & 1/2 \\ 1/4 & N_c/4 - 1/(2N_c)\end{pmatrix}. \end{aligned} \tag{26}$$

If operators transform in an irreducible representation of a given symmetry, they only mix with other operators transforming in the same irreducible representation. In the case of the chiral basis the decomposition of operators according to irreducible representations of $\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$ is given in Eqs. (10). The set of operators used in Eq. (15) for the chiral basis is given by $\{O_i\} = \{Q_1', Q_2', Q_3', Q_5', \ldots, Q_8'\}$ and $\{F_k\} = \{E_1, \ldots, E_{13}\}$, where the operators E_{11}, E_{12}, and E_{13} are defined in Eq. (8). The matrix $a^{\overline{\rm MS}}$ is then again block-diagonal with

$$\begin{aligned} (a_1^{\overline{\rm MS}}) &= \frac{\alpha_s}{4\pi\varepsilon}(3 - 3/N_c)\,,\\ (a_{2,3}^{\overline{\rm MS}}) &= \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} -3/N_c & 3 \\ 3 & -3/N_c\end{pmatrix},\\ (a_{5,6}^{\overline{\rm MS}}) &= (a_{7,8}^{\overline{\rm MS}}) = \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} 3/N_c & -3 \\ 0 & 3/N_c - 3N_c\end{pmatrix}, \end{aligned} \tag{27}$$

and

$$c^{\overline{\rm MS}} = \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & 1 & 0 & 0\end{pmatrix}^T. \tag{28}$$

The nonzero elements of $b^{\overline{\rm MS}}$ are given in Tab. I. In the chiral basis the on-shell limit of G_1 is given by

$$\begin{aligned} Q_p &= (2 - 3/N_c) Q_2' + (3 - 2/N_c) Q_3' - Q_5'/N_c + Q_6'\\ &\quad - E_{11} - E_{12} - E_{13}\,. \end{aligned} \tag{29}$$

C. Regularization-independent schemes - The mixing of four-quark operators

In the following we define RI schemes for the $\Delta S = 1$ four-quark operator bases. The RI schemes are defined non-perturbatively, so that they can be used in lattice

(i,j)	$b^{\overline{\mathrm{MS}}}_{ij}/\frac{\alpha_s}{4\pi\varepsilon}$	(i,j)	$b^{\overline{\mathrm{MS}}}_{ij}/\frac{\alpha_s}{4\pi\varepsilon}$
$(1,1)$	1	$(3,1)$	$3/10$
$(1,2)$	$-1/N_c$	$(3,2)$	$-3/(10N_c)$
$(1,9)$	$-1/N_c$	$(3,9)$	$1/(5N_c)$
$(1,10)$	1	$(3,10)$	$-1/5$
$(1,11)$	9	$(3,11)$	$6/5$
$(1,12)$	-6	$(3,12)$	$6/5$
$(1,13)$	3	$(3,13)$	$-3/5$
$(2,1)$	$-1/5$	$(5,5)$	$-1/(2N_c)$
$(2,2)$	$1/(5N_c)$	$(5,6)$	$1/2$
$(2,3)$	$-1/(6N_c)$	$(6,5)$	$1/4$
$(2,4)$	$1/6$	$(6,6)$	$N_c/4-1/(2N_c)$
$(2,9)$	$-2/(15N_c)$	$(7,7)$	$-1/(2N_c)$
$(2,10)$	$2/15$	$(7,8)$	$1/2$
$(2,11)$	$-9/5$	$(8,7)$	$1/4$
$(2,12)$	$-9/5$	$(8,8)$	$N_c/4-1/(2N_c)$
$(2,13)$	$-3/5$		

TABLE I. The nonzero elements of $b^{\overline{\mathrm{MS}}}$ for the chiral basis.

simulations as well as in continuum perturbation theory. In lattice calculations the RI schemes serve as intermediate schemes and allow for a straightforward conversion of the studied quantity to the $\overline{\mathrm{MS}}$ scheme. In this subsection we focus on the mixing of four-quark operators O_i among themselves. In terms of Eq. (11) this means that we provide the RI renormalization conditions to determine the renormalization matrix Z^{RI}. The renormalization conditions which determine the mixing of two-quark operators G_l and N_m with the four-quark operators O_i, i.e., c^{RI} and d^{RI}, are discussed in subsection III D. While the content of this subsection thus suffices to define the RI schemes for the $(27,1)$ and $(8,8)$ operators of the chiral basis, the discussion of subsection III D is necessary to complete the RI schemes for the $(8,1)$ operators.

Let us consider a set of bare operators $\{\mathcal{O}_i\}$ that is closed under renormalization and contains the set of four-quark operators $\{O_i\}$, i.e., $\{O_i\}\subset\{\mathcal{O}_i\}$. The renormalized operators $\mathcal{O}^x_i$ in the $\overline{\mathrm{MS}}$ or an RI scheme can be expressed in terms of the bare operators by

$$\mathcal{O}^x_i = Z^x_{ij}\mathcal{O}_j, \quad x\in\{\overline{\mathrm{MS}},\mathrm{RI}\}. \tag{30}$$

The renormalization conditions of the RI schemes are formulated in terms of renormalized amputated Green's functions $\Gamma^y_n(\mathcal{O}^x_i)$ with a single insertion of such an operator $\mathcal{O}^x_i$, where y denotes the wave function renormalization scheme and n indicates the external states of the Green's function. In this subsection we only consider Green's functions with four external quarks which we denote by $n=4$.

The renormalized amputated Green's function Γ^y_4 is related to the bare amputated Green's function Γ_4 by

$$\Gamma^y_4(\mathcal{O}^x_i) = \frac{1}{(Z^y_q)^2} Z^x_{ij}\Gamma_4(\mathcal{O}_j)\,, \tag{31}$$

where Z^y_q is the quark wave function renormalization constant, which relates the bare quark field f to the renormalized quark field $f^y=(Z^y_q)^{1/2}f$ in the scheme y.

Various RI/SMOM wave function renormalization schemes have been proposed in Ref. [19] and will be used later. In order to convert the quark fields from the RI scheme to the $\overline{\mathrm{MS}}$ scheme matching factors

$$C^y_q = \frac{Z^{\overline{\mathrm{MS}}}_q}{Z^y_q} \tag{32}$$

have been computed with $f^{\overline{\mathrm{MS}}} = (C^y_q)^{1/2}f^y$.

The operators which are renormalized in an RI scheme can be converted to the $\overline{\mathrm{MS}}$ scheme

$$\mathcal{O}^{\overline{\mathrm{MS}}}_i = S^{\mathrm{RI}\to\overline{\mathrm{MS}}}_{ij}\mathcal{O}^{\mathrm{RI}}_j \tag{33}$$

using the conversion matrix

$$S^{\mathrm{RI}\to\overline{\mathrm{MS}}} = Z^{\overline{\mathrm{MS}}}(Z^{\mathrm{RI}})^{-1}\,. \tag{34}$$

The renormalization matrix $Z^{\overline{\mathrm{MS}}}$ has been discussed in the previous Sec. III B, whereas the matrix Z^{RI} is determined by the renormalization conditions of the RI scheme. Using Eqs. (32) and (34) one can express Eq. (31) completely in terms of matching factors and renormalized Green's functions

$$\Gamma^y_4(\mathcal{O}^{\mathrm{RI}}_i) = (C^y_q)^2 (S^{\mathrm{RI}\to\overline{\mathrm{MS}}})^{-1}_{ij}\Gamma^{\overline{\mathrm{MS}}}_4(\mathcal{O}^{\overline{\mathrm{MS}}}_j)\,. \tag{35}$$

In our case the set of operators $\{\mathcal{O}_i\}$ is given by $\{O_i, G_l, N_m, F_k\}$, where $\{O_i\}$ are the four-quark operators of basis I, II, or the chiral basis and $\{F_k\}$ are the corresponding evanescent operators used to define the $\overline{\mathrm{MS}}$ scheme in the previous section. Since the evanescent operators are an artifact of dimensional regularization and escape a regularization-independent definition, their contribution to the right-hand side of Eq. (33) should be avoided. In order to achieve this, one defines new subtracted operators $\{\mathcal{O}^{\mathrm{sub}}_i\}$ from the set of bare operators $\{\mathcal{O}_i\}=\{O_i, G_l, N_m\}$.

In general for any regulator one first has to subtract all contributions specific to the regularization from a given bare operator. Hence, in dimensional regularization we perform modified minimal subtraction of the evanescent operators [31, 32], i.e., the subtracted operator is defined as

$$\mathcal{O}^{\mathrm{sub}}_i = \mathcal{O}_i + s^{\overline{\mathrm{MS}}}_{ik}F_k\,, \tag{36}$$

where $s^{\overline{\mathrm{MS}}}$ is chosen such that it cancels all contributions of F_k proportional to a pole in ε. In principle the choice of evanescent operators $\{F_k\}$ used on the right-hand side of Eq. (36) is not unique. A useful choice is the basis $\{F_k\}$ given in the previous section to define the $\overline{\mathrm{MS}}$ scheme for operator basis I, II, and the chiral basis, and for which we therefore have

$$\mathcal{O}^{\mathrm{sub},\overline{\mathrm{MS}}}_i = \mathcal{O}^{\overline{\mathrm{MS}}}_i\,. \tag{37}$$

For convenience we adopt this definition of the subtracted operator in the following. In a lattice regularization one has to perform a similar subtraction of lower-dimensional two-quark operators [2] which do not occur in dimensional regularization. From now on we only consider such subtracted operators and therefore drop the explicit notation of the superscript "sub".

In the RI schemes one imposes the renormalization condition [14] that amputated Green's functions with given off-shell external states at a given momentum point and in a fixed gauge coincide with their tree-level value. This condition is made explicit by choosing a certain set of projectors $\{P_{4j}\}$ in spinor, color, and flavor space that is applied to the four-quark amputated Green's function and imposing

$$P_{4j}\Gamma_4^y(O_i^{\mathrm{RI}})\big|_{\mathrm{mom.\ conf.}} = P_{4j}\Gamma_4^{\mathrm{tree}}(O_i)\,, \tag{38}$$

where $\Gamma_4^{\mathrm{tree}}(O_i)$ denotes the insertion of the operator O_i in the amputated Green's function Γ_4 at tree level. Since we are only interested in RI-renormalized operators O_i^{RI}, we do not provide RI conditions for the operators G_l and N_m in this work. For a set of n operators $O_i \in \{O_1,\ldots,O_n\}$ we will provide n projectors $P_{4j} \in \{P_{41},\ldots,P_{4n}\}$ to determine the $n\times n$ elements of the renormalization matrix Z^{RI}. If no two-quark operators mix with the four-quark operators O_i, i.e., $c^{\mathrm{RI}}=0$ and $d^{\mathrm{RI}}=0$ in Eq. (11), the renormalization matrix is given by [2]

$$Z^{\mathrm{RI}} = (Z_q^y)^2 F_4[M_4(O)]^{-1}\,, \tag{39}$$

where

$$[M_4(O)]_{ij} = P_{4j}\Gamma_4(O_i)\big|_{\mathrm{mom.\ conf.}}\,. \tag{40}$$

and $F_4 = \lim_{\alpha_s\to 0} M_4(O)$. If two-quark operators mix with the operators O_i, Eq. (39) has to be modified slightly, which is discussed in subsection III D.

The matrix Z^{RI} given in Eq. (39) depends on the regulator used to define the RI scheme, i.e., the matrix Z^{RI} obtained using a lattice regulator is different from the matrix Z^{RI} obtained in dimensional regularization. The RI condition of Eq. (38), however, fixes the physical amplitudes of the RI-renormalized operators at a certain off-shell momentum point to its tree-level value, which is independent of the choice of the regulator. Therefore, the physical amplitudes of the RI-renormalized operators agree for all choices of the regulator.

We define the RI scheme in the limit of vanishing quark masses. The choice of projectors $\{P_{4j}\}$, the gauge fixing, and the momentum configuration of the off-shell amputated Green's functions Γ_4^y defines the scheme up to mixing with two-quark operators. The explicit form of the projectors will be discussed later in Sec. IV. Note that Eq. (38) matches the amputated Green's function of a certain physical process with insertion of RI operators at a certain off-shell momentum point.

In particular for the $\Delta S=1$ operators we consider the off-shell amputated Green's functions $\Gamma_4^{\alpha\beta\gamma\delta;ijkl;f}$ of the process

$$d(p_1)\bar{s}(-p_2) \to \bar{f}(-p_3)f(p_4) \tag{41}$$

with quarks $f=u,d,s$, momenta p_1, p_2, p_3, p_4, color indices i,j,k,l, and spinor indices α, β, γ, δ. Using crossing symmetry we could equally well consider the scattering amplitude

$$d(p_1)f(p_3) \to s(p_2)f(p_4)\,. \tag{42}$$

The momentum configuration used in Eq. (38) to define the RI/SMOM scheme is then given by

$$p_3 = p_1\,, \qquad p_4 = p_2 \tag{43}$$

with

$$p_1^2 = p_2^2 = q^2 = -\mu_s^2\,, \qquad q = p_1 - p_2\,, \tag{44}$$

in Minkowski space, where μ_s is the subtraction scale. This momentum configuration and our convention for open indices for the amputated Green's functions Γ_4 is shown in Fig. 3. In Fig. 4 we explicitly show the corresponding penguin diagrams at one-loop order, where a momentum transfer $2q$ leaves the operator. This choice of momenta is non-exceptional (no partial sum of incoming external momenta vanishes) which has the advantage of suppressing unwanted infrared effects in the lattice simulation.

In Ref. [32] a RI/MOM scheme was defined which uses exceptional kinematics and a different momentum point for current-current and penguin diagrams, see Fig. 5 of Ref. [32]. The momentum configuration in the RI/MOM scheme is given by $p=p_1=p_2=p_3=p_4$ at $\mu_s^2=-p^2$ for current-current diagrams and $p=p_1=p_4$ and $p'=p_2=p_3$ at $\mu_s^2=-q^2$ for penguin diagrams. We consider the RI/MOM scheme in the following for completeness, illustration, and check of our calculation.

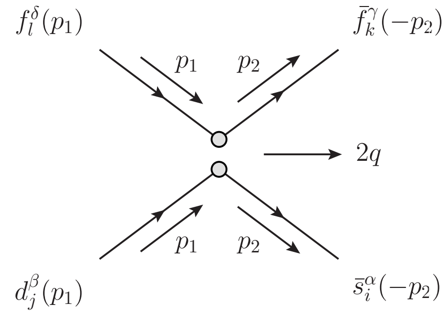

FIG. 3. Momentum flow and open indices for the four-quark Green's functions Γ_4. We explicitly write the quark fields $\bar{s}_i^\alpha$, d_j^β, $\bar{f}_k^\gamma$, and f_l^δ with spinor indices α, β, γ, and δ and color indices i, j, k, and l. The momenta of the quark fields are given in brackets, which are counted as incoming, i.e., they flow towards the four-quark operator. An additional momentum of $2q$ leaves the operator as indicated by the arrow.

