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We develop an analytical technique to evaluate the magnetic helicity in the debris from sphaleron decay. We show that baryon number production leads to left-handed magnetic fields, and that the magnetic helicity is conserved at late times. Our analysis explicitly demonstrates the connection between sphaleron-mediated cosmic baryogenesis and cosmic magnetogenesis.

I. INTRODUCTION

A cosmic phase transition in which magnetic monopoles are produced necessarily produces cosmic magnetic fields, as these are sourced by the monopoles. The electroweak model also contains magnetic monopoles [1] that can source magnetic fields [2, 3]. However, a quantitative estimate of the magnetic field is difficult to obtain because electroweak monopoles are confined to electroweak antimonopoles by Z-strings [4], and estimates of the monopole density based on topological considerations cannot be applied. Fortunately there is a way to sidestep this difficulty because magnetic monopole-antimonopole pairs, in the guise of "sphalerons" [6], play a crucial role in the violation of baryon number in the electroweak model [5]. Hence monopole-antimonopole pairs are a common factor in baryogenesis and magnetogenesis. This fact allows one to estimate the helicity of the magnetic field by relating it to the baryonic density [7, 8]. The connection between baryons and magnetic fields may provide an opportunity for tests of fundamental particle physics and cosmology at the electroweak epoch via observations of the topological and spectral properties of a primordial magnetic field.

To test the production of magnetic fields during baryon number violating processes, sphaleron decay was studied in [9] (also see [10]) by numerically evolving the electroweak equations of motion on a lattice. Debris from sphaleron decay was analysed for magnetic helicity and the connection between baryon number violation and generation of magnetic helicity was confirmed. The aim of the present paper is to *analytically* study the sphaleron decay process and to derive the generation of magnetic helicity in a transparent manner.

Although the full dynamics of sphaleron decay involves the complete electroweak equations of motion, we are able to extract the dynamics relevant to the electromagnetic sector following a scheme similar to Ref. [11]. Our strategy is to first consider the "$SU(2)$ sphaleron" which is an explicitly known saddle-point solution in the electroweak model with the hypercharge coupling, g', set to zero [6]. Not only is the $SU(2)$ sphaleron known explicitly but a path in field space that connects the sphaleron solution to the vacuum solution has also been constructed [6]. Let μ be the parameter along this path. We will consider the decay of the sphaleron along this particular path in field space. Hence the time dependence of the fields is given by the (unspecified) function $\mu(t)$. With this model of the decaying sphaleron, we now turn on the hypercharge coupling, g', assuming it to be small. Then the decaying sphaleron produces electromagnetic currents, which produce electromagnetic fields. We calculate the magnetic helicity in these fields and show analytically that it is conserved at late times and asymptotes to some non-zero value which depends on $\mu(t)$. We can also show that the sign of the magnetic helicity depends on which side of the saddle the sphaleron rolls down. Thus the sign of the magnetic helicity is directly related to whether baryons or antibaryons are produced during the decay of the sphaleron. Since we know excess baryons are produced in the universe, cosmic magnetic fields carry left-handed magnetic helicity. Furthermore, the magnitude of the helicity is directly proportional to the cosmic baryon number density as discussed in [7–9].

II. $SU(2)$ SPHALERON

The $SU(2)$ sphaleron is a saddle-point solution to the static electroweak equations with the hypercharge coupling, g', set to zero. For our purpose, it is necessary to consider a path of field configurations that starts with the sphaleron and ends at the vacuum. Such a path was constructed by Manton [6] in order to implement a Morse theory argument indicating the existence of a sphaleron solution. The path in field space is now written in spherical coordinates, (r, θ, ϕ), and μ is a parameter along the path

$$\Phi(\mu, r, \theta, \phi) = \frac{v}{\sqrt{2}} \left[(1 - h(r)) \begin{pmatrix} 0 \\ e^{-i\mu} \cos\mu \end{pmatrix} + h(r)\Phi^\infty(\mu, \theta, \phi) \right]$$

$$W_i(\mu, r, \theta, \phi) = i\frac{2}{g} f(r) W_i^\infty(\mu, \theta, \phi) \tag{1}$$

where g is the $SU(2)$ coupling, $v/\sqrt{2}$ is the vacuum expectation value of the Higgs field, $h(r)$ is the Higgs profile function and $f(r)$ the gauge profile function.

