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# General Helicity Formalism for Semi-Inclusive Deep Inelastic Scattering

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We study polarized and unpolarized Semi-Inclusive Deep Inelastic Scattering (SIDIS) processes, $\ell(S_\ell) + p(S) \to \ell'\, h\, X$, within a QCD parton model motivated by a generalized QCD factorization scheme. We take into account all transverse motions, of partons inside the initial proton and of hadrons inside the fragmenting partons and use the helicity formalism. The elementary interactions are computed at LO with non collinear exact kinematics, which introduces phases in the expressions of their helicity amplitudes. Several Transverse Momentum Dependent (TMD) distribution and fragmentation functions appear and contribute to the cross sections and to spin asymmetries. Our results agree with those obtained with different formalisms, showing the consistency of our approach. The full expression for single and double spin asymmetries $A_{S_\ell S}$ is derived. Simplified, explicit analytical expressions, convenient for phenomenological studies, are obtained assuming a factorized Gaussian dependence on intrinsic momenta for the TMDs.

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## I. INTRODUCTION

Experiments with inclusive Deep Inelastic Scattering (DIS) processes, $\ell\, N \to \ell'\, X$, have been performed for decades and have been interpreted as the most common way to investigate the internal structure of protons and neutrons. At large energy and momentum transfer the leptons interact with the nucleon constituents; by detecting the angle and the energy of the scattered lepton one obtains information on the partonic content of the nucleons. This information is encoded in the Parton Distribution Functions (PDFs) which give the number density of partons moving collinearly with the nucleon and carrying a fraction $x$ of its momentum at a certain value of the squared momentum transfer $Q^2$. The prediction of the $Q^2$ dependence of the PDFs has been one of the great successes of pQCD. Although successful, such an approach only offers information on the longitudinal degrees of freedom of quarks and gluons, giving no information on the transverse motion, which is integrated over. This transverse motion − transverse with respect to the parent nucleon direction − is related to intrinsic properties of the partons, like orbital motion, and reveals new aspects of the nucleon structure.

In the last years, driven by unexpected spin effects and azimuthal dependences, the study of the intrinsic motion of partons has made enormous progress; indeed, a new phase in the exploration of the proton and neutron composition has begun. The leading role in such an effort is played by Semi-Inclusive Deep Inelastic Scattering (SIDIS) processes, $\ell\, N \to \ell'\, h\, X$, in which, in addition to the scattered lepton, also a final hadron is detected; this hadron is generated in the fragmentation of the scattered quark (or gluon) − the so-called current fragmentation region − and, as such, yields some new information on the parton primordial motion. This new information is encoded in the so-called Transverse Momentum Dependent partonic distribution and fragmentation functions (TMD-PDFs and TMD-FFs, or, shortly, TMDs), $\hat f_{a/p}(x, \boldsymbol{k}_\perp)$ and $\hat D_{h/a}(z, \boldsymbol{p}_\perp)$. The TMD-PDFs give the number density of quarks ($a = q$), antiquarks ($a = \bar q$) or gluons ($a = g$) with light-cone momentum fraction $x$ and transverse momentum $\boldsymbol{k}_\perp$ inside a fast moving proton; the TMD-FFs give the number density of hadrons $h$ resulting in the fragmentation of parton $a$, with a light-cone momentum fraction $z$ and a transverse momentum $\boldsymbol{p}_\perp$, relative to the original parton motion. At leading-twist, taking into account the parton and the nucleon spins, there are eight independent TMD-PDFs [1, 2]; if the final hadron is unpolarized or spinless, say a pion, there are two TMD-FFs. All these quantities combine into physical observables and by gathering information about them one accesses the momentum distribution of partons inside the nucleons.

The theoretical framework used to analyze the experimental data is the QCD factorization scheme, according to which the SIDIS cross section is written as a convolution of TMDs and elementary interactions:

$$d\sigma^{\ell p\to\ell' hX} = \sum_q \hat f_{q/p}(x, \boldsymbol{k}_\perp; Q^2) \otimes d\hat\sigma^{\ell q\to\ell q} \otimes \hat D_{h/q}(z, \boldsymbol{p}_\perp; Q^2)\,. \tag{1}$$

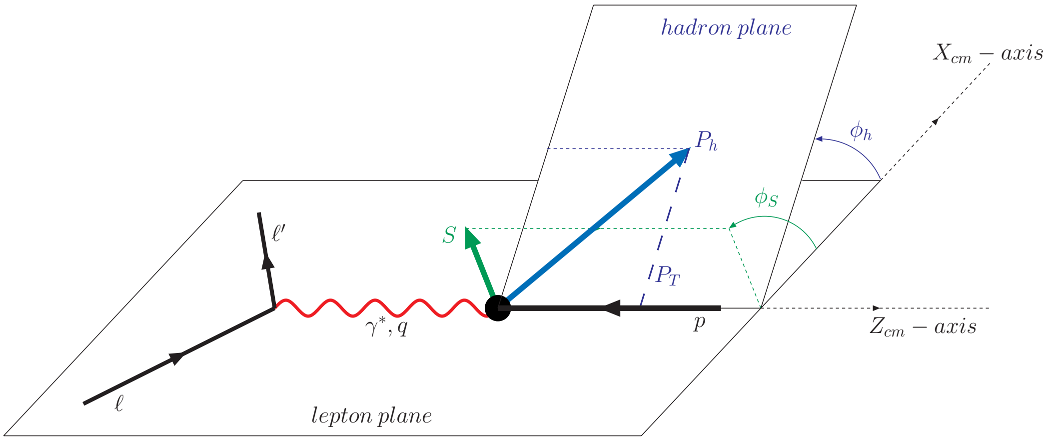


FIG. 1: Kinematical configuration and conventions for SIDIS processes. The initial and final lepton momenta define the $(X - Z)_{cm}$ plane.

In the $\gamma^* - p$ c.m. frame, see Fig. 1, the measured transverse momentum, $\boldsymbol{P}_T$, of the final hadron is generated by the transverse momentum of the quark in the target proton, $\boldsymbol{k}_\perp$, and of the final hadron with respect to the fragmenting quark, $\boldsymbol{p}_\perp$. At order $k_\perp/Q$ it is simply given by

$$\boldsymbol{P}_T = z\,\boldsymbol{k}_\perp + \boldsymbol{p}_\perp \,. \tag{2}$$

There is a general consensus [3–7] that such a scheme holds in the kinematical region defined by

$$P_T \simeq \Lambda_{\rm QCD} \ll Q \,. \tag{3}$$

The presence of the two scales, small $P_T$ and large $Q$, allows to identify the contribution from the unintegrated partonic distribution ($P_T \simeq k_\perp$), while remaining in the region of validity of the QCD parton model. At larger values of $P_T$ other mechanisms, like quark-gluon correlations and higher order pQCD contributions become important [7–9]. Corrections at subleading order in $1/Q$ might spoil the factorization scheme [10]. A similar situation [4, 6, 11–17] holds for Drell-Yan processes, $AB \to \ell^+\ell^- X$, where the two scales are the small transverse momentum, $q_T$, and the large invariant mass, $M$, of the dilepton pair.

Let us elaborate now on Eq. (1). We consider the SIDIS cross section at the leading $\alpha_{\rm em}$ order – *i.e.* one-photon exchange – and in the "standard" [18] kinematical configuration of Fig. 1, which defines the azimuthal angles $\phi_h$ and $\phi_S$ in the $\gamma^* - p$ c.m. frame. The most general dependence on these angles has been discussed in several seminal papers [1, 19–21], both in a model independent scheme and in the parton model. According to the usual derivation, the polarization states of the virtual photon, as emitted by the lepton in a certain direction, contains azimuthal dependences [19, 20]; within the parton model, the virtual photon scatters off a quark – which subsequently fragments into the final hadron – and each term of the azimuthal dependences can be written as a convolution of distribution and fragmentation functions [1, 20–23].

We re-derive here the same general expression of the cross section, and its parton model content, by assuming from the beginning the validity of the TMD factorization (1); we use the helicity basis to compute the elementary interaction and to introduce transverse momentum dependent distribution and fragmentation functions. In such an approach the full azimuthal dependence is simply generated by the properties of the helicity spinors and amplitudes. Our final results coincide with the existing ones, showing the full equivalence of the two procedures. Our formalism is based on a physical and intuitive picture, which somehow factorizes the physical process in different steps: the "emission" of a parton by the interacting hadron ($p \to q + X$), the interaction of the parton with the lepton ($\ell\, q \to \ell\, q$), and the "emission" of the final hadron by the scattered quark ($q \to h + X$); each step is described by the corresponding helicity amplitudes. For SIDIS processes this factorization has been formally proven and expressed in terms of TMDs, Eq. (1). Such a procedure can naturally be extended to other processes, and indeed this has been done for the large $P_T$ production of a single particle in inclusive hadronic interactions, $A\,B \to C\,X$ [2]. The point, however, is that, despite the natural simplicity of the approach, the TMD factorization has not been proven for processes with a single large scale, like $A\,B \to C\,X$. Due to this, the study of dijet production at large $P_T$ in hadronic processes was proposed [24–27], where the second small scale is the total $q_T$ of the two jets, which is of the order of the intrinsic partonic momentum $k_\perp$. This procedure leads to a modified TMD factorization approach, with the inclusion in the elementary processes of gauge link color factors [28–31]. However, some doubts on the validity of such a factorization scheme have been recently cast [32–34]. A possible experimental test of the TMD factorization for processes with only one large scale has been proposed in Ref. [35]. We limit our discussion in this paper to SIDIS processes, in the kinematical region (3) for which TMD factorization holds, and obtain the most general expression for the polarized cross section, with our helicity formalism. A similar study can be done, with the same validity, for Drell-Yan processes [13, 17, 36]. We introduce only leading-twist TMDs and take into account exact kinematics, often simplifying results by only keeping terms up to $\mathcal{O}\,(k_\perp/Q)$.

The paper is organized as follows. In Section II we present our formalism and compute the polarized SIDIS cross section. In Section III we give the explicit general expressions of all independent single and double spin asymmetries, in terms of the TMDs. In Section IV we give explicit analytical formulae for the spin and azimuthal asymmetries, assuming a factorized Gaussian dependence of the TMDs on $k_\perp$ and $p_\perp$. In Section V we draw our conclusions. Useful results are derived and collected in Appendices A–E.

## II. CROSS SECTIONS IN POLARIZED SIDIS

According to Refs. [37] and [2] the full differential cross section for the polarized SIDIS process, $\ell(S_\ell) + p(S) \to \ell' \, h \, X$, can be written, within TMD factorization, as

$$\begin{aligned}\frac{d\sigma^{\ell(S_\ell)+p(S)\to\ell' hX}}{dx_{\scriptscriptstyle B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} \;&=\; \frac{1}{2\pi}\sum_q\sum_{\{\lambda\}}\frac{1}{16\,\pi\,(x_{\scriptscriptstyle B}\, s)^2}\int d^2\boldsymbol{k}_\perp\,\frac{z}{z_h}\,J \\ &\times\;\rho^{\ell,S_\ell}_{\lambda_\ell\lambda'_\ell}\,\rho^{q_i/p,S}_{\lambda_{q_i}\lambda'_{q_i}}\,\hat f_{q_i/p,S}(x,\boldsymbol{k}_\perp)\,\hat M_{\lambda_{\ell'}\lambda_{q_f};\lambda_\ell\lambda_{q_i}}\,\hat M^*_{\lambda_{\ell'}\lambda'_{q_f};\lambda'_\ell\lambda'_{q_i}}\,\hat D^{\lambda_h,\lambda_h}_{\lambda_{q_f},\lambda'_{q_f}}(z,\boldsymbol{p}_\perp)\;,\end{aligned} \tag{4}$$

where we adopt the kinematical configuration of Fig. 1, and, as usual:

$$s = (\ell+p)^2 \qquad Q^2 = -q^2 = -(\ell-\ell')^2 \qquad x_{\scriptscriptstyle B} = \frac{Q^2}{2p\cdot q} \qquad z_h = \frac{p\cdot P_h}{p\cdot q}\;\cdot \tag{5}$$

The variables $x$, $z$ and $\boldsymbol{p}_\perp$ which appear under integration in Eq. (4) are related to the final observed variables $x_{\scriptscriptstyle B}$, $z_h$ and $\boldsymbol{P}_T$ and to the integration variable $\boldsymbol{k}_\perp$. The exact relations can be found in Ref. [37]; at $\mathcal{O}(k_\perp/Q)$ one simply has

$$x = x_{\scriptscriptstyle B} \qquad z = z_h \qquad \boldsymbol{p}_\perp = \boldsymbol{P}_T - z_h \boldsymbol{k}_\perp\;. \tag{6}$$

$J$ includes some non-planar kinematical factors [37]:

$$J = \frac{x_{\scriptscriptstyle B}}{x}\left(1+\frac{x_{\scriptscriptstyle B}^2}{x^2}\frac{k_\perp^2}{Q^2}\right)^{-1} \simeq 1\,, \tag{7}$$

where the last relation holds at $\mathcal{O}(k_\perp/Q)$. At this order Eq. (4) can be written as:

$$\begin{aligned}\frac{d\sigma^{\ell(S_\ell)+p(S)\to\ell' hX}}{dx_{\scriptscriptstyle B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} \;&\simeq\; \frac{1}{2\pi}\sum_q\sum_{\{\lambda\}}\frac{1}{16\,\pi\,(x_{\scriptscriptstyle B}\, s)^2}\int d^2\boldsymbol{k}_\perp\, d^2\boldsymbol{p}_\perp\;\delta^{(2)}(\boldsymbol{P}_T - z_h\boldsymbol{k}_\perp - \boldsymbol{p}_\perp) \\ &\times\;\rho^{\ell,S_\ell}_{\lambda_\ell\lambda'_\ell}\,\rho^{q/p,S}_{\lambda_{q_i}\lambda'_{q_i}}\,\hat f_{q_i/p,S}(x,\boldsymbol{k}_\perp)\,\hat M_{\lambda_{\ell'}\lambda_{q_f};\lambda_\ell\lambda_{q_i}}\,\hat M^*_{\lambda_{\ell'}\lambda'_{q_f};\lambda'_\ell\lambda'_{q_i}}\,\hat D^{\lambda_h,\lambda_h}_{\lambda_{q_f},\lambda'_{q_f}}(z,\boldsymbol{p}_\perp)\;,\end{aligned} \tag{8}$$

where we have explicitly shown the integration over $\boldsymbol{p}_\perp$ for clarity and further use. In Eqs. (4) and (8) the sums are performed over all quark flavors ($q = u, \bar u, d, \bar d, s, \bar s$) and all quark, lepton and hadron helicity indices; $\rho^{\ell,S_\ell}_{\lambda_\ell\lambda'_\ell}$ is the initial lepton helicity density matrix, which describes the spin state of the lepton beam; for unpolarized leptons one simply has $\rho^{\ell}_{\lambda_\ell\lambda'_\ell} = \frac{1}{2}\,\delta_{\lambda_\ell\lambda'_\ell}$. It might be helpful, and useful for physical interpretations, to recall that, in general, for a spin 1/2 Dirac particle one has:

$$\rho_{\lambda\,\lambda'} = \frac{1}{2}\begin{pmatrix} 1+P_z & P_x - iP_y \\ P_x + iP_y & 1-P_z\end{pmatrix}, \tag{9}$$

where $P_j = P_x, P_y, P_z$ are the components of the particle polarization vector in its helicity frame (throughout the paper we follow the definitions and conventions for helicity states of Ref. [38]).

Let us discuss in detail the different "factors" in Eq. (4): they represent the distribution of polarized partons (only quarks at LO) inside the proton, their interaction with the lepton and the fragmentation of the (polarized) final quark into the observed unpolarized hadron $h$. We follow, and adapt to the case of SIDIS, the discussion of Ref. [2]. We describe the three stages of the process – quark emission, interaction and fragmentation – within the helicity formalism, which allows us to introduce in a natural way, at each step, several phases; these, when combined into the expression for the physical cross section (4) give its full azimuthal dependence, in agreement with results in the literature derived in a more formal and somewhat less intuitive way [23].

### A. TMD partonic distribution functions

$\rho^{q_i/p,S}_{\lambda_{q_i}\lambda'_{q_i}}\,\hat{f}_{q_i/p,S}(x,\boldsymbol{k}_\perp)$ counts the number of polarized quarks inside a polarized proton; it is the polarized distribution function of the initial quark $q_i$ with light-cone momentum fraction $x$ and intrinsic transverse momentum $\boldsymbol{k}_\perp$, inside the target proton $p$ in a pure spin state $S$. Using Eq. (9) and parity invariance one can see that there are eight independent distribution functions, which can be defined as:

$$P^q_j\,\hat{f}_{q/p,S_T}(x,\boldsymbol{k}_\perp) = \hat{f}^q_{s_j/S_T}(x,\boldsymbol{k}_\perp) - \hat{f}^q_{-s_j/S_T}(x,\boldsymbol{k}_\perp) \equiv \Delta\hat{f}^q_{s_j/S_T}(x,\boldsymbol{k}_\perp) \tag{10}$$

$$P^q_j\,\hat{f}_{q/p,S_L}(x,\boldsymbol{k}_\perp) = \hat{f}^q_{s_j/S_L}(x,\boldsymbol{k}_\perp) - \hat{f}^q_{-s_j/S_L}(x,\boldsymbol{k}_\perp) \equiv \Delta\hat{f}^q_{s_j/S_L}(x,\boldsymbol{k}_\perp) \tag{11}$$

$$\hat{f}_{q/p,S_T}(x,\boldsymbol{k}_\perp) \equiv f_{q/p}(x,k_\perp) + \frac{1}{2}\,\Delta\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp)\,, \tag{12}$$

with

$$\Delta\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp) \equiv \hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp) - \hat{f}_{q/-S_T}(x,\boldsymbol{k}_\perp)\,. \tag{13}$$

We define, for further use,

$$\frac{1}{2}\,[\hat{f}_{s_y/S_T}(x,\boldsymbol{k}_\perp) - \hat{f}_{s_y/-S_T}(x,\boldsymbol{k}_\perp)] \equiv \Delta^-\hat{f}_{s_y/S_T}(x,\boldsymbol{k}_\perp). \tag{14}$$

In Eqs. (10) and (11), $j = x, y, z$ are the coordinate-axes in the quark helicity frame and $S_{L,T}$ are respectively the longitudinal and transverse components of the proton polarization vector, with respect to its direction of motion.

Different notations can be found in the literature for these functions, in particular those introduced by the Amsterdam group [1, 39, 40], which are largely adopted. The relationships between the two sets can be found in Ref. [2], and will be repeated for convenience in Eqs. (22)–(25).