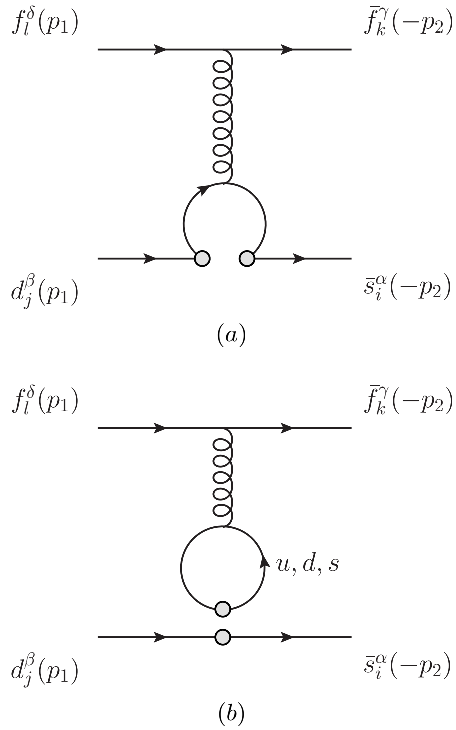

FIG. 4. Penguin contributions at one loop and the respective momentum configuration. We explicitly write the fields and momentum configuration as in Fig. 3. We do not explicitly show the flow of the additional momentum $2q$ here.

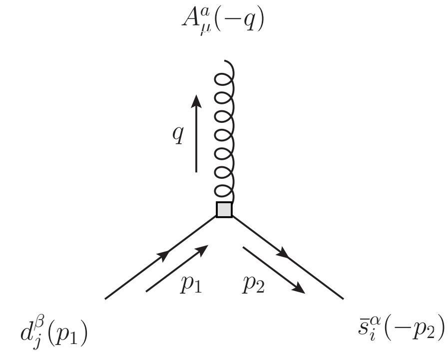

FIG. 5. Momentum flow and open indices for the two-quark Green's functions Γ_2. We explicitly write the quark fields $\bar{s}_i^\alpha$, d_j^β and the gluon field A_μ^a with spinor indices α, β, color indices i, j, and Lorentz index μ. The index $a = 1, \ldots, N_c^2 - 1$ is contracted with the generators of SU(N_c). The incoming momenta of the fields are also given in brackets.

D. Regularization-independent schemes - The mixing of two-quark operators

In the following we discuss the mixing of the two-quark operators G_l and N_m with the four-quark operators O_i in the RI schemes. Such mixing occurs, e.g., for the $(8, 1)$ operators of the chiral basis. The two-quark operators mix through the penguin diagrams, and therefore their mixing should be determined from amplitudes which only receive contributions from penguin-type contractions.

Two kinds of such amplitudes will be considered in the following: (a) amputated Green's functions Γ_2 with two external quarks and one external gluon and (b) amputated Green's functions Γ_{4p} with four external quarks corresponding to the process $df \to sf$, where the quark flavor $f \notin \{u, d, s\}$. The momentum flow and our convention for open indices in case (a) is shown in Fig. 5. The corresponding momenta p_1, p_2, and q for the RI/SMOM scheme satisfy Eq. (44). This momentum configuration is also non-exceptional. In case (b) we adhere to the momentum configuration and choice of indices shown in Figs. 3 and 4. In the following we define the RI schemes in such a way that both cases, (a) and (b), lead to identical one-loop conversion factors from the RI scheme to the $\overline{\text{MS}}$[NDR] scheme. At higher loops, however, the RI schemes defined by (a) differ from the RI schemes defined by (b). Case (b) can be implemented in lattice simulations in a straightforward way [2], while case (a) requires an external gluonic state. Nevertheless, the availability of both cases should be beneficial for lattice simulations, in particular to estimate higher-loop effects that are neglected in this work.

Case (a) is formulated in terms of renormalized amputated Green's functions Γ_2^y which are related to the bare amputated Green's functions Γ_2 by

$$\Gamma_2^y(\mathcal{O}_i^x) = \frac{1}{Z_q^y (Z_A^y)^{1/2}} Z_{ij}^x \Gamma_2(\mathcal{O}_j) \,, \tag{45}$$

where Z_A^y is the gluon wave function renormalization constant, which relates the bare gluon field A_μ to the renormalized gluon field $A_\mu^y = (Z_A^y)^{1/2} A_\mu$ in the scheme y. Using Eqs. (32) and (34) one can express Eq. (45) as

$$\begin{aligned}\Gamma_2^y(\mathcal{O}_i^{\text{RI}}) = {}& (Z_A^{\overline{\text{MS}}}/Z_A^y)^{1/2} C_q^y \\ & \times \left(S^{\text{RI}\to\overline{\text{MS}}}\right)^{-1}_{ij} \Gamma_2^{\overline{\text{MS}}}(\mathcal{O}_j^{\overline{\text{MS}}}) \,. \end{aligned} \tag{46}$$

We then determine the renormalization coefficients c^{RI} and d^{RI} by imposing

$$P_{2k} \Gamma_2^y(O_i^{\text{RI}})\big|_{\text{mom. conf.}} = P_{2k} \Gamma_2^{\text{tree}}(O_i) = 0 \tag{47}$$

for a certain set of projectors $\{P_{2k}\}$. Since only tree-level insertions of G_l and N_m need to be considered in this work, we do not provide renormalization conditions to define RI-renormalized operators G_l^{RI} and N_m^{RI} . The definition of RI schemes for the two-quark operators G_l and N_m is beyond the scope of this work.

The RI conditions given in Eqs. (38) and (47) then allow for a non-perturbative determination of the renormalization matrices Z^{RI}, c^{RI}, and d^{RI} defined in Eq. (11). Without loss of generality we set $d^{\text{RI}} = 0$ in the following. We find

$$\begin{aligned} Z^{\text{RI}} &= (Z_q^y)^2 F_4 [M_4(O)]^{-1} (\mathbb{1} - z^{\text{RI}})^{-1} \,, \\ c^{\text{RI}} &= -Z^{\text{RI}} M_2(O) [M_2(G)]^{-1} \end{aligned} \tag{48}$$

with

$$z^{\rm RI} = M_2(O)[M_2(G)]^{-1} M_4(G) [M_4(O)]^{-1} \tag{49}$$

and

$$[M_2(\mathcal{O})]_{ik} = P_{2k} \Gamma_2(\mathcal{O}_i)\big|_{\text{mom. conf.}} . \tag{50}$$

Case (b) is formulated in terms of the renormalized amputated Green's function Γ^y_{4p} corresponding to the process $sf \to df$ that is shown in Fig. 4 at one loop. Note that the quark flavor $f \notin \{u,d,s\}$ for Γ^y_{4p} as opposed to $f \in \{u,d,s\}$ for Γ^y_4 which was used in subsection III C. We then determine the mixing of the two-quark operators by imposing

$$P_{4p,k} \Gamma^y_{4p}(O_i^{\rm RI})\big|_{\text{mom. conf.}} = P_{4p,k} \Gamma^{\rm tree}_{4p}(O_i) = 0 \tag{51}$$

for a certain set of projectors $\{P_{4p,k}\}$. The renormalization coefficients for case (b) can be obtained from Eqs. (48) and (49) by replacing $M_2 \to M_{4p}$ with

$$[M_{4p}(\mathcal{O})]_{ik} = P_{4p,k} \Gamma_{4p}(\mathcal{O}_i)\big|_{\text{mom. conf.}} . \tag{52}$$

We provide the explicit projectors used to define the RI/SMOM schemes in Sec. IV. We will define the set of projectors $\{P_{2k}\}$ and $\{P_{4p,k}\}$ such that the resulting RI/SMOM schemes agree at one loop. From now on we refer to RI/SMOM schemes defined using case (a) as RI/SMOM$_2$ schemes, while we do not write a subscript for RI/SMOM schemes defined using case (b).

Since in the physical application of the RI scheme one is not interested in the conversion of the operators G_l and N_m from the RI scheme to the $\overline{\rm MS}$ scheme, it is useful to decompose the matrix $S_{ij}^{\rm RI\to\overline{MS}}$ of Eq. (34) into several blocks. The conversion relation of Eq. (33) for subtracted operators is then given by

$$\begin{aligned} O_i^{\overline{\rm MS}} = O_i^{\rm RI} + \Delta a_{ij}^{\rm RI\to\overline{MS}} O_j^{\rm RI} + \Delta c_{il}^{\rm RI\to\overline{MS}} G_l^{\rm RI} \\ + \Delta d_{im}^{\rm RI\to\overline{MS}} N_m^{\rm RI} , \end{aligned} \tag{53}$$

where the operators O_i are either of basis I, II, or the chiral basis. The three blocks $\Delta a_{ij}^{\rm RI\to\overline{MS}}$, $\Delta c_{il}^{\rm RI\to\overline{MS}}$, and $\Delta d_{im}^{\rm RI\to\overline{MS}}$ are obtained by imposing the renormalization condition of Eqs. (38) and (47), i.e., we have to evaluate $\Gamma_n^{\overline{\rm MS}}(O_i^{\overline{\rm MS}})$ with off-shell external legs, see Eq. (35).

We denote on-shell matrix elements of the operators $O_i^{\overline{\rm MS}}$ with a general external momentum setting by $\left\langle O_i^{\overline{\rm MS}} \right\rangle$ for which one finds

$$\left\langle O_i^{\overline{\rm MS}} \right\rangle = R_{ij}^{\rm RI\to\overline{MS}} \left\langle O_j^{\rm RI} \right\rangle \tag{54}$$

since the on-shell matrix elements of the two-quark operators G_l and N_m are related to the on-shell matrix elements of O_i. At the one-loop level only the two-quark operator G_1 contributes to the right-hand side of Eq. (53), and one has

$$\langle G_1 \rangle = \langle Q_p \rangle . \tag{55}$$

In Sec. IV we compute $\Delta a^{\rm RI\to\overline{MS}}$ and $\Delta c^{\rm RI\to\overline{MS}}$ as well as the on-shell conversion factors

$$\Delta r^{\rm RI\to\overline{MS}} = R^{\rm RI\to\overline{MS}} - \mathbb{1} \tag{56}$$

at one loop for different RI schemes and for operator basis I, II, and the chiral basis.

E. The Wilson coefficients of the chiral basis

For lattice calculations it is advantageous to consider the chiral operator basis that uses the classification of operators according to irreducible representations of the chiral symmetry. In this basis the operators of each irreducible representation can be renormalized independently. Therefore the effective Hamiltonian should be expressed in terms of the operators ${Q'}_i^{\rm RI}$.

On-shell matrix elements of the effective Hamiltonian defined in Eq. (1) in terms of ${Q'}_i^{\rm RI}$ read

$$\begin{aligned} \left\langle \mathcal{H}_{\rm eff}^{\Delta S=1} \right\rangle &= \frac{G_F}{\sqrt{2}} \sum_i {C'}_i^{\overline{\rm MS}}(\mu) \left\langle {Q'}_i^{\overline{\rm MS}}(\mu) \right\rangle \\ &= \frac{G_F}{\sqrt{2}} \sum_{i,j} {C'}_i^{\overline{\rm MS}}(\mu) {R'}_{ij}^{\rm RI\to\overline{MS}}(\mu) \\ &\qquad \times \left\langle {Q'}_j^{\rm RI}(\mu) \right\rangle , \end{aligned} \tag{57}$$

where μ is the renormalization scale and ${R'}^{\rm RI\to\overline{MS}}$ is the conversion matrix of Eq. (54) for the chiral basis. The Wilson coefficients are, however, typically given for the traditional operator bases in Refs. [33, 36, 50, 51]. Therefore it remains to relate the above Wilson coefficients ${C'}_i^{\overline{\rm MS}}$ to the known Wilson coefficients $C_i^{\overline{\rm MS}}$ corresponding to the operators O_i of basis I or II.

To this end we first note that Eqs. (9) and (10) hold for RI-renormalized operators without contributions of evanescent operators, i.e.,

$$O_i^{\rm RI} - T_{ij} {Q'}_j^{\rm RI} = 0 \tag{58}$$

with

$$T = \begin{pmatrix} 1/5 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1/5 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & 3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 3/10 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ 3/10 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} , \tag{59}$$

where vanishing matrix elements are denoted by dots, and the matrix T can be read off from Eqs. (9) and (10). This equation holds for the operators O_i of basis I and II. Note that Q_5–Q_8 are equal to Q'_5–Q'_8, see Eqs. (10), and therefore the respective sub-block in T is proportional to the unit matrix. In the $\overline{\text{MS}}$ scheme, however, finite contributions of evanescent operators modify Eq. (58) to

$$O_i^{\overline{\text{MS}}} - T_{ij} {Q'}_j^{\overline{\text{MS}}} = \Delta t_i G_1 \tag{60}$$

at the one-loop level. The coefficients Δt_i are given by

$$\Delta t_4 = -\frac{\alpha_s}{4\pi}\,, \qquad \Delta t_i = 0 \quad \text{for} \quad i \neq 4 \tag{61}$$

for operators O_i of basis I and

$$\begin{aligned} \Delta t_2 &= -\frac{1}{3}\frac{\alpha_s}{4\pi}\,, \qquad \Delta t_4 = -\frac{\alpha_s}{4\pi}\,, \\ \Delta t_i &= 0 \quad \text{for} \quad i \neq 2, 4 \end{aligned} \tag{62}$$

for operators O_i of basis II.

Since the Hamiltonian is independent of the choice of operator basis, we have

$$\sum_{i=1}^{7} {C'}_i^{\overline{\text{MS}}}(\mu) \left\langle {Q'}_i^{\overline{\text{MS}}}(\mu) \right\rangle = \sum_{i=1}^{10} C_i^{\overline{\text{MS}}}(\mu) \left\langle O_i^{\overline{\text{MS}}}(\mu) \right\rangle \,. \tag{63}$$

Therefore using Eqs. (60) and $\langle G_1 \rangle = \langle Q_p \rangle$ we can determine ΔT in

$${C'}_j^{\overline{\text{MS}}}(\mu) = C_i^{\overline{\text{MS}}}(\mu)(T_{ij} + \Delta T_{ij})\,. \tag{64}$$

At one-loop order one finds [33]

$$\Delta T_{\text{I}}^{\overline{\text{MS}}} = \frac{\alpha_s}{4\pi} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{3}{N_c} - 2 & \frac{2}{N_c} - 3 & \frac{1}{N_c} & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{65}$$

for the Wilson coefficients $C_i^{\overline{\text{MS}}}$ corresponding to basis I and

$$\Delta T_{\text{II}}^{\overline{\text{MS}}} = \frac{\alpha_s}{4\pi} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{N_c} - \frac{2}{3} & \frac{2}{3N_c} - 1 & \frac{1}{3N_c} & -\frac{1}{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{3}{N_c} - 2 & \frac{2}{N_c} - 3 & \frac{1}{N_c} & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{66}$$

for the Wilson coefficients $C_i^{\overline{\text{MS}}}$ corresponding to basis II, where we again denote vanishing matrix elements by dots.

In App. B we give an alternative way to determine ΔT using our results for an RI scheme derived in the next section.

IV. CALCULATION AND RESULTS

In the following we give the main results of this work. We first express the finite part of the $\overline{\text{MS}}$[NDR] amplitudes defined in Secs. III C and III D in a compact and instructive way in Sec. IV A. We then discuss the general matching procedure in Sec. IV B before giving the conversion matrices for different RI/MOM and RI/SMOM schemes in Secs. IV C and IV D.

A. A compact expression for the amplitudes

The RI schemes are defined by applying projectors P_{nj} in spinor, color, and flavor space to the renormalized amputated Green's functions $\Gamma_n^y(O_i^{\text{RI}})$, see Eqs. (38), (47), and (51), and thus to $\Gamma_n^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}})$, see Eqs. (35) and (46). In this section we calculate $\Gamma_2^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}})$, $\Gamma_4^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}})$, and $\Gamma_{4p}^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}})$ as defined in Secs. III C and III D. The diagrams are generated with the program QGRAF [52], and the symbolic manipulations are carried out using FORM [53]. We give results for the respective exceptional momentum configuration of the RI/MOM scheme as well as the non-exceptional momentum configuration of the RI/SMOM schemes. We consider the operators $O_i^{\overline{\text{MS}}}$ of basis I, II, and the chiral basis.