The asymptotic fields are given by

$$\Phi^\infty(\mu, \theta, \phi) = \begin{pmatrix} \sin\mu \ \sin\theta \ e^{i\phi} \\ e^{-i\mu}(\cos\mu + i\sin\mu \ \cos\theta) \end{pmatrix} \tag{2}$$

$$W_i^\infty = -\partial_i U^\infty \ (U^\infty)^{-1} \tag{3}$$

where

$$U^\infty = \begin{pmatrix} \Phi_2^{\infty*} & \Phi_1^\infty \\ -\Phi_1^{\infty*} & \Phi_2^\infty \end{pmatrix} \ . \tag{4}$$

The sphaleron solution is obtained if we set $\mu = \pi/2$ and the vacuum is obtained for $\mu = 0$ or $\mu = \pi$. The path from $\mu = \pi/2$ to $\mu = 0$ describes the sphaleron rolling down one side of the saddle, and the path from $\mu = \pi/2$ to $\mu = \pi$ describes roll down on the other side of the saddle. The former path describes the creation of antibaryons while the latter describes the creation of baryons.

The dimensionless sphaleron profile functions are found by numerically solving the static electroweak equations of motion. However, the essential features of the profiles are captured by the simple ansatz called "Ansatz a" in [11],

$$f^a(r) = \begin{cases} (r/r_f)^2, & \text{if } r < r_f \\ 1, & \text{if } r \geq r_f \end{cases} \tag{5}$$

$$h^a(r) = \begin{cases} r/r_h, & \text{if } r < r_h \\ 1, & \text{if } r \geq r_h \end{cases} \ . \tag{6}$$

The length scales r_f and r_h are related to the vector and scalar masses in the model. For simplicity we will take $r_f = r_h$, when we numerically evalute certain integrals in Sec. VI.

III.   ELECTROMAGNETIC CURRENTS

To model the decay of the sphaleron, we take the parameter μ to be a function of time, $\mu(t)$, with $\mu(0) = \pi/2$ and $\mu(\infty) = \pi$. The Higgs and $SU(2)$ gauge fields are still given by Eqs. (1). As discussed in [11], we can now consider a small hypercharge coupling constant, g', and evaluate the electromagnetic field perturbatively in g'. Then the decaying sphaleron will be a source for an electromagnetic field. If we denote the electromagnetic gauge field by A_μ, Maxwell equations in the Lorenz gauge give

$$(\partial_t^2 - \nabla^2)A^\mu = J^\mu \ , \tag{7}$$

where J^μ is the electromagnetic current produced by the decaying sphaleron. To leading order in the hypercharge coupling constant, g', the electromagnetic current is equal to the hypercharge current given by

$$J^\mu = -i\frac{g'}{2}\left[\Phi^\dagger D^\mu\Phi - (D^\mu\Phi)^\dagger\Phi\right] \ . \tag{8}$$

To calculate the magnetic helicity produced by sphaleron decay we only need to find the spatial components of A_μ. Hence we only need the spatial components of the current J^μ. Eq. (8) gives the contravariant current vector components; we prefer to work with the "ordinary" components (see Sec. 4.8 of [12]) that are obtained by dividing by the square root of the metric factors, namely, r and $r\sin\theta$ in the θ and ϕ components. The ordinary components

of the current are

$$J_r = \frac{g'v^2}{4} \cos\theta \ h'(r)\sin(2\mu)$$

$$\equiv \frac{g'v^2}{4} \cos\theta \ \mathcal{J}_r(t,r) \tag{9}$$

$$J_\theta = \frac{g'v^2}{4} \sin\theta \ \frac{\sin(2\mu)}{r} \left[(1-h)f\cos^2\mu - h(1-f) + hf(1-h) \ \sin^2\mu \right]$$