According to the physical interpretation of the factorization scheme, as outlined above, these quantities can be introduced by making use of the helicity amplitudes $\hat{\mathcal{F}}_{\lambda_q,\lambda_X;\lambda_p}$, which describe the soft process $p \to q + X$. Since the partonic distribution is usually regarded, at LO, as the inclusive cross section for this process, the helicity density matrix of a quark $q$ inside the proton $p$ with spin $S$ can be written as

$$\begin{aligned}\rho^{q/p,S}_{\lambda_q\lambda'_q}\,\hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp) &= \sum_{\lambda_p,\lambda'_p}\rho^{p,S}_{\lambda_p\lambda'_p}\;\sum\!\!\!\!\!\!\!\int_{X,\lambda_X}\hat{\mathcal{F}}_{\lambda_q,\lambda_X;\lambda_p}\,\hat{\mathcal{F}}^*_{\lambda'_q,\lambda_X;\lambda'_p}\\ &\equiv \sum_{\lambda_p,\lambda'_p}\rho^{p,S}_{\lambda_p\lambda'_p}\,\hat{F}^{\lambda_q,\lambda'_q}_{\lambda_p,\lambda'_p}\,,\end{aligned} \tag{15}$$

having defined

$$\hat{F}^{\lambda_q,\lambda'_q}_{\lambda_p,\lambda'_p} \equiv \sum\!\!\!\!\!\!\!\int_{X,\lambda_X}\hat{\mathcal{F}}_{\lambda_q,\lambda_X;\lambda_p}\,\hat{\mathcal{F}}^*_{\lambda'_q,\lambda_X;\lambda'_p}\,, \tag{16}$$

where the $\sum\!\!\!\!\!\!\!\int_{X,\lambda_X}$ stands for a spin sum and phase-space integration over all the undetected remnants of the proton, considered as a system $X$, and the $\hat{\mathcal{F}}$'s are the *helicity distribution amplitudes* for the $p \to q + X$ process. Eq. (15) relates, via the unknown distribution amplitudes, the helicity density matrix of the parton $q$,

$$\rho^{q/p,S}_{\lambda_q\lambda'_q} = \frac{1}{2}\begin{pmatrix}1+P^q_z & P^q_x - iP^q_y\\ P^q_x + iP^q_y & 1-P^q_z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+P^q_z & P^q_T\,e^{-i\varphi_{s_q}}\\ P^q_T\,e^{i\varphi_{s_q}} & 1-P^q_z\end{pmatrix}, \tag{17}$$

to the helicity density matrix of the polarized parent proton,

$$\rho^{p,S}_{\lambda_p\lambda'_p} = \frac{1}{2}\begin{pmatrix} 1+S_Z & S_X - iS_Y \\ S_X + iS_Y & 1-S_Z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+S_L & S_T\, e^{-i\varphi_S} \\ S_T\, e^{i\varphi_S} & 1-S_L \end{pmatrix}. \tag{18}$$

In the above equations $\boldsymbol{S} = (S_X, S_Y, S_Z) = (S_T\cos\varphi_S, S_T\sin\varphi_S, S_L)$ is the proton polarization vector and $\varphi_S$ its azimuthal angle, defined in the helicity reference frame of the proton $p$. Similarly, $\boldsymbol{P}^q = (P^q_x, P^q_y, P^q_z) = (P^q_T\cos\varphi_{s_q}, P^q_T\sin\varphi_{s_q}, P^q_z)$ is the quark polarization vector defined in the quark helicity frame and $\varphi_{s_q}$ its azimuthal angle. For the kinematical configuration of Fig. 1, one has $\varphi_S = 2\pi - \phi_S$ (see Appendix B), so that:

$$\rho^{p,S}_{\lambda_p\lambda'_p} = \frac{1}{2}\begin{pmatrix} 1+S_L & S_T\, e^{i\phi_S} \\ S_T\, e^{-i\phi_S} & 1-S_L \end{pmatrix}. \tag{19}$$

Notice that, in general, we denote by $\varphi$ angles defined in the proton or quark helicity frames, while the symbol $\phi$ is used for the corresponding angles measured in the $\gamma^* - p$ c.m. frame.

The distribution amplitudes $\hat{\mathcal{F}}$ depend on the parton light-cone momentum fraction $x$ and on its intrinsic transverse momentum $\boldsymbol{k}_\perp$, with modulus $k_\perp$ and azimuthal angle $\phi_\perp$, in a precise way [2, 38], which, again referred to the kinematical configuration of Fig. 1, reads:

$$\hat{\mathcal{F}}_{\lambda_q,\lambda_X;\lambda_p}(x,\boldsymbol{k}_\perp) = \mathcal{F}_{\lambda_q,\lambda_X;\lambda_p}(x,k_\perp)\exp[-i\lambda_p\phi_\perp]\,, \tag{20}$$

so that

$$\hat{F}^{\lambda_q,\lambda'_q}_{\lambda_p,\lambda'_p}(x,\boldsymbol{k}_\perp) = F^{\lambda_q,\lambda'_q}_{\lambda_p,\lambda'_p}(x,k_\perp)\exp[i(\lambda'_p-\lambda_p)\phi_\perp]\,. \tag{21}$$

$F^{\lambda_q,\lambda'_q}_{\lambda_p,\lambda'_p}(x,k_\perp)$ has the same definition as $\hat{F}^{\lambda_q,\lambda'_q}_{\lambda_p,\lambda'_p}(x,\boldsymbol{k}_\perp)$, Eq. (16), with $\hat{\mathcal{F}}$ replaced by $\mathcal{F}$, and does not depend on phases anymore. Notice that we have chosen, throughout the paper, to denote with a hat all soft quantities which depend on both the modulus and the phase of the $\boldsymbol{k}_\perp$ and $\boldsymbol{p}_\perp$ intrinsic momentum vectors, while we drop the hat for quantities which only depend on the modulus of these vectors and not on their phases.

Eqs. (15), (17), (19) and (21), together with parity properties and the arguments collected in Appendix B, allow to extract the explicit phase dependence of the eight independent distribution functions which appear in Eqs. (10)–(12), with the result (more details can be found in Ref. [2]):

$$\begin{aligned}
\hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp) &= F^{++}_{++}(x,k_\perp) + F^{++}_{--}(x,k_\perp) - 2\,S_T\,\mathrm{Im}F^{++}_{+-}(x,k_\perp)\,\sin(\phi_S-\phi_\perp) &(22)\\
&= f_{q/p}(x,k_\perp) - \frac{1}{2}\,S_T\,\Delta f_{q/S_T}(x,k_\perp)\sin(\phi_S-\phi_\perp)\\
&= f_1(x,k_\perp) + S_T\,\frac{k_\perp}{M}\,f_{1T}^\perp(x,k_\perp)\,\sin(\phi_S-\phi_\perp)\\
P^q_z\,\hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp) &= S_L\left[F^{++}_{++}(x,k_\perp) - F^{++}_{--}(x,k_\perp)\right] + 2\,S_T\,\mathrm{Re}F^{++}_{+-}(x,k_\perp)\,\cos(\phi_S-\phi_\perp) &(23)\\
&= S_L\,\Delta f^q_{s_z/S_L}(x,k_\perp) + S_T\,\Delta f^q_{s_z/S_T}(x,k_\perp)\,\cos(\phi_S-\phi_\perp)\\
&= S_L\,g_{1L}(x,k_\perp) + S_T\,\frac{k_\perp}{M}\,g_{1T}^\perp(x,k_\perp)\,\cos(\phi_S-\phi_\perp)\\
P^q_x\,\hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp) &= -2\,S_L\,\mathrm{Re}F^{+-}_{++}(x,k_\perp) - S_T\left[F^{+-}_{+-}(x,k_\perp) + F^{-+}_{+-}(x,k_\perp)\right]\,\cos(\phi_S-\phi_\perp) &(24)\\
&= -S_L\,\Delta f^q_{s_x/S_L}(x,k_\perp) - S_T\,\Delta f^q_{s_x/S_T}(x,k_\perp)\,\cos(\phi_S-\phi_\perp)\\
&= -S_L\,\frac{k_\perp}{M}\,h_{1L}^\perp(x,k_\perp) - S_T\left[h_1(x,k_\perp) + \frac{k_\perp^2}{2M^2}\,h_{1T}^\perp(x,k_\perp)\right]\,\cos(\phi_S-\phi_\perp)\\
P^q_y\,\hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp) &= 2\,\mathrm{Im}F^{+-}_{++}(x,k_\perp) + S_T\left[F^{+-}_{+-}(x,k_\perp) - F^{-+}_{+-}(x,k_\perp)\right]\,\sin(\phi_S-\phi_\perp) &(25)\\
&= -\Delta f^q_{s_y/p}(x,k_\perp) + S_T\,\Delta^- f^q_{s_y/S_T}(x,k_\perp)\,\sin(\phi_S-\phi_\perp)\\
&= \frac{k_\perp}{M}\,h_1^\perp(x,k_\perp) + S_T\left[h_1(x,k_\perp) - \frac{k_\perp^2}{2M^2}\,h_{1T}^\perp(x,k_\perp)\right]\,\sin(\phi_S-\phi_\perp)\,.
\end{aligned}$$

As already stated, $\phi_S$ and $\phi_\perp$ are respectively the azimuthal angle of the proton polarization vector $\boldsymbol{S}$ and of the quark intrinsic momentum $\boldsymbol{k}_\perp$ measured in the $\gamma^* - p$ c.m. frame of Fig. 1. Also the quark polarization vector components $P^q_i$ $(i = x,y,z)$ refer to the helicity frame of the quark, as reached from the $\gamma^* - p$ frame: this explains the sign differences between Eqs. (22, 24–25) and Eqs. (B12, B14–B15) of Ref. [2] (in the latter case the polarized proton was moving along $Z_{cm}$ rather than $-Z_{cm}$. Further comments are given in Appendix B). Notice that, while $P_y^q\, f_{q/p} \neq 0$, one has $P_x^q\, f_{q/p} = P_z^q\, f_{q/p} = 0$.

The above equations, which will be soon used, deserve some further explanation. In each equation the first line expresses the partonic distributions in terms of the $F_{\lambda_p,\lambda'_p}^{\lambda_q,\lambda'_q}(x,k_\perp)$'s and shows their exact phase dependence. The second line gives the same quantities using our notations for the TMD-PDFs. According to our "hat convention", quantities like $\Delta f^q_{s_j/S}(x,k_\perp)$ do not depend on phases anymore, as such dependence has been explicitly extracted out; comparing with Eqs. (10)–(12) one has (always referred to the variables and kinematical configuration of Fig. 1):

$$\begin{aligned}
\Delta\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp) &= -\Delta f_{q/S_T}(x,k_\perp)\,\sin(\phi_S-\phi_\perp) && (26)\\
\Delta\hat{f}^q_{s_x/S_L}(x,\boldsymbol{k}_\perp) &= -\Delta f^q_{s_x/S_L}(x,k_\perp) && (27)\\
\Delta\hat{f}^q_{s_x/S_T}(x,\boldsymbol{k}_\perp) &= -\Delta f^q_{s_x/S_T}(x,k_\perp)\,\cos(\phi_S-\phi_\perp) && (28)\\
\Delta\hat{f}^q_{s_y/S_L}(x,\boldsymbol{k}_\perp) &= -\Delta f^q_{s_y/S_L}(x,k_\perp) = -\Delta f^q_{s_y/p}(x,k_\perp) && (29)\\
\Delta\hat{f}^q_{s_y/S_T}(x,\boldsymbol{k}_\perp) &= -\Delta f^q_{s_y/p}(x,k_\perp) + \Delta^- f^q_{s_y/S_T}(x,k_\perp)\,\sin(\phi_S-\phi_\perp) && (30)\\
&\equiv -\Delta f^q_{s_y/p}(x,k_\perp) + \Delta^- \hat{f}^q_{s_y/S_T}(x,\boldsymbol{k}_\perp) &&\\
\Delta\hat{f}^q_{s_z/S_L}(x,\boldsymbol{k}_\perp) &= \Delta f^q_{s_z/S_L}(x,k_\perp) && (31)\\
\Delta\hat{f}^q_{s_z/S_T}(x,\boldsymbol{k}_\perp) &= \Delta f^q_{s_z/S_T}(x,k_\perp)\,\cos(\phi_S-\phi_\perp)\,. && (32)
\end{aligned}$$

According to our choice the $\Delta f^q_{s_j/S_T,S_L}(x,k_\perp)$ introduced here are the same as in Ref. [2].

The last line of Eqs. (22)–(25) gives the connection with the Amsterdam group notations; $M$ is taken as the proton mass. These last relationships hold at leading twist; notice also that, when comparing with the results of the Amsterdam group, one should take into account other differences in conventions and notations. In particular:

$$\begin{aligned}
(\boldsymbol{p}_T)_{\rm Amsterdam} &= \boldsymbol{k}_\perp && (33)\\
(-z\,\boldsymbol{k}_T)_{\rm Amsterdam} &= \boldsymbol{p}_\perp = (\boldsymbol{P}_T - z_h\,\boldsymbol{k}_\perp) && (34)\\
(\hat{\boldsymbol{h}})_{\rm Amsterdam} &= \frac{\boldsymbol{P}_T}{P_T} = \hat{\boldsymbol{P}}_T\,. && (35)
\end{aligned}$$

Finally, we recall some other notations widely used in the literature:

$$\Delta^N f_{q/p^\uparrow}(x,k_\perp) \equiv \Delta f_{q/S_T}(x,k_\perp) = 4\,{\rm Im}F_{+-}^{++}(x,k_\perp) = -\frac{2k_\perp}{M}\,f_{1T}^\perp(x,k_\perp) \tag{36}$$

$$\Delta^N f_{q^\uparrow/p}(x,k_\perp) \equiv \Delta f^q_{s_y/p}(x,k_\perp) = -2\,{\rm Im}F_{++}^{+-}(x,k_\perp) = -\frac{k_\perp}{M}\,h_1^\perp(x,k_\perp) \tag{37}$$

$$\begin{aligned}
\frac{1}{2}\left[\Delta f^q_{s_x/S_T}(x,k_\perp) + \Delta^- f^q_{s_y/S_T}(x,k_\perp)\right] &= F_{+-}^{+-}(x,k_\perp) = h_{1T}(x,k_\perp) + \frac{k_\perp^2}{2M^2}h_{1T}^\perp(x,k_\perp) \equiv h_1(x,k_\perp) && (38)\\
\frac{1}{2}\left[\Delta f^q_{s_x/S_T}(x,k_\perp) - \Delta^- f^q_{s_y/S_T}(x,k_\perp)\right] &= F_{+-}^{-+}(x,k_\perp) = \frac{k_\perp^2}{2M^2}h_{1T}^\perp(x,k_\perp) && (39)
\end{aligned}$$

$$\Delta_T q(x) = h_1(x) = \int d^2\boldsymbol{k}_\perp\, h_1(x,k_\perp) = \int d^2\boldsymbol{k}_\perp \left[h_{1T}(x,k_\perp) + \frac{k_\perp^2}{2M^2}h_{1T}^\perp(x,k_\perp)\right]. \tag{40}$$

Eqs. (36), (37) and (40) refer, respectively, to the Sivers, the Boer-Mulders and the transversity distributions.

### B. TMD fragmentation functions

The quantity $\hat{D}^{\lambda_h,\lambda'_h}_{\lambda_{q_f},\lambda'_{q_f}}(z,\boldsymbol{p}_\perp)$ describes the hadronization of the quark $q_f$ into the observed final hadron $h$, which carries, with respect to the fragmenting quark, the light-cone momentum fraction $z$ and the intrinsic transverse momentum $\boldsymbol{p}_\perp$. Similarly to the distribution functions, also $\hat{D}^{\lambda_h,\lambda'_h}_{\lambda_q,\lambda'_q}(z,\boldsymbol{p}_\perp)$ can be written as the product of *fragmentation amplitudes* for the $q \to h + X$ process:

$$\hat{D}^{\lambda_h,\lambda'_h}_{\lambda_q,\lambda'_q} = \sum\!\!\!\!\!\!\!\int_{X,\lambda_X} \hat{\mathcal{D}}_{\lambda_h,\,\lambda_X;\lambda_q}\,\hat{\mathcal{D}}^*_{\lambda'_h,\,\lambda_X;\lambda'_q}\,, \tag{41}$$

where the $\displaystyle\sum\!\!\!\!\!\!\!\int_{X,\lambda_X}$ stands for a spin sum and phase space integration over all undetected particles, considered as a system $X$. The usual unpolarized fragmentation function $D_{h/q}(z)$, *i.e.* the number density of hadrons $h$ resulting from the fragmentation of an unpolarized parton $q$ and carrying a light-cone momentum fraction $z$, is given by

$$D_{h/q}(z) = \frac{1}{2} \sum_{\lambda_q, \lambda_h} \int d^2\boldsymbol{p}_\perp \, \hat{D}^{\lambda_h, \lambda_h}_{\lambda_q, \lambda_q}(z, \boldsymbol{p}_\perp) \,. \tag{42}$$

We consider only the cases in which the final particle is either spinless ($\lambda_h = 0$) or its polarization is not observed,

$$D^{h/q}_{\lambda_q, \lambda'_q}(z, \boldsymbol{p}_\perp) = \sum_{\lambda_h} \hat{D}^{\lambda_h, \lambda_h}_{\lambda_q, \lambda'_q}(z, \boldsymbol{p}_\perp) \,. \tag{43}$$

In such a case, parity invariance reduces to two the number of independent $\hat{D}^{h/q}_{\lambda_q, \lambda'_q}(z, \boldsymbol{p}_\perp)$. These, in general, may depend on the azimuthal angle of the final hadron momentum $\boldsymbol{P}_h$ around the direction of the fragmenting quark $q$, *as defined in the quark $q$ helicity frame*, which we denote by $\varphi_q^h$ (it was actually denoted as $\phi_q^h$ in Ref. [2]):

$$\hat{D}^{h/q}_{++}(z, \boldsymbol{p}_\perp) = \hat{D}^{h/q}_{--}(z, \boldsymbol{p}_\perp) = D_{h/q}(z, p_\perp) \tag{44}$$

$$\hat{D}^{h/q}_{+-}(z, \boldsymbol{p}_\perp) = D^{h/q}_{+-}(z, p_\perp) \, e^{i\varphi_q^h} \tag{45}$$

$$\hat{D}^{h/q}_{-+}(z, \boldsymbol{p}_\perp) = [D^{h/q}_{+-}(z, \boldsymbol{p}_\perp)]^* = -D^{h/q}_{+-}(z, p_\perp) \, e^{-i\varphi_q^h} \,. \tag{46}$$

In Appendix C it is shown how to express $\varphi_q^h$ in terms of integration and external variables (defined in the $\gamma^* - p$ c.m. frame), with the result, at leading order in the $(k_\perp/Q)$ expansion:

$$\cos\varphi_q^h = \frac{P_T}{p_\perp} \left[ \cos(\phi_h - \phi_\perp) - z_h \frac{k_\perp}{P_T} \right] \tag{47}$$

$$\sin\varphi_q^h = \frac{P_T}{p_\perp} \sin(\phi_h - \phi_\perp) \,. \tag{48}$$

In Eq. (44) $D_{h/q}(z, p_\perp)$ is the unintegrated unpolarized fragmentation function. Other common notations used in the literature are:

$$\Delta^N D_{h/q^\uparrow}(z, p_\perp) \equiv -2i \, D^{h/q}_{+-}(z, p_\perp) = 2 \, \mathrm{Im} D^{h/q}_{+-}(z, p_\perp) = \frac{2 p_\perp}{z M_h} H_1^\perp(z, p_\perp) \,, \tag{49}$$

referred to the Collins fragmentation function. $M_h$ is the mass of the produced hadron.