We simplify the $\overline{\text{MS}}$-renormalized amplitudes using the decomposition [30]

$$\gamma_\mu \gamma_\nu \gamma_\rho = g_{\mu\nu}\gamma_\rho + g_{\nu\rho}\gamma_\mu - g_{\mu\rho}\gamma_\nu + i\varepsilon_{\alpha\mu\nu\rho}\gamma^\alpha\gamma_5 \tag{67}$$

and the Fierz identities

$$\begin{aligned} [\not{q}(1-\gamma_5)]_{ij}[\not{q}(1-\gamma_5)]_{kl} &= [\not{q}(1-\gamma_5)]_{il}[\not{q}(1-\gamma_5)]_{kj} \\ &\quad - (q^2/2)[\gamma_\rho(1-\gamma_5)]_{il}[\gamma^\rho(1-\gamma_5)]_{kj}\,, \end{aligned} \tag{68}$$

$$\begin{aligned} [\gamma_\rho(1-\gamma_5)]_{ij}[\gamma^\rho(1-\gamma_5)]_{kl} &= \\ &\quad - [\gamma_\rho(1-\gamma_5)]_{il}[\gamma^\rho(1-\gamma_5)]_{kj}\,. \end{aligned} \tag{69}$$

The resulting one-loop expressions are written as

$$\begin{aligned} \Gamma_4^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}}) &= \Gamma_4^{\text{tree}}(O_i) + \frac{\alpha_s}{4\pi}\Big(\Upsilon_{ij}^{\gamma}\Gamma_4^{\text{tree}}(O_j) \\ &\quad + \sum_{k=q,p_1,p_2} \Upsilon_{ij}^{\not{k}}\Gamma_4^{\text{tree}}(X_j^{\not{k}}) \\ &\quad + \Upsilon_{ij}^{\varepsilon}\Gamma_4^{\text{tree}}(Y_j) + \Upsilon_i^{G_1}\Gamma_4^{\text{tree}}(G_1)\Big)\,, \end{aligned} \tag{70}$$

$$\Gamma_2^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}}) = \frac{\alpha_s}{4\pi}\Upsilon_i^{G_1}\Gamma_2^{\text{tree}}(G_1)\,, \tag{71}$$

$$\Gamma_{4p}^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}}) = \frac{\alpha_s}{4\pi}\Upsilon_i^{G_1}\Gamma_{4p}^{\text{tree}}(G_1)\,, \tag{72}$$

where $\Gamma_n^{\text{tree}}(O_j)$ denotes the insertion of the operator O_j of basis I, II or the chiral basis at tree level and the sum over repeated indices is implied. The operators $X_j^{\not{k}}$ are obtained from the operators O_j by replacing the gamma structure $\gamma_\mu \otimes \gamma^\mu$ with $\not{k} \otimes \not{k}/k^2$, i.e.,

$$X_1^{\not{q}} = (\bar{s}_a \not{q}(1-\gamma_5)u_b)(\bar{u}_b \not{q}(1-\gamma_5)d_a)/q^2 \tag{73}$$

for the case of Q_1 and $k = q$ and analog for all other O_j and $k \in \{q, p_1, p_2\}$. Similarly the operators Y_j are obtained from the operators O_j by replacing the gamma structure $\gamma_\mu \otimes \gamma^\mu$ with $i\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu \otimes \gamma^\nu p_1^\alpha p_2^\beta / q^2$, i.e.,

$$Y_1 = i\varepsilon_{\mu\nu\alpha\beta}(\bar{s}_a \gamma^\mu (1-\gamma_5) u_b) \times (\bar{u}_b \gamma^\nu (1-\gamma_5) d_a) p_1^\alpha p_2^\beta / q^2 \quad (74)$$

for the case of Q_1 and analog for all other O_j. In Tabs. II–IV we give the coefficients $\Upsilon^z_{\dots}$ with $z \in \{\gamma, \not{q}, \not{p}_1, \not{p}_2, \varepsilon, G_1\}$ at one loop for the chiral basis and the exceptional and non-exceptional momentum configuration. The corresponding results for basis I and II can be obtained from these tables using Eq. (60). The constant C_0 is defined as

$$C_0 = (2/3)\Psi'(1/3) - (2\pi/3)^2 \approx 2.34391\,, \quad (75)$$

where $\Psi(x)$ is the digamma function [54].

In order to determine the conversion factors from the RI schemes to the $\overline{\text{MS}}$[NDR] scheme we need to study projected Green's functions

$$\Lambda_{nst} = P_{nt} \Gamma_n^{\overline{\text{MS}}}(O_s^{\overline{\text{MS}}})\,, \quad (76)$$

where $n \in \{2, 4, 4p\}$ and both P_{nt} and $\Gamma_n^{\overline{\text{MS}}}$ have open spinor, color, and flavor indices. The action of P_{nt} on $\Gamma_n^{\overline{\text{MS}}}$ for $n \in \{4, 4p\}$ is thus given by

$$\Lambda_{nst} = P_{nt}^{\alpha\beta\gamma\delta;ijkl;f} \Gamma^{\overline{\text{MS}}}_{n;\alpha\beta\gamma\delta;ijkl;f}(O_s^{\overline{\text{MS}}}) \quad (77)$$

with spinor indices $\alpha, \beta, \gamma, \delta$, color indices i, j, k, l, and flavor index f, as shown in Fig. 3. Similarly

$$\Lambda_{2st} = P_{2t}^{\alpha\beta;\mu;a;ij} \Gamma^{\overline{\text{MS}}}_{2;\alpha\beta;\mu;a;ij}(O_s^{\overline{\text{MS}}})\,, \quad (78)$$

where μ is a Lorentz index and $a \in \{1, \dots, N_c^2 - 1\}$ enumerates the generators of the SU(N_c) algebra, see Fig. 5. The summation over repeated indices is implied. The resulting expression Λ_{nst} naturally depends on the choice of the projectors P_{nt}. In the following we make certain assumptions about the structure of the projectors, i.e., we discuss different classes of projectors. For each class of projectors we can then give a very compact form of the amputated Green's function that can, without loss of generality, be used to calculate all projections Λ of the respective class.

First we restrict the discussion to projectors which contain no external momentum, except for at most the external momentum

$$q = p_1 - p_2\,.$$

We also do not consider projectors that contain vectors such as $\delta_{\mu 1}$, $\delta_{\mu 2}$, $\delta_{\mu 3}$, or $\delta_{\mu 4}$ which break Lorentz symmetry explicitly. The corresponding class of projectors shall be denoted by $P^{\not{q}}$. We will also study the sub-class $P^{\gamma_\mu} \subset P^{\not{q}}$ of projectors that additionally do not contain the external momentum q.

Let us focus on the amputated Green's functions with four external quarks and examine the projection of a spinor structure such as $\Omega_1 \otimes \Omega_2$ under projectors of class $P^{\not{q}}$, i.e.,

$$P^{\not{q}}[\Omega_1 \otimes \Omega_2]\,,$$

where the two factors belong to the two fermion lines, and the combination $\Omega_1 \otimes \Omega_2$ contains strings of gamma matrices without remaining open Lorentz indices. Then the ansatz

$$P^{\not{q}}[\gamma_\mu \Omega_1 \otimes \gamma_\nu \Omega_2] = t_1 g_{\mu\nu} + t_2 q_\mu q_\nu / q^2 \quad (79)$$

with Lorentz indices μ and ν is justified by the Lorentz-transformation properties of both sides since $P^{\not{q}}$ does not contain any Lorentz vectors apart from the momentum q. The coefficients t_1 and t_2 can be determined by contraction with $g_{\mu\nu}$ and $q_\mu q_\nu$, respectively. We find

$$\begin{aligned} P^{\not{q}}[\gamma_\mu \Omega_1 \otimes \gamma_\nu \Omega_2] = & \\ + \frac{q_\mu q_\nu}{3q^2} & \left(-P^{\not{q}}[\gamma_\rho \Omega_1 \otimes \gamma^\rho \Omega_2] + 4P^{\not{q}}[\not{q}\Omega_1 \otimes \not{q}\Omega_2]/q^2 \right) \\ + \frac{g_{\mu\nu}}{3} & \left(P^{\not{q}}[\gamma_\rho \Omega_1 \otimes \gamma^\rho \Omega_2] - P^{\not{q}}[\not{q}\Omega_1 \otimes \not{q}\Omega_2]/q^2 \right). \end{aligned} \quad (80)$$

Therefore not all of the spinor structures that appear in the general $\overline{\text{MS}}$ amplitude of Eq. (70) are independent under projection with $P^{\not{q}}$, i.e.,

$$\begin{aligned} P^{\not{q}}[\Gamma_n^{\text{tree}}(X_j^{\not{p}_1})] = P^{\not{q}}[\Gamma_n^{\text{tree}}(X_j^{\not{p}_2})] = \frac{1}{4} P^{\not{q}}[\Gamma_n^{\text{tree}}(O_j)]\,, \\ P^{\not{q}}[\Gamma_n^{\text{tree}}(Y_j)] = 0 \end{aligned} \quad (81)$$

with $n \in \{4, 4p\}$. The resulting projected amputated Green's functions for $n = 4$ up to one-loop order can be written as

$$\begin{aligned} P^{\not{q}}[\Gamma_4^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}})] &= P^{\not{q}}[\Gamma_4^{\text{tree}}(O_i)] \\ &+ \frac{\alpha_s}{4\pi}\Big((\Upsilon^\gamma_{ij} + \frac{1}{4}\Upsilon^{\not{p}_1}_{ij} + \frac{1}{4}\Upsilon^{\not{p}_2}_{ij}) P^{\not{q}}[\Gamma_4^{\text{tree}}(O_j)] \\ &+ \Upsilon^{\not{q}}_{ij} P^{\not{q}}[\Gamma_4^{\text{tree}}(X_j)] + \Upsilon^{G_1}_i P^{\not{q}}[\Gamma_4^{\text{tree}}(G_1)] \Big)\,. \end{aligned} \quad (82)$$

If we further restrict the discussion to projectors of the sub-class $P^{\gamma_\mu} \subset P^{\not{q}}$, we find

$$\begin{aligned} P^{\gamma_\mu}[\Gamma_n^{\text{tree}}(X_j^{\not{q}})] &= \frac{1}{4} P^{\gamma_\mu}[\Gamma_n^{\text{tree}}(O_j)]\,, \\ P^{\gamma_\mu}[\Gamma_n^{\text{tree}}(G_1)] &= \frac{3}{4} P^{\gamma_\mu}[\Gamma_n^{\text{tree}}(Q_p)] \end{aligned} \quad (83)$$

with $n \in \{4, 4p\}$ using the same arguments as in Eq. (79) without the term proportional to t_2. Therefore, for projectors of class P^{γ_μ} and $n = 4$ we can write

$$\begin{aligned} P^{\gamma_\mu}[\Gamma_4^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}})] = P^{\gamma_\mu}[\Gamma_4^{\text{tree}}(O_i)] + \frac{\alpha_s}{4\pi}\Big((\Upsilon^\gamma_{ij} + \frac{1}{4}\Upsilon^{\not{p}_1}_{ij} \\ + \frac{1}{4}\Upsilon^{\not{p}_2}_{ij} + \frac{1}{4}\Upsilon^{\not{q}}_{ij} + \frac{3}{4}\Upsilon^{G_1}_i \tau_j) P^{\gamma_\mu}[\Gamma_4^{\text{tree}}(O_j)] \Big) \end{aligned} \quad (84)$$

with $\Gamma_4^{\text{tree}}(Q_p) = \tau_j \Gamma_4^{\text{tree}}(O_j)$, where we can read off the coefficients τ_j from Eq. (14) for basis I and II or from Eq. (29) for the chiral basis.

(i,j)	Υ^{γ}_{ij}	$\Upsilon^{\not p_1}_{ij}$	$\Upsilon^{\not q}_{ij}$	$\Upsilon^{\varepsilon}_{ij}$
$(1,1)$	$\xi\big(-\frac{4\log(2)}{N_c}+4\log(2)+N_c+\frac{1}{N_c}-1\big)-\frac{12\log(2)}{N_c}+12\log(2)+\frac{7}{N_c}-7$	$(-2N_c-2)\xi$	0	0
$(2,2)$	$\xi\big(-\frac{4\log(2)}{N_c}+N_c+\frac{1}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{7}{N_c}$	$-2N_c\xi$	0	0
$(2,3)$	$(4\log(2)-1)\xi+12\log(2)-7$	-2ξ	0	0
$(3,2)$	$(4\log(2)-1)\xi+12\log(2)-7$	-2ξ	0	0
$(3,3)$	$\xi\big(-\frac{4\log(2)}{N_c}+N_c+\frac{1}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{7}{N_c}$	$-2N_c\xi$	0	0
$(5,5)$	$\xi\big(-\frac{4\log(2)}{3N_c}+N_c-\frac{5}{3N_c}\big)-\frac{4\log(2)}{3N_c}-\frac{5}{3N_c}$	$\xi\big(-\frac{8\log(2)}{3N_c}-2N_c+\frac{8}{3N_c}\big)-\frac{8\log(2)}{3N_c}-\frac{4}{3N_c}$	0	0
$(5,6)$	$\big(\frac{4\log(2)}{3}+\frac{2}{3}\big)\xi+\frac{4\log(2)}{3}+\frac{5}{3}$	$\big(\frac{8\log(2)}{3}-\frac{2}{3}\big)\xi+\frac{8\log(2)}{3}+\frac{4}{3}$	0	0
$(6,5)$	$\big(\frac{4\log(2)}{3}-\frac{1}{3}\big)\xi+\frac{4\log(2)}{3}-\frac{7}{3}$	$\big(\frac{8\log(2)}{3}-\frac{8}{3}\big)\xi+\frac{8\log(2)}{3}+\frac{4}{3}$	0	0
$(6,6)$	$\xi\big(-\frac{4\log(2)}{3N_c}+2N_c-\frac{5}{3N_c}\big)-\frac{4\log(2)}{3N_c}+4N_c-\frac{5}{3N_c}$	$\xi\big(\frac{8}{3N_c}-\frac{8\log(2)}{3N_c}\big)-\frac{8\log(2)}{3N_c}-\frac{4}{3N_c}$	0	0

TABLE II. One-loop coefficients Υ^{γ}, $\Upsilon^{\not p_1}$, $\Upsilon^{\not q}$, and Υ^{ε} for the exceptional momentum configuration and the chiral basis. The coefficient $\Upsilon^{\not p_2}_{ij}=0$ for the exceptional momentum configuration. The coefficients for Q_7 and Q_8 are identical to the coefficients for Q_5 and Q_6. The remaining matrix elements not given here are zero.

