Finite volume effects in B_{K} with improved staggered fermions

Jangho Kim, Chulwoo Jung, Hyung-Jin Kim, Weonjong Lee, and Stephen R. Sharpe (SWME Collaboration)

Phys. Rev. D **83**, 117501 — Published 1 June 2011

DOI: 10.1103/PhysRevD.83.117501

Finite volume effects in B_K with improved staggered fermions

Jangho Kim,[1] Chulwoo Jung,[2] Hyung-Jin Kim,[1] Weonjong Lee,[1] and Stephen R. Sharpe[3]
(SWME Collaboration)
[1] *Lattice Gauge Theory Research Center, FPRD, and CTP, Department of Physics and Astronomy, Seoul National University, Seoul, 151-747, South Korea*
[2] *Physics Department, Brookhaven National Laboratory, Upton, NY11973, USA*
[3] *Physics Department, University of Washington, Seattle, WA 98195-1560, USA*

We extend our recent unquenched ($N_f = 2+1$ flavor) calculation of B_K using improved staggered fermions by including in the fits the finite volume shift predicted by one-loop staggered chiral perturbation theory. The net result is to lower the result in the continuum limit by 0.6%. This shift is slightly smaller than our previous estimate of finite volume effects based on a direct comparison between different volumes.

PACS numbers: 11.15.Ha, 12.38.Gc, 12.38.Aw
Keywords: lattice QCD, B_K, CP violation

We have recently reported a result for B_K using improved staggered fermions with all errors controlled and with a total error of 6% [1]. (We refer to this paper as SWME in the following; a recent update including an additional lattice spacing is given in Ref. [2].) Although the dominant error in this result is from uncertainty in the matching factors, an important subdominant source of error is that arising from our use of a finite volume. In SWME we estimated this error to be 0.85% by comparing the result obtained on two lattices of different volumes (ensembles C3 and C3-2, as discussed below). Here we revisit the finite volume (FV) error using an alternative approach: repeating our chiral fits using forms predicted by one-loop SU(2) staggered chiral perturbation theory (SChPT) but now including FV effects from pion loops.

We did not previously include one-loop FV effects because we found that fitting with the FV form using CPUs on our extensive dataset took too long (of order two months). We now use GPUs (Graphics Processing Units) which allows us to reduce the fitting time down to a few hours, while maintaining double-precision throughout. Thus we can undertake the many analyses needed to estimate systematic errors.

To extrapolate our data to the physical light quark masses and to the continuum limit we use the functional form predicted by SChPT[3, 4]. Specifically, our central result in SWME used partial next-to-next-to-leading order (NNLO) SU(2) SChPT fits, and we consider here only such fits. The key extrapolation is in the valence down-quark mass, m_x, or more precisely in the mass-squared of the valence pion with flavor $\bar{x}x$, which we call X_P. The NNLO form contains 3 parameters (at a fixed lattice spacing a and fixed sea-quark masses): the coefficients of (i) the constant + chiral logarithm term, (ii) the term linear in X_P and (iii) the term quadratic in X_P. For details see Eqs. (41-45) of SWME. Finite volume corrections only impact the chiral logarithms. In SWME these are written in terms of the functions:

$$\ell(X) = X\left[\log(X/\mu_{\rm DR}^2) + \delta_1^{\rm FV}(X)\right] , \quad (1)$$

$$\tilde{\ell}(X) = -\frac{d\ell(X)}{dX} = -\log(X/\mu_{\rm DR}^2) - 1 + \delta_3^{\rm FV}(X) , \quad (2)$$

where $\mu_{\rm DR}$ is the scale introduced by dimensional regularization. The FV parts, left out in SWME but included here, are

$$\delta_1^{\rm FV}(M^2) = \frac{4}{ML}\sum_{n\neq 0}\frac{K_1(|n|ML)}{|n|} \quad (3)$$

$$\delta_3^{\rm FV}(M^2) = 2\sum_{n\neq 0}K_0(|n|ML) , \quad (4)$$

where L is the spatial box size and $n = (n_1, n_2, n_3, n_4)$ is a vector of integers labeling image positions. The norm $|n|$ is defined as

$$|n| \equiv \sqrt{n_1^2 + n_2^2 + n_3^2 + \left(\frac{L_T}{L}n_4\right)^2} , \quad (5)$$

with L_T is the (Euclidean) temporal box size. K_0 and K_1 are the standard modified Bessel functions of the second kind, which fall exponentially for large x.