### C. Elementary interaction

The $\hat{M}_{\lambda_{\ell'} \lambda_{q_f}; \lambda_\ell \lambda_{q_i}}$ are the helicity amplitudes for the elementary process $\ell \, q_i \to \ell' q_f$, computed at LO in the $\gamma^* - p$ c.m. frame, taking into account the quark intrinsic motion; the amplitudes are normalized so that the unpolarized cross section, for a collinear collision, is given by

$$\frac{d\hat{\sigma}^{\ell q_i \to \ell' q_f}}{d\hat{t}} = \frac{1}{16\pi \hat{s}^2} \, \frac{1}{4} \sum_{\{\lambda\}} |\hat{M}_{\lambda_{\ell'} \lambda_{q_f}; \lambda_\ell \lambda_{q_i}}|^2 \,, \tag{50}$$

where $\hat{t} = -Q^2$ and $\hat{s} = x_B \, s$.

Helicity conservation for massless particles requires $\lambda_\ell = \lambda_{\ell'}$, $\lambda_{q_i} = \lambda_{q_f} = \lambda_q$, which implies that there are only two independent non-vanishing amplitudes, explicitly computed in Appendix A, with the result:

$$\hat{M}_1 \equiv \hat{M}_{++;++} = \hat{M}^*_{--;--} = e_q \, e^2 \left[ \frac{1}{y} \, A_+ \, e^{+i\phi_\perp} \; - \; \frac{1-y}{y} \, A_- \, e^{-i\phi_\perp} \; - \; 4 \, \frac{\sqrt{1-y}}{y} \, \frac{k_\perp}{Q} \right] \tag{51}$$

$$\hat{M}_2 \equiv \hat{M}_{+-;+-} = \hat{M}^*_{-+;-+} = e_q \, e^2 \left[ \frac{1-y}{y} \, A_+ \, e^{-i\phi_\perp} \; - \; \frac{1}{y} \, A_- \, e^{+i\phi_\perp} \; - \; 4 \, \frac{\sqrt{1-y}}{y} \, \frac{k_\perp}{Q} \right] \,, \tag{52}$$

where $y = \frac{Q^2}{x_B s}$ and

$$A_\pm = \left(1 \pm \sqrt{1 + 4\frac{k_\perp^2}{Q^2}}\right) . \tag{53}$$

These are exact LO results, holding at all orders in the $k_\perp/Q$ expansion. By truncating this expansion at first order in $k_\perp/Q$, one obtains much simpler expressions, which will be useful later,

$$\hat{M}_1 = \hat{M}_{++;++} \simeq 2\, e_q e^2 \left[\frac{1}{y}\, e^{+i\phi_\perp} \; - \; 2\, \frac{\sqrt{1-y}}{y}\, \frac{k_\perp}{Q}\right] \tag{54}$$

$$\hat{M}_2 = \hat{M}_{+-;+-} \simeq 2\, e_q e^2 \left[\frac{(1-y)}{y}\, e^{-i\phi_\perp} \; - \; 2\, \frac{\sqrt{1-y}}{y}\, \frac{k_\perp}{Q}\right] . \tag{55}$$

We can now assemble the expression of the different factors - each corresponding to a physical step - into Eqs. (4) or (8) to obtain the SIDIS cross section in terms of the TMDs. This can be done in several ways. The most direct one is that of performing the helicity sums in Eq. (4) taking into account Eqs. (17), (44)–(46), (49), (51) and (52). It yields:

$$\begin{aligned}
\frac{d\sigma^{\ell(S_\ell)+p(S)\to\ell' h X}}{dx_B\, dQ^2 dz_h\, d^2\boldsymbol{P}_T d\phi_S} \;=\; & \frac{1}{2\pi}\sum_q \frac{1}{16\,\pi\,(x_B\, s)^2}\int d^2\boldsymbol{k}_\perp\, \frac{z}{z_h}\, J \\
\times\, & \frac{1}{2}\Big\{\hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp)\,\left(|\hat{M}_1|^2 + |\hat{M}_2|^2\right)\, D_{h/q}(z,p_\perp) \\
& + P_z^\ell\, P_z^q\, \hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp)\,\left(|\hat{M}_1|^2 - |\hat{M}_2|^2\right)\, D_{h/q}(z,p_\perp) \\
& + \Big[P_y^q\, \hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp)\,\left(\mathrm{Re}(\hat{M}_1\hat{M}_2^*)\cos\varphi_q^h - \mathrm{Im}(\hat{M}_1\hat{M}_2^*)\sin\varphi_q^h\right) \\
& - P_x^q\, \hat{f}_{q/p,S}(x,\boldsymbol{k}_\perp)\,\left(\mathrm{Im}(\hat{M}_1\hat{M}_2^*)\cos\varphi_q^h + \mathrm{Re}(\hat{M}_1\hat{M}_2^*)\sin\varphi_q^h\right)\Big]\Delta^N D_{h/q^\uparrow}(z,p_\perp)\Big\} ,
\end{aligned} \tag{56}$$

which expresses the cross section in terms of the lepton and the quark polarization vectors, the helicity amplitudes of the elementary interaction and either the unpolarized or the Collins fragmentation functions. The intrinsic transverse momentum of the produced hadron, $\boldsymbol{p}_\perp$, is related to $\boldsymbol{k}_\perp$ and the other kinematical variables as shown in Eq. (28) of Ref. [37]. The exact expressions of $\cos\varphi_q^h$ and $\sin\varphi_q^h$ can be obtained from Eqs. (C3) and (C4).

We now continue our computation, in this Section, at $\mathcal{O}\,(k_\perp/Q)$. From Eqs. (54), (55), (47) and (48), we have:

$$|\hat{M}_1|^2 + |\hat{M}_2|^2 = \frac{4e_q^2 e^4}{y^2}\left[1 + (1-y)^2 - 4(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}\cos\phi_\perp\right] \tag{57}$$

$$|\hat{M}_1|^2 - |\hat{M}_2|^2 = \frac{4e_q^2 e^4}{y^2}\left[1 - (1-y)^2 - 4y\sqrt{1-y}\,\frac{k_\perp}{Q}\cos\phi_\perp\right] \tag{58}$$

$$\begin{aligned}
\mathrm{Im}(\hat{M}_1\hat{M}_2^*)\,\cos\varphi_q^h + \mathrm{Re}(\hat{M}_1\hat{M}_2^*)\,\sin\varphi_q^h \;=\; & \frac{P_T}{p_\perp}\frac{4e_q^2 e^4}{y^2}\left\{(1-y)\left[\sin(\phi_h+\phi_\perp) - z_h\frac{k_\perp}{P_T}\sin 2\phi_\perp\right]\right. \\
- \; & \left. 2\sqrt{1-y}(2-y)\frac{k_\perp}{Q}\left[\sin\phi_h - z_h\frac{k_\perp}{P_T}\sin\phi_\perp\right]\right\}
\end{aligned} \tag{59}$$

$$\begin{aligned}
\mathrm{Re}(\hat{M}_1\hat{M}_2^*)\,\cos\varphi_q^h - \mathrm{Im}(\hat{M}_1\hat{M}_2^*)\,\sin\varphi_q^h \;=\; & \frac{P_T}{p_\perp}\frac{4e_q^2 e^4}{y^2}\left\{(1-y)\left[\cos(\phi_h+\phi_\perp) - z_h\frac{k_\perp}{P_T}\cos 2\phi_\perp\right]\right. \\
- \; & \left. 2\sqrt{1-y}(2-y)\frac{k_\perp}{Q}\left[\cos\phi_h - z_h\frac{k_\perp}{P_T}\cos\phi_\perp\right]\right\} .
\end{aligned} \tag{60}$$

Inserting these results, together with Eqs. (22)–(25), into Eq. (56), gives, at order $k_\perp/Q$, the following expression for the SIDIS cross section in the TMD factorization scheme:

$$
\begin{aligned}
&\frac{d\sigma^{\ell(S_\ell)+p(S)\to\ell' hX}}{dx_{\scriptscriptstyle B}\,dQ^2\,dz_h\,d^2\boldsymbol{P}_T\,d\phi_S} = \frac{1}{2\pi}\sum_q \frac{1}{16\,\pi\,(x_{\scriptscriptstyle B}\,s)^2}\int d^2\boldsymbol{k}_\perp\,d^2\boldsymbol{p}_\perp\,\delta^{(2)}(\boldsymbol{P}_T - z_h\boldsymbol{k}_\perp - \boldsymbol{p}_\perp)\,\frac{4e_q^2e^4}{y^2}\\
&\Bigg\{\frac{1}{2}f_{q/p}\left[1+(1-y)^2\right]D_{h/q} - \frac{1}{2}\Delta f^q_{s_y/p}\,\frac{P_T}{p_\perp}(1-y)\left[\cos(\phi_h+\phi_\perp) - z_h\,\frac{k_\perp}{P_T}\,\cos 2\phi_\perp\right]\Delta^N D_{h/q^\uparrow}\\
&-2(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}\left[f_{q/p}\cos\phi_\perp D_{h/q} - \frac{1}{2}\Delta f^q_{s_y/p}\,\frac{P_T}{p_\perp}\left(\cos\phi_h - z_h\,\frac{k_\perp}{P_T}\,\cos\phi_\perp\right)\Delta^N D_{h/q^\uparrow}\right]\\
&+\frac{1}{2}S_L\Bigg[\frac{P_T}{p_\perp}(1-y)\,\Delta f^q_{s_x/S_L}\left(\sin(\phi_h+\phi_\perp) - z_h\,\frac{k_\perp}{P_T}\,\sin 2\phi_\perp\right)\Delta^N D_{h/q^\uparrow}\\
&\qquad -2(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}\,\frac{P_T}{p_\perp}\,\Delta f^q_{s_x/S_L}\left(\sin\phi_h - z_h\,\frac{k_\perp}{P_T}\,\sin\phi_\perp\right)\Delta^N D_{h/q^\uparrow}\\
&\qquad +P_z^\ell\left(\left[1-(1-y)^2\right]\Delta f^q_{s_z/S_L}D_{h/q} - 4y\sqrt{1-y}\,\frac{k_\perp}{Q}\,\Delta f^q_{s_z/S_L}\cos\phi_\perp D_{h/q}\right)\Bigg]\\
&+\frac{1}{2}S_T\Bigg[\frac{1}{2}\left[1+(1-y)^2\right]\Delta f_{q/S_T}\sin(\phi_\perp-\phi_S)D_{h/q}\\
&\qquad +P_z^\ell\left[1-(1-y)^2\right]\Delta f^q_{s_z/S_T}\cos(\phi_\perp-\phi_S)D_{h/q}\\
&\qquad -P_z^\ell\,2y\sqrt{1-y}\,\frac{k_\perp}{Q}\,\Delta f^q_{s_z/S_T}\Big(\cos\phi_S + \cos(2\phi_\perp-\phi_S)\Big)D_{h/q}\\
&\qquad +\frac{P_T}{2p_\perp}\,(1-y)\,(\Delta f^q_{s_x/S_T} + \Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+\phi_S) - z_h\,\frac{k_\perp}{P_T}\,\sin(\phi_\perp+\phi_S)\Big)\Delta^N D_{h/q^\uparrow}\\
&\qquad +\frac{P_T}{2p_\perp}\,(1-y)\,(\Delta f^q_{s_x/S_T} - \Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+2\phi_\perp-\phi_S) - z_h\frac{k_\perp}{P_T}\sin(3\phi_\perp-\phi_S)\Big)\Delta^N D_{h/q^\uparrow}\\
&\qquad -\frac{P_T}{p_\perp}\,(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}(\Delta f^q_{s_x/S_T} + \Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h-\phi_\perp+\phi_S) - z_h\,\frac{k_\perp}{P_T}\,\sin\phi_S\Big)\Delta^N D_{h/q^\uparrow}\\
&\qquad -\frac{P_T}{p_\perp}\,(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}(\Delta f^q_{s_x/S_T} - \Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+\phi_\perp-\phi_S) - z_h\,\frac{k_\perp}{P_T}\,\sin(2\phi_\perp-\phi_S)\Big)\Delta^N D_{h/q^\uparrow}\\
&\qquad +(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}\,\Delta f_{q/S_T}\Big(\sin\phi_S - \sin(2\phi_\perp-\phi_S)\Big)D_{h/q}\Bigg]\Bigg\}\,. \qquad (61)
\end{aligned}
$$

The first three terms of Eq. (61) correspond to the contribution of the unpolarized proton to the SIDIS cross section; they contain either the unpolarized or the Boer-Mulders distribution functions. The following three terms correspond to the longitudinally-polarized proton contributions; they depend either on the helicity distribution $\Delta f^q_{s_z/S_L}[=\Delta q = g_1]$ or on the $\Delta f^q_{s_x/S_L}[=(k_\perp/M)\,h_{1L}^\perp]$ transverse momentum dependent distribution. Finally, the last eight terms correspond to the transversely-polarized proton contributions; they may originate from the Sivers function, from $\Delta f^q_{s_z/S_T}[=(k_\perp/M)\,g_{1T}^\perp]$, and from the transversity distribution functions, related to the combinations $(\Delta f^q_{s_x/S_T} \pm \Delta^- f^q_{s_y/S_T})$ as shown in Eqs. (38) and (39). The partonic distributions couple either to the unpolarized or to the Collins fragmentation functions, depending on whether they are, respectively, chiral even or odd.

Notice that we have intentionally grouped all terms according to their phases, so that this expression can be easily compared with the analogous formulae of Ref. [23], which have the same structure. To make the comparison fully explicit, apart from converting our notation to the Amsterdam group notation, we need to extract from the integration over the intrinsic transverse momentum $\boldsymbol{k}_\perp$ the dependence on the azimuthal angles $\phi_h$ and $\phi_S$. On the basis of a simple tensorial analysis, which is described in detail in Appendices D and E, we can recover Eqs. (4.2)-(4.19) of Ref. [23], without formulating any particular assumption on the $x$ $(z)$ and $k_\perp$ $(p_\perp)$ dependence of the distribution (fragmentation) functions.

In analogy with the Amsterdam notation, Ref. [23], we define the convolution on transverse momenta in the following way

$$
\mathcal{C}[w\,f\,D] = \sum_q e_q^2\int d^2\boldsymbol{k}_\perp\,d^2\boldsymbol{p}_\perp\,\delta^{(2)}(\boldsymbol{P}_T - z_h\boldsymbol{k}_\perp - \boldsymbol{p}_\perp)\,w(\boldsymbol{k}_\perp,\boldsymbol{P}_T)\,f(x_{\scriptscriptstyle B},k_\perp)\,D\,(z_h,p_\perp)\ . \qquad (62)
$$

Notice that this definition differs from Eq. (41) of Ref. [23] by a factor $x_{\scriptscriptstyle B}$ and for the definition of the parton momenta, see Eqs. (33)–(35).

The convolutions on intrinsic transverse momenta in the single terms of Eq. (61) can in fact be written as:

$$
\begin{aligned}
F_{UU} &= \sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, f_{q/p}\, D_{h/q} = \mathcal{C}[f_1\, D_1] && (63)\\
\cos 2\phi_h\, F_{UU}^{\cos 2\phi_h} &= -\sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, \Delta f^q_{s_y/p} \frac{P_T}{2\,p_\perp}\left[\cos(\phi_h+\phi_\perp) - z_h \frac{k_\perp}{P_T}\cos 2\phi_\perp\right] \Delta^N D_{h/q^\uparrow} \\
&= \cos 2\phi_h\, \mathcal{C}\left[\frac{(\boldsymbol{P}_T\cdot\boldsymbol{k}_\perp) - 2 z_h(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp)^2 + z_h k_\perp^2}{z_h M_h M} h_1^\perp\, H_1^\perp\right] && (64)\\
\cos\phi_h\, F_{UU}^{\cos\phi_h} &= -2\sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, \frac{k_\perp}{Q}\Big\{\cos\phi_\perp f_{q/p} D_{h/q} \\
&\qquad - \frac{P_T}{2\,p_\perp}\left[\cos\phi_h - z_h\frac{k_\perp}{P_T}\cos\phi_\perp\right]\Delta f^q_{s_y/p}\Delta^N D_{h/q^\uparrow}\Big\} \\
&= \cos\phi_h\left(-\frac{2}{Q}\right)\, \mathcal{C}\left[(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp) f_1\, D_1 + \frac{k_\perp^2\left(P_T - z_h\,\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp\right)}{z_h M_h M}\, h_1^\perp\, H_1^\perp\right] && (65)\\
\sin 2\phi_h\, F_{UL}^{\sin 2\phi_h} &= \sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, \frac{P_T}{2p_\perp}\Delta f^q_{s_x/S_L}\left(\sin(\phi_h+\phi_\perp) - z_h\frac{k_\perp}{P_T}\sin 2\phi_\perp\right)\Delta^N D_{h/q^\uparrow} \\
&= \sin 2\phi_h\, \mathcal{C}\left[\frac{(\boldsymbol{P}_T\cdot\boldsymbol{k}_\perp) - 2 z_h(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp)^2 + z_h k_\perp^2}{z_h\, M_h\, M}\, h_{1L}^\perp\, H_1^\perp\right] && (66)\\
\sin\phi_h\, F_{UL}^{\sin\phi_h} &= -2\sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, \frac{k_\perp}{Q}\frac{P_T}{2p_\perp}\Delta f^q_{s_x/S_L}\left(\sin\phi_h - z_h\frac{k_\perp}{P_T}\sin\phi_\perp\right)\Delta^N D_{h/q^\uparrow} \\
&= \sin\phi_h\left(-\frac{2}{Q}\right)\, \mathcal{C}\left[\frac{k_\perp^2\Big(P_T - z_h(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp)\Big)}{z_h\, M_h\, M}\, h_{1L}^\perp\, H_1^\perp\right] && (67)\\
\sin\phi_h\, F_{LU}^{\sin\phi_h} &= 0 \quad \text{(no contribution from twist-2 TMDs)} && (68)\\
F_{LL} &= \sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, \Delta f^q_{s_z/S_L}\, D_{h/q} = \mathcal{C}[g_{1L}\, D_1] && (69)\\
\cos\phi_h\, F_{LL}^{\cos\phi_h} &= -2\sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, \frac{k_\perp}{Q}\Delta f^q_{s_z/S_L}\cos\phi_\perp D_{h/q} \\
&= \cos\phi_h\left(-\frac{2}{Q}\right)\, \mathcal{C}\left[(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp) g_{1L} D_1\right] && (70)\\
\sin(\phi_h-\phi_S)\, F_{UT}^{\sin(\phi_h-\phi_S)} &= \frac{1}{2}\sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, \Delta f_{q/S_T}\sin(\phi_\perp-\phi_S) D_{h/q} \\
&= \sin(\phi_h-\phi_S)\, \mathcal{C}\left[\frac{-(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp)}{M}\, f_{1T}^\perp\, D_1\right] && (71)\\
\cos(\phi_h-\phi_S)\, F_{LT}^{\cos(\phi_h-\phi_S)} &= \sum_q e_q^2 \int d^2\boldsymbol{k}_\perp\, \Delta f^q_{s_z/S_T}\cos(\phi_\perp-\phi_S) D_{h/q} \\
&= \cos(\phi_h-\phi_S)\, \mathcal{C}\left[\frac{(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp)}{M}\, g_{1T}^\perp\, D_1\right] && (72)\\
\cos\phi_S\, F_{LT}^{\cos\phi_S} &= -\sum_q e_q^2 \int d^2\boldsymbol{k}_\perp \frac{k_\perp}{Q}\Delta f^q_{s_z/S_T}\cos\phi_S\, D_{h/q} \\
&= \cos\phi_S\,\left(-\frac{1}{Q}\right)\, \mathcal{C}\left[\frac{k_\perp^2}{M}\, g_{1T}^\perp\, D_1\right] && (73)
\end{aligned}
$$