(i,j)	Υ^{γ}_{ij}	$\Upsilon^{\not p_1}_{ij}$, $\Upsilon^{\not p_2}_{ij}$	$\Upsilon^{\not q}_{ij}$	$\Upsilon^{\varepsilon}_{ij}$
$(1,1)$	$\xi\big(-\frac{2C_0N_c}{3}+\frac{C_0}{N_c}-\frac{2C_0}{3}-\frac{4\log(2)}{N_c}+4\log(2)+2N_c-\frac{1}{N_c}\big)+\frac{C_0N_c}{3}+\frac{C_0}{3}-\frac{12\log(2)}{N_c}+12\log(2)-2N_c+\frac{9}{N_c}-9$	$-\frac{2C_0N_c}{3}-\frac{2C_0}{3}+\big(-\frac{2N_c}{3}-\frac{2}{3}\big)\xi+\frac{4N_c}{3}+\frac{4}{3}$	$\xi\big(\frac{2C_0N_c}{3}+\frac{2C_0}{3}-\frac{2N_c}{3}-\frac{2}{3}\big)+\frac{4N_c}{3}+\frac{4}{3}$	0
$(2,2)$	$\xi\big(-\frac{2C_0N_c}{3}+\frac{C_0}{N_c}-\frac{4\log(2)}{N_c}+2N_c-\frac{1}{N_c}\big)+\frac{C_0N_c}{3}-\frac{12\log(2)}{N_c}-2N_c+\frac{9}{N_c}$	$-\frac{2C_0N_c}{3}-\frac{2N_c\xi}{3}+\frac{4N_c}{3}$	$\xi\big(\frac{2C_0N_c}{3}-\frac{2N_c}{3}\big)+\frac{4N_c}{3}$	0
$(2,3)$	$\xi\big(4\log(2)-\frac{2C_0}{3}\big)+\frac{C_0}{3}+12\log(2)-9$	$-\frac{2C_0}{3}-\frac{2\xi}{3}+\frac{4}{3}$	$\big(\frac{2C_0}{3}-\frac{2}{3}\big)\xi+\frac{4}{3}$	0
$(3,2)$	$\xi\big(4\log(2)-\frac{2C_0}{3}\big)+\frac{C_0}{3}+12\log(2)-9$	$-\frac{2C_0}{3}-\frac{2\xi}{3}+\frac{4}{3}$	$\big(\frac{2C_0}{3}-\frac{2}{3}\big)\xi+\frac{4}{3}$	0
$(3,3)$	$\xi\big(-\frac{2C_0N_c}{3}+\frac{C_0}{N_c}-\frac{4\log(2)}{N_c}+2N_c-\frac{1}{N_c}\big)+\frac{C_0N_c}{3}-\frac{12\log(2)}{N_c}-2N_c+\frac{9}{N_c}$	$-\frac{2C_0N_c}{3}-\frac{2N_c\xi}{3}+\frac{4N_c}{3}$	$\xi\big(\frac{2C_0N_c}{3}-\frac{2N_c}{3}\big)+\frac{4N_c}{3}$	0
$(5,5)$	$\xi\big(-\frac{2C_0N_c}{3}+\frac{7C_0}{6N_c}-\frac{4\log(2)}{3N_c}+2N_c-\frac{8}{3N_c}\big)+\frac{C_0N_c}{3}+\frac{7C_0}{6N_c}-\frac{4\log(2)}{3N_c}-2N_c+\frac{1}{3N_c}$	$-\frac{2C_0N_c}{3}+\frac{2C_0}{3N_c}+\xi\big(-\frac{4\log(2)}{3N_c}-\frac{2N_c}{3}+\frac{1}{N_c}\big)-\frac{4\log(2)}{3N_c}+\frac{4N_c}{3}-\frac{2}{N_c}$	$\xi\big(\frac{2C_0N_c}{3}-\frac{2C_0}{3N_c}-\frac{2N_c}{3}+\frac{2}{3N_c}\big)+\frac{4N_c}{3}-\frac{4}{3N_c}$	$-\frac{C_0\xi}{3N_c}-\frac{2C_0N_c}{3}+\frac{C_0}{N_c}$
$(5,6)$	$\xi\big(-\frac{C_0}{2}+\frac{4\log(2)}{3}+\frac{2}{3}\big)-\frac{3C_0}{2}+\frac{4\log(2)}{3}+\frac{5}{3}$	$\big(\frac{4\log(2)}{3}-\frac{1}{3}\big)\xi+\frac{4\log(2)}{3}+\frac{2}{3}$	0	$\frac{C_0\xi}{3}-\frac{C_0}{3}$
$(6,5)$	$\xi\big(-\frac{2C_0}{3}+\frac{4\log(2)}{3}+\frac{2}{3}\big)+\frac{C_0}{3}+\frac{4\log(2)}{3}-\frac{13}{3}$	$-\frac{2C_0}{3}+\big(\frac{4\log(2)}{3}-1\big)\xi+\frac{4\log(2)}{3}+2$	$\big(\frac{2C_0}{3}-\frac{2}{3}\big)\xi+\frac{4}{3}$	$-\frac{2C_0}{3}$
$(6,6)$	$\xi\big(-\frac{C_0N_c}{2}+\frac{7C_0}{6N_c}-\frac{4\log(2)}{3N_c}+2N_c-\frac{8}{3N_c}\big)-\frac{3C_0N_c}{2}+\frac{7C_0}{6N_c}-\frac{4\log(2)}{3N_c}+4N_c+\frac{1}{3N_c}$	$\frac{2C_0}{3N_c}+\xi\big(\frac{1}{N_c}-\frac{4\log(2)}{3N_c}\big)-\frac{4\log(2)}{3N_c}-\frac{2}{N_c}$	$\xi\big(\frac{2}{3N_c}-\frac{2C_0}{3N_c}\big)-\frac{4}{3N_c}$	$\xi\big(\frac{C_0N_c}{3}-\frac{C_0}{3N_c}\big)-\frac{C_0N_c}{3}+\frac{C_0}{N_c}$

TABLE III. One-loop coefficients Υ^{γ}, $\Upsilon^{\not p_1}$, $\Upsilon^{\not p_2}$, $\Upsilon^{\not q}$, and Υ^{ε} for the non-exceptional momentum configuration and the chiral basis. The coefficient $\Upsilon^{\not p_1}_{ij}=\Upsilon^{\not p_2}_{ij}$ for the exceptional momentum configuration. The coefficients for Q_7 and Q_8 are identical to the coefficients for Q_5 and Q_6. The remaining matrix elements not given here are zero.

$\Upsilon^{G_1}_1$	$\Upsilon^{G_1}_2$	$\Upsilon^{G_1}_3$	$\Upsilon^{G_1}_5$	$\Upsilon^{G_1}_6$	$\Upsilon^{G_1}_7$	$\Upsilon^{G_1}_8$
0	0	$-\frac{2}{9}$	0	$-\frac{5}{3}$	0	0

TABLE IV. One-loop coefficients Υ^{G_1} for the chiral basis. The results are identical for the exceptional and non-exceptional momentum configuration.

B. The matching procedure

In the following sections we give the one-loop conversion coefficients from different RI schemes to the $\overline{\text{MS}}$[NDR] scheme. We first explain the general procedure of calculating the conversion coefficients and then give results for explicit RI/MOM and RI/SMOM schemes.

At one loop the conversion from RI schemes to the $\overline{\text{MS}}$[NDR] scheme in terms of renormalized amputated Green's functions reads

$$C^y\Gamma_n^{\overline{\text{MS}}}(O_i^{\overline{\text{MS}}}) = \Gamma_n^y(O_i^{\text{RI}}) + \Delta a_{ij}^{\text{RI}\to\overline{\text{MS}}}\Gamma_n^y(O_j^{\text{RI}}) + \Delta c_i^{\text{RI}\to\overline{\text{MS}}}\Gamma_n^y(G_1^{\text{RI}}) \tag{85}$$

with wave function conversion factor $C^y=(C_q^y)^2$ for $n\in\{4,4p\}$ and $C^y=C_q^y(Z_A^{\overline{\text{MS}}}/Z_A^y)^{1/2}$ for $n=2$, see Eq. (53). In order to define the RI scheme, and hence to determine the coefficients $\Delta a_{ij}^{\text{RI}\to\overline{\text{MS}}}$ and $\Delta c_i^{\text{RI}\to\overline{\text{MS}}}$, we apply projectors P_{nl} to Eq. (85) and use the RI conditions of Eqs. (38) and (47) for the RI/SMOM$_2$ schemes and of Eqs. (38) and (51) for the RI/SMOM schemes, see Sec. III D.

From Eqs. (71) and (72) and the RI conditions of

Eqs. (47) and (51) we can already conclude that

$$\Delta c^{\mathrm{RI}\to\overline{\mathrm{MS}}} = \frac{\alpha_s}{4\pi}\Upsilon^{G_1} \tag{86}$$

is the only possible result at one loop. For convenience we provide a specific projector to determine the mixing of the two-quark operator G_1. In an RI/SMOM$_2$ scheme, we can use

$$P_{2,G_1} = \lambda^a_{ji}(\gamma_\mu)_{\beta\alpha}\,, \tag{87}$$

where λ^a is a generator of the SU(N_c) group algebra and the convention of open indices is given in Fig. 5. In an RI/SMOM scheme, we can use

$$P_{4p,G_1} = \delta_{il}\delta_{kj}(\gamma^\mu)_{\beta\alpha}(\gamma_\mu)_{\delta\gamma}\,. \tag{88}$$

At higher loops the additional operators [35]

$$\begin{aligned} G_2 &= \frac{4}{ig^2}\bar{s}\{D_\mu D^\mu, D^\nu\}\gamma_\nu(1-\gamma_5)d\,, \qquad (89)\\ G_3 &= \frac{4}{ig^2}\bar{s}D_\mu D_\nu D_\lambda S^{\mu\nu\lambda}(1-\gamma_5)d\,,\\ G_4 &= \frac{4}{ig^2}\bar{s}[D_\mu D^\mu, D^\nu]\gamma_\nu(1-\gamma_5)d\,, \end{aligned}$$

with $S^{\mu\nu\lambda} = \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\lambda\gamma^\nu\gamma^\mu$ and possibly other non-gauge-invariant two-quark operators N_m mix with the $(8,1)$ operators. We therefore need to provide additional renormalization conditions at higher loops in order to define the RI schemes for the $(8,1)$ operators uniquely, i.e., we need to specify additional projectors. The specific choice of projectors to determine the mixing with the operator G_2 and additional operators which occur at higher loops is not relevant for the one-loop conversion factors given in this work. At one loop the conversion factors can thus be given without specifying the details of higher-loop contributions.

In the following we use the RI′/MOM, RI/SMOM, and RI/SMOM$_{\gamma_\mu}$ wave function renormalizations defined in Refs. [14, 16, 19]. The corresponding conversion factors up to one-loop order

$$C^y_q = 1 + \frac{\alpha_s}{4\pi}\Delta^y_q \tag{90}$$

are given by

$$\Delta_q^{\mathrm{RI}'/\mathrm{MOM}} = \Delta_q^{\mathrm{RI/SMOM}} = -\frac{\xi}{2}\left(N_c - \frac{1}{N_c}\right) \tag{91}$$

for the RI′/MOM and RI/SMOM wave function renormalization schemes as well as

$$\Delta_q^{\mathrm{RI/SMOM}_{\gamma_\mu}} = -\frac{1}{2}\left(N_c - \frac{1}{N_c}\right)\left(-1 + \frac{\xi}{2}(3 - C_0)\right) \tag{92}$$

for the RI/SMOM$_{\gamma_\mu}$ scheme.

C. The RI/MOM and RI/SMOM(γ_μ, y) schemes

We first give the conversion coefficients for RI schemes that either (i) use only projectors of class P^{γ_μ} or (ii) use the exceptional momentum configuration and projectors of class $P^{\not{q}}$. The coefficients $\Delta c^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ are already determined in Eq. (86), and it remains to obtain the coefficients $\Delta a^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ by applying projectors P_{4k} to the amputated Green's function $\Gamma_4^{\overline{\mathrm{MS}}}$.

In case (i) the amputated Green's function $\Gamma_4^{\overline{\mathrm{MS}}}$ simplifies to

$$\begin{aligned} P^{\gamma_\mu}[(C^y_q)^2\Gamma_4^{\overline{\mathrm{MS}}}(O_i^{\overline{\mathrm{MS}}})] &= P^{\gamma_\mu}[\Gamma_4^{\mathrm{tree}}(O_i)] + \frac{\alpha_s}{4\pi}\Big((\Upsilon^\gamma_{ij}\\ &+ \frac{1}{4}\Upsilon^{\not{p}_1}_{ij} + \frac{1}{4}\Upsilon^{\not{p}_2}_{ij} + \frac{1}{4}\Upsilon^{\not{q}}_{ij} + 2\Delta^y_q\delta_{ij} + \frac{3}{4}\Upsilon^{G_1}_i\tau_j)\\ &\times P^{\gamma_\mu}[\Gamma_4^{\mathrm{tree}}(O_j)]\Big)\,, \end{aligned} \tag{93}$$

under projectors $P_{4k} \in P^{\gamma_\mu}$, see Eq. (84). This should be compared to Eq. (85), i.e.,

$$\begin{aligned} P^{\gamma_\mu}[(C^y_q)^2\Gamma_4^{\overline{\mathrm{MS}}}(O_i^{\overline{\mathrm{MS}}})] &= P^{\gamma_\mu}[\Gamma_4^y(O_i^{\mathrm{RI}})]\\ &+ \Big(\Delta a_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}} + \frac{3}{4}\Delta c_i^{\mathrm{RI}\to\overline{\mathrm{MS}}}\tau_j\Big)P^{\gamma_\mu}[\Gamma_4^y(O_j^{\mathrm{RI}})]\,, \end{aligned} \tag{94}$$

where we used Eqs. (83). If we impose the RI condition of Eq. (38) and insert Eq. (86), we find

$$\begin{aligned} \Delta a^{\mathrm{RI}\to\overline{\mathrm{MS}}} = \frac{\alpha_s}{4\pi}\Big(&\Upsilon^\gamma + \frac{1}{4}(\Upsilon^{\not{p}_1} + \Upsilon^{\not{p}_2} + \Upsilon^{\not{q}})\\ &+ 2\Delta^y_q\,\mathbb{1}\Big) \end{aligned} \tag{95}$$

with identity matrix $\mathbb{1}$.

In case (ii) we have $\Upsilon^{\not{q}} = 0$, and therefore Eq. (82) simplifies to

$$\begin{aligned} P^{\not{q}}[\Gamma_4^{\overline{\mathrm{MS}}}(O_i^{\overline{\mathrm{MS}}})] &= P^{\not{q}}[\Gamma_4^{\mathrm{tree}}(O_i)] + \frac{\alpha_s}{4\pi}\Big((\Upsilon^\gamma_{ij} + \frac{1}{4}\Upsilon^{\not{p}_1}_{ij}\\ &+ \frac{1}{4}\Upsilon^{\not{p}_2}_{ij})P^{\not{q}}[\Gamma_4^{\mathrm{tree}}(O_j)] + \Upsilon_i^{G_1}P^{\not{q}}[\Gamma_4^{\mathrm{tree}}(G_1)]\Big)\,. \end{aligned} \tag{96}$$

One can read off the conversion matrix $\Delta a^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ by comparing Eq. (85) to Eq. (96) with

$$\Delta a^{\mathrm{RI}\to\overline{\mathrm{MS}}} = \frac{\alpha_s}{4\pi}\Big(\Upsilon^\gamma + \frac{1}{4}\Upsilon^{\not{p}_1} + \frac{1}{4}\Upsilon^{\not{p}_2} + 2\Delta^y_q\,\mathbb{1}\Big)\,. \tag{97}$$

In both cases the conversion coefficients are unique up to the definition of the RI quark wave function renormalization Δ^y_q. In other words, in these cases the details of the projectors do not matter, as long as one uses a sufficient number of independent projectors to determine all elements of the conversion matrices or Z-factors, respectively.

The choice of the RI wave function renormalization scheme only affects the diagonal elements of $\Delta a^{\mathrm{RI}\to\overline{\mathrm{MS}}}$. One can also combine different RI wave function renormalization schemes with different RI schemes for the operators O_i in a straightforward way using Eq. (95).