$$\equiv \frac{g'v^2}{4} \sin\theta \ \mathcal{J}_\theta(t,r) \tag{10}$$

$$J_\phi = \frac{g'v^2}{4} \sin\theta \ \frac{2\sin^2\mu}{r} \left[f(1-h)^2 \cos^2\mu + h^2(1-f) \right]$$

$$\equiv \frac{g'v^2}{4} \sin\theta \ \mathcal{J}_\phi(t,r) \ , \tag{11}$$

where the prime on h denotes differentiation with respect to r.

The spherical components of the current, J_i, as given in Eq. (8) only involve spatial derivatives of Φ and the spacelike components of the gauge fields. Hence the three-current does not involve time derivatives of $\mu(t)$ or the time-component of the gauge fields. Also, the zeroth (time) component of the current is unspecified since we will not need it in our calculation. We assume it is such as to ensure current conservation.

The angular form of the currents is of interest since there is an azimuthal component, J_ϕ, and the J_r, J_θ components circulate in the constant ϕ plane. Hence the J_ϕ current resembles circular current-carrying wires in planes parallel to the xy−plane and the J_r, J_θ components resemble current-carrying wires wrapped on a toroidal solenoid. The two components of the current are linked and hence the current is helical as measured by $\mathbf{J} \cdot \nabla \times \mathbf{J}$.

Already we can see the relation of the sign of the helicity and the decay path taken by the sphaleron. The path from $\mu = \pi/2$ to $\mu = \pi$ is related to the path from $\mu = \pi/2$ to $\mu = 0$ by the transformation $\mu \to \pi - \mu$. Under this transformation J_ϕ is unchanged but J_r and J_θ change sign. Thus the linking number of the current components is reversed. This implies that the magnetic helicity produced by the currents will also be reversed.

We will eventually need to Fourier transform the current. For this we find it most convenient to work with Cartesian components

$$J_x(t,\mathbf{x}) = J_r \sin\theta \ \cos\phi + J_\theta \cos\theta \ \cos\phi - J_\phi \sin\phi \tag{12}$$

$$J_y(t,\mathbf{x}) = J_r \sin\theta \ \sin\phi + J_\theta \cos\theta \ \sin\phi + J_\phi \cos\phi \tag{13}$$

$$J_z(t,\mathbf{x}) = J_r \cos\theta - J_\theta \sin\theta \ . \tag{14}$$

The Fourier transform is implemented by using the plane wave expansion (see Sec. 11.3 of [13])

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{n=0}^{\infty} (2n+1)i^n \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos(m(\phi-v)) P_n^m(\cos u) P_n^m(\cos\theta) j_n(kr) \ , \tag{15}$$

where

$$\mathbf{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\mathbf{k} = k(\sin u\cos v, \sin u\sin v, \cos u) \ , \tag{16}$$

and $j_n(kr)$ is the spherical Bessel function of order n, $P_n^m(\cdot)$ is the associated Legendre function, and $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m \geq 1$.

Now, for example, the Fourier transform of the x−component of the current is

$$\tilde{J}_x(t,\mathbf{k}) = \int d\Omega \int_0^\infty dr \ r^2 \ e^{i\mathbf{k}\cdot\mathbf{x}} J_x(t,\mathbf{x}) \ . \tag{17}$$