The expressions (3) and (4) for the FV corrections cease to be valid when $ML \lesssim 1$. One then moves from the so-called "p-regime" into the "ϵ-regime", and a different power-counting applies. Our calculations are done with valence pions satisfying $ML \gtrsim 3$, which, as the following results indicate, appears to be above the minimum value at which it is appropriate to use the above expressions.

The lattice ensembles used in this paper are those used in SWME to do the continuum extrapolation, namely (in the notation of SWME) C3 (coarse), F1 (fine), S1 (superfine), together with a new ensemble U1 (ultrafine). The latter has a nominal lattice spacing of 0.045 fm, sea quark masses $am_\ell = 0.0028$ and $am_s = 0.014$, and size $64^3 \times 192$. Our results for this fourth lattice spacing are preliminary, based on only 305 configurations. We also use the larger volume coarse ensemble, C3-2.

On each ensemble we calculate B_K using 10 different sea quark masses, ranging down from $\approx m_s^{\rm phys}$ to $\approx m_s^{\rm phys}/10$. The precise values are given in SWME for the S1, F1, C3 and C3-2 ensembles, while for the U1 ensemble they are $0.0014 \times n$ with $n = 1, 2, 3, \ldots, 10$. For the chiral extrapolation we use the lightest four valence masses for m_x, while for the valence strange quark (which we call m_y) we use the three heaviest valence masses to extrapolate to the physical strange-quark mass. We do the fit in two stages, first (the "X fit") extrapolating in the valence d quark mass to the physical value while holding the valence s quark mass fixed, and second (the "Y fit") extrapolating linearly to the physical valence s quark mass. Altogether this procedure is labeled a "4X3Y" fit.

To calculate the finite volume corrections $\delta_1^{\rm FV}$ and $\delta_3^{\rm FV}$, we must truncate the sums in Eqs. (3) and (4). We keep image vectors satisfying $|n| \le n_{\rm max}$, with $n_{\rm max}$ determined separately for each sum and for each value of ML, using the following criterion. For $|n| < n_{\rm max}$, a shell in image space of radius $|n|$ and unit thickness should give a contribution larger than the desired precision times the leading ($|n| = 1$) term. Thus we keep all n satisfying

$$[4\pi|n|^2] \times \frac{K_1(|n|ML)}{|n|} \ge \epsilon \times [6K_1(ML)] \quad \text{for } \delta_1^{\rm FV}\,, \quad (6)$$

$$[4\pi|n|^2] \times K_0(|n|ML) \ge \epsilon \times [6K_0(ML)] \quad \text{for } \delta_3^{\rm FV}\,, \quad (7)$$

with $\epsilon = 1.0 \times 10^{-14}$ since we want double-precision accuracy. Note that on the left-hand side we are assuming that the dominant contribution is from spatial images, which is a reasonable approximation.

The values of $n_{\rm max}$ depend, of course, on ML. For $\delta_1^{\rm FV}$ we find, for example, that for $ML = 0.4$, 2.0 and 4.0 that $n_{\rm max} = 93$, 18 and 10, respectively. The values for $\delta_3^{\rm FV}$ are slightly larger, although comparable.

Using a single core of the Intel i7 920 CPU, it takes about two months to run the analysis code on our full dataset in double precision including FV corrections, at a sustained speed of about 0.5 Gigaflop/s. By contrast, using the Nvidia GTX 480 GPU, and optimizing our code using the CUDA library, we obtain a sustained performance in double precision of 64.3 Gigaflop/s, 38% of peak.