We call the RI scheme of case (i) RI/SMOM(γ_μ, y) scheme, where $y = \not{q}$ corresponds to the RI/SMOM wave function renormalization scheme and $y = \gamma_\mu$ corresponds to the RI/SMOM$_{\gamma_\mu}$ wave function renormalization scheme. The name RI/SMOM(γ_μ, y) reflects that we restrict the choice of projectors to the class P^{γ_μ}.

The scheme corresponding to case (ii) is the RI/MOM scheme, and we give results only for the RI′/MOM wave function renormalization.

In Tabs. V–VII we give the values for $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$, which is defined in Eq. (56), for the RI/MOM scheme with RI′/MOM wave function renormalization and operators of basis I, II, and the chiral basis. In Tabs. VIII and IX we present the results for $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the RI/SMOM($\gamma_\mu, \not{q}$) and the RI/SMOM(γ_μ, γ_μ) scheme and operators of the chiral basis.

We would like to point out that the result of Tab. V for $N_c = 3$ and $\xi = 0$ or $\xi = 1$ agrees with the result of Ref. [32] for the conversion from the RI/MOM scheme to the $\overline{\mathrm{MS}}$[NDR] scheme. In Ref. [30] the current-current contributions in the RI/MOM scheme have been considered. Our result for these contributions agrees with the one of Ref. [30], which can be seen by using the results of Eq. (60) and Tab. II and combining them with the quark wave function renormalization constant of Eq. (5.3) of Ref. [30].

Furthermore, the result for the (27, 1) operator Q_1' in Tabs. VIII and IX agrees with the result of Ref. [28] for the conversion from the RI/SMOM($\gamma_\mu, \not{q}$) and the RI/SMOM(γ_μ, γ_μ) scheme to the $\overline{\mathrm{MS}}$[NDR] scheme for the (VV+AA)–($\Delta S = 2$) operator.

D. Results for the RI/SMOM($\not{q}, y$) schemes

In the following we discuss the non-exceptional momentum configuration and RI/SMOM schemes defined by projectors of the class $P^{\not{q}}$. This allows for the definition of new and independent RI schemes which we call RI/SMOM($\not{q}, y$) schemes, where y denotes the choice of the wave function renormalization scheme as in the previous section. We give results only for the chiral basis which does not contain linearly dependent operators. Since mixing only occurs within the blocks of the (27, 1), (8, 1), and (8, 8) operators, we renormalize each block separately.

We first note that for a scheme with projectors of the class $P^{\not{q}}$ the conversion matrices cannot be simply read off from Tabs. II–IV due to the nonzero contribution of $\Upsilon^{\not{q}}$. This means that the schemes defined in this section, unlike the schemes discussed in the previous section IV C, depend on the specific choice of the projectors. However, any non-degenerate linear transformation of a given set of projectors leaves the conversion matrix of Eq. (34) invariant. We give the projectors used to define the RI/SMOM($\not{q}, y$) schemes in the following. The

(i,j)	$\Delta r_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$	$\Delta r^*_{ij}/\frac{\alpha_s}{4\pi}$
$(1,1)$	$\xi\big(-\frac{4\log(2)}{N_c}-\frac{N_c}{2}+\frac{2}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{7}{N_c}$	-0.43926
$(1,2)$	$\big(4\log(2)-\frac{3}{2}\big)\xi+12\log(2)-7$	1.31777
$(2,1)$	$\big(4\log(2)-\frac{3}{2}\big)\xi+12\log(2)-7$	1.31777
$(2,2)$	$\xi\big(-\frac{4\log(2)}{N_c}-\frac{N_c}{2}+\frac{2}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{7}{N_c}$	-0.43926
$(2,3)$	$\frac{2}{9N_c}$	0.07407
$(2,4)$	$-\frac{2}{9}$	-0.22222
$(2,5)$	$\frac{2}{9N_c}$	0.07407
$(2,6)$	$-\frac{2}{9}$	-0.22222
$(3,3)$	$\xi\big(-\frac{4\log(2)}{N_c}-\frac{N_c}{2}+\frac{2}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{67}{9N_c}$	-0.29111
$(3,4)$	$\big(4\log(2)-\frac{3}{2}\big)\xi+12\log(2)-\frac{67}{9}$	0.87332
$(3,5)$	$\frac{4}{9N_c}$	0.14815
$(3,6)$	$-\frac{4}{9}$	-0.44444
$(4,3)$	$\big(4\log(2)-\frac{3}{2}\big)\xi+12\log(2)+\frac{5}{3N_c}-7$	1.87332
$(4,4)$	$\xi\big(-\frac{4\log(2)}{N_c}-\frac{N_c}{2}+\frac{2}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{7}{N_c}-\frac{5}{3}$	-2.10592
$(4,5)$	$\frac{5}{3N_c}$	0.55556
$(4,6)$	$-\frac{5}{3}$	-1.66667
$(5,5)$	$\xi\big(-\frac{2\log(2)}{N_c}-\frac{N_c}{2}\big)-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$	-1.12876
$(5,6)$	$\big(2\log(2)+\frac{1}{2}\big)\xi+2\log(2)+2$	3.38629
$(6,3)$	$\frac{5}{3N_c}$	0.55556
$(6,4)$	$-\frac{5}{3}$	-1.66667
$(6,5)$	$(2\log(2)-1)\xi+2\log(2)+\frac{5}{3N_c}-2$	-0.05815
$(6,6)$	$\xi\big(N_c-\frac{2\log(2)}{N_c}\big)-\frac{2\log(2)}{N_c}+4N_c-\frac{2}{N_c}-\frac{5}{3}$	9.20457
$(7,7)$	$\xi\big(-\frac{2\log(2)}{N_c}-\frac{N_c}{2}\big)-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$	-1.12876
$(7,8)$	$\big(2\log(2)+\frac{1}{2}\big)\xi+2\log(2)+2$	3.38629
$(8,7)$	$(2\log(2)-1)\xi+2\log(2)-2$	-0.61371
$(8,8)$	$\xi\big(N_c-\frac{2\log(2)}{N_c}\big)-\frac{2\log(2)}{N_c}+4N_c-\frac{2}{N_c}$	10.87124
$(9,3)$	$-\frac{2}{9N_c}$	-0.07407
$(9,4)$	$\frac{2}{9}$	0.22222
$(9,5)$	$-\frac{2}{9N_c}$	-0.07407
$(9,6)$	$\frac{2}{9}$	0.22222
$(9,9)$	$\xi\big(-\frac{4\log(2)}{N_c}-\frac{N_c}{2}+\frac{2}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{7}{N_c}$	-0.43926
$(9,10)$	$\big(4\log(2)-\frac{3}{2}\big)\xi+12\log(2)-7$	1.31777
$(10,9)$	$\big(4\log(2)-\frac{3}{2}\big)\xi+12\log(2)-7$	1.31777
$(10,10)$	$\xi\big(-\frac{4\log(2)}{N_c}-\frac{N_c}{2}+\frac{2}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{7}{N_c}$	-0.43926

TABLE V. One-loop conversion matrix $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for basis I in the RI/MOM scheme with RI′/MOM wave function renormalization. We also give numerical values Δr^* for $N_c = 3$ and $\xi = 0$.

(i,j)	$\Delta r_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$	$\Delta r^*_{ij}/\frac{\alpha_s}{4\pi}$
$(2,3)$	$\frac{5}{9N_c}$	0.18519
$(2,4)$	$-\frac{5}{9}$	-0.55556
$(2,5)$	$\frac{5}{9N_c}$	0.18519
$(2,6)$	$-\frac{5}{9}$	-0.55556

TABLE VI. One-loop conversion matrix $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for basis II in the RI/MOM scheme with RI′/MOM wave function renormalization. All elements not given here are identical to Tab. V.

(i,j)	$\Delta r_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$	$\Delta r^*_{ij}/\frac{\alpha_s}{4\pi}$
$(1,1)$	$\xi\big(-\frac{4\log(2)}{N_c}+4\log(2)-\frac{N_c}{2}+\frac{2}{N_c}-\frac{3}{2}\big)-\frac{12\log(2)}{N_c}+12\log(2)+\frac{7}{N_c}-7$	0.87851
$(2,2)$	$\xi\big(-\frac{4\log(2)}{N_c}-\frac{N_c}{2}+\frac{2}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{7}{N_c}$	−0.43926
$(2,3)$	$\big(4\log(2)-\frac{3}{2}\big)\xi+12\log(2)-7$	1.31777
$(3,2)$	$\big(4\log(2)-\frac{3}{2}\big)\xi+12\log(2)+\frac{2}{3N_c}-\frac{67}{9}$	1.09554
$(3,3)$	$\xi\big(-\frac{4\log(2)}{N_c}-\frac{N_c}{2}+\frac{2}{N_c}\big)-\frac{12\log(2)}{N_c}+\frac{67}{9N_c}-\frac{2}{3}$	−0.95777
$(3,5)$	$\frac{2}{9N_c}$	0.07407
$(3,6)$	$-\frac{2}{9}$	−0.22222
$(5,5)$	$\xi\big(-\frac{2\log(2)}{N_c}-\frac{N_c}{2}\big)-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$	−1.12876
$(5,6)$	$\big(2\log(2)+\frac{1}{2}\big)\xi+2\log(2)+2$	3.38629
$(6,2)$	$\frac{5}{N_c}-\frac{10}{3}$	−1.66667
$(6,3)$	$\frac{10}{3N_c}-5$	−3.88889
$(6,5)$	$(2\log(2)-1)\xi+2\log(2)+\frac{5}{3N_c}-2$	−0.05815
$(6,6)$	$\xi\big(N_c-\frac{2\log(2)}{N_c}\big)-\frac{2\log(2)}{N_c}+4N_c-\frac{2}{N_c}-\frac{5}{3}$	9.20457
$(7,7)$	$\xi\big(-\frac{2\log(2)}{N_c}-\frac{N_c}{2}\big)-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$	−1.12876
$(7,8)$	$\big(2\log(2)+\frac{1}{2}\big)\xi+2\log(2)+2$	3.38629
$(8,7)$	$(2\log(2)-1)\xi+2\log(2)-2$	−0.61371
$(8,8)$	$\xi\big(N_c-\frac{2\log(2)}{N_c}\big)-\frac{2\log(2)}{N_c}+4N_c-\frac{2}{N_c}$	10.87124

TABLE VII. One-loop conversion matrix $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the chiral basis in the RI/MOM scheme with RI′/MOM wave function renormalization. We also give numerical values Δr^* for $N_c = 3$ and $\xi = 0$.

projectors will be expressed in terms of

$$
\begin{aligned}
P^{VV\pm AA,\not{q}}_{(1),f'} &= \delta_{ff'}\delta_{ij}\delta_{kl} \\
&\quad\times [(\not{q})_{\beta\alpha}(\not{q})_{\delta\gamma} \pm (\not{q}\gamma_5)_{\beta\alpha}(\not{q}\gamma_5)_{\delta\gamma}]/q^2\,, \\
P^{VV\pm AA,\not{q}}_{(2),f'} &= \delta_{ff'}\delta_{il}\delta_{kj} \\
&\quad\times [(\not{q})_{\beta\alpha}(\not{q})_{\delta\gamma} \pm (\not{q}\gamma_5)_{\beta\alpha}(\not{q}\gamma_5)_{\delta\gamma}]/q^2\,, \\
P^{VV\pm AA,\not{q}}_{(3),f'} &= \delta_{ff'}\delta_{il}\delta_{kj} \\
&\quad\times [(\not{q})_{\beta\gamma}(\not{q})_{\delta\alpha} \pm (\not{q}\gamma_5)_{\beta\gamma}(\not{q}\gamma_5)_{\delta\alpha}]/q^2\,, \\
P^{VV\pm AA,\not{q}}_{(4),f'} &= \delta_{ff'}\delta_{ij}\delta_{kl} \\
&\quad\times [(\not{q})_{\beta\gamma}(\not{q})_{\delta\alpha} \pm (\not{q}\gamma_5)_{\beta\gamma}(\not{q}\gamma_5)_{\delta\alpha}]/q^2\,, \quad (98)
\end{aligned}
$$

where all indices are as shown in Fig. 3.

The $(27,1)$ operator Q'_1 only mixes with itself, so we only need to provide a single projector $P^{\not{q}}_{(27,1)}$. We adopt the RI/SMOM($\not{q},\not{q}$) and RI/SMOM($\not{q},\gamma_\mu$) schemes defined in Ref. [28] so that for the $(27,1)$ operator we use

$$
P^{\not{q}}_{(27,1),1} = \frac{P^{VV+AA,\not{q}}_{(1),u}}{64N_c(N_c+1)}\,. \quad (99)
$$

The $(8,8)$ operators Q'_7 and Q'_8 also only mix with themselves under renormalization, and therefore we have to provide two projectors. We choose

$$
\begin{aligned}
P^{\not{q}}_{(8,8),7} &= \frac{N_c P^{VV-AA,\not{q}}_{(1),u} - P^{VV-AA,\not{q}}_{(2),u}}{32N_c(N_c^2-1)}\,, \\
P^{\not{q}}_{(8,8),8} &= \frac{-P^{VV-AA,\not{q}}_{(1),u} + N_c P^{VV-AA,\not{q}}_{(2),u}}{32N_c(N_c^2-1)} \quad (100)
\end{aligned}
$$

with the property

$$
P^{\not{q}}_{(8,8),i}\Gamma_4^{\mathrm{tree}}(Q'_j) = \delta_{ij}\,, \quad i,j = 7,8\,. \quad (101)
$$

The one-loop mixing coefficients for the $(8,1)$ operators shall be determined using the projectors

$$
\begin{aligned}
P^{\not{q}}_{(8,1),2} &= \frac{(3N_c-2)P^{VV+AA,\not{q}}_{(1),u} + (2N_c-3)P^{VV+AA,\not{q}}_{(2),u}}{32N_c(N_c^2-1)}\,, \\
P^{\not{q}}_{(8,1),3} &= \frac{(2N_c-3)P^{VV+AA,\not{q}}_{(1),u} + (3N_c-2)P^{VV+AA,\not{q}}_{(2),u}}{32N_c(N_c^2-1)}\,,
\end{aligned}
$$

(i,j)	$\Delta r_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$	$\Delta r^*_{ij}/\frac{\alpha_s}{4\pi}$
$(1,1)$	$\xi\big(-\frac{C_0N_c}{2}+\frac{C_0}{N_c}-\frac{C_0}{2}-\frac{4\log(2)}{N_c}+4\log(2)+\frac{N_c}{2}-\frac{1}{2}\big)-\frac{12\log(2)}{N_c}+12\log(2)-N_c+\frac{9}{N_c}-8$	−2.45482
$(2,2)$	$\xi\big(-\frac{C_0N_c}{2}+\frac{C_0}{N_c}-\frac{4\log(2)}{N_c}+\frac{N_c}{2}\big)-\frac{12\log(2)}{N_c}-N_c+\frac{9}{N_c}$	−2.77259
$(2,3)$	$\xi\big(-\frac{C_0}{2}+4\log(2)-\frac{1}{2}\big)+12\log(2)-8$	0.31777
$(3,2)$	$\xi\big(-\frac{C_0}{2}+4\log(2)-\frac{1}{2}\big)+12\log(2)+\frac{2}{3N_c}-\frac{76}{9}$	0.09554
$(3,3)$	$\xi\big(-\frac{C_0N_c}{2}+\frac{C_0}{N_c}-\frac{4\log(2)}{N_c}+\frac{N_c}{2}\big)-\frac{12\log(2)}{N_c}-N_c+\frac{85}{9N_c}-\frac{2}{3}$	−3.29111
$(3,5)$	$\frac{2}{9N_c}$	0.07407
$(3,6)$	$-\frac{2}{9}$	−0.22222
$(5,5)$	$\xi\big(-\frac{C_0N_c}{2}+\frac{C_0}{N_c}-\frac{2\log(2)}{N_c}+\frac{N_c}{2}-\frac{1}{N_c}\big)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}-N_c-\frac{1}{N_c}$	−2.62348
$(5,6)$	$\xi\big(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\big)-\frac{3C_0}{2}+2\log(2)+2$	−0.12957
$(6,2)$	$\frac{5}{N_c}-\frac{10}{3}$	−1.66667
$(6,3)$	$\frac{10}{3N_c}-5$	−3.88889
$(6,5)$	$\xi\big(2\log(2)-\frac{C_0}{2}\big)+2\log(2)+\frac{5}{3N_c}-3$	−1.05815
$(6,6)$	$\xi\big(-\frac{C_0N_c}{2}+\frac{C_0}{N_c}-\frac{2\log(2)}{N_c}+N_c-\frac{1}{N_c}\big)-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+4N_c-\frac{1}{N_c}-\frac{5}{3}$	0.16227
$(7,7)$	$\xi\big(-\frac{C_0N_c}{2}+\frac{C_0}{N_c}-\frac{2\log(2)}{N_c}+\frac{N_c}{2}-\frac{1}{N_c}\big)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}-N_c-\frac{1}{N_c}$	−2.62348
$(7,8)$	$\xi\big(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\big)-\frac{3C_0}{2}+2\log(2)+2$	−0.12957
$(8,7)$	$\xi\big(2\log(2)-\frac{C_0}{2}\big)+2\log(2)-3$	−1.61371
$(8,8)$	$\xi\big(-\frac{C_0N_c}{2}+\frac{C_0}{N_c}-\frac{2\log(2)}{N_c}+N_c-\frac{1}{N_c}\big)-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+4N_c-\frac{1}{N_c}$	1.82894

TABLE VIII. One-loop conversion matrix $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the chiral basis in the RI/SMOM($\gamma_\mu,\not{q}$) scheme. We also give numerical values Δr^* for $N_c = 3$ and $\xi = 0$.