$$\begin{aligned}
\cos(2\phi_h-\phi_S)\,F_{LT}^{\cos(2\phi_h-\phi_S)} &= -\sum_q e_q^2 \int d^2\boldsymbol{k}_\perp \frac{k_\perp}{Q}\Delta f^q_{s_z/S_T}\cos(2\phi_\perp-\phi_S)\,D_{h/q} \\
&= \cos(2\phi_h-\phi_S)\,\frac{1}{Q}\,\mathcal{C}\left[\frac{\Big(k_\perp^2-2(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp)^2\Big)}{M}\,g_{1T}^\perp\,D_1\right] \qquad (74)\\
\sin(\phi_h+\phi_S)F_{UT}^{\sin(\phi_h+\phi_S)} &= \sum_q e_q^2\int d^2\boldsymbol{k}_\perp\frac{P_T}{2p_\perp} \\
&\times\ (\Delta f^q_{s_x/S_T}+\Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+\phi_S)-z_h\frac{k_\perp}{P_T}\sin(\phi_\perp+\phi_S)\Big)\Delta^N D_{h/q^\uparrow} \\
&= \sin(\phi_h+\phi_S)\,\mathcal{C}\left[\frac{\Big(P_T-z_hk_\perp(\hat{\boldsymbol{P}}_T\cdot\hat{\boldsymbol{k}}_\perp)\Big)}{z_h\,M_h}\,h_1\,H_1^\perp\right] \qquad (75)\\
\sin(3\phi_h-\phi_S)F_{UT}^{\sin(3\phi_h-\phi_S)} &=
\end{aligned}$$

$$\begin{aligned}
&= \sum_q e_q^2\int d^2\boldsymbol{k}_\perp\frac{P_T}{2p_\perp}(\Delta f^q_{s_x/S_T}-\Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+2\phi_\perp-\phi_S)-z_h\frac{k_\perp}{P_T}\sin(3\phi_\perp-\phi_S)\Big)\Delta^N D_{h/q^\uparrow} \\
&= \sin(3\phi_h-\phi_S)\,\mathcal{C}\left[\frac{k_\perp^2\Big\{-P_T+2\,P_T\,(\hat{\boldsymbol{P}}_T\cdot\hat{\boldsymbol{k}}_\perp)^2-z_hk_\perp\Big[4(\hat{\boldsymbol{P}}_T\cdot\hat{\boldsymbol{k}}_\perp)^3+3(\hat{\boldsymbol{P}}_T\cdot\hat{\boldsymbol{k}}_\perp)\Big]\Big\}}{2z_h\,M_h\,M^2}h_{1T}^\perp\,H_1^\perp\right] \qquad (76)
\end{aligned}$$

$$\begin{aligned}
\sin\phi_S F_{UT}^{\sin\phi_S} &= -\sum_q e_q^2\int d^2\boldsymbol{k}_\perp\frac{P_T}{p_\perp}\,\frac{k_\perp}{Q} \\
&\quad\times(\Delta f^q_{s_x/S_T}+\Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h-\phi_\perp+\phi_S)-z_h\frac{k_\perp}{P_T}\sin\phi_S\Big)\Delta^N D_{h/q^\uparrow} \\
&+\ \frac{1}{2}\sum_q e_q^2\int d^2\boldsymbol{k}_\perp\frac{k_\perp}{Q}\Delta f_{q/S_T}\sin\phi_S\,D_{h/q} \\
&= \sin\phi_S\,\left(-\frac{2}{Q}\right)\,\mathcal{C}\left[\frac{(\boldsymbol{P}_T\cdot\boldsymbol{k}_\perp-z_hk_\perp^2)}{z_h\,M_h}h_1\,H_1^\perp\ +\ \frac{k_\perp^2}{2M}f_{1T}^\perp\,D_1\right] \qquad (77)\\
\sin(2\phi_h-\phi_S)F_{UT}^{\sin(2\phi_h-\phi_S)} &= -\sum_q e_q^2\int d^2\boldsymbol{k}_\perp\frac{P_T}{p_\perp}\,\frac{k_\perp}{Q} \\
&\quad\times(\Delta f^q_{s_x/S_T}-\Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+\phi_\perp-\phi_S)-z_h\frac{k_\perp}{P_T}\sin(2\phi_\perp-\phi_S)\Big)\Delta^N D_{h/q^\uparrow} \\
&-\ \frac{1}{2}\sum_q e_q^2\int d^2\boldsymbol{k}_\perp\frac{k_\perp}{Q}\Delta f_{q/S_T}\sin(2\phi_\perp-\phi_S)\,D_{h/q} \\
&= \sin(2\phi_h-\phi_S)\,\left(-\frac{1}{Q}\right)\,\mathcal{C}\Big[\frac{k_\perp^2\,\Big((\boldsymbol{P}_T\cdot\boldsymbol{k}_\perp)+z_hk_\perp^2\,\Big(1-2(\hat{\boldsymbol{P}}_T\cdot\hat{\boldsymbol{k}}_\perp)^2\Big)\Big)}{z_h\,M_h\,M^2}h_{1T}^\perp\,H_1^\perp \\
&\qquad -\frac{\Big(2(\hat{\boldsymbol{P}}_T\cdot\boldsymbol{k}_\perp)^2-k_\perp^2\Big)}{M}\,f_{1T}^\perp\,D_1\Big]\,. \qquad (78)
\end{aligned}$$

These "$F_{S_\ell S}$ structure functions" are the same as those defined in Ref. [23], apart from an overall factor $x_{\scriptscriptstyle B}$ which appears in the latter. In the comparison one should consider only leading twist TMDs and remember the different notations of Ref. [23], Eqs. (33)–(35). Using the above $F$'s in Eq. (61) one obtains the full expression of the SIDIS polarized cross section, valid with leading twist TMDs and at kinematical order $k_\perp/Q$:

$$\begin{aligned}
\frac{d\sigma^{\ell(S_\ell)+p(S)\to\ell' hX}}{dx_{\scriptscriptstyle B}\,dQ^2\,dz_h\,d^2\boldsymbol{P}_T\,d\phi_S} &= \frac{2\,\alpha^2}{Q^4} \\
&\times\ \Big\{\frac{1+(1-y)^2}{2}F_{UU}+(2-y)\sqrt{1-y}\,\cos\phi_h\,F_{UU}^{\cos\phi_h}+(1-y)\,\cos2\phi_h\,F_{UU}^{\cos2\phi_h} \\
&\quad+S_L\Big[(1-y)\,\sin2\phi_h\,F_{UL}^{\sin2\phi_h}+(2-y)\sqrt{1-y}\,\sin\phi_h\,F_{UL}^{\sin\phi_h}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ S_L\, P_z^\ell \Big[\frac{1-(1-y)^2}{2}\, F_{LL} + y\sqrt{1-y}\, \cos\phi_h\, F_{LL}^{\cos\phi_h}\Big] \\
&+ S_T \Big[\frac{1+(1-y)^2}{2}\, \sin(\phi_h-\phi_S)\, F_{UT}^{\sin(\phi_h-\phi_S)} \\
&\qquad + (1-y)\Big(\sin(\phi_h+\phi_S)\, F_{UT}^{\sin(\phi_h+\phi_S)} + \sin(3\phi_h-\phi_S)\, F_{UT}^{\sin(3\phi_h-\phi_S)}\Big) \\
&\qquad + (2-y)\sqrt{1-y}\Big(\sin\phi_S\, F_{UT}^{\sin\phi_S} + \sin(2\phi_h-\phi_S)\, F_{UT}^{\sin(2\phi_h-\phi_S)}\Big)\Big] \\
&+ S_T\, P_z^\ell \Big[\frac{1-(1-y)^2}{2}\, \cos(\phi_h-\phi_S)\, F_{LT}^{\cos(\phi_h-\phi_S)} \\
&\qquad\quad + y\sqrt{1-y}\Big(\cos\phi_S\, F_{LT}^{\cos\phi_S} + \cos(2\phi_h-\phi_S)\, F_{LT}^{\cos(2\phi_h-\phi_S)}\Big)\Big]\Bigg\}\,. \qquad (79)
\end{aligned}
$$

This expression agrees with Eq. (2.7) of Ref. [23], bearing in mind Eqs. (2.8–2.13) and that, at leading twist, $F_{UU,L} = F_{LU}^{\sin\phi_h} = 0$. In general, our results agree with the leading order results of Refs. [1, 23, 39] and reproduce part of the subleading order results of Refs. [1, 23], in particular those obtained in the so-called Wandzura-Wilczek type approximation [41].

In obtaining the general cross section structure of Eq. (79) we started from the TMD factorization, Eq. (4); then we have simply exploited the properties of the helicity amplitudes, which essentially originate from the phase dependence of the Dirac spinors and their non collinear kinematics. Each step of the factorization scheme contributes some phases, including the elementary interactions.

Some of the final azimuthal dependences have a clear and direct physical interpretation. For example, the phase of $F_{UT}^{\sin(\phi_h-\phi_S)}$, Eq. (71), originates from the phase dependence of the $\Delta\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp)$ distribution, Eq. (26). This is the Sivers effect [42, 43], which relates the number of unpolarized quarks with intrinsic momentum $\boldsymbol{k}_\perp$ to the spin of the proton; such an effect, due to parity invariance, can only be of the form $\boldsymbol{S}\cdot(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{k}}_\perp) = S_T\,\sin(\phi_\perp-\phi_S)$. Similarly, the phase in the first term of $F_{UU}^{\cos\phi_h}$, Eq. (65), being associated with unpolarized distribution and fragmentation functions, can only come from the $\boldsymbol{k}_\perp$ dependence of the elementary interaction, the so-called Cahn effect [37].

## III. SINGLE AND DOUBLE SPIN ASYMMETRIES IN SIDIS

From the expression of the SIDIS polarized cross section we can now compute all spin asymmetries which have been, or can be, measured. We can restart from Eq. (56), inserting into it the expressions of the polarized quark distributions, as given in Eqs. (22)–(32):

$$
\begin{aligned}
&\frac{d\sigma^{\ell(S_\ell)+p(S)\to\ell' hX}}{dx_{\!B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} = \frac{1}{2\pi}\sum_q \frac{1}{16\,\pi\,(x_{\!B}\, s)^2}\int d^2\boldsymbol{k}_\perp\, \frac{z}{z_h}\, J \\
&\times \frac{1}{2}\Bigg\{\Big(f_{q/p}(x,k_\perp) + \frac{1}{2} S_T\, \Delta\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp)\Big)\Big(|\hat{M}_1|^2 + |\hat{M}_2|^2\Big)\, D_{h/q}(z,p_\perp) \\
&+\; P_z^\ell\Big(S_L\, \Delta\hat{f}^q_{s_z/S_L}(x,\boldsymbol{k}_\perp) + S_T\, \Delta\hat{f}^q_{s_z/S_T}(x,\boldsymbol{k}_\perp)\Big)\Big(|\hat{M}_1|^2 - |\hat{M}_2|^2\Big)\, D_{h/q}(z,p_\perp) \qquad (80)\\
&-\;\Big[\Big(\Delta f^q_{s_y/p}(x,k_\perp) - S_T\, \Delta^-\hat{f}^q_{s_y/S_T}(x,\boldsymbol{k}_\perp)\Big)\Big(\mathrm{Re}(\hat{M}_1\hat{M}_2^*)\cos\varphi_q^h - \mathrm{Im}(\hat{M}_1\hat{M}_2^*)\sin\varphi_q^h\Big) \\
&\quad + \Big(S_L\, \Delta f^q_{s_x/S_L}(x,k_\perp) + S_T\, \Delta\hat{f}^q_{s_x/S_T}(x,\boldsymbol{k}_\perp)\Big)\Big(\mathrm{Im}(\hat{M}_1\hat{M}_2^*)\cos\varphi_q^h + \mathrm{Re}(\hat{M}_1\hat{M}_2^*)\sin\varphi_q^h\Big)\Big]\Delta^N D_{h/q^\uparrow}(z,p_\perp)\Bigg\}\,.
\end{aligned}
$$

Notice that this expression, at leading twist, is exact at all orders in $k_\perp/Q$. We list here some properties of the polarized distribution functions which are useful in computing the asymmetries [2]:

$$
\begin{aligned}
&\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp) + \hat{f}_{q/-S_T}(x,\boldsymbol{k}_\perp) = 2 f_{q/p}(x,k_\perp) \\
&\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp) - \hat{f}_{q/-S_T}(x,\boldsymbol{k}_\perp) = \Delta\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp) \\
&\Delta\hat{f}_{s_x/S_T}(x,\boldsymbol{k}_\perp) = -\Delta\hat{f}_{s_x/-S_T}(x,\boldsymbol{k}_\perp) \\
&\Delta\hat{f}_{s_y/S_T}(x,\boldsymbol{k}_\perp) - \Delta\hat{f}_{s_y/-S_T}(x,\boldsymbol{k}_\perp) = 2\,\Delta^-\hat{f}^q_{s_y/S_T}(x,\boldsymbol{k}_\perp) \qquad (81)
\end{aligned}
$$

$$\Delta\hat{f}_{s_y/S_T}(x,\boldsymbol{k}_\perp) + \Delta\hat{f}_{s_y/-S_T}(x,\boldsymbol{k}_\perp) = -2\,\Delta f^q_{s_y/p}(x,k_\perp)$$
$$\Delta\hat{f}_{s_z/S_T}(x,\boldsymbol{k}_\perp) = -\Delta\hat{f}_{s_z/-S_T}(x,\boldsymbol{k}_\perp)$$
$$\Delta\hat{f}_{s_i/S_L}(x,\boldsymbol{k}_\perp) = \Delta\hat{f}_{s_i/-S_L}(x,\boldsymbol{k}_\perp) \qquad (i = x,y,z)\,.$$

Let us now consider Eq. (80) in several particular cases. In the sequel, transverse and longitudinal always refer, both for the protons and the leptons, to their (different) directions of motion in the $\gamma^* - p$ c.m. frame. Longitudinal states coincide with helicity states.

### A. Nucleon transverse single spin asymmetry, $A_{UT}$

Let us start with one of the most common SIDIS single spin asymmetries, $A_{S_\ell S}$, with unpolarized leptons (U) and transversely polarized protons (T):

$$A_{UT} \equiv \frac{d^6\sigma^{\ell p^\uparrow \to \ell' h X} - d^6\sigma^{\ell p^\downarrow \to \ell' h X}}{d^6\sigma^{\ell p^\uparrow \to \ell' h X} + d^6\sigma^{\ell p^\downarrow \to \ell' h X}} = \frac{d^6\sigma^{\ell + p(S_T) \to \ell' h X} - d^6\sigma^{\ell + p(-S_T) \to \ell' h X}}{d^6\sigma^{\ell + p(S_T) \to \ell' h X} + d^6\sigma^{\ell + p(-S_T) \to \ell' h X}}. \tag{82}$$

For the numerator of $A_{UT}$ we have:

$$\begin{aligned}
\frac{d\sigma^{\ell p^\uparrow \to \ell' h X} - d\sigma^{\ell p^\downarrow \to \ell' h X}}{dx_{_B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} &= \frac{1}{2\pi}\sum_q \frac{1}{16\,\pi\,(x_{_B}\,s)^2}\int d^2\boldsymbol{k}_\perp\,\frac{z}{z_h}\,J \qquad (83)\\
&\times \left\{\frac{1}{2}\Delta\hat{f}_{q/S_T}(x,\boldsymbol{k}_\perp)\,(|\hat{M}_1|^2 + |\hat{M}_2|^2)\,D_{h/q}(z,p_\perp)\right.\\
&+ \Big[\Delta^- \hat{f}^q_{s_y/S_T}(x,\boldsymbol{k}_\perp)\left(\mathrm{Re}(\hat{M}_1\hat{M}_2^*)\,\cos\phi_q^h - \mathrm{Im}(\hat{M}_1\hat{M}_2^*)\,\sin\phi_q^h)\right)\\
&\left. -\,\Delta\hat{f}^q_{s_x/S_T}(x,\boldsymbol{k}_\perp)\left(\mathrm{Re}(\hat{M}_1\hat{M}_2^*)\,\sin\phi_q^h + \mathrm{Im}(\hat{M}_1\hat{M}_2^*)\,\cos\phi_q^h)\right)\Big]\,\Delta^N D_{h/q^\uparrow}(z,p_\perp)\right\}.
\end{aligned}$$

The first term in Eq. (83) corresponds to the Sivers effect, whereas the second and the third terms correspond to the Collins effect, coupled to the transversity distributions.