We first compare the results of fitting with and without FV corrections on the C3 ensemble. We note that on this ensemble (and on C3-2) we have about 9 times the number of measurements as on the finer ensembles and thus considerably smaller statistical errors. Figure 1 shows the X-fits for our heaviest valence strange quark ($am_s^{\rm val} = 0.05$). Although our lightest valence pion mass corresponds to $ML = 2.7$ on these lattices, the difference between the infinite and finite volume fits is not significant until much smaller values of ML. This is in apparent contradiction with rule-of-thumb that one has large FV effects when $ML < 3$. This conundrum is resolved by the presence of taste breaking. Almost all the pions which appear in the loops have non-Goldstone tastes, and thus are heavier than the Goldstone pion by shifts of $O(a^2)$. These shifts push ML up to values considerably larger than 3, except for a small contribution from the Goldstone pion itself.

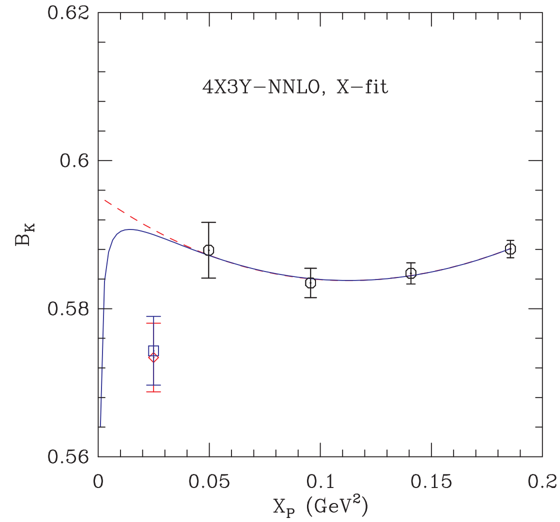

FIG. 1. $B_K(1/a)$ vs. X_P on the C3 ensemble for $am_y = 0.05$. The fit type is 4X3Y-NNLO. The dashed [red] line shows the result of fitting to the infinite volume form, while the solid [blue] line shows the fit with FV corrections included. The [red] diamond and [blue] square correspond to the "physical" B_K values obtained (as described in the text) from the infinite and finite volume fits respectively.

The net result is that fitting with the FV form leads to a very small shift in the extrapolated "physical" values of B_K, as shown in the first row of Table I. These values are obtained by using the fit function to (a) extrapolate to the physical valence d mass, (b) set all taste-splittings to zero, (c) set the light sea-quark pion mass-squared to its physical value, and (d), in the case of the FV fit, set the volume to infinity. We use quotes around "physical" because these values of B_K have yet to be extrapolated to the physical strange quark mass and to the continuum limit. Note also that B_K is here matched to a continuum operator renormalized at a scale $1/a$, so that results from different lattice spacings are not directly comparable.

TABLE I. "Physical" $B_K(\text{NDR}, 1/a)$ obtained as described in the text from fits without and with finite volume corrections, and the percentage difference between the two. The valence strange quark mass is set to to its heaviest value on each ensemble. Errors are statistical.

ID	B_K	B_K(FV)	ΔB_K
C3	0.5734(46)	0.5743(46)	+0.159(2)%
C3-2	0.5784(46)	0.5785(46)	+0.0319(3)%
F1	0.5225(111)	0.5199(110)	-0.505(14)%
S1	0.4914(65)	0.4898(65)	-0.329(6)%
U1	0.4780(92)	0.4757(92)	-0.474(14)%

From the Table, we see that the FV fit leads to a result that is 0.16% higher on the C3 ensemble. Although this difference is much smaller than the statistical errors in the individual results, it is statistically highly significant. This is possible because of the high degree of correlation between the infinite volume and FV fits. The same holds true on the other ensembles.

Figure 2 shows the corresponding fits on ensemble of larger lattices, C3-2. As expected, the difference between the infinite and finite volume curves does not become significant until smaller values of the pion masses than in Fig. 1. The "physical" B_K values resulting from these two fits are essentially the same. This supports our claim in SWME that we can essentially treat the C3-2 ensemble as having infinite volume when calculating B_K.