(i,j)	$\Delta r_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$	$\Delta r^*_{ij}/\frac{\alpha_s}{4\pi}$
$(1,1)$	$\xi\big(\frac{C_0}{2N_c}-\frac{C_0}{2}-\frac{4\log(2)}{N_c}+4\log(2)+\frac{1}{2N_c}-\frac{1}{2}\big)-\frac{12\log(2)}{N_c}+12\log(2)+\frac{8}{N_c}-8$	0.21184
$(2,2)$	$\xi\big(\frac{C_0}{2N_c}-\frac{4\log(2)}{N_c}+\frac{1}{2N_c}\big)-\frac{12\log(2)}{N_c}+\frac{8}{N_c}$	−0.10592
$(2,3)$	$\xi\big(-\frac{C_0}{2}+4\log(2)-\frac{1}{2}\big)+12\log(2)-8$	0.31777
$(3,2)$	$\xi\big(-\frac{C_0}{2}+4\log(2)-\frac{1}{2}\big)+12\log(2)+\frac{2}{3N_c}-\frac{76}{9}$	0.09554
$(3,3)$	$\xi\big(\frac{C_0}{2N_c}-\frac{4\log(2)}{N_c}+\frac{1}{2N_c}\big)-\frac{12\log(2)}{N_c}+\frac{76}{9N_c}-\frac{2}{3}$	−0.62444
$(3,5)$	$\frac{2}{9N_c}$	0.07407
$(3,6)$	$-\frac{2}{9}$	−0.22222
$(5,5)$	$\xi\big(\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{1}{2N_c}\big)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$	0.04319
$(5,6)$	$\xi\big(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\big)-\frac{3C_0}{2}+2\log(2)+2$	−0.12957
$(6,2)$	$\frac{5}{N_c}-\frac{10}{3}$	−1.66667
$(6,3)$	$\frac{10}{3N_c}-5$	−3.88889
$(6,5)$	$\xi\big(2\log(2)-\frac{C_0}{2}\big)+2\log(2)+\frac{5}{3N_c}-3$	−1.05815
$(6,6)$	$\xi\big(\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}+\frac{N_c}{2}-\frac{1}{2N_c}\big)-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+5N_c-\frac{2}{N_c}-\frac{5}{3}$	2.82894
$(7,7)$	$\xi\big(\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{1}{2N_c}\big)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$	0.04319
$(7,8)$	$\xi\big(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\big)-\frac{3C_0}{2}+2\log(2)+2$	−0.12957
$(8,7)$	$\xi\big(2\log(2)-\frac{C_0}{2}\big)+2\log(2)-3$	−1.61371
$(8,8)$	$\xi\big(\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}+\frac{N_c}{2}-\frac{1}{2N_c}\big)-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+5N_c-\frac{2}{N_c}$	4.49561

TABLE IX. One-loop conversion matrix $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the chiral basis in the RI/SMOM(γ_μ,γ_μ) scheme. We also give numerical values Δr^* for $N_c=3$ and $\xi=0$.

$$P^{\not{q}}_{(8,1),5}=P^{\not{q}}_{(8,8),7}\,,\qquad P^{\not{q}}_{(8,1),6}=P^{\not{q}}_{(8,8),8}\,, \tag{102}$$

where

$$P^{\not{q}}_{(8,1),i}\Gamma_4^{\mathrm{tree}}(Q'_j)=\delta_{ij}\,,\qquad i,j=2,3,5,6\,. \tag{103}$$

Other choices of projectors within class $P^{\not{q}}$, which are not linear combinations of the projectors given above, are possible and might lead to smaller conversion factors and a better convergence of the perturbative expansion. The reader can easily obtain the respective projections using the results which we provide in Tabs. II–IV.

In Tabs. X and XI we provide the resulting one-loop conversion matrices $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the RI/SMOM$(\not{q},\gamma_\mu)$ and RI/SMOM$(\not{q},\not{q})$ schemes. We note that the result for the $(27,1)$ operator agrees with the result of Ref. [28] for the conversion of the (VV+AA)–($\Delta S=2$) operator. In Tabs. XII–XIII we give the individual conversion matrices $\Delta a^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the $(8,1)$ operators in the same schemes.

(i,j)	$\Delta r_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$	$\Delta r^*_{ij}/\frac{\alpha_s}{4\pi}$
$(1,1)$	$\xi\big(\frac{C_0N_c}{2}+\frac{C_0}{2N_c}-C_0-\frac{4\log(2)}{N_c}+4\log(2)-\frac{N_c}{2}+\frac{1}{2N_c}\big)-\frac{12\log(2)}{N_c}+12\log(2)+N_c+\frac{8}{N_c}-9$	2.21184
$(2,2)$	$\xi\big(\frac{13C_0N_c}{10}+\frac{C_0}{2N_c}-\frac{6C_0}{5}-\frac{4\log(2)}{N_c}-\frac{13N_c}{10}+\frac{1}{2N_c}+\frac{6}{5}\big)-\frac{12\log(2)}{N_c}+\frac{13N_c}{5}+\frac{8}{N_c}-\frac{12}{5}$	5.29408
$(2,3)$	$\xi\big(\frac{6C_0N_c}{5}-\frac{9C_0}{5}+4\log(2)-\frac{6N_c}{5}+\frac{4}{5}\big)+12\log(2)+\frac{12N_c}{5}-\frac{53}{5}$	4.91777
$(3,2)$	$\xi\big(-\frac{6C_0N_c}{5}+\frac{4C_0}{5}+4\log(2)+\frac{6N_c}{5}-\frac{9}{5}\big)+12\log(2)-\frac{12N_c}{5}+\frac{2}{3N_c}-\frac{263}{45}$	−4.50446
$(3,3)$	$\xi\big(-\frac{13C_0N_c}{10}+\frac{C_0}{2N_c}+\frac{6C_0}{5}-\frac{4\log(2)}{N_c}+\frac{13N_c}{10}+\frac{1}{2N_c}-\frac{6}{5}\big)-\frac{12\log(2)}{N_c}-\frac{13N_c}{5}+\frac{76}{9N_c}+\frac{26}{15}$	−6.02444
$(3,5)$	$\frac{2}{9N_c}$	0.07407
$(3,6)$	$-\frac{2}{9}$	−0.22222
$(5,5)$	$\xi\big(\frac{C_0N_c}{2}-\frac{2\log(2)}{N_c}-\frac{N_c}{2}\big)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+N_c-\frac{3}{N_c}$	2.70986
$(5,6)$	$\xi\big(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\big)-\frac{3C_0}{2}+2\log(2)+2$	−0.12957
$(6,2)$	$\frac{5}{N_c}-\frac{10}{3}$	−1.66667
$(6,3)$	$\frac{10}{3N_c}-5$	−3.88889
$(6,5)$	$\big(2\log(2)-\frac{1}{2}\big)\xi+2\log(2)+\frac{5}{3N_c}-2$	−0.05815
$(6,6)$	$-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}+\xi\big(\frac{N_c}{2}-\frac{2\log(2)}{N_c}\big)-\frac{2\log(2)}{N_c}+5N_c-\frac{3}{N_c}-\frac{5}{3}$	2.49561
$(7,7)$	$\xi\big(\frac{C_0N_c}{2}-\frac{2\log(2)}{N_c}-\frac{N_c}{2}\big)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+N_c-\frac{3}{N_c}$	2.70986
$(7,8)$	$\xi\big(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\big)-\frac{3C_0}{2}+2\log(2)+2$	−0.12957
$(8,7)$	$\big(2\log(2)-\frac{1}{2}\big)\xi+2\log(2)-2$	−0.61371
$(8,8)$	$-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}+\xi\big(\frac{N_c}{2}-\frac{2\log(2)}{N_c}\big)-\frac{2\log(2)}{N_c}+5N_c-\frac{3}{N_c}$	4.16227

TABLE X. One-loop conversion matrix $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the chiral basis in the RI/SMOM$(\not{q},\gamma_\mu)$ scheme. We also give numerical values Δr^* for $N_c=3$ and $\xi=0$.

E. Discussion

In the previous sections we provided results for the conversion from different RI/MOM and RI/SMOM schemes to the $\overline{\mathrm{MS}}$[NDR] scheme. In general the different schemes have a different magnitude of one-loop corrections $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ and a different rate of convergence of the perturbative expansion of $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ in α_s. In the context of lattice QCD and non-perturbative renormalization it is thus useful to have several RI schemes to choose from in order to estimate the effects of missing higher-order terms in the perturbative expansion. In this work we give four different RI/SMOM schemes for each operator of the chiral $\Delta S=1$ operator basis.

Since we only give one-loop results, we cannot estimate the rate of convergence of the different schemes presented in this paper. In some cases where higher-order results in RI/SMOM schemes are known, such as the conver-

(i,j)	$\Delta r_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$	$\Delta r_{ij}^{*}/\frac{\alpha_s}{4\pi}$
$(1,1)$	$\xi\left(\frac{C_0}{N_c}-C_0-\frac{4\log(2)}{N_c}+4\log(2)\right)-\frac{12\log(2)}{N_c}+12\log(2)+\frac{9}{N_c}-9$	-0.45482
$(2,2)$	$\xi\left(\frac{4C_0N_c}{5}+\frac{C_0}{N_c}-\frac{6C_0}{5}-\frac{4\log(2)}{N_c}-\frac{4N_c}{5}+\frac{6}{5}\right)-\frac{12\log(2)}{N_c}+\frac{8N_c}{5}+\frac{9}{N_c}-\frac{12}{5}$	2.62741
$(2,3)$	$\xi\left(\frac{6C_0N_c}{5}-\frac{9C_0}{5}+4\log(2)-\frac{6N_c}{5}+\frac{4}{5}\right)+12\log(2)+\frac{12N_c}{5}-\frac{53}{5}$	4.91777
$(3,2)$	$\xi\left(-\frac{6C_0N_c}{5}+\frac{4C_0}{5}+4\log(2)+\frac{6N_c}{5}-\frac{9}{5}\right)+12\log(2)-\frac{12N_c}{5}+\frac{2}{3N_c}-\frac{263}{45}$	-4.50446
$(3,3)$	$\xi\left(-\frac{9C_0N_c}{5}+\frac{C_0}{N_c}+\frac{6C_0}{5}-\frac{4\log(2)}{N_c}+\frac{9N_c}{5}-\frac{6}{5}\right)-\frac{12\log(2)}{N_c}-\frac{18N_c}{5}+\frac{85}{9N_c}+\frac{26}{15}$	-8.69111
$(3,5)$	$\frac{2}{9N_c}$	0.07407
$(3,6)$	$-\frac{2}{9}$	-0.22222
$(5,5)$	$\xi\left(\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{1}{2N_c}\right)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$	0.04319
$(5,6)$	$\xi\left(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\right)-\frac{3C_0}{2}+2\log(2)+2$	-0.12957
$(6,2)$	$\frac{5}{N_c}-\frac{10}{3}$	-1.66667
$(6,3)$	$\frac{10}{3N_c}-5$	-3.88889
$(6,5)$	$\left(2\log(2)-\frac{1}{2}\right)\xi+2\log(2)+\frac{5}{3N_c}-2$	-0.05815
$(6,6)$	$\xi\left(-\frac{C_0N_c}{2}+\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}+N_c-\frac{1}{2N_c}\right)-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+4N_c-\frac{2}{N_c}-\frac{5}{3}$	-0.17106
$(7,7)$	$\xi\left(\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{1}{2N_c}\right)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$	0.04319
$(7,8)$	$\xi\left(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\right)-\frac{3C_0}{2}+2\log(2)+2$	-0.12957
$(8,7)$	$\left(2\log(2)-\frac{1}{2}\right)\xi+2\log(2)-2$	-0.61371
$(8,8)$	$\xi\left(-\frac{C_0N_c}{2}+\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}+N_c-\frac{1}{2N_c}\right)-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+4N_c-\frac{2}{N_c}$	1.49561

TABLE XI. One-loop conversion matrix $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the chiral basis in the RI/SMOM($\not{q},\not{q}$) scheme. We also give numerical values Δr^* for $N_c=3$ and $\xi=0$.

(i,j)	$\Delta a_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$
$(2,2)$	$\xi\left(\frac{13C_0N_c}{10}+\frac{C_0}{2N_c}-\frac{6C_0}{5}-\frac{4\log(2)}{N_c}-\frac{13N_c}{10}+\frac{1}{2N_c}+\frac{6}{5}\right)-\frac{12\log(2)}{N_c}+\frac{13N_c}{5}+\frac{8}{N_c}-\frac{12}{5}$
$(2,3)$	$\xi\left(\frac{6C_0N_c}{5}-\frac{9C_0}{5}+4\log(2)-\frac{6N_c}{5}+\frac{4}{5}\right)+12\log(2)+\frac{12N_c}{5}-\frac{53}{5}$
$(3,2)$	$\xi\left(-\frac{6C_0N_c}{5}+\frac{4C_0}{5}+4\log(2)+\frac{6N_c}{5}-\frac{9}{5}\right)+12\log(2)-\frac{12N_c}{5}-\frac{27}{5}$
$(3,3)$	$\xi\left(-\frac{13C_0N_c}{10}+\frac{C_0}{2N_c}+\frac{6C_0}{5}-\frac{4\log(2)}{N_c}+\frac{13N_c}{10}+\frac{1}{2N_c}-\frac{6}{5}\right)-\frac{12\log(2)}{N_c}-\frac{13N_c}{5}+\frac{8}{N_c}+\frac{12}{5}$
$(5,5)$	$\xi\left(\frac{C_0N_c}{2}-\frac{2\log(2)}{N_c}-\frac{N_c}{2}\right)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+N_c-\frac{3}{N_c}$
$(5,6)$	$\xi\left(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\right)-\frac{3C_0}{2}+2\log(2)+2$
$(6,5)$	$\left(2\log(2)-\frac{1}{2}\right)\xi+2\log(2)-2$
$(6,6)$	$-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}+\xi\left(\frac{N_c}{2}-\frac{2\log(2)}{N_c}\right)-\frac{2\log(2)}{N_c}+5N_c-\frac{3}{N_c}$

TABLE XII. One-loop conversion matrix $\Delta a^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the $(8,1)$ operators of the chiral basis in the RI/SMOM($\not{q},\gamma_\mu$) scheme.