In this we insert $e^{i\mathbf{k}\cdot\mathbf{x}}$ from Eq. (15) and the current from (12), using (9)-(11). After quite a bit of algebra, we find

$$\tilde{J}_x(t,\mathbf{k}) = -\frac{g'v^2}{4}4\pi \sin u \ [\cos u\cos v \ K_2(t,k) + i\sin v \ K_1(t,k)] \tag{18}$$

$$\tilde{J}_y(t,\mathbf{k}) = -\frac{g'v^2}{4}4\pi \sin u \ [\cos u\sin v \ K_2(t,k) - i\cos v \ K_1(t,k)] \tag{19}$$

$$\tilde{J}_z(t,\mathbf{k}) = \frac{g'v^2}{4}\frac{4\pi}{3}[\ K_0(t,k) + (1 - 3\cos^2 u) \ K_2(t,k)] \ , \tag{20}$$

where

$$K_0 = \int_0^\infty dr r^2 j_0(kr)[\mathcal{J}_r(t,r) - 2\mathcal{J}_\theta(t,r)]$$

$$K_1 = \int_0^\infty dr r^2 j_1(kr)\mathcal{J}_\phi(t,r)$$

$$K_2 = \int_0^\infty dr r^2 j_2(kr)\mathcal{J}_{r+\theta}(t,r) \ , \tag{21}$$

and

$$\mathcal{J}_{r+\theta}(t,r) \equiv \mathcal{J}_r(t,r) + \mathcal{J}_\theta(t,r) \ . \tag{22}$$

IV. MAGNETIC HELICITY

The magnetic helicity is defined by

$$\mathcal{H}(t) = \int d^3x \ \mathbf{A} \cdot \mathbf{B} \ . \tag{23}$$

The procedure to evaluate magnetic helicity is now straightforward though cumbersome. We find $\mathbf{A}$ from Maxwell's equation, then the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, and finally the resulting magnetic helicity. In terms of the fields in Fourier space (denoted by over tilde's)

$$\tilde{\mathbf{A}}(k) = -\frac{\tilde{\mathbf{J}}(k)}{\omega^2 - \mathbf{k}^2} \ , \tag{24}$$

where $\tilde{\mathbf{J}}(k)$ is given in Eqs. (18)-(20) , and

$$\tilde{\mathbf{B}}(k) = -i\mathbf{k} \times \mathbf{A}(k) \ . \tag{25}$$

For us it is more useful to work with the three dimensional (spatial) Fourier transform of the fields while not transforming their time dependence. This is because the time dependence occurs via the unknown function $\mu(t)$. To go from the frequency dependent function to the time dependent function, we use

$$\int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega(t-\tau)}}{\omega^2 - \mathbf{k}^2} = \frac{2\pi \sin(k(t-\tau))}{k}\Theta(t-\tau) \ , \tag{26}$$

where we have chosen a contour prescription so as to get retarded solutions.

We use Eqs. (24) and (25) in the helicity integral (23) and then integrate over the frequencies using (26). The result is

$$\mathcal{H}(t) = i \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^t d\tau \int_{-\infty}^t d\tau' \frac{\sin(k(t-\tau))}{k}\frac{\sin(k(t-\tau'))}{k}\mathbf{k} \cdot \tilde{\mathbf{J}}(\tau, -\mathbf{k}) \times \tilde{\mathbf{J}}(\tau', \mathbf{k}) \ . \tag{27}$$

The triple product can be evaluated using the explicit expressions for the current components in Eqs. (18)-(20).

Further simplification involves considerable algebra and integration over trigonometric functions. We also use the spherical Bessel function relation

$$j_2(x) = \frac{3}{x}j_1(x) - j_0(x) \ , \tag{28}$$

and perform the integrations over the angular coordinates u, v in $k-$space to eventually get

$$\mathcal{H}(t) = \frac{16}{3}\left(\frac{g'v^2}{4}\right)^2 [\mathcal{H}_0(t) + \mathcal{H}_1(t)] \ , \tag{29}$$

where

$$\mathcal{H}_0(t) = -\int_{-\infty}^t d\tau \int_{-\infty}^t d\tau' \int_0^\infty dr \ r^2 \int_0^\infty dr' \ r'^2 \mathcal{J}_\phi(\tau, r)\mathcal{J}_\theta(\tau', r')$$

$$\times \int_0^\infty dk \ k \left[\cos(k(\tau - \tau')) - \cos(k(2t - \tau - \tau'))\right] j_1(kr)j_0(kr') \tag{30}$$