Similarly, for the denominator we find:

$$\begin{aligned}
\frac{d\sigma^{\ell p^\uparrow \to \ell' h X} + d\sigma^{\ell p^\downarrow \to \ell' h X}}{dx_{_B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} &= \frac{1}{2\pi}\sum_q \frac{1}{16\,\pi\,(x_{_B}\,s)^2}\int d^2\boldsymbol{k}_\perp\,\frac{z}{z_h}\,J\\
&\times \left\{f_{q/p}(x,k_\perp)\,(|\hat{M}_1|^2 + |\hat{M}_2|^2)\,D_{h/q}(z,p_\perp)\right.\\
&\left. -\,\Delta f^q_{s_y/p}(x,k_\perp)\left(\mathrm{Re}(\hat{M}_1\hat{M}_2^*)\,\cos\phi_q^h - \mathrm{Im}(\hat{M}_1\hat{M}_2^*)\,\sin\phi_q^h\right)\Delta^N D_{h/q^\uparrow}\right\}. \qquad (84)
\end{aligned}$$

Here, the first term corresponds to the usual unpolarized cross section (which survives in the collinear limit) whereas the second term is an effect obtained combining the Boer-Mulders distribution function, $\Delta f^q_{s_y/p}(x,k_\perp)$, with the Collins fragmentation function, $\Delta^N D_{h/q^\uparrow}(z,p_\perp)$.

If we insert the exact relations for $\hat{M}_1$ and $\hat{M}_2$ – given in Eqs. (51) and (52) – and for $\cos\varphi_q^h$, $\sin\varphi_q^h$ – given in Eq. (C3) – into Eqs. (83) and (84), we obtain an exact expression for the $A_{UT}$ asymmetry. As already mentioned, the numerator is given by two different contributions, the Sivers and the Collins effect. Similarly, the denominator, which is simply twice the unpolarized cross section for the $\ell\, p \to \ell' h X$ process, receive most contribution from the first term, proportional to the unpolarized distribution and fragmentation functions, with a further contribution from a combination of the Boer-Mulders and Collins effects.

Much simpler, and often quite accurate, expressions can be obtained at $\mathcal{O}(k_\perp/Q)$, neglecting higher order corrections. Using Eqs. (57)–(60) and (26)–(32) in Eqs. (83) and (84), one has:

$$\begin{aligned}
&\frac{d\sigma^{\ell p^\uparrow \to \ell' h X} - d\sigma^{\ell p^\downarrow \to \ell' h X}}{dx_{_B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} =\\
&\quad\frac{2\alpha^2}{Q^4}\sum_q e_q^2 \int d^2\boldsymbol{k}_\perp \left\{\frac{1}{2}\Delta f_{q/S_T}\sin(\phi_\perp - \phi_S)[1 + (1-y)^2]\,D_{h/q}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{P_T}{2p_\perp}\,(1-y)\,(\Delta f^q_{s_x/S_T}+\Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+\phi_S)-z_h\,\frac{k_\perp}{P_T}\,\sin(\phi_\perp+\phi_S)\Big)\Delta^N D_{h/q^\uparrow}\\
&+\frac{P_T}{2p_\perp}\,(1-y)\,(\Delta f^q_{s_x/S_T}-\Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+2\phi_\perp-\phi_S)-z_h\frac{k_\perp}{P_T}\sin(3\phi_\perp-\phi_S)\Big)\Delta^N D_{h/q^\uparrow}\\
&-\frac{P_T}{p_\perp}\,(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}(\Delta f^q_{s_x/S_T}+\Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h-\phi_\perp+\phi_S)-z_h\,\frac{k_\perp}{P_T}\,\sin\phi_S\Big)\Delta^N D_{h/q^\uparrow}\\
&-\frac{P_T}{p_\perp}\,(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}(\Delta f^q_{s_x/S_T}-\Delta^- f^q_{s_y/S_T})\Big(\sin(\phi_h+\phi_\perp-\phi_S)-z_h\,\frac{k_\perp}{P_T}\,\sin(2\phi_\perp-\phi_S)\Big)\Delta^N D_{h/q^\uparrow}\\
&+(2-y)\sqrt{1-y}\,\frac{k_\perp}{Q}\,\Delta f_{q/S_T}\Big(\sin\phi_S-\sin(2\phi_\perp-\phi_S)\Big)D_{h/q}\Big\}\;.
\end{aligned}
$$

$$
\begin{aligned}
=\frac{2\alpha^2}{Q^4}\,\Big\{&\big[1+(1-y)^2\big]\sin(\phi_h-\phi_S)F_{UT}^{\sin(\phi_h-\phi_S)}\\
&+2(1-y)\,\Big[\sin(\phi_h+\phi_S)\,F_{UT}^{\sin(\phi_h+\phi_S)}+\sin(3\phi_h-\phi_S)\,F_{UT}^{\sin(3\phi_h-\phi_S)}\Big]\\
&+2(2-y)\sqrt{1-y}\,\Big[\sin\phi_S\,F_{UT}^{\sin\phi_S}+\sin(2\phi_h-\phi_S)\,F_{UT}^{\sin(2\phi_h-\phi_S)}\Big]\Big\}
\end{aligned}
\tag{85}
$$

and

$$
\begin{aligned}
&\frac{d\sigma^{\ell p^\uparrow\to\ell' hX}+d\sigma^{\ell p^\downarrow\to\ell' hX}}{dx_{\scriptscriptstyle B}\,dQ^2\,dz_h\,d^2\boldsymbol{P}_T\,d\phi_S}=\\
&\frac{2\alpha^2}{Q^4}\sum_q e_q^2\int d^2\boldsymbol{k}_\perp\Big\{f_{q/p}\left[1+(1-y)^2-4(2-y)\,\sqrt{1-y}\,\frac{k_\perp}{Q}\,\cos\phi_\perp\right]\,D_{h/q}\\
&-\Delta f^q_{s_y/p}\left[(1-y)\left(\cos(\phi_h+\phi_\perp)-z_h\,\frac{k_\perp}{P_T}\,\cos(2\phi_\perp)\right)\right.\\
&\qquad\left.-2(2-y)\,\sqrt{1-y}\,\frac{k_\perp}{Q}\,\left(\cos\phi_h-z_h\,\frac{k_\perp}{P_T}\,\cos\phi_\perp\right)\right]\,\frac{P_T}{p_\perp}\,\Delta^N D_{h/q^\uparrow}\Big\}\\
=&\;\frac{2\alpha^2}{Q^4}\Big\{\big[1+(1-y)^2\big]\;F_{UU}+2(1-y)\cos2\phi_h\,F_{UU}^{\cos2\phi_h}+2(2-y)\sqrt{1-y}\,\cos\phi_h\,F_{UU}^{\cos\phi_h}\Big\}\,,
\end{aligned}
\tag{86}
$$

where we have also exploited the definitions of the $F$ structure functions, Eqs. (63)–(78). These last expressions, Eqs. (85) and (86), can also be obtained directly from Eq. (79). We recall that, at $\mathcal{O}(k_\perp/Q)$, one has $x=x_{\scriptscriptstyle B}$, $z=z_h$, $\boldsymbol{p}_\perp=\boldsymbol{P}_T-z_h\boldsymbol{k}_\perp$ and $J=1$.

The first term in Eq. (85) corresponds to the SIDIS Sivers asymmetry, which we analyzed in Refs. [37, 44–46] for the extraction of the Sivers function, while the second term corresponds to the SIDIS Collins asymmetry, studied in Refs. [47, 48] and used for the simultaneous extraction of the Collins and transversity functions.

### B. Nucleon longitudinal single spin asymmetry, $A_{UL}$

This asymmetry is defined for unpolarized leptons and a longitudinally polarized proton target:

$$
A_{UL}\equiv\frac{d^6\sigma^{\ell p^\rightarrow\to\ell' hX}-d^6\sigma^{\ell p^\leftarrow\to\ell' hX}}{d^6\sigma^{\ell p^\rightarrow\to\ell' hX}+d^6\sigma^{\ell p^\leftarrow\to\ell' hX}}=\frac{d^6\sigma^{\ell+p(S_L)\to\ell' hX}-d^6\sigma^{\ell+p(-S_L)\to\ell' hX}}{d^6\sigma^{\ell+p(S_L)\to\ell' hX}+d^6\sigma^{\ell+p(-S_L)\to\ell' hX}}\,.
\tag{87}
$$

We give explicit results, for this and the next asymmetries, only valid at $\mathcal{O}(k_\perp/Q)$. The denominator, as in the previous asymmetry, is twice the unpolarized cross section and is given in Eq. (86). For the numerator we have:

$$
\frac{d\sigma^{\ell+p(S_L)\to\ell' hX}-d\sigma^{\ell+p(-S_L)\to\ell' hX}}{dx_{\scriptscriptstyle B}\,dQ^2\,dz_h\,d^2\boldsymbol{P}_T\,d\phi_S}=\frac{4\,\alpha^2}{Q^4}\,\Big\{(1-y)\sin2\phi_hF_{UL}^{\sin2\phi_h}+\sqrt{1-y}(2-y)\sin\phi_hF_{UL}^{\sin\phi_h}\Big\}\,,
\tag{88}
$$

as can be easily checked from Eq. (79).

### C. Nucleon longitudinal double spin asymmetry, $A_{LL}$

This asymmetry is defined by keeping fixed the longitudinal polarization of the lepton, while flipping the direction of the proton target longitudinal polarization:

$$A_{LL} = \frac{d^6\sigma^{\ell^\rightarrow p^\rightarrow \to \ell' h X} - d^6\sigma^{\ell^\rightarrow p^\leftarrow \to \ell' h X}}{d^6\sigma^{\ell^\rightarrow p^\rightarrow \to \ell' h X} + d^6\sigma^{\ell^\rightarrow p^\leftarrow \to \ell' h X}} = \frac{d^6\sigma^{\ell(S_\ell)+p(S_L)\to \ell' h X} - d^6\sigma^{\ell(S_\ell)+p(-S_L)\to \ell' h X}}{d^6\sigma^{\ell(S_\ell)+p(S_L)\to \ell' h X} + d^6\sigma^{\ell(S_\ell)+p(-S_L)\to \ell' h X}} \,. \tag{89}$$

The denominator is the same as given in Eq. (86), while for the numerator we have

$$\frac{d\sigma^{\ell(S_\ell)+p(S_L)\to \ell' h X} - d\sigma^{\ell(S_\ell)+p(-S_L)\to \ell' h X}}{dx_{\!_B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} =$$
$$\frac{2\alpha^2}{Q^4}\left\{[1-(1-y)^2]F_{LL} + 2y\sqrt{1-y}\,\cos\phi_h F_{LL}^{\cos\phi_h} + 2(1-y)\sin 2\phi_h F_{UL}^{\sin 2\phi_h} + 2(2-y)\sqrt{1-y}\sin\phi_h F_{UL}^{\sin\phi_h}\right\}. \tag{90}$$

### D. Lepton longitudinal double spin asymmetry, $\tilde{A}_{LL}$

This asymmetry is defined by keeping fixed the longitudinal polarization of the proton target, while flipping the lepton longitudinal polarization:

$$\tilde{A}_{LL} = \frac{d^6\sigma^{\ell^\rightarrow p^\rightarrow \to \ell' h X} - d^6\sigma^{\ell^\leftarrow p^\rightarrow \to \ell' h X}}{d^6\sigma^{\ell^\rightarrow p^\rightarrow \to \ell' h X} + d^6\sigma^{\ell^\leftarrow p^\rightarrow \to \ell' h X}} = \frac{d^6\sigma^{\ell(S_\ell)+p(S_L)\to \ell' h X} - d^6\sigma^{\ell(-S_\ell)+p(S_L)\to \ell' h X}}{d^6\sigma^{\ell(S_\ell)+p(S_L)\to \ell' h X} + d^6\sigma^{\ell(-S_\ell)+p(S_L)\to \ell' h X}} \,. \tag{91}$$

For the numerator we have

$$\frac{d\sigma^{\ell(S_\ell)+p(S_L)\to \ell' h X} - d\sigma^{\ell(-S_\ell)+p(S_L)\to \ell' h X}}{dx_{\!_B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} = \frac{2\alpha^2}{Q^4}\left\{[1-(1-y)^2]F_{LL} + 2y\sqrt{1-y}\,\cos\phi_h F_{LL}^{\cos\phi_h}\right\}. \tag{92}$$

Notice that, in this case, the denominator differs from that given in Eqs. (86), as it acquires additional terms (generated by $\Delta f^q_{s_x/S_L}$):

$$\begin{aligned}
&\frac{d\sigma^{\ell(S_\ell)+p(S_L)\to \ell' h X} + d\sigma^{\ell(-S_\ell)+p(S_L)\to \ell' h X}}{dx_{\!_B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} = \\
&\frac{2\alpha^2}{Q^4}\Bigg\{\left[1+(1-y)^2\right]\, F_{UU} + 2(1-y)[\cos 2\phi_h\, F_{UU}^{\cos 2\phi_h} + \sin 2\phi_h\, F_{UL}^{\sin 2\phi_h}] \\
&\qquad + 2(2-y)\sqrt{1-y}\,[\cos\phi_h\, F_{UU}^{\cos\phi_h} + \sin\phi_h\, F_{UL}^{\sin\phi_h}]\Bigg\}.
\end{aligned} \tag{93}$$

### E. Nucleon longitudinal-transverse double spin asymmetry, $A_{LT}$

This asymmetry is defined by keeping fixed the longitudinal polarization of the lepton, while flipping the proton target transverse polarization:

$$A_{LT} = \frac{d^6\sigma^{\ell^\rightarrow p^\uparrow \to \ell' h X} - d^6\sigma^{\ell^\rightarrow p^\downarrow \to \ell' h X}}{d^6\sigma^{\ell^\rightarrow p^\uparrow \to \ell' h X} + d^6\sigma^{\ell^\rightarrow p^\downarrow \to \ell' h X}} = \frac{d^6\sigma^{\ell(S_\ell)+p(S_T)\to \ell' h X} - d^6\sigma^{\ell(S_\ell)+p(-S_T)\to \ell' h X}}{d^6\sigma^{\ell(S_\ell)+p(S_T)\to \ell' h X} + d^6\sigma^{\ell(S_\ell)+p(-S_T)\to \ell' h X}} \,. \tag{94}$$

The denominator is given in Eq. (86), while for the numerator we have

$$\frac{d\sigma^{\ell(S_\ell)+p(S_T)\to \ell' h X} - d\sigma^{\ell(S_\ell)+p(-S_T)\to \ell' h X}}{dx_{\!_B}\, dQ^2\, dz_h\, d^2\boldsymbol{P}_T\, d\phi_S} =$$
$$\frac{2\alpha^2}{Q^4}\left\{\Big[1+(1-y)^2\Big]\sin(\phi_h-\phi_S)F_{UT}^{\sin(\phi_h-\phi_S)}\right.$$

$$
\begin{aligned}
&+\Big[1-(1-y)^2\Big]\cos(\phi_h-\phi_S)F_{LT}^{\cos(\phi_h-\phi_S)}\\
&+2y\sqrt{1-y}\Big[\cos\phi_S F_{LT}^{\cos\phi_S}+\cos(2\phi_h-\phi_S)F_{LT}^{\cos(2\phi_h-\phi_S)}\Big]\\
&+2(1-y)\Big[\sin(\phi_h+\phi_S)F_{UT}^{\sin(\phi_h+\phi_S)}+\sin(3\phi_h-\phi_S)F_{UT}^{\sin(3\phi_h-\phi_S)}\Big]\\
&+2(2-y)\sqrt{1-y}\Big[\cos\phi_S F_{LT}^{\cos\phi_S}+\cos(\phi_h-\phi_S)F_{LT}^{\cos(\phi_h-\phi_S)}\\
&\qquad\qquad+\sin\phi_S F_{UT}^{\sin\phi_S}+\sin(2\phi_h-\phi_S)F_{UT}^{\sin(2\phi_h-\phi_S)}\Big]\Big\}\,. \qquad (95)
\end{aligned}
$$

### F. Lepton longitudinal-transverse double spin asymmetry $\tilde{A}_{LT}$

This asymmetry is defined by flipping the direction of the longitudinal polarization of the lepton, while keeping fixed the proton target transverse polarization:

$$
\tilde{A}_{LT}=\frac{d^6\sigma^{\ell^{\rightarrow}p^{\uparrow}\to\ell' hX}-d^6\sigma^{\ell^{\leftarrow}p^{\uparrow}\to\ell' hX}}{d^6\sigma^{\ell^{\rightarrow}p^{\uparrow}\to\ell' hX}+d^6\sigma^{\ell^{\leftarrow}p^{\uparrow}\to\ell' hX}}=\frac{d^6\sigma^{\ell(S_\ell)+p(S_T)\to\ell' hX}-d^6\sigma^{\ell(-S_\ell)+p(S_T)\to\ell' hX}}{d^6\sigma^{\ell(S_\ell)+p(S_T)\to\ell' hX}+d^6\sigma^{\ell(-S_\ell)+p(S_T)\to\ell' hX}}\,. \qquad (96)
$$

For the numerator we have

$$
\begin{aligned}
&\frac{d\sigma^{\ell(S_\ell)+p(S_T)\to\ell' hX}-d\sigma^{\ell(-S_\ell)+p(S_T)\to\ell' hX}}{dx_{\scriptscriptstyle B}\,dQ^2\,dz_h\,d^2\boldsymbol{P}_T\,d\phi_S}=\\
&\frac{2\alpha^2}{Q^4}\Big\{[1-(1-y)^2]\cos(\phi_h-\phi_S)F_{LT}^{\cos(\phi_h-\phi_S)}+2y\sqrt{1-y}\Big[\cos\phi_S F_{LT}^{\cos\phi_S}+\cos(2\phi_h-\phi_S)F_{LT}^{\cos(2\phi_h-\phi_S)}\Big]\Big\}\,. \qquad (97)
\end{aligned}
$$

The denominator differs from that given in Eq. (86), as it acquires several additional terms, which also appear in the numerator of $A_{UT}$:

$$
\begin{aligned}
&\frac{d\sigma^{\ell(S_\ell)+p(S_T)\to\ell' hX}+d\sigma^{\ell(-S_\ell)+p(S_T)\to\ell' hX}}{dx_{\scriptscriptstyle B}\,dQ^2\,dz_h\,d^2\boldsymbol{P}_T\,d\phi_S}=\\
&=\frac{2\alpha^2}{Q^4}\Big\{\big[1+(1-y)^2\big]\,[F_{UU}+\sin(\phi_h-\phi_S)F_{UT}^{\sin(\phi_h-\phi_S)}]\\
&\qquad+2(1-y)[\cos 2\phi_h\,F_{UU}^{\cos 2\phi_h}+\sin(\phi_h+\phi_S)\,F_{UT}^{\sin(\phi_h+\phi_S)}+\sin(3\phi_h-\phi_S)\,F_{UT}^{\sin(3\phi_h-\phi_S)}]\\
&\qquad+2(2-y)\sqrt{1-y}\,[\cos\phi_h\,F_{UU}^{\cos\phi_h}+\sin\phi_S\,F_{UT}^{\sin\phi_S}+\sin(2\phi_h-\phi_S)\,F_{UT}^{\sin(2\phi_h-\phi_S)}]\Big\}\,. \qquad (98)
\end{aligned}
$$

### G. Other asymmetries

All the other single and double spin asymmetries are either zero or related to those already shown above. In particular, all the single spin asymmetries generated by the lepton polarization vanish: $A_{LU}=0$ as $F_{LU}=0$ to leading order in $k_\perp/Q$ and $A_{TU}=0$ as we have no access to the transverse polarization of the lepton and therefore there are no terms proportional to either $P_x^\ell$ or $P_y^\ell$ in Eqs. (4) or (79). For the same reason we have $A_{TT}=A_{UT}$ and $A_{TL}=A_{UL}$. Despite its possible presence, the transverse polarization of the lepton plays no role in SIDIS; $P_x^\ell$ and $P_y^\ell$ appear in the off-diagonal terms of $\rho^{\ell,S_\ell}_{\lambda_\ell\lambda'_\ell}$ in Eq. (4), but, due to helicity conservation and the fact that the final lepton polarization cannot be observed, one is forced to have $\lambda_{\ell'}=\lambda_\ell=\lambda'_\ell$.