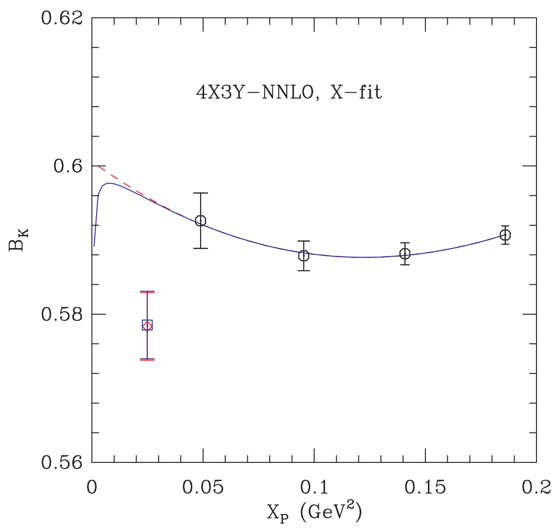

FIG. 2. $B_K(1/a)$ on the C3-2 ensemble. Notation as in Fig. 1.

A natural question is whether the FV fit leads to better agreement between the C3 and C3-2 ensembles. The answer is a qualified yes. The FV shift does bring the two results closer, although it removes only 20% of the difference. It should be noted, however, that the results on these two ensembles, which are statistically independent, agree within 1-σ already before the FV correction. Thus the improvement is marginal.

Figure 3 shows the fits with and without FV corrections on the U1 ensemble. The results on F1 and S1 ensembles are qualitatively similar, although the FV effects are smaller than on the U1 ensemble. We see that the FV corrections are of opposite sign to those on the C3 and C3-2 ensembles, that FV effects set in at a larger value of X_P as one reduces a, and that FV effects at the physical value of X_P grow as a is reduced. All these changes are due to the reduction in the size of taste breaking as a is decreased, so that the values of ML for non-Goldstone pions are much closer to those of the Goldstone pion, enhancing FV effects. It turns out that the enhancement is greater for terms contributing positively than those contributing negatively, leading to the change in sign. It is clear from the curves that our lightest valence pions are close to the smallest values that can be used without encountering large FV corrections.

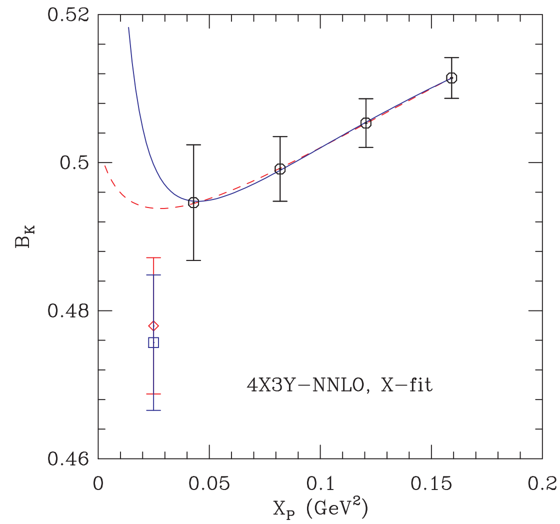

FIG. 3. $B_K(1/a)$ on the U1 ensemble. Notation as in Fig. 1.

The corresponding values of "physical" B_K are given in Table I. The FV shifts are now larger, as large as 0.5% in magnitude, although still small compared to other systematic errors. We stress that one cannot gauge the size of the FV shift at the physical value of X_P by looking at the difference between the two curves for this value. This is because the fits are being done at larger values of X_P, where the FV corrections are smaller.

In Fig. 4, we show how the continuum limit is impacted by the inclusion of FV corrections. Here we plot B_K after extrapolation to the physical valence strange quark mass (using 3Y fits), and after running in the continuum to a common scale of 2 GeV. The overall effect of fitting with FV corrections is to reduce the continuum result from 0.5260(73) to 0.5229(72), i.e. by $0.59 \pm 0.01\%$.