$$\mathcal{H}_1(t) = \int_{-\infty}^{t} d\tau \int_{-\infty}^{t} d\tau' \int_{0}^{\infty} dr\, r^2 \int_{0}^{\infty} dr'\, r'^2 \mathcal{J}_\phi(\tau, r) \mathcal{J}_{r+\theta}(\tau', r')$$

$$\times \frac{1}{r'} \int_{0}^{\infty} dk\, [\cos(k(\tau - \tau')) - \cos(k(2t - \tau - \tau'))]\, j_1(kr) j_1(kr') \ . \tag{31}$$

The integrations over k are performed using the explicit expressions for the spherical Bessel functions in terms of trigonometric functions

$$j_0(x) = \frac{\sin(x)}{x} \tag{32}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \ . \tag{33}$$

We find

$$\int_{0}^{\infty} dk\, k\, \cos(k\delta) j_1(kr) j_0(kr') = \frac{\pi}{8r^2 r'^2} P_0(\delta, r, r') \tag{34}$$

$$P_0(\delta, r, r') = r'[S_0(\delta, r, y) - S_0(\delta, -r, r') - S_0(\delta, r, -r') + S_0(\delta, -r, -r')] \ , \tag{35}$$

with

$$S_0(\delta, r, r') = (\delta + r')\, \text{sign}(\delta + r + r') \tag{36}$$

and

$$\int_{0}^{\infty} dk\, \cos(k\delta) j_1(kr) j_1(kr') = \frac{-\pi}{48 r^2 r'^2} P_1(\delta, r, r') \ , \tag{37}$$

where

$$P_1(\delta, r, r') = S_1(\delta, r, r') - S_1(\delta, -r, r') - S_1(\delta, r, -r') + S_1(\delta, -r, -r') \ , \tag{38}$$

with

$$S_1(\delta, r, r') = (\delta^3 - 3(r^2 + r'^2)\delta - 2(r^3 + r'^3))\, \text{sign}(\delta + r + r') \ . \tag{39}$$

With these integrations, the final expressions for $\mathcal{H}_0$ and $\mathcal{H}_1$ are

$$\mathcal{H}_0(t) = -\frac{\pi}{8} \int_{-\infty}^{t} d\tau \int_{-\infty}^{t} d\tau' \int_{0}^{\infty} dr \int_{0}^{\infty} dr'\, \mathcal{J}_\phi(\tau, r) \mathcal{J}_\theta(\tau', r')\, [P_0(\tau - \tau', r, r') - P_0(2t - \tau - \tau', r, r')] \tag{40}$$

$$\mathcal{H}_1(t) = -\frac{\pi}{48} \int_{-\infty}^{t} d\tau \int_{-\infty}^{t} d\tau' \int_{0}^{\infty} dr \int_{0}^{\infty} \frac{dr'}{r'}\, \mathcal{J}_\phi(\tau, r) \mathcal{J}_{r+\theta}(\tau', r')\, [P_1(\tau - \tau', r, r') - P_1(2t - \tau - \tau', r, r')] \ . \tag{41}$$

V.   ASYMPTOTIC MAGNETIC HELICITY

In the integrations for $\mathcal{H}_0(t)$ and $\mathcal{H}_1(t)$ in Eqs. (40) and (41), the factors $\mathcal{J}_r$, $\mathcal{J}_\theta$ and $\mathcal{J}_\phi$ have compact support. So the integrand is non-vanishing for some finite domain of τ, τ', r and r'. If we now take t to be large, while keeping τ, τ', r and r' finite, then $2t - \tau - \tau'$ is large and also large compared to r and r'. Then the "sign" factors in Eqs. (36) and (39) can be replaced by $+1$ and it can be checked that $P_0(2t - \tau - \tau', r, r') = 0$ and $P_1(2t - \tau - \tau', r, r') = 0$. Hence at late times, only the time independent pieces in $\mathcal{H}_0$ and $\mathcal{H}_1$ can survive, and we get