## IV. PHENOMENOLOGY OF SPIN ASYMMETRIES

To leading order in $(k_\perp/Q)$, all terms contributing to the polarized SIDIS cross section and to the spin asymmetries can be integrated analytically, provided we adopt a simple $k_\perp$ and $p_\perp$ dependence for the distribution and fragmentation functions. As usual, we assume the $x$ and $k_\perp$ dependences to be factorized and we assign the $k_\perp$ dependence a

Gaussian distribution with one free parameter to fix the Gaussian width. For the unpolarized and helicity distribution functions and for the fragmentation function we simply use

$$f_{q/p}(x,k_\perp) = f_{q/p}(x)\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle}}{\pi\langle k_\perp^2\rangle} \tag{99}$$

$$\Delta f^q_{s_z/S_L}(x,k_\perp) = \Delta f^q_{s_z/S_L}(x)\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle_L}}{\pi\langle k_\perp^2\rangle_L} \tag{100}$$

$$D_{h/q}(z,p_\perp) = D_{h/q}(z)\,\frac{e^{-p_\perp^2/\langle p_\perp^2\rangle}}{\pi\langle p_\perp^2\rangle}\,, \tag{101}$$

where $f_{q/p}(x)$, $\Delta f^q_{s_z/S_L}(x)$ and $D_{h/q}(z)$ can be taken from the available fits of the world data. In general, we allow for different widths of the Gaussians for the different distributions, but take them to be constant and flavor independent. For the Sivers and Boer-Mulders functions, we assume a similar parametrization, with an extra multiplicative factor $k_\perp$ to give them the appropriate behavior in the small $k_\perp$ region [44]:

$$\begin{aligned}\Delta f_{q/S_T}(x,k_\perp) &= \Delta f_{q/S_T}(x)\,\sqrt{2e}\,\frac{k_\perp}{M_S}\,e^{-k_\perp^2/M_S^2}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle}}{\pi\langle k_\perp^2\rangle}\\ &= \Delta f_{q/S_T}(x)\,\sqrt{2e}\,\frac{k_\perp}{M_S}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle_S}}{\pi\langle k_\perp^2\rangle}\end{aligned} \tag{102}$$

$$\begin{aligned}\Delta f^q_{s_y/p}(x,k_\perp) &= \Delta f^q_{s_y/p}(x)\,\sqrt{2e}\,\frac{k_\perp}{M_{BM}}\,e^{-k_\perp^2/M_{BM}^2}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle}}{\pi\langle k_\perp^2\rangle}\\ &= \Delta f^q_{s_y/p}(x)\,\sqrt{2e}\,\frac{k_\perp}{M_{BM}}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle_{BM}}}{\pi\langle k_\perp^2\rangle}\,,\end{aligned} \tag{103}$$

where the $x$-dependent functions $\Delta f_{q/S_T}(x)$ and $\Delta f^q_{s_y/p}(x)$ are not known, and should be determined phenomenologically by fitting the available data on azimuthal asymmetries and moments; the $k_\perp$ dependent Gaussians have been assigned a reduced width to make sure they fulfill the appropriate positivity bounds [49]:

$$\langle k_\perp^2\rangle_S = \frac{\langle k_\perp^2\rangle\,M_S^2}{\langle k_\perp^2\rangle + M_S^2} \tag{104}$$

$$\langle k_\perp^2\rangle_{BM} = \frac{\langle k_\perp^2\rangle\,M_{BM}^2}{\langle k_\perp^2\rangle + M_{BM}^2}\,. \tag{105}$$

Similarly, for the distribution of longitudinally polarized quarks inside a transversely polarized proton, $\Delta f^q_{s_z/S_T}$, and of transversely polarized quarks inside a longitudinally polarized proton, $\Delta f^q_{s_x/S_L}$, we set:

$$\begin{aligned}\Delta f^q_{s_z/S_T}(x,k_\perp) &= \Delta f^q_{s_z/S_T}(x)\,\sqrt{2e}\,\frac{k_\perp}{M_{LT}}\,e^{-k_\perp^2/M_{LT}^2}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle}}{\pi\langle k_\perp^2\rangle}\\ &= \Delta f^q_{s_z/S_T}(x)\,\sqrt{2e}\,\frac{k_\perp}{M_{LT}}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle_{LT}}}{\pi\langle k_\perp^2\rangle}\end{aligned} \tag{106}$$

$$\begin{aligned}\Delta f^q_{s_x/S_L}(x,k_\perp) &= \Delta f^q_{s_x/S_L}(x)\,\sqrt{2e}\,\frac{k_\perp}{M_{TL}}\,e^{-k_\perp^2/M_{TL}^2}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle}}{\pi\langle k_\perp^2\rangle}\\ &= \Delta f^q_{s_x/S_L}(x)\,\sqrt{2e}\,\frac{k_\perp}{M_{TL}}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle_{TL}}}{\pi\langle k_\perp^2\rangle}\,,\end{aligned} \tag{107}$$

with

$$\langle k_\perp^2\rangle_{LT} = \frac{\langle k_\perp^2\rangle\,M_{LT}^2}{\langle k_\perp^2\rangle + M_{LT}^2} \tag{108}$$

$$\langle k_\perp^2\rangle_{TL} = \frac{\langle k_\perp^2\rangle\,M_{TL}^2}{\langle k_\perp^2\rangle + M_{TL}^2}\,. \tag{109}$$

For the transversity distribution function, it is most convenient to parametrize the following combinations

$$\frac{1}{2}\left(\Delta f^q_{s_x/S_T}(x,k_\perp) + \Delta^- f^q_{s_y/S_T}(x,k_\perp)\right) \;=\; h_1(x,k_\perp) = \; h_1(x)\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle_T}}{\pi\langle k_\perp^2\rangle_T} \tag{110}$$

$$\begin{aligned}\frac{1}{2}\left(\Delta f^q_{s_x/S_T}(x,k_\perp) - \Delta^- f^q_{s_y/S_T}(x,k_\perp)\right) \;&=\; \frac{k_\perp^2}{2M_{TT}^2}\,h_{1T}^\perp(x,k_\perp) = \; h_{1T}^\perp(x)\,\frac{e\,k_\perp^2}{M_{TT}^2}\,e^{-k_\perp^2/M_{TT}^2}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle}}{\pi\langle k_\perp^2\rangle}\\ &=\; h_{1T}^\perp(x)\,\frac{e\,k_\perp^2}{M_{TT}^2}\,\frac{e^{-k_\perp^2/\langle k_\perp^2\rangle_{TT}}}{\pi\langle k_\perp^2\rangle}\,,\end{aligned} \tag{111}$$

as these are the quantities which appear in the polarized cross section and in the spin asymmetries. Notice that for $h_1(x,k_\perp)$ and $h_{1T}^\perp(x,k_\perp)$, as for each of the other TMDs, we introduce their own reduced Gaussian widths

$$\langle k_\perp^2\rangle_T \qquad\qquad \langle k_\perp^2\rangle_{TT} = \frac{\langle k_\perp^2\rangle M_{TT}^2}{\langle k_\perp^2\rangle + M_{TT}^2}\,. \tag{112}$$

Finally, for the Collins fragmentation function we choose

$$\begin{aligned}\Delta^N D_{h/q^\uparrow}(z,p_\perp) \;&=\; \Delta^N D_{h/q^\uparrow}(z)\,\sqrt{2e}\,\frac{p_\perp}{M_h}\,e^{-p_\perp^2/M_h^2}\,\frac{e^{-p_\perp^2/\langle p_\perp^2\rangle}}{\pi\langle p_\perp^2\rangle}\\ &=\; \Delta^N D_{h/q^\uparrow}(z)\,\sqrt{2e}\,\frac{p_\perp}{M_h}\,\frac{e^{-p_\perp^2/\langle p_\perp^2\rangle_C}}{\pi\langle p_\perp^2\rangle}\,,\end{aligned} \tag{113}$$

having defined

$$\langle p_\perp^2\rangle_C = \frac{\langle p_\perp^2\rangle\,M_h^2}{\langle p_\perp^2\rangle + M_h^2}\,. \tag{114}$$

Using the parametrizations in Eqs. (99-114) we can perform the $\boldsymbol{k}_\perp$ integrations analytically in Eqs. (63-78), and re-express all the $F$ structure functions in terms of the Gaussian parameters (some of these results were already given in Refs. [1, 50]):

$$F_{UU} \;=\; \sum_q e_q^2\, f_{q/p}(x_B)\, D_{h/q}(z_h)\frac{e^{-P_T^2/\langle P_T^2\rangle}}{\pi\langle P_T^2\rangle} \tag{115}$$

$$F_{UU}^{\cos 2\phi_h} \;=\; -P_T^2\sum_q e_q^2\,\frac{\Delta f^q_{s_y/p}(x_B)}{M_{BM}}\,\frac{\Delta^N D_{h/q^\uparrow}(z_h)}{M_h}\,\frac{e^{1-P_T^2/\langle P_T^2\rangle_{BM}}}{\pi\langle P_T^2\rangle_{BM}^3}\,\frac{z_h\,\langle k_\perp^2\rangle_{BM}^2\,\langle p_\perp^2\rangle_C^2}{\langle k_\perp^2\rangle\langle p_\perp^2\rangle} \tag{116}$$

$$\begin{aligned}F_{UU}^{\cos\phi_h} \;&=\; -2\,\frac{P_T}{Q}\sum_q e_q^2\, f_{q/p}(x_B)\, D_{h/q}(z_h)\,\frac{e^{-P_T^2/\langle P_T^2\rangle}}{\pi\langle P_T^2\rangle^2}\,z_h\langle k_\perp^2\rangle\\ &\quad +2\,\frac{P_T}{Q}\sum_q e_q^2\,\frac{\Delta f^q_{s_y/p}(x_B)}{M_{BM}}\,\frac{\Delta^N D_{h/q^\uparrow}(z_h)}{M_h}\,\frac{e^{1-P_T^2/\langle P_T^2\rangle_{BM}}}{\pi\langle P_T^2\rangle_{BM}^4}\\ &\quad\times\frac{\langle k_\perp^2\rangle_{BM}^2\,\langle p_\perp^2\rangle_C^2}{\langle k_\perp^2\rangle\langle p_\perp^2\rangle}\left[z_h^2\langle k_\perp^2\rangle_{BM}\left(P_T^2-\langle P_T^2\rangle_{BM}\right)+\langle p_\perp^2\rangle_C\langle P_T^2\rangle_{BM}\right]\end{aligned} \tag{117}$$

$$F_{UL}^{\sin 2\phi_h} \;=\; P_T^2\sum_q e_q^2\,\frac{\Delta f^q_{s_x/S_L}(x_B)}{M_{TL}}\,\frac{\Delta^N D_{h/q^\uparrow}(z_h)}{M_h}\,\frac{e^{1-P_T^2/\langle P_T^2\rangle_{TL}}}{\pi\langle P_T^2\rangle_{TL}^3}\,\frac{z_h\,\langle k_\perp^2\rangle_{TL}^2\,\langle p_\perp^2\rangle_C^2}{\langle k_\perp^2\rangle\langle p_\perp^2\rangle} \tag{118}$$

$$\begin{aligned}F_{UL}^{\sin\phi_h} \;&=\; -2\,\frac{P_T}{Q}\sum_q e_q^2\,\frac{\Delta f^q_{s_x/S_L}(x_B)}{M_{TL}}\,\frac{\Delta^N D_{h/q^\uparrow}(z_h)}{M_h}\,\frac{e^{1-P_T^2/\langle P_T^2\rangle_{TL}}}{\pi\langle P_T^2\rangle_{TL}^4}\\ &\quad\times\frac{\langle k_\perp^2\rangle_{TL}^2\,\langle p_\perp^2\rangle_C^2}{\langle k_\perp^2\rangle\langle p_\perp^2\rangle}\left[z_h^2\langle k_\perp^2\rangle_{TL}\left(P_T^2-\langle P_T^2\rangle_{TL}\right)+\langle p_\perp^2\rangle_C\langle P_T^2\rangle_{TL}\right]\end{aligned} \tag{119}$$

$$F_{LU}^{\sin\phi_h} \;=\; 0 \quad \text{at leading twist} \tag{120}$$

$$F_{LL} \;=\; \sum_q e_q^2\Delta f^q_{s_z/S_L}(x_B)\, D_{h/q}(z_h)\,\frac{e^{-P_T^2/\langle P_T^2\rangle_L}}{\pi\langle P_T^2\rangle_L} \tag{121}$$

$$F_{LL}^{\cos\phi_h} = -2\,\frac{P_T}{Q}\sum_q e_q^2 \Delta f^q_{s_z/S_L}(x_B)\, D_{h/q}(z_h)\,\frac{e^{-P_T^2/\langle P_T^2\rangle_L}}{\pi\langle P_T^2\rangle_L^2}\, z_h\langle k_\perp^2\rangle_L \tag{122}$$

$$F_{UT}^{\sin(\phi_h-\phi_S)} = \frac{P_T}{\sqrt{2}}\sum_q e_q^2\,\frac{\Delta f_{q/S_T}(x_B)}{M_S} D_{h/q}(z_h)\frac{e^{1/2-P_T^2/\langle P_T^2\rangle_S}}{\pi\langle P_T^2\rangle_S^2}\,\frac{z_h\langle k_\perp^2\rangle_S^2}{\langle k_\perp^2\rangle} \tag{123}$$

$$F_{LT}^{\cos(\phi_h-\phi_S)} = P_T\sum_q e_q^2\,\frac{\Delta f^q_{s_z/S_T}(x_B)}{M_{LT}}\, D_{h/q}(z_h)\frac{e^{-P_T^2/\langle P_T^2\rangle_{LT}}}{\pi\langle P_T^2\rangle_{LT}^2}\,\frac{z_h\langle k_\perp^2\rangle_{LT}^2}{\langle k_\perp^2\rangle} \tag{124}$$

$$F_{LT}^{\cos\phi_S} = -\frac{1}{Q}\sum_q e_q^2\,\frac{\Delta f^q_{s_z/S_T}(x_B)}{M_{LT}}\, D_{h/q}(z_h)\,\frac{e^{-P_T^2/\langle P_T^2\rangle_{LT}}}{\pi\langle P_T^2\rangle_{LT}^3}\,\frac{\langle k_\perp^2\rangle_{LT}^2[\langle p_\perp^2\rangle\langle P_T^2\rangle_{LT} + z_h^2 P_T^2\langle k_\perp^2\rangle_{LT}]}{\langle k_\perp^2\rangle} \tag{125}$$

$$F_{LT}^{\cos(2\phi_h-\phi_S)} = -\frac{P_T^2}{Q}\sum_q e_q^2\,\frac{\Delta f^q_{s_z/S_T}(x_B)}{M_{LT}}\, D_{h/q}(z_h)\,\frac{e^{-P_T^2/\langle P_T^2\rangle_{LT}}}{\pi\langle P_T^2\rangle_{LT}^3}\,\frac{z_h^2\langle k_\perp^2\rangle_{LT}^3}{\langle k_\perp^2\rangle} \tag{126}$$

$$F_{UT}^{\sin(\phi_h+\phi_S)} = \frac{P_T}{\sqrt{2}}\sum_q e_q^2\, h_1(x_B)\,\frac{\Delta^N D_{h/q^\uparrow}(z_h)}{M_h}\,\frac{e^{1/2-P_T^2/\langle P_T^2\rangle_T}}{\pi\langle P_T^2\rangle_T^2}\,\frac{\langle p_\perp^2\rangle_C^2}{\langle p_\perp^2\rangle} \tag{127}$$