In conclusion, using GPUs we have been able to improve the chiral extrapolation of our data for B_K by the inclusion of FV effects arising from pion loop diagrams. The net result is that, after continuum extrapolation, we find that including the FV shifts leads to an (infinite volume) result for B_K which is 0.59% smaller than that obtained when fitting with the infinite volume forms. The smallness of this shift confirms that our values of ML (which range down to ≈ 2.7) are (barely) large enough. In this regard, we are helped by the presence of taste breaking, so that most of the pions appearing in the loops are heavier than our lightest pion.

As is well known, one-loop ChPT forms can underestimate the size of FV shifts, due to the contributions from higher-order terms. This underestimate can be by as much as a factor of 2 for our range of ML (see, e.g., Ref. [5]). Thus, our alternative procedure for accounting for FV errors is to take the central value from the fits including FV effects, and take the systematic error to be the size of the FV shift, namely 0.6%. This estimate is a little smaller than our previous estimate of the FV error, 0.89%, which was based on the difference between

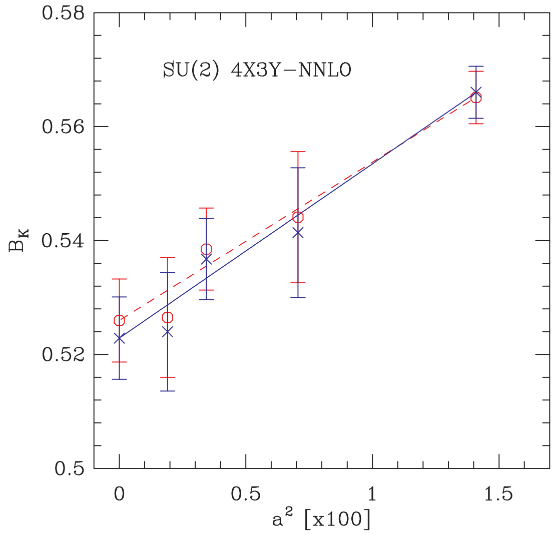

FIG. 4. Continuum extrapolation of $B_K(\mathrm{NDR}, 2\ \mathrm{GeV})$. [Red] octagons show results obtained using SU(2) 4X3Y-NNLO SChPT fits without the finite volume corrections, while [blue] crosses show the results obtained with FV fits. The points at $a = 0$ are obtained by linear extrapolation in a^2.

the results on the C3 and C3-2 ensembles. Given the uncertainties in the two error estimates, however, they are certainly consistent.[1]

We conclude that our previous finding that the FV error is subdominant relative to the matching error is supported by our new analysis.

ACKNOWLEDGMENTS

C. Jung is supported by the US DOE under contract DE-AC02-98CH10886. The research of W. Lee is supported by the Creative Research Initiatives program (3348-20090015) of the NRF grant funded by the Korean government (MEST). The work of S. Sharpe is supported in part by the US DOE grant no. DE-FG02-96ER40956. Computations for this work were carried out in part on QCDOC computers of the USQCD Collaboration at Brookhaven National Laboratory. The USQCD Collaboration are funded by the Office of Science of the U.S. Department of Energy.

[1] T. Bae, Y.-C. Jang, C. Jung, H.-J. Kim, J. Kim, J. Kim, K. Kim, W. Lee, S. R. Sharpe, and B. Yoon, Phys.Rev. **D82**, 114509 (2010), arXiv:1008.5179.
[2] T. B. Boram Yoon, Y.-C. Jang, H.-J. Kim, J. Kim, J. Kim, K. Kim, W. Lee, C. Jung, and S. R. Sharpe, PoS **(Lattice 2010)**, 319 (2010), arXiv:1010.4778.
[3] W. Lee and S. R. Sharpe, Phys. Rev. **D60**, 114503 (1999), hep-lat/9905023.
[4] C. Aubin and C. Bernard, Phys. Rev. D **68**, 034014 (2003), arXiv:hep-lat/0304014.
[5] G. Colangelo, S. Durr, and C. Haefeli, Nucl.Phys. **B721**, 136 (2005), hep-lat/0503014.

[1] The main uncertainty in our previous estimate is the fact that the difference between the results on ensembles C3 and C3-2 could be partly or wholly statistical.