$$\mathcal{H}_0(\infty) = -\frac{\pi}{8} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \int_{0}^{\infty} dr \int_{0}^{\infty} dr'\, \mathcal{J}_\phi(\tau, r) \mathcal{J}_\theta(\tau', r') P_0(\tau - \tau', r, r') \tag{42}$$

$$\mathcal{H}_1(\infty) = -\frac{\pi}{48} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \int_0^{\infty} dr \int_0^{\infty} \frac{dr'}{r'} \mathcal{J}_\phi(\tau, r) \mathcal{J}_{r+\theta}(\tau', r') P_1(\tau - \tau', r, r') \tag{43}$$

and

$$\mathcal{H}(\infty) = \frac{16}{3} \left(\frac{g'v^2}{4} \right)^2 [\mathcal{H}_0(\infty) + \mathcal{H}_1(\infty)] . \tag{44}$$

In terms of the mass of the W-boson, $m_W = gv/2$, the helicity can be written as

$$\mathcal{H}(\infty) = \frac{16}{3} \frac{\sin^2 \theta_w}{g^2} m_W^4 [\mathcal{H}_0(\infty) + \mathcal{H}_1(\infty)] , \tag{45}$$

where θ_w is the weak mixing angle and for $g' \ll g$, $\sin \theta_w \sim \theta_w \sim g'/g$. The quantities $\mathcal{H}_0$ and $\mathcal{H}_1$ have dimensions of L^4 where L is a length scale that enters the profile functions. Assuming $r_f \sim r_h$ in the profile functions and left-handed helicity (see sections below), the parametric dependence of the helicity is

$$\mathcal{H}(\infty) \sim -\frac{\sin^2 \theta_w}{g^2} (m_W r_f)^4 . \tag{46}$$

The result can also be expressed in terms of the fine structure constant, $\alpha = e^2/4\pi \approx 1/137$, using $e = \sqrt{g^2 + g'^2} \sin \theta_w \cos \theta_w$.

The asymptotic helicity depends on the time evolution chosen in $\mu(t)$ and also on the profile functions. There is no apparent symmetry reason for the integrals to vanish, and hence the decay of the sphaleron will lead to a magnetic field with non-vanishing magnetic helicity. Below we will numerically evaluate the time-dependent magnetic helicity for a choice of $\mu(t)$ and the ansatz for the sphaleron profile functions in Eqs. (5) and (6). This evaluation also gives the asymptotic numerical value of the magnetic helicity.

VI. TIME DEPENDENT MAGNETIC HELICITY

Eqs. (40) and (41) give the time dependent magnetic helicity but the integrations are difficult to do analytically, especially because the true profile functions are not known in closed form, and the time evolution function, $\mu(t)$, needs to be chosen. Even with the very simple profile Ansatz of Eqs. (5) and (6), the integrations lead to a lot of different terms. So we have evaluated the integrals numerically but this also requires a choice of the time dependent function, $\mu(t)$.

One possible scheme to find a suitable $\mu(t)$ is to derive an effective equation of motion by substituting the field ansatz for the $SU(2)$ sphaleron in Sec. II in the energy functional. This evaluation was done in Ref. [6] by taking μ to be a parameter and not a dynamical variable. The result was a "potential energy" function

$$V(\mu) = m_W (C_1 \sin^2 \mu + C_2 \sin^4 \mu) , \tag{47}$$

where C_1 and C_2 are numerical constants. If we promote μ to a dynamical variable, the energy will also depend on $\dot{\mu}$ and we can obtain an equation of motion for μ. However, this exercise is moot because it leads to a conservative equation of motion for μ, while the true evolution is dissipative, and once μ gets to the true vacuum value (0 or π) it should stay there. One way to introduce dissipation is to replace second time derivatives by first time derivatives in the equation of motion. This suggests an equation of motion

$$\dot{\mu} = A_1 \sin \mu + A_2 \sin^3 \mu , \tag{48}$$

where A_1 and A_2 are some coefficients. The equation can be integrated and its messy solution is very similar to the much simpler

$$\mu(t) = \frac{\pi}{2} \left[1 + \tanh \left(\frac{t}{r_t} \right) \right] . \tag{49}$$

Hence we adopt this functional form for $\mu(t)$ and proceed with our numerical evaluation of the magnetic helicity. Note that $\mu(0) = \pi/2$ and $\mu(\infty) = \pi$, and hence this choice of $\mu(t)$ describes creation of baryons.