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = \frac{P_T^3}{\sqrt{2}}\sum_q e_q^2\,\frac{h_{1T}^\perp(x_B)}{M_{TT}^2}\,\frac{\Delta^N D_{h/q^\uparrow}(z_h)}{M_h}\,\frac{e^{3/2-P_T^2/\langle P_T^2\rangle_{TT}}}{\pi\langle P_T^2\rangle_{TT}^4}\,\frac{z_h^2\langle k_\perp^2\rangle_{TT}^3\langle p_\perp^2\rangle_C^2}{\langle k_\perp^2\rangle\langle p_\perp^2\rangle} \tag{128}$$

$$\begin{aligned} F_{UT}^{\sin\phi_S} &= \sqrt{2}\frac{1}{Q}\sum_q e_q^2 h_1(x_B)\,\frac{\Delta^N D_{h/q^\uparrow}(z_h)}{M_h}\,\frac{e^{1/2-P_T^2/\langle P_T^2\rangle_T}}{\pi\langle P_T^2\rangle_T^3}\,\frac{z_h\langle k_\perp^2\rangle\langle p_\perp^2\rangle_C^2(\langle P_T^2\rangle_T - P_T^2)}{\langle p_\perp^2\rangle} \\ &+ \frac{1}{\sqrt{2}}\frac{1}{Q}\sum_q e_q^2\,\frac{\Delta f_{q/S_T}(x_B)}{M_S} D_{h/q}(z_h)\frac{e^{1/2-P_T^2/\langle P_T^2\rangle_S}}{\pi\langle P_T^2\rangle_S^3}\,\frac{\langle k_\perp^2\rangle_S^2(\langle p_\perp^2\rangle\langle P_T^2\rangle_S + z_h^2 P_T^2\langle k_\perp^2\rangle_S)}{\langle k_\perp^2\rangle} \end{aligned} \tag{129}$$

$$\begin{aligned} F_{UT}^{\sin(2\phi_h-\phi_S)} &= -\sqrt{2}\,\frac{P_T^2}{Q}\sum_q e_q^2\,\frac{h_{1T}^\perp(x_B)}{M_{TT}^2}\,\frac{\Delta^N D_{h/q^\uparrow}(z_h)}{M_h} \\ &\times\frac{e^{3/2-P_T^2/\langle P_T^2\rangle_{TT}}}{\pi\langle P_T^2\rangle_{TT}^5}\,\frac{z_h\langle k_\perp^2\rangle_{TT}^3\langle p_\perp^2\rangle_C^2\left[z_h^2\langle k_\perp^2\rangle_{TT}(P_T^2-\langle P_T^2\rangle_{TT}) + 2\langle p_\perp^2\rangle_C\langle P_T^2\rangle_{TT}\right]}{\langle k_\perp^2\rangle\langle p_\perp^2\rangle} \\ &- \frac{1}{\sqrt{2}}\frac{P_T^2}{Q}\sum_q e_q^2\,\frac{\Delta f_{q/S_T}(x_B)}{M_S} D_{h/q}(z_h)\frac{e^{1/2-P_T^2/\langle P_T^2\rangle_S}}{\pi\langle P_T^2\rangle_S^3}\,\frac{z_h^2\langle k_\perp^2\rangle_S^3}{\langle k_\perp^2\rangle} \end{aligned} \tag{130}$$

where

$$\begin{aligned} \langle P_T^2\rangle &= \langle p_\perp^2\rangle + z_h^2\,\langle k_\perp^2\rangle \\ \langle P_T^2\rangle_I &= \langle p_\perp^2\rangle + z_h^2\,\langle k_\perp^2\rangle_I \quad (I=S,L,LT) \\ \langle P_T^2\rangle_J &= \langle p_\perp^2\rangle_C + z_h^2\,\langle k_\perp^2\rangle_J \quad (J=T,BM,TL,TT)\,. \end{aligned} \tag{131}$$

The unpolarized SIDIS cross section and all the asymmetries presented in Section III can now be rewritten in terms of the Gaussian-integrated $F$'s, which depend on the TMDs. In order to single out information on a particular TMD from the measurements of the asymmetries, one has to disentangle the different azimuthal dependences. For example, the unpolarized cross section, see Eq. (86), includes the usual unpolarized collinear SIDIS cross section, the Cahn effect proportional to $\cos\phi_h$ (studied in Ref. [37]), and a contribution generated by a combined Boer-Mulders and Collins effect, which appears in terms proportional to $\cos 2\phi_h$ and $\cos\phi_h$. Similarly, in the numerator of the $A_{UT}$ single spin asymmetry, Eq. (85), the Sivers and Collins effects are both simultaneously at work, together with other azimuthal modulations. To extract single effects, one introduces appropriate azimuthal moments of the asymmetries, defined as

$$A_{S_\ell S}^{W(\phi_h,\phi_S)} \equiv 2\,\frac{\displaystyle\int d\phi_h\, d\phi_S\,[d\sigma^{\ell(S_\ell)+p(S)\to\ell' hX} - d\sigma^{\ell(S_\ell)+p(-S)\to\ell' hX}]\,W(\phi_h,\phi_S)}{\displaystyle\int d\phi_h\, d\phi_S\,[d\sigma^{\ell(S_\ell)+p(S)\to\ell' hX} + d\sigma^{\ell(S_\ell)+p(-S)\to\ell' hX}]}\,, \tag{132}$$

where the function $W(\phi_h,\phi_S)$ is an appropriate "weighting phase" which, upon integration, singles out one individual term of the asymmetry. For instance, to isolate the Sivers effect one can consider the $\sin(\phi_h-\phi_S)$ azimuthal moment of the $A_{UT}$ asymmetry:

$$A_{UT}^{\sin(\phi_h-\phi_S)} = 2\,\frac{\displaystyle\int d\phi_h\, d\phi_S\,[d\sigma^{\ell\, p^\uparrow\to\ell' hX} - d\sigma^{\ell\, p^\downarrow\to\ell' hX}]\,\sin(\phi_h-\phi_S)}{\displaystyle\int d\phi_h\, d\phi_S\,[d\sigma^{\ell\, p^\uparrow\to\ell' hX} + d\sigma^{\ell\, p^\downarrow\to\ell' hX}]}\,. \tag{133}$$

The $W$ weight selects the Sivers term of the asymmetry in the numerator, while the integration over the azimuthal angles $\phi_S$ and $\phi_h$ leaves only the first term of the unpolarized cross section, Eq. (86), in the denominator: thus, this azimuthal moment is simply proportional to the ratio $\int F_{UT}^{\sin(\phi_h-\phi_S)}/\int F_{UU}$.

Furthermore, experimental data deliver these azimuthal moments as a function of one variable at a time, either $x_{_B}$, $z_h$ or $P_T$. Therefore, one has to integrate the numerator and denominator *separately* over all variables but one, in order to obtain the appropriate expression to be compared with the data. Clearly, no simplification of common terms in the numerator and denominator can be made before the integrations have been performed (notice also that $y$ is a function of both $x_{_B}$ and $Q^2$).

Let us consider, as an explicit example, the Sivers azimuthal moment $A_{UT}^{\sin(\phi_h-\phi_S)}(z_h)$, as function of $z_h$ alone. Using the Gaussian-integrated expression of $F_{UT}^{\sin(\phi_h-\phi_S)}$ of Eq. (123) and integrating analytically over $\boldsymbol{P}_T$ we obtain

$$A_{UT}^{\sin(\phi_h-\phi_S)}(z_h) = \,A_{_S}\,\frac{\displaystyle\int dx_{_B}\,dQ^2\,\frac{1+(1-y)^2}{Q^4}\sum_q e_q^2\,\Delta^N f_{q/S_T}(x_{_B})\,D_{h/q}(z_h)}{\displaystyle\int dx_{_B}\,dQ^2\,\frac{1+(1-y)^2}{Q^4}\sum_q e_q^2\,f_{q/p}(x_{_B})\,D_{h/q}(z_h)}\,, \tag{134}$$

where $A_{_S}$ is a factor which only depends on $z_h$ and on the free parameters which give the Gaussian widths for the distribution and fragmentation functions

$$A_{_S} = \frac{z_h}{4\,M_{_S}}\sqrt{\frac{2\,e\,\pi}{\langle P_T^2\rangle_{_S}}}\,\frac{\langle k_\perp^2\rangle_{_S}^2}{\langle k_\perp^2\rangle}\,. \tag{135}$$

Notice the further dependence on $z_h$ hidden in $\langle P_T^2\rangle_{_S}$, Eq. (131).

Repeating similar procedures one can extract information on the other TMDs. The azimuthal moment $A_{UT}^{\sin(\phi_h+\phi_S)}$, obtained using the weighting phase $W(\phi_h,\phi_S)=\sin(\phi_h+\phi_S)$ in Eq. (132) with unpolarized leptons, selects the Collins effect, coupled to the transversity distribution $F_{+-}^{+-}(x) = \Delta_T q(x) = h_1(x)$. In this case, the azimuthal moment is sensitive to the ratio $F_{UT}^{\sin(\phi_h+\phi_S)}/F_{UU}$, and precisely:

$$A_{UT}^{\sin(\phi_h+\phi_S)}(z_h) = A_{_C}\,\frac{\displaystyle\int dx_{_B}\,dQ^2\,\frac{2(1-y)}{Q^4}\sum_q e_q^2\,h_1(x_{_B})\,\Delta^N D_{h/q^\uparrow}(z_h)}{\displaystyle\int dx_{_B}\,dQ^2\,\frac{1+(1-y)^2}{Q^4}\sum_q e_q^2\,f_{q/p}(x_{_B})\,D_{h/q}(z_h)}\,, \tag{136}$$

with

$$A_{_C} = \frac{1}{4\,M_h}\sqrt{\frac{2\,e\,\pi}{\langle P_T^2\rangle_{_C}}}\,\frac{\langle p_\perp^2\rangle_{_C}^2}{\langle p_\perp^2\rangle}\,. \tag{137}$$

One can further exploit the $A_{UT}$ asymmetry, to isolate and measure the transverse distribution function $F_{+-}^{-+}(x) = h_{1T}^\perp(x)$, by weighting the single spin asymmetry numerator with the phase $W(\phi_h,\phi_S)=\sin(3\phi_h-\phi_S)$, obtaining:

$$A_{UT}^{\sin(3\phi_h-\phi_S)}(z_h) = A_{_{TT}}\,\frac{\displaystyle\int dx_{_B}\,dQ^2\,\frac{2(1-y)}{Q^4}\sum_q e_q^2\,h_{1T}^\perp(x_{_B})\,\Delta^N D_{h/q^\uparrow}(z_h)}{\displaystyle\int dx_{_B}\,dQ^2\,\frac{1+(1-y)^2}{Q^4}\sum_q e_q^2\,f_{q/p}(x_{_B})\,D_{h/q}(z_h)}\,, \tag{138}$$

where

$$A_{_{TT}} = \frac{3\,e\,z_h^2}{8\,M_{_{TT}}^2\,M_h\,\langle P_T^2\rangle_{_{TT}}}\sqrt{\frac{2\,e\,\pi}{\langle P_T^2\rangle_{_{TT}}}}\,\frac{\langle k_\perp^2\rangle_{_{TT}}^3}{\langle k_\perp^2\rangle}\,\frac{\langle p_\perp^2\rangle_{_C}^2}{\langle p_\perp^2\rangle}\,. \tag{139}$$

One can write similar expressions for all other asymmetries, which we do not report here. From $A_{UL}^{\sin\phi_h}$ and $A_{UL}^{\sin 2\phi_h}$ one can obtain information on $\Delta f_{s_x/S_L}$, while $A_{LT}^{\cos\phi_S}$, $A_{LT}^{\cos(\phi_h-\phi_S)}$ and $A_{LT}^{\cos(2\phi_h-\phi_S)}$ depend on $\Delta f_{s_z/S_T}$. $A_{UT}^{\sin\phi_S}$ and $A_{UT}^{\sin(2\phi_h-\phi_S)}$ are more complicated to analyze as they receive contributions from the Sivers distribution function (both of them) and, in addition, from the transversity distribution $h_1(x)$ ($A_{UT}^{\sin\phi_S}$) and from $h_{1T}^{\perp}$ ($A_{UT}^{\sin(2\phi_h-\phi_S)}$).

Let us consider in more details the unpolarized cross section, to which, remarkably, a similar "weighting" procedure can be applied. In fact, one can introduce the average value of $W(\phi_h)$ with an expression similar to Eq. (132) in which the unpolarized cross section appears in the numerator as well as in the denominator

$$\langle W(\phi_h)\rangle = \frac{\displaystyle\int d\phi_h\, d\phi_S\, [d\sigma^{\ell p^\uparrow\to\ell' hX} + d\sigma^{\ell p^\downarrow\to\ell' hX}]\, W(\phi_h)}{\displaystyle\int d\phi_h\, d\phi_S\, [d\sigma^{\ell p^\uparrow\to\ell' hX} + d\sigma^{\ell p^\downarrow\to\ell' hX}]}\,. \tag{140}$$

For instance, weighting the unpolarized cross section with $W(\phi_h) = \cos 2\phi_h$ one can gain direct access to the Boer-Mulders function, coupled to the Collins function (on which independent information can be obtained):

$$\langle\cos 2\phi_h\rangle = A_{_{BM}}\,\frac{\displaystyle\int dx_{_B}\, dQ^2\, \frac{(1-y)}{Q^4}\sum_q e_q^2\, \Delta f^q_{s_y/p}(x_{_B})\, \Delta^N D_{h/q^\uparrow}(z_h)}{\displaystyle\int dx_{_B}\, dQ^2\, \frac{1+(1-y)^2}{Q^4}\sum_q e_q^2\, f_{q/p}(x_{_B})\, D_{h/q}(z_h)}\,, \tag{141}$$

with

$$A_{_{BM}} = -\frac{e\, z_h}{M_{_{BM}}\, M_h\, \langle P_T^2\rangle_{_{BM}}}\,\frac{\langle k_\perp^2\rangle^2_{_{BM}}}{\langle k_\perp^2\rangle}\,\frac{\langle p_\perp^2\rangle^2_{_C}}{\langle p_\perp^2\rangle}\,. \tag{142}$$

Analogously, using $W(\phi_h) = \cos\phi_h$, one has

$$\langle\cos\phi_h\rangle = \frac{\displaystyle\int dx_{_B}\, dQ^2\, \frac{(2-y)\sqrt{1-y}}{Q^4}\sum_q e_q^2\left[A_{\rm unp}\, f_{q/p}(x_{_B})\, D_{h/q}(z_h) + B_{_{BM}}\Delta f^q_{s_y/p}(x_{_B})\, \Delta^N D_{h/q^\uparrow}(z_h)\right]}{\displaystyle\int dx_{_B}\, dQ^2\, \frac{1+(1-y)^2}{Q^4}\sum_q e_q^2\, f_{q/p}(x_{_B})\, D_{h/q}(z_h)} \tag{143}$$

with

$$A_{\rm unp} = -\, z_h\, \frac{\langle k_\perp^2\rangle}{Q}\sqrt{\frac{\pi}{\langle P_T^2\rangle}}\,, \qquad B_{_{BM}} = \frac{e\sqrt{\pi}}{2\, Q\, M_{_{BM}}\, M_h}\,\frac{\langle k_\perp^2\rangle^2_{_{BM}}}{\langle k_\perp^2\rangle}\,\frac{\langle p_\perp^2\rangle^2_{_C}}{\langle p_\perp^2\rangle}\,\frac{[\langle p_\perp^2\rangle_{_C} + \langle P_T^2\rangle_{_{BM}}]}{\langle P_T^2\rangle^{3/2}_{_{BM}}}\,. \tag{144}$$

## V. CONCLUSIONS AND FURTHER REMARKS

The study of the 3-dimensional structure of protons and neutrons is one of the central issues in hadron physics, with many dedicated experiments, either running (COMPASS at CERN, CLASS at JLab, STAR and PHENIX at RHIC), approved (JLab upgrade) or being planned (ENC/EIC Colliders). The transverse momentum dependent partonic distribution and fragmentation functions, together with the generalized parton distributions, play a crucial role in gathering and interpreting information towards a true 3-dimensional imaging of the nucleons. TMDs can be accessed in several experiments, but the main source of information is semi-inclusive deep inelastic scattering of leptons off polarized nucleons. The theoretical framework in which the experimental information is analyzed is the QCD factorization scheme.

We have used here an intuitive approach to TMD factorization in SIDIS and shown that one can re-derive, at leading order, the most general expression of the polarized cross section, obtained within the QCD factorization scheme by other authors [1, 20, 23]. All azimuthal dependences are precisely generated by the properties of the helicity amplitudes, which we use to describe the factorized steps of the process: the partonic distributions, the elementary interaction and the quark fragmentation.

We have obtained explicit expressions for all the SIDIS spin asymmetries and the cross section azimuthal dependences which allow to extract information on the TMDs. Indeed, some of them have already been used to study the Sivers [42, 43], the Cahn [51, 52] and the Collins [3] effects. Simplified expressions, based on a Gaussian $k_\perp$ and $p_\perp$ dependence of the distribution and fragmentation functions, recently supported by data [53], have been given; they might be useful for fast and simple analyses of the experimental data.

We wonder, at this stage, whether the same approach can be used for other processes. It works, with the same validity as for SIDIS, for Drell-Yan processes (D-Y) [36], where our helicity amplitudes for the different factorized steps reproduce the most general azimuthal structure of the cross section as obtained in the TMD factorization [16]. As commented in the Introduction, both in SIDIS and D-Y the presence of two different natural scales, a small and a large one, is crucial for the validity of the QCD TMD factorization.

Our approach was actually first introduced for processes with a single large scale, like $p\,p \to \pi X$, with large $P_T$ pions [2]. These are the processes for which the largest single spin asymmetries have been observed and might be generated by TMDs [54–56]. However, TMD factorization has not been proven in these cases. Despite that, an extension of the intuitive approach used for SIDIS – *and shown to be perfectly equivalent to the QCD TMD factorization scheme* – is natural. That was the guiding idea in Ref. [2]; each proton "emits" a parton, the two partons interacts and one of the final parton fragments into the observed hadron. All intrinsic motions are taken into account and phases appear in the helicity amplitudes. The difference with SIDIS processes is that, in this case, the measured large $P_T$ of the final hadron is generated by the hard elementary scattering, and all intrinsic motions are integrated over. As a consequence, the phase integrations strongly suppress the relevance of most TMDs, with the exception of the Sivers and Collins effects [57, 58], which combine into the observed asymmetry, and cannot be separated unless one could resolve the internal structure of the final jet [59].

A global simultaneous phenomenological analysis of single spin asymmetries in SIDIS and $pp$ interactions is, at the moment, rather difficult. Apart from the validity of the factorization scheme in both cases, another important open point is the universality of the Sivers functions; it is not clear whether or not they should be the same in the two processes or should be corrected by some gauge color factors [30, 31]. In any case it is worth trying to explore the possibility to have a unique description of SSAs in different processes, based on TMDs; work in this direction is in progress and will be presented elsewhere.

## VI. ACKNOWLEDGMENTS

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### Appendix A: Helicity Amplitudes

We show the explicit computation of the helicity amplitudes $\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ for the non-planar process $\ell(k_1,\lambda_1) + q(k_2,\lambda_2) \to \ell'(k_3,\lambda_3) + q'(k_4,\lambda_4)$, in the $\gamma^* - p$ c.m. frame of Fig. 1. We exploit the spinor helicity technique, adopting the conventions of Ref. [38]. At LO in QED, when neglecting all masses, there are two independent helicity amplitudes:

$$\hat{M}_{++;++} = \frac{e_q\, e^2}{\hat{t}} \langle q'^+|\gamma^\mu|q^+\rangle \langle \ell'^+|\gamma_\mu|\ell^+\rangle = \frac{e_q\, e^2}{\hat{t}} \langle 4^+|\gamma^\mu|2^+\rangle \langle 3^+|\gamma_\mu|1^+\rangle \tag{A1}$$

$$\hat{M}_{+-;+-} = \frac{e_q\, e^2}{\hat{t}} \langle q'^-|\gamma^\mu|q^-\rangle \langle \ell'^+|\gamma_\mu|\ell^+\rangle = \frac{e_q\, e^2}{\hat{t}} \langle 4^-|\gamma^\mu|2^-\rangle \langle 3^+|\gamma_\mu|1^+\rangle\,, \tag{A2}$$

which can be written as

$$\hat{M}_{++;++} = 2\,\frac{e_q\, e^2}{\hat{t}}\,[43]\,\langle 12\rangle \tag{A3}$$

$$\hat{M}_{+-;+-} = 2\,\frac{e_q\, e^2}{\hat{t}}\,[23]\,\langle 14\rangle\,, \tag{A4}$$

where

$$\bar{u}_-(k_i)\,u_+(k_j) \equiv \langle ij\rangle = -[ij]^* = \sqrt{k_i^+ k_j^-}\,e^{-i(\phi_i-\phi_j)/2} - \sqrt{k_i^- k_j^+}\,e^{i(\phi_i-\phi_j)/2} \tag{A5}$$

$$\bar{u}_+(k_i)\,u_-(k_j) \equiv [ij] = -\langle ij\rangle^*\,, \tag{A6}$$

with $k^\pm = k^0 \pm k^3$.