(i,j)	$\Delta a_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}/\frac{\alpha_s}{4\pi}$
$(2,2)$	$\xi\left(\frac{4C_0N_c}{5}+\frac{C_0}{N_c}-\frac{6C_0}{5}-\frac{4\log(2)}{N_c}-\frac{4N_c}{5}+\frac{6}{5}\right)-\frac{12\log(2)}{N_c}+\frac{8N_c}{5}+\frac{9}{N_c}-\frac{12}{5}$
$(2,3)$	$\xi\left(\frac{6C_0N_c}{5}-\frac{9C_0}{5}+4\log(2)-\frac{6N_c}{5}+\frac{4}{5}\right)+12\log(2)+\frac{12N_c}{5}-\frac{53}{5}$
$(3,2)$	$\xi\left(-\frac{6C_0N_c}{5}+\frac{4C_0}{5}+4\log(2)+\frac{6N_c}{5}-\frac{9}{5}\right)+12\log(2)-\frac{12N_c}{5}-\frac{27}{5}$
$(3,3)$	$\xi\left(-\frac{9C_0N_c}{5}+\frac{C_0}{N_c}+\frac{6C_0}{5}-\frac{4\log(2)}{N_c}+\frac{9N_c}{5}-\frac{6}{5}\right)-\frac{12\log(2)}{N_c}-\frac{18N_c}{5}+\frac{9}{N_c}+\frac{12}{5}$
$(5,5)$	$\xi\left(\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{1}{2N_c}\right)+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}-\frac{2}{N_c}$
$(5,6)$	$\xi\left(-\frac{C_0}{2}+2\log(2)+\frac{1}{2}\right)-\frac{3C_0}{2}+2\log(2)+2$
$(6,5)$	$\left(2\log(2)-\frac{1}{2}\right)\xi+2\log(2)-2$
$(6,6)$	$\xi\left(-\frac{C_0N_c}{2}+\frac{C_0}{2N_c}-\frac{2\log(2)}{N_c}+N_c-\frac{1}{2N_c}\right)-\frac{3C_0N_c}{2}+\frac{3C_0}{2N_c}-\frac{2\log(2)}{N_c}+4N_c-\frac{2}{N_c}$

TABLE XIII. One-loop conversion matrix $\Delta a^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ for the $(8,1)$ operators of the chiral basis in the RI/SMOM($\not{q},\not{q}$) scheme.

sion relation for light quark mass determinations, the convergence in the RI/SMOM schemes is significantly faster than the convergence in the traditional RI/MOM schemes.

It is, however, interesting to compare the magnitude of the one-loop corrections $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ of different RI schemes. In Tab. XIV we give the spectral matrix norm and the maximum norm for the conversion matrices Δr^* at $N_c=3$ and $\xi=0$ for the different blocks of operators in the chiral basis. The spectral norm of matrix M is defined as the maximum singular value of the matrix M and the maximum norm of matrix M is defined as the maximum absolute value of the matrix elements. Therefore the spectral norm gives a good estimate of the general magnitude of the one-loop corrections since it is invariant under a change of operator basis. The maximum norm gives a good estimate of especially large individual mixing coefficients.

One observes that for the $(27,1)$ operator the one-loop coefficients for the RI/MOM scheme with RI′/MOM wave function renormalization are of order 1, while the RI/SMOM results vary from order 1/10 to order 1. The RI/SMOM(x,y) schemes with $x=y$ have especially small one-loop coefficients. In this case the spectral norm is, of course, equal to the maximum norm. For the $(8,1)$ operators the spectral norm and the maximum norm for the RI/MOM, RI′/MOM scheme and the RI/SMOM($\not{q},y$) schemes is approximately twice the size of the RI/SMOM(γ_μ,y) schemes. In the case of the $(8,8)$ operators the maximum norm as well as the spectral norm in the RI/MOM, RI′/MOM scheme is significantly larger than the respective norm in the RI/SMOM schemes. The difference is especially pronounced for the RI/SMOM($\not{q},\not{q}$) scheme.

We conclude that the magnitude of the one-loop contributions to $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ varies significantly between the

Scheme	(L,R)	$\lVert\Delta r^*/\frac{\alpha_s}{4\pi}\rVert_s$	$\lVert\Delta r^*/\frac{\alpha_s}{4\pi}\rVert_\infty$
RI/MOM, RI′/MOM	(27,1)	0.87851	
RI/SMOM(γ_μ, $\not q$)	(27,1)	2.45482	
RI/SMOM(γ_μ, γ_μ)	(27,1)	0.21184	
RI/SMOM($\not q$, $\not q$)	(27,1)	0.45482	
RI/SMOM($\not q$, γ_μ)	(27,1)	2.21184	
RI/MOM, RI′/MOM	(8,1)	10.61021	9.20457
RI/SMOM(γ_μ, $\not q$)	(8,1)	5.34301	3.88889
RI/SMOM(γ_μ, γ_μ)	(8,1)	5.21189	3.88889
RI/SMOM($\not q$, $\not q$)	(8,1)	12.03242	8.69111
RI/SMOM($\not q$, γ_μ)	(8,1)	11.17384	6.02444
RI/MOM, RI′/MOM	(8,8)	11.42390	10.87124
RI/SMOM(γ_μ, $\not q$)	(8,8)	3.23249	2.62348
RI/SMOM(γ_μ, γ_μ)	(8,8)	4.77841	4.49561
RI/SMOM($\not q$, $\not q$)	(8,8)	1.62236	1.49561
RI/SMOM($\not q$, γ_μ)	(8,8)	4.26036	4.16227

TABLE XIV. Magnitude of the one-loop conversion matrices $\Delta r^*/\frac{\alpha_s}{4\pi}$ for operators in the chiral basis and $N_c = 3$, $\xi = 0$. We give the spectral norm $\lVert\Delta r^*/\frac{\alpha_s}{4\pi}\rVert_s$ as well as the maximum norm $\lVert\Delta r^*/\frac{\alpha_s}{4\pi}\rVert_\infty$ for operators in the (L,R) representation of $SU(3)_L \otimes SU(3)_R$.

different schemes, and we expect that the same holds for the rate of convergence of the perturbative expansion of $\Delta r^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ in α_s. Note that Tab. XIV only gives the numerical values for Landau gauge and that the qualitative analysis of this section may be different for other gauges.

V. SUMMARY AND CONCLUSION

Physical processes that change the strangeness by one unit play an important role in the field of flavor phenomenology. These processes can be studied in lattice simulations using an effective $\Delta S = 1$ Hamiltonian of electroweak interactions that is formulated in terms of $\Delta S = 1$ flavor-changing four-quark operators. In order to renormalize these operators non-perturbatively and to convert measured matrix elements to the $\overline{\mathrm{MS}}$ scheme it is necessary to define renormalization schemes that are independent of the specific regulator. In this work we define RI/SMOM renormalization schemes for different $\Delta S = 1$ operator bases and provide one-loop matching factors to the $\overline{\mathrm{MS}}$[NDR] scheme.

Since the different RI schemes project out different components of the perturbative series, the variety of RI schemes can be used to estimate the effects of higher-order corrections to the conversion matrices which are presently unknown. In this work we define four different RI/SMOM schemes for each $\Delta S = 1$ operator. We also provide a compact expression for the finite one-loop amplitudes that can be used by the reader to define further RI/SMOM schemes in a straightforward way.

The numerical size of the one-loop contributions to the matching factors is discussed briefly for different RI schemes in the Landau gauge. We find that their magnitude varies significantly between the different RI/MOM and RI/SMOM schemes.

For convenience we supply the results of Tabs. V–XI in a format suitable for use in computer algebra software in Ref. [55].

In future work two-loop conversion factors will be calculated which will further reduce the systematic uncertainties in lattice calculations involving the effective $\Delta S = 1$ Hamiltonian of electroweak interactions.

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Appendix A: Flavor and isospin decomposition of four-quark operators

In this appendix we give the flavor and isospin decomposition of the four-quark operators defined in Eqs. (2)-(5). We proceed along the lines of App. B of Ref. [2], carefully avoiding the use of Fierz transformations. We can then convert operators $\tilde{Q}_i$ to operators Q_i including the evanescent operators of Eq. (8).

1. Left-left operators

The left-left operators $Q_1, \ldots, Q_4, Q_9, Q_{10}$ can be written as

$$Q_{LL} = (T_{LL})^{ij}_{kl} (\bar{q}^i_L \otimes q^k_L)(\bar{q}^j_L \otimes q^l_L) \tag{A1}$$

with left-handed quark fields q^i_L and flavor indices i, j, k, l. We identify q^1_L with an up quark, q^2_L with a down quark, and q^3_L with a strange quark. The color and spinor contractions are denoted by $\otimes$. The individual quark fields transform in the fundamental representation of $SU(3)_L$, i.e., under $V \in SU(3)_L$ we find

$$\begin{aligned} Q'_{LL} &= (V^\dagger)_{ia}(V^\dagger)_{jb}(T_{LL})^{ab}_{cd} V_{ck} V_{dl} (\bar{q}^i_L \otimes q^k_L)(\bar{q}^j_L \otimes q^l_L) \\ &= (T'_{LL})^{ij}_{kl} (\bar{q}^i_L \otimes q^k_L)(\bar{q}^j_L \otimes q^l_L) \end{aligned} \tag{A2}$$

with

$$(T'_{LL})^{ij}_{kl} = (V^\dagger)_{ia}(V^\dagger)_{jb}(T_{LL})^{ab}_{cd} V_{ck} V_{dl} \,. \tag{A3}$$

This transformation corresponds to the 81-dimensional representation $(\bar{3} \otimes 3) \otimes (\bar{3} \otimes 3)$ of $SU(3)_L$. We can decompose the tensor of this 81-dimensional representation as

$$(T_{LL})^{ij}_{kl} = (T_{LL})^{\{i,j\}}_{\{k,l\}} + (T_{LL})^{\{i,j\}}_{[k,l]} + (T_{LL})^{[i,j]}_{\{k,l\}} + (T_{LL})^{[i,j]}_{[k,l]}\,, \quad \text{(A4)}$$

where for a general tensor t_{ij} we have $2t_{\{i,j\}} = t_{ij} + t_{ji}$ and $2t_{[i,j]} = t_{ij} - t_{ji}$. It is straightforward to check that the respective subspaces are invariant under Eq. (A3) and that their dimensionality is given by

$$\dim\left[(T_{LL})^{\{i,j\}}_{\{k,l\}}\right] = 36\,, \quad \dim\left[(T_{LL})^{[i,j]}_{\{k,l\}}\right] = 18\,, \quad \text{(A5)}$$

$$\dim\left[(T_{LL})^{\{i,j\}}_{[k,l]}\right] = 18\,, \quad \dim\left[(T_{LL})^{[i,j]}_{[k,l]}\right] = 9\,. \quad \text{(A6)}$$

Since Q_{LL} is symmetric under simultaneous exchange of $(i,k) \leftrightarrow (j,l)$ we only need to consider the completely symmetric case $(T_{LL})^{\{i,j\}}_{\{k,l\}}$ and the completely antisymmetric case $(T_{LL})^{[i,j]}_{[k,l]}$.

We can further decompose the remaining subspaces by considering the trace of a pair of upper and lower indices. If such a trace vanishes, it also vanishes after applying the transformation of Eq. (A3), and therefore such a constraint defines an invariant subspace. For the completely symmetric case we distinguish between

$$\text{(i)} \quad (T_{LL})^{\{i,j\}}_{\{i,l\}} = 0 \quad \Rightarrow \quad (T_{LL})^{\{i,j\}}_{\{i,j\}} = 0\,, \quad \text{(A7)}$$

$$\text{(ii)} \quad (T_{LL})^{\{i,j\}}_{\{i,l\}} \neq 0 \quad \wedge \quad (T_{LL})^{\{i,j\}}_{\{i,j\}} = 0\,, \quad \text{(A8)}$$

$$\text{(iii)} \quad (T_{LL})^{\{i,j\}}_{\{i,l\}} \neq 0 \quad \wedge \quad (T_{LL})^{\{i,j\}}_{\{i,j\}} \neq 0\,, \quad \text{(A9)}$$

where we sum over repeated indices. The subspace (i) has nine constraints and is thus 27-dimensional, the subspace (ii) is orthogonal to (i) and has one constraint and is thus 8-dimensional, and the remaining subspace (iii) is 1-dimensional. Therefore the completely symmetric case can be decomposed as

$$27 \oplus 8 \oplus 1\,.$$

For the completely antisymmetric case the analog definition of subspaces leads to a zero-dimensional space (i), a 8-dimensional space (ii), and a 1-dimensional space (iii), i.e., the completely antisymmetric case can be decomposed as

$$0 \oplus 8 \oplus 1\,.$$

This completes the classification of the left-left operators of the form given in Eq. (A1) according to representations of $\mathrm{SU}(3)_L$. From now on we restrict the discussion to Q_{LL} with either (I) exactly one $\bar{s}$ field or (II) exactly one u or d field. In the case (I) only the elements $(T_{LL})^{3j}_{kl}$ and $(T_{LL})^{j3}_{kl}$ with $j,k,l = 1,2$ are nonzero. Therefore they live in a 8-dimensional representation of SU(2) isospin, and we can classify them according to irreducible isospin representations in the following. In the completely symmetric case (which is especially symmetric in $k \leftrightarrow l$) they live in a 6-dimensional isospin representation, and we distinguish between

$$\text{(Ia)} \quad (T_{LL})^{\{3,j\}}_{\{k,j\}} = 0\,, \quad \text{(A10)}$$

$$\text{(Ib)} \quad (T_{LL})^{\{3,j\}}_{\{k,j\}} \neq 0\,, \quad \text{(A11)}$$

where we sum over $j = 1,2$. The subspace (Ia) has two constraints and is thus 4-dimensional ($I = 3/2$), the subspace (Ib) is orthogonal to (Ia) and is thus 2-dimensional ($I = 1/2$). In the completely antisymmetric case T_{LL} lives in a 2-dimensional isospin representation. The respective subspace (Ia) has two constraints and is thus zero-dimensional, the respective subspace (Ib) is orthogonal to (Ia) and is thus two-dimensional ($I = 1/2$). In the case (II) the tensor transforms in the fundamental isospin representation ($I = 1/2$).