Next we write

$$\mathcal{H}(t) = \frac{\sin^2 \theta_w}{g^2} (m_W r_f)^4 \mathcal{F}(t) , \tag{50}$$
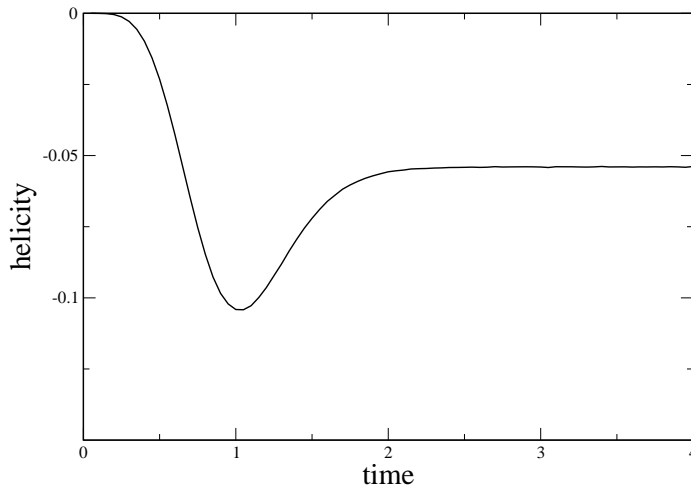
FIG. 1: $\mathcal{F}(t)$ versus t/r_t for $r_t = r_f = r_h$.

with the dimensionless function $\mathcal{F}(t)$ defined as

$$\mathcal{F}(t) = \frac{16}{3r_f^4} \left[\mathcal{H}_0(t) + \mathcal{H}_1(t) \right] \ . \tag{51}$$

The numerically evaluated $\mathcal{F}(t)$ vs. t is shown in Fig. 1 for $r_h = r_f = r_t$. An interesting feature of the plot is that the magnetic helicity is negative and conserved at late times.

The plot in Fig. 1 corresponds to Fig. 1 of Ref. [9] where it was obtained by solving the electroweak field equations on a spatial lattice starting with a perturbed sphaleron.

VII. SIGN OF HELICITY

An important virtue of the present calculation is that it can relate the sign of the asymptotic magnetic helicity to the direction in which the sphaleron decays, which in turn is also related to whether baryons or antibaryons are produced. So far we have only seen this connection by numerically evaluating the helicity integrals in the limit $t \to \infty$. Here we will analytically establish the connection for small times, just as the sphaleron starts to decay.

The sphaleron is located at $\mu = \pi/2$. Therefore we consider

$$\mu = \frac{\pi}{2} + \epsilon(t) = \frac{\pi}{2} + \dot{\epsilon}_0 \delta t + O((\delta t)^2) \ , \tag{52}$$

where δt is a small time interval and $\dot{\epsilon}_0$ denotes the velocity with which the sphaleron starts to roll. Now we can expand $\mathcal{J}_r$, $\mathcal{J}_\theta$ and $\mathcal{J}_\phi$ to lowest order in δt. This gives

$$\mathcal{J}_r \sim F_r(r)\dot{\epsilon}_0 \delta t + O((\delta t)^2) \tag{53}$$

$$\mathcal{J}_\theta \sim F_\theta(r)\dot{\epsilon}_0 \delta t + O((\delta t)^2) \tag{54}$$