In the $\gamma^* - p$ c.m. frame we have (see Ref. [37] for details):

$$\begin{aligned}
&k_1 = E(1,\sin\theta,0,\cos\theta)\\
&q = \frac{1}{2}\Big(W - \frac{Q^2}{W}, 0, 0, W + \frac{Q^2}{W}\Big)\\
&k_2 = \Big(xP_0 + \frac{k_\perp^2}{4xP_0}, \boldsymbol{k}_\perp, -xP_0 + \frac{k_\perp^2}{4xP_0}\Big)\\
&k_3 = k_1 - q\\
&k_4 = k_2 + q\\
&\phi_{1,3} = 0\,, \quad \phi_{2,4} = \phi_\perp\,,
\end{aligned} \tag{A7}$$

where, neglecting the proton mass:

$$\begin{aligned}
&x = \frac{1}{2}x_{\scriptscriptstyle B}\left(1 + \sqrt{1 + 4\frac{k_\perp^2}{Q^2}}\right)\\
&E = \frac{s - Q^2}{2W} = \frac{\sqrt{s}}{2}\,\frac{1 - x_{\scriptscriptstyle B}\,y}{\sqrt{y(1-x_{\scriptscriptstyle B})}}\\
&Q^2 = x_{\scriptscriptstyle B}\,y\,s \qquad W = \sqrt{y(1-x_{\scriptscriptstyle B})s}\\
&P_0 = \frac{1}{2}\Big(W + \frac{Q^2}{W}\Big) = \frac{\sqrt{s}}{2}\sqrt{\frac{y}{1-x_{\scriptscriptstyle B}}}\\
&\frac{1}{2}\Big(W - \frac{Q^2}{W}\Big) = \frac{\sqrt{s}}{2}\sqrt{\frac{y}{1-x_{\scriptscriptstyle B}}}(1-2x_{\scriptscriptstyle B})\\
&\cos\theta = \frac{1 + (y-2)x_{\scriptscriptstyle B}}{1 - yx_{\scriptscriptstyle B}} \qquad \sin\theta = \frac{2\sqrt{x_{\scriptscriptstyle B}(1-x_{\scriptscriptstyle B})(1-y)}}{1 - yx_{\scriptscriptstyle B}}\,.
\end{aligned} \tag{A8}$$

These relations allow us to express all the $k_i^\pm$ components in terms of $x_{\scriptscriptstyle B}$ and $y$ [37]:

$$k_1^+ = E(1 + \cos\theta) = \sqrt{s}\sqrt{\frac{1 - x_{\scriptscriptstyle B}}{y}}$$

$$
\begin{aligned}
k_1^- &= E(1-\cos\theta) = \sqrt{s}\,\frac{x_{\scriptscriptstyle B}(1-y)}{\sqrt{y(1-x_{\scriptscriptstyle B})}} \\
k_3^+ &= E(1+\cos\theta) - W = \sqrt{s}\sqrt{\frac{1-x_{\scriptscriptstyle B}}{y}}(1-y) \\
k_3^- &= E(1-\cos\theta) - \frac{Q^2}{W} = \sqrt{s}\,\frac{x_{\scriptscriptstyle B}}{\sqrt{y(1-x_{\scriptscriptstyle B})}} \\
k_2^+ &= \frac{k_\perp^2}{2xP_0} = \frac{k_\perp^2}{x\sqrt{s}}\sqrt{\frac{1-x_{\scriptscriptstyle B}}{y}} \\
k_2^- &= 2xP_0 = x\sqrt{s}\sqrt{\frac{y}{1-x_{\scriptscriptstyle B}}} \\
k_4^+ &= \frac{k_\perp^2}{2xP_0} + W = \sqrt{s}\sqrt{\frac{1-x_{\scriptscriptstyle B}}{y}}\left[\frac{k_\perp^2}{xs} + y\right] \\
k_4^- &= 2xP_0 - \frac{Q^2}{W} = \sqrt{s}\sqrt{\frac{y}{1-x_{\scriptscriptstyle B}}}[x - x_{\scriptscriptstyle B}] \\
\phi_1 &= \phi_3 = 0\,, \qquad \phi_2 = \phi_4 = \phi_\perp\,.
\end{aligned}
\tag{A9}
$$

From Eqs. (A3)–(A6) we get:

$$
\hat{M}_{++;++} = 2\frac{e_q e^2}{\hat{t}}\left[\sqrt{k_1^- k_2^+} - \sqrt{k_1^+ k_2^-}\,e^{i\phi_\perp}\right] \times \left[\sqrt{k_3^- k_4^+} - \sqrt{k_3^+ k_4^-}\,e^{-i\phi_\perp}\right] \tag{A10}
$$

$$
\hat{M}_{+-;+-} = 2\frac{e_q e^2}{\hat{t}}\left[\sqrt{k_2^- k_3^+} - \sqrt{k_2^+ k_3^-}\,e^{i\phi_\perp}\right] \times \left[\sqrt{k_1^+ k_4^-} - \sqrt{k_1^- k_4^+}\,e^{-i\phi_\perp}\right]\,. \tag{A11}
$$

Exploiting Eqs. (A9) we can finally compute the amplitudes as function of $y$, $Q^2$ and $k_\perp$:

$$
\hat{M}_{++;++} = e_q\,e^2\left[\frac{1}{y}\left(1+\sqrt{1+4\frac{k_\perp^2}{Q^2}}\right)e^{+i\phi_\perp} - \frac{1-y}{y}\left(1-\sqrt{1+4\frac{k_\perp^2}{Q^2}}\right)e^{-i\phi_\perp} - 4\;\frac{\sqrt{1-y}}{y}\,\frac{k_\perp}{Q}\right] \tag{A12}
$$

$$
\hat{M}_{+-;+-} = e_q\,e^2\left[\frac{1-y}{y}\left(1+\sqrt{1+4\frac{k_\perp^2}{Q^2}}\right)e^{-i\phi_\perp} - \frac{1}{y}\left(1-\sqrt{1+4\frac{k_\perp^2}{Q^2}}\right)e^{+i\phi_\perp} - 4\frac{\sqrt{1-y}}{y}\,\frac{k_\perp}{Q}\right]\,. \tag{A13}
$$

## Appendix B: Helicity formalism and helicity transformations

All our analytical and numerical computations of the SIDIS cross section, Eq. (4), are performed in the $\gamma^* - p$ center of mass frame (c.m.), with the kinematics represented in Fig. 1. However, in our helicity formalism all components of the polarization vectors (like in Eqs. (17) and (18)) and of the transverse momenta which enter the definition of the TMDs, refer to the appropriate helicity frame of the corresponding particle. Then, in order to perform our calculations, we have to express the helicity frame variables in terms of the c.m. ones, which requires some care.

For the proton, which moves along $-\hat{\boldsymbol{Z}}_{cm}$, the helicity frame $(\hat{\boldsymbol{X}}_p, \hat{\boldsymbol{Y}}_p, \hat{\boldsymbol{Z}}_p)$, as reached from the $\gamma^* - p$ c.m. frame, is given by (as discussed in Appendix D of Ref. [2]):

$$
\hat{\boldsymbol{X}}_p = \hat{\boldsymbol{X}}_{cm} \qquad \hat{\boldsymbol{Y}}_p = -\hat{\boldsymbol{Y}}_{cm} \qquad \hat{\boldsymbol{Z}}_p = -\hat{\boldsymbol{Z}}_{cm}\,, \tag{B1}
$$

so that

$$
\begin{aligned}
&\hat{\boldsymbol{k}}_\perp = \cos\varphi_\perp\,\hat{\boldsymbol{X}}_p + \sin\varphi_\perp\,\hat{\boldsymbol{Y}}_p = \cos\phi_\perp\,\hat{\boldsymbol{X}}_{cm} + \sin\phi_\perp\,\hat{\boldsymbol{Y}}_{cm} = \cos\varphi_\perp\,\hat{\boldsymbol{X}}_{cm} - \sin\varphi_\perp\,\hat{\boldsymbol{Y}}_{cm} \\
&\boldsymbol{k}_2 = \boldsymbol{k}_\perp - \left(x_{\scriptscriptstyle B}P_0 - \frac{k_\perp^2}{4x_{\scriptscriptstyle B}P_0}\right)\hat{\boldsymbol{Z}}_{cm} \\
&\boldsymbol{S}_T = \cos\varphi_S\,\hat{\boldsymbol{X}}_p + \sin\varphi_S\,\hat{\boldsymbol{Y}}_p = \cos\phi_S\,\hat{\boldsymbol{X}}_{cm} + \sin\phi_S\,\hat{\boldsymbol{Y}}_{cm} = \cos\varphi_S\,\hat{\boldsymbol{X}}_{cm} - \sin\varphi_S\,\hat{\boldsymbol{Y}}_{cm}\,,
\end{aligned}
\tag{B2}
$$

which implies $\varphi_{\perp,S} = 2\pi - \phi_{\perp,S}$. As long as there is no ambiguity we use $\varphi$ for angles defined in the helicity frames and $\phi$ for angles defined in the c.m. frame, following the notations of Fig. 1.

It is less straightforward to deal with the quark polarization vector, $\boldsymbol{P}^q = (P^q_x, P^q_y, P^q_z)$, which describes intrinsic properties of the proton constituents, and is defined in the *quark helicity frame*. In order to keep the same definitions, through the helicity formalism, of the polarized TMDs as in Ref. [2], we have to define $\boldsymbol{P}^q$ in the quark helicity frame as reached from the proton helicity frame. The axes $\hat{\boldsymbol{x}}_q, \hat{\boldsymbol{y}}_q, \hat{\boldsymbol{z}}_q$ of the quark helicity frame are then given by [2, 38]:

$$\hat{\boldsymbol{z}}_q = \hat{\boldsymbol{k}}_2 \tag{B3}$$

$$\hat{\boldsymbol{y}}_q = \hat{\boldsymbol{Z}}_p \times \hat{\boldsymbol{k}}_\perp = -\hat{\boldsymbol{Z}}_{cm} \times \hat{\boldsymbol{k}}_\perp \tag{B4}$$

$$\hat{\boldsymbol{x}}_q = \hat{\boldsymbol{y}}_q \times \hat{\boldsymbol{z}}_q = (\hat{\boldsymbol{Z}}_p \times \hat{\boldsymbol{k}}_\perp) \times \hat{\boldsymbol{k}}_2 = -(\hat{\boldsymbol{Z}}_{cm} \times \hat{\boldsymbol{k}}_\perp) \times \hat{\boldsymbol{k}}_2\,. \tag{B5}$$

Notice that the quark helicity frame as reached from the c.m. frame ($\hat{\boldsymbol{Z}}_{cm}$) is different from the quark helicity frame as reached from its parent proton helicity frame ($\hat{\boldsymbol{Z}}_p$); although the $\hat{\boldsymbol{z}}_q$ axes obviously coincide, $\hat{\boldsymbol{x}}_q$ and $\hat{\boldsymbol{y}}_q$ have opposite signs, Eqs. (B5) and (B4). Therefore, when referring to the kinematical configuration of Fig. 1, which we use throughout the paper, we have to take the $x$ and $y$ component of the quark polarization vector, $P^q_x$ and $P^q_y$, with opposite signs with respect to those obtained from Eq. (15); this has been done in Eqs. (24) and (25).

## Appendix C: Analysis of the fragmentation process

Let us now focus on the azimuthal angle $\varphi^h_q$ involved in the fragmentation process. This is the azimuthal angle of the momentum $\boldsymbol{P}_h$ of the final hadron around the direction $\boldsymbol{k}_4$ of the fragmenting quark $q$, *as defined in the quark $q$ helicity frame*, see Fig. 2. Notice that the fragmenting quark, in the $\gamma^* - p$ c.m. frame, has a longitudinal component along the positive $Z_{cm}$ axis. Its helicity frame, as reached from the $\gamma^* - p$ c.m. frame, is given by Ref. [2]:

$$\begin{aligned} &\hat{\boldsymbol{z}} = \hat{\boldsymbol{k}}_4 \\ &\hat{\boldsymbol{y}} = \hat{\boldsymbol{Z}}_{cm} \times \hat{\boldsymbol{k}}_\perp \\ &\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}\,, \end{aligned} \tag{C1}$$

where $\hat{\boldsymbol{k}}_\perp$ is the unit transverse component – with respect to the $Z_{cm}$ direction – of the outgoing quark, $\hat{\boldsymbol{k}}_4$.

In the quark helicity frame, $\varphi^h_q$ coincides with the azimuthal angle which identifies the hadron transverse momentum $\boldsymbol{p}_\perp$, therefore

$$\begin{aligned} \cos\varphi^h_q &= \hat{\boldsymbol{p}}_\perp \cdot \hat{\boldsymbol{x}} \\ \sin\varphi^h_q &= \hat{\boldsymbol{p}}_\perp \cdot \hat{\boldsymbol{y}}\,. \end{aligned} \tag{C2}$$

By using the SIDIS kinematics as reported in Ref. [37], one finds

$$\begin{aligned} \cos\varphi^h_q &= \frac{1}{p_\perp\,|\boldsymbol{k}_4|}\,[P_T\,k_4^Z\,\cos(\phi_h - \phi_\perp) - P_h^Z\,k_\perp] \\ \sin\varphi^h_q &= \frac{P_T}{p_\perp}\,\sin(\phi_h - \phi_\perp)\,, \end{aligned} \tag{C3}$$

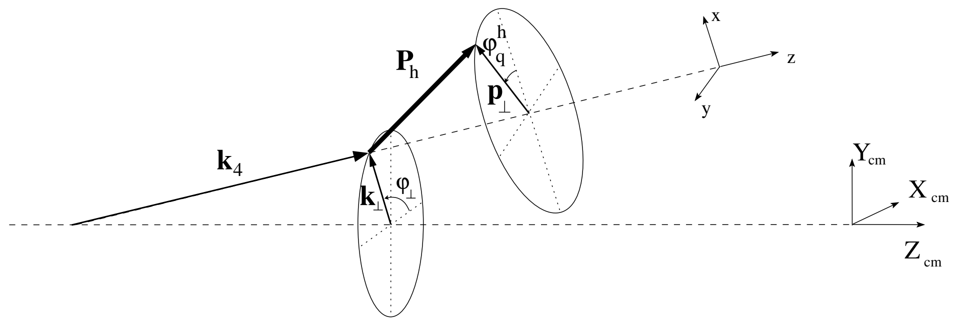


FIG. 2: Kinematics of the fragmentation process.

where the superscript $Z$ refers to the $\gamma^* - p$ c.m. frame, where one measures $\boldsymbol{P}_h = (P_T \cos\phi_h, P_T \sin\phi_h, P_h^Z)$, and

$$
\begin{aligned}
P_h^Z &= \frac{z_h^2\, W^2 - P_T^2}{2\, z_h\, W} \\
k_4^Z &= \frac{W}{2}\left(\frac{1-x}{1-x_B} + \frac{x_B}{x}\frac{k_\perp^2}{Q^2}\right) \\
|\boldsymbol{k}_4| &= \sqrt{\frac{W^2}{4}\left(\frac{1-x}{1-x_B} + \frac{x_B}{x}\frac{k_\perp^2}{Q^2}\right)^2 + k_\perp^2}\;,
\end{aligned}
\tag{C4}
$$

as derived in Ref. [37].

At $\mathcal{O}(k_\perp/Q)$ one simply has

$$
\begin{aligned}
\cos\varphi_q^h &= \frac{P_T}{p_\perp}\left[\cos(\phi_h - \phi_\perp) - z_h\frac{k_\perp}{P_T}\right] \\
\sin\varphi_q^h &= \frac{P_T}{p_\perp}\,\sin(\phi_h - \phi_\perp)\,,
\end{aligned}
\tag{C5}
$$

having neglected terms $\mathcal{O}(k_\perp^2/W^2)$ and $\mathcal{O}(P_T^2/W^2)$.

## Appendix D: Tensorial Analysis

Eqs. (63)-(78) are obtained using a simple euclidean tensorial analysis, as outlined in what follows. In general, the tensorial structure of each of the $F$'s functions defined in Eqs. (63)-(78) can be reduced to a linear combination of the convolutions

$$T^i \;=\; \int d^2\boldsymbol{k}_\perp\; \Delta f(x,k_\perp)\, k_\perp^i\, \Delta D(z,p_\perp) \tag{D1}$$

$$T^{ij} \;=\; \int d^2\boldsymbol{k}_\perp\; \Delta f(x,k_\perp)\, k_\perp^i\, k_\perp^j\, \Delta D(z,p_\perp) \tag{D2}$$

$$T^{ijl} \;=\; \int d^2\boldsymbol{k}_\perp\; \Delta f(x,k_\perp)\, k_\perp^i\, k_\perp^j\, k_\perp^l\, \Delta D(z,p_\perp)\,, \tag{D3}$$

where we have denoted by $\Delta f$ ($\Delta D$) any distribution (fragmentation) function appearing in the definition of the particular $F$ function one is considering, while the $k_\perp^i$, $i = X, Y$ ($X$ and $Y$ refer to the $\gamma^* - p$ c.m. frame, we have dropped the *cm* subscript) are the components of the $\boldsymbol{k}_\perp$ transverse momentum vector, $k_\perp^X = k_\perp \cos\phi_\perp$, $k_\perp^Y = k_\perp \sin\phi_\perp$. One should bear in mind that $p_\perp$ is not an independent quantity, as it can be expressed in terms of $k_\perp$ and $P_T$. Notice that $T^i$, $T^{ij}$ and $T^{ijl}$ are symmetric, rank 1, 2, 3 euclidean tensors respectively. Once the integration over $d^2\boldsymbol{k}_\perp$ is performed, the $T^i$, $T^{ij}$ and $T^{ijl}$ can only depend on the observable quantities $P_T$ and $\phi_h$, *i.e.* the measured modulus and azimuthal phase of the final observed hadron transverse momentum $\boldsymbol{P}_T$. Therefore, in a completely general way, it must be

$$T^i \;=\; P_T