In the remainder of this subsection we give a list of left-left operators with $\Delta S = 1$ transforming in irreducible representations of $\mathrm{SU}(3)_L$ and isospin in which the operator

$$(\bar{s}d)_{V-A}(\bar{u}u)_{V-A} \quad \text{(A12)}$$

enters. The completely symmetric operators are given by

$$\begin{aligned}
Q_S^{(27,1),3/2} &= (\bar{s}d)_{V-A}(\bar{u}u)_{V-A} + (\bar{s}u)_{V-A}(\bar{u}d)_{V-A} \\
&\quad - (\bar{s}d)_{V-A}(\bar{d}d)_{V-A}\,, \\
Q_S^{(27,1),1/2} &= (\bar{s}d)_{V-A}(\bar{u}u)_{V-A} + (\bar{s}u)_{V-A}(\bar{u}d)_{V-A} \\
&\quad + 2(\bar{s}d)_{V-A}(\bar{d}d)_{V-A} - 3(\bar{s}d)_{V-A}(\bar{s}s)_{V-A}\,, \\
Q_S^{(8,1),1/2} &= (\bar{s}d)_{V-A}(\bar{u}u)_{V-A} + (\bar{s}u)_{V-A}(\bar{u}d)_{V-A} \\
&\quad + 2(\bar{s}d)_{V-A}(\bar{d}d)_{V-A} \\
&\quad + 2(\bar{s}d)_{V-A}(\bar{s}s)_{V-A}\,,
\end{aligned} \quad \text{(A13)}$$

where (L,R) indicates the representation of $\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$, and 1/2 (3/2) indicates the isospin. The completely antisymmetric operator is given by

$$Q_A^{(8,1),1/2} = (\bar{s}d)_{V-A}(\bar{u}u)_{V-A} - (\bar{s}u)_{V-A}(\bar{u}d)_{V-A}\,. \quad \text{(A14)}$$

2. Left-right operators

The left-right operators $Q_5, \ldots, Q_8$ can be written as

$$Q_{LR} = (T_{LR})^{ij}_{kl}(\bar{q}^i_L \otimes q^k_L)(\bar{q}^j_R \otimes q^l_R) \quad \text{(A15)}$$

with right-handed quark fields q^i_R. The individual quark fields transform in the fundamental representation of $\mathrm{SU}(3)_{L/R}$, i.e., under $V_L \in \mathrm{SU}(3)_L$ and $V_R \in \mathrm{SU}(3)_R$ we find

$$\begin{aligned}
Q'_{LR} &= (V_L^\dagger)_{ia}(V_R^\dagger)_{jb}(T_{LR})^{ab}_{cd}(V_L)_{ck}(V_R)_{dl} \\
&\quad \times (\bar{q}^i_L \otimes q^k_L)(\bar{q}^j_R \otimes q^l_R)
\end{aligned}$$

$$= (T'_{LR})^{ij}_{kl}(\bar{q}^i_L \otimes q^k_L)(\bar{q}^j_R \otimes q^l_R) \tag{A16}$$

with

$$(T'_{LR})^{ij}_{kl} = (V^\dagger_L)_{ia}(V^\dagger_R)_{jb}(T_{LR})^{ab}_{cd}(V_L)_{ck}(V_R)_{dl}\,. \tag{A17}$$

This transformation corresponds to the 9-dimensional $\bar{3} \otimes 3$ representations of $\mathrm{SU}(3)_L$ and $\mathrm{SU}(3)_R$. These 9-dimensional representations can be decomposed in a 8-dimensional subspace with vanishing trace of left (right) indices and in a 1-dimensional subspace with nonzero trace, i.e.,

$$\bar{3} \otimes 3 = 8 \oplus 1\,. \tag{A18}$$

This completes the classification of the left-right operators of the form given in Eq. (A15) according to representations of $\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$. From now on we restrict the discussion to Q_{LR} with $(T_{LR})^{ij}_{kl} \neq 0$ only for (I) $i = 3$ and $j, k, l = 1, 2$ or (II) $i = j = l = 3$, $k = 1, 2$. The classification according to irreducible representations of isospin now follows from the analog discussion of left-left operators.

In the remainder of this subsection we give a list of left-right operators with $\Delta S = 1$ transforming in irreducible representations of $\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$ and isospin in which the operator

$$(\bar{s}d)_{V-A}(\bar{u}u)_{V+A} \tag{A19}$$

enters. It is straightforward to see that Q_5 and Q_6 transform in the $(8, 1)$ representation of $\mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$ and in isospin representation $I = 1/2$, i.e., we define

$$Q^{(8,1),1/2}_{5,6} = Q_{5,6}\,. \tag{A20}$$

Operators symmetric under $k \leftrightarrow l$ are given by

$$\begin{aligned} Q^{(8,8),3/2}_S &= (\bar{s}d)_{V-A}(\bar{u}u)_{V+A} + (\bar{s}u)_{V-A}(\bar{u}d)_{V+A} \\ &\quad - (\bar{s}d)_{V-A}(\bar{d}d)_{V+A}\,, \\ Q^{(8,8),1/2}_S &= (\bar{s}d)_{V-A}(\bar{u}u)_{V+A} + (\bar{s}u)_{V-A}(\bar{u}d)_{V+A} \\ &\quad + 2(\bar{s}d)_{V-A}(\bar{d}d)_{V+A} \\ &\quad - 3(\bar{s}d)_{V-A}(\bar{s}s)_{V+A}\,. \end{aligned} \tag{A21}$$

One can also construct an operator that is antisymmetric under $k \leftrightarrow l$ for $k, l = 1, 2$ with well-defined transformation under flavor and isospin:

$$\begin{aligned} Q^{(8,8),1/2}_A &= (\bar{s}d)_{V-A}(\bar{u}u)_{V+A} - (\bar{s}u)_{V-A}(\bar{u}d)_{V+A} \\ &\quad - (\bar{s}d)_{V-A}(\bar{s}s)_{V+A}\,. \end{aligned} \tag{A22}$$

3. Change of basis

Operators of the physical basis I and II with spinor structure $(V - A)$–$(V - A)$ can be decomposed in the color-diagonal operators of Eqs. (A13) and (A14). We find

$$\begin{aligned} \tilde{Q}_1 &= \frac{1}{10}Q^{(8,1),1/2}_S + \frac{1}{15}Q^{(27,1),1/2}_S \\ &\quad + \frac{1}{3}Q^{(27,1),3/2}_S + \frac{1}{2}Q^{(8,1),1/2}_A\,, \end{aligned} \tag{A23}$$

$$Q_3 = \frac{1}{2}Q^{(8,1),1/2}_S + \frac{1}{2}Q^{(8,1),1/2}_A\,, \tag{A24}$$

$$\begin{aligned} Q_9 &= -\frac{1}{10}Q^{(8,1),1/2}_S + \frac{1}{10}Q^{(27,1),1/2}_S \\ &\quad + \frac{1}{2}Q^{(27,1),3/2}_S + \frac{1}{2}Q^{(8,1),1/2}_A \end{aligned} \tag{A25}$$

for the color-diagonal operators and

$$\begin{aligned} Q_2 &= \frac{1}{10}Q^{(8,1),1/2}_S + \frac{1}{15}Q^{(27,1),1/2}_S \\ &\quad + \frac{1}{3}Q^{(27,1),3/2}_S - \frac{1}{2}Q^{(8,1),1/2}_A\,, \end{aligned} \tag{A26}$$

$$\tilde{Q}_4 = \frac{1}{2}Q^{(8,1),1/2}_S - \frac{1}{2}Q^{(8,1),1/2}_A\,, \tag{A27}$$

$$\begin{aligned} \tilde{Q}_{10} &= -\frac{1}{10}Q^{(8,1),1/2}_S + \frac{1}{10}Q^{(27,1),1/2}_S \\ &\quad + \frac{1}{2}Q^{(27,1),3/2}_S - \frac{1}{2}Q^{(8,1),1/2}_A \end{aligned} \tag{A28}$$

for the color-mixed operators with

$$\begin{aligned} \tilde{Q}_4 &= \sum_{q=u,d,s}(\bar{s}_a q_a)_{V-A}(\bar{q}_b d_b)_{V-A}\,, \\ \tilde{Q}_{10} &= \frac{3}{2}\sum_{q=u,d,s}e_q(\bar{s}_a q_a)_{V-A}(\bar{q}_b d_b)_{V-A}\,. \end{aligned} \tag{A29}$$

Operators with spinor structure $(V - A)$–$(V + A)$ are already in explicit $(8, 1)$ and $(8, 8)$ representations, i.e.,

$$Q_{5,6} = Q^{(8,1),1/2}_{5,6}\,, \tag{A30}$$

$$Q_7 = \frac{1}{2}Q^{(8,8),3/2}_S + \frac{1}{2}Q^{(8,8),1/2}_A\,, \tag{A31}$$

$$Q_8 = \frac{1}{2}{Q'}^{(8,8),3/2}_S + \frac{1}{2}{Q'}^{(8,8),1/2}_A \tag{A32}$$

with

$$\begin{aligned} {Q'}^{(8,8),3/2}_S &= (\bar{s}_a d_b)_{V-A}(\bar{u}_b u_a)_{V+A} \\ &\quad + (\bar{s}_a u_b)_{V-A}(\bar{u}_b d_a)_{V+A} \\ &\quad - (\bar{s}_a d_b)_{V-A}(\bar{d}_b d_a)_{V+A}\,, \end{aligned} \tag{A33}$$

$$\begin{aligned} {Q'}^{(8,8),1/2}_A &= (\bar{s}_a d_b)_{V-A}(\bar{u}_b u_a)_{V+A} \\ &\quad - (\bar{s}_a u_b)_{V-A}(\bar{u}_b d_a)_{V+A} \\ &\quad - (\bar{s}_a d_b)_{V-A}(\bar{s}_b s_a)_{V+A}\,. \end{aligned} \tag{A34}$$

Appendix B: Alternative determination of ΔT

In this appendix we provide an alternative method to determine ΔT defined in Eq. (64) relating the Wilson coefficients of the $\overline{\mathrm{MS}}$-renormalized operator basis I, II to the respective coefficients of the chiral basis.

To this end we consider an arbitrary RI scheme and express on-shell matrix elements of the effective Hamiltonian in terms of operators ${Q'}^{\mathrm{RI}}_j$ and first in terms of

Wilson coefficients ${C'}_i^{\overline{\rm MS}}$,

$$\left\langle \mathcal{H}_{\rm eff}^{\Delta S=1} \right\rangle = \frac{G_F}{\sqrt{2}} \sum_{i,j} {C'}_i^{\overline{\rm MS}}(\mu) {R'}_{ij}^{{\rm RI}\to\overline{\rm MS}}(\mu) \times \left\langle {Q'}_j^{\rm RI}(\mu) \right\rangle , \tag{B1}$$

and then in terms of Wilson coefficients $C_i^{\overline{\rm MS}}$,

$$\left\langle \mathcal{H}_{\rm eff}^{\Delta S=1} \right\rangle = \frac{G_F}{\sqrt{2}} \sum_{i,j,l} C_i^{\overline{\rm MS}}(\mu) R_{ij}^{{\rm RI}\to\overline{\rm MS}}(\mu) \times T_{jl} \left\langle {Q'}_l^{\rm RI}(\mu) \right\rangle . \tag{B2}$$

Since these equations hold for an arbitrary matrix element, we find

$$({C'}^{\overline{\rm MS}})^T = (C^{\overline{\rm MS}})^T R^{{\rm RI}\to\overline{\rm MS}} T ({R'}^{{\rm RI}\to\overline{\rm MS}})^{-1} \tag{B3}$$

where T is defined in Eq. (58). A comparison with Eq. (64) yields

$$T + \Delta T = R^{{\rm RI}\to\overline{\rm MS}} T ({R'}^{{\rm RI}\to\overline{\rm MS}})^{-1} . \tag{B4}$$

Note that the left-hand side is just a conversion factor of Wilson coefficients in the $\overline{\rm MS}$ scheme, and thus the right-hand side must also be independent of the RI scheme used to calculate $R^{{\rm RI}\to\overline{\rm MS}}$ and ${R'}^{{\rm RI}\to\overline{\rm MS}}$.

We calculate ΔT using Eq. (B4) for the non-exceptional momentum configuration with projectors

$$P_1 = \frac{P_{(1),u}^{VV+AA,\not{q}} - P_{(3),u}^{VV+AA,\not{q}} - P_{(1),d}^{VV+AA,\not{q}}}{160 N_c (N_c+1)} , \tag{B5}$$

$$\begin{aligned} P_2 = {} & \left(3P_{(1),u}^{VV+AA,\not{q}} + 2P_{(1),d}^{VV+AA,\not{q}} + 2P_{(3),u}^{VV+AA,\not{q}}\right) \times \frac{3N_c-2}{160N_c(N_c^2-1)} \\ & + \left(3P_{(2),u}^{VV+AA,\not{q}} + 2P_{(2),d}^{VV+AA,\not{q}} + 2P_{(4),u}^{VV+AA,\not{q}}\right) \times \frac{2N_c-3}{160N_c(N_c^2-1)} , \end{aligned} \tag{B6}$$

$$\begin{aligned} P_3 = {} & \left(3P_{(1),u}^{VV+AA,\not{q}} + 2P_{(1),d}^{VV+AA,\not{q}} + 2P_{(3),u}^{VV+AA,\not{q}}\right) \times \frac{2N_c-3}{160N_c(N_c^2-1)} \\ & + \left(3P_{(2),u}^{VV+AA,\not{q}} + 2P_{(2),d}^{VV+AA,\not{q}} + 2P_{(4),u}^{VV+AA,\not{q}}\right) \times \frac{3N_c-2}{160N_c(N_c^2-1)} , \end{aligned} \tag{B7}$$

$$P_5 = \frac{P_{(1),u}^{VV-AA,\not{q}} + 2P_{(1),d}^{VV-AA,\not{q}}}{96(N_c^2-1)} - \frac{P_{(2),u}^{VV-AA,\not{q}} + 2P_{(2),d}^{VV-AA,\not{q}}}{96N_c(N_c^2-1)} , \tag{B8}$$

$$P_6 = \frac{P_{(2),u}^{VV-AA,\not{q}} + 2P_{(2),d}^{VV-AA,\not{q}}}{96(N_c^2-1)} - \frac{P_{(1),u}^{VV-AA,\not{q}} + 2P_{(1),d}^{VV-AA,\not{q}}}{96N_c(N_c^2-1)} , \tag{B9}$$

$$P_7 = \frac{P_{(1),u}^{VV-AA,\not{q}} - P_{(1),d}^{VV-AA,\not{q}}}{48(N_c^2-1)} - \frac{P_{(2),u}^{VV-AA,\not{q}} - P_{(2),d}^{VV-AA,\not{q}}}{48N_c(N_c^2-1)} , \tag{B10}$$

$$P_8 = \frac{P_{(2),u}^{VV-AA,\not{q}} - P_{(2),d}^{VV-AA,\not{q}}}{48(N_c^2-1)} - \frac{P_{(1),u}^{VV-AA,\not{q}} - P_{(1),d}^{VV-AA,\not{q}}}{48N_c(N_c^2-1)} \tag{B11}$$

with

$$P_i \Gamma_4^{\rm tree}({Q'}_j) = \delta_{ij} , \quad i,j = 1,2,3,5,6,7,8 . \tag{B12}$$

This choice of projectors recovers the conversion matrices of the RI/SMOM($\not{q}, y$) schemes for the non-exceptional momentum configuration. It makes use of projectors with a more complex flavor structure which allows for a definition of this scheme without considering each irreducible representation of operators separately. We use the basis of projectors $\{P_{(i),f'}^{VV\pm AA}\}$ with $i = 1,2,3,4$ defined in Eqs. (98). The details of the wave function renormalization given in Δ_q^y drop out in Eq. (B4) and hence we do not need to specify the wave function renormalization scheme y. The result for ΔT using this method agrees with ΔT given in Sec. III E.

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