$$\mathcal{J}_\phi = F_\phi(r) + O((\delta t)^2) \ , \tag{55}$$

where F_r, F_θ and F_ϕ are some time-independent functions. Next we count powers of δt in the helicity integrals. We get two powers from the two time integrals, one from the factors of the currents as given above, and one power from the terms in square brackets in Eqs. (40) and (41). Therefore

$$\mathcal{H}(\delta t) \sim -\dot{\epsilon}_0 (\delta t)^4 \times \mathcal{I} \ , \tag{56}$$

where $\mathcal{I}$ is an integral that does not depend on $\mu(t)$. Since $\mathcal{I}$ still depends on the profile functions, we can only evaluate it numerically and, as is clear from Fig. 1, $\mathcal{I} > 0$. This tells us that an increase of μ ($\dot{\epsilon}_0 > 0$), corresponding to baryon production, yields left-handed magnetic helicity, and a decrease of μ ($\dot{\epsilon}_0 < 0$) leads to right-handed magnetic helicity. In the cosmological context, since we know that baryons prevail, this predicts that left-handed magnetic helicity is prevalent in the universe [7].

VIII. CONCLUSIONS

We have developed an analytical technique to study the magnetic debris produced during baryon number violating processes that occur via sphalerons. We have calculated the time-dependent helicity of the magnetic fields that are produced and found the asymptotic helicity in terms of the decay path of the sphaleron up to quadrature. The sign of the helicity depends on whether baryons or antibaryons are produced and baryon production leads to left-handed magnetic helicity. Our final result for the time evolution of magnetic helicity is shown in Fig. 1 and compares well with the numerical computations of Ref. [9]. These figures demonstrate that magnetic helicity is conserved at late times. This is novel because the conservation of magnetic helicity is usually discussed in the magneto-hydrodynamic context where a highly conducting plasma is present. In our case there is no external plasma. The (approximate) conservation of helicity has also been noted in solutions to the *vacuum* Maxwell equations in Ref. [14].

The wider implication of our results is that there is likely to be a unified origin of cosmic matter and cosmic magnetic fields. This possibility has been discussed before [2, 3, 5, 7–9]. The analytical technique we have developed makes the connection between magnetic helicity and baryon number more transparent and has the potential for further development. For example, one could consider an ensemble of possible decay paths described by the function $\mu(t)$, and the mean magnetic helicity could then be evaluated as an average over this ensemble. Further one could incorporate our results in a realistic baryogenesis scenario and derive spectral properties of the magnetic field. A simple analysis of this kind was done in Ref. [15], where a model for the magnetic fields produced during sphaleron decay was assumed. Based on our present results, it appears that the model captures most of the features of the magnetic fields produced in sphaleron debris, and so it is worth re-displaying the magnetic field expressions here. In spherical coordinates with origin at the location of the sphaleron, the model for the magnetic field is

$$
\begin{aligned}
B_r &= -\frac{a\cos\theta}{(a^2+r^2)^{3/2}} \\
B_\theta &= +\frac{a\sin\theta}{2}\frac{2a^2-r^2}{(a^2+r^2)^{5/2}} \\
B_\phi &= \frac{r}{a^3}e^{-r/a}\sin\theta \ .
\end{aligned}
\tag{57}
$$

The extent of the magnetic field is determined by the length scale a, which is time-dependent. The simplest choice is

$$
a(t) = t - t_0 \ ,
\tag{58}
$$

where t_0 is the time at which the sphaleron started rolling down. As the magnetic field expands, just as in the case of the sphaleron, the helicity stays constant at

$$
\mathcal{H} = -\frac{8\pi}{3}0.57 = -4.8 \ .
\tag{59}
$$

The numerical value of the asymptotic helicity can be adjusted by suitably rescaling the magnetic field.

Once a space-filling helical magnetic field has been produced at the baryogenesis epoch, further evolution will be subject to magneto-hydrodynamical evolution in a cosmological setting e.g. [7, 16]. We expect the presence of magnetic helicity to have significant impact on the evolution. This topic, as well as the possibility of observation of early magnetic fields, deserves further investigation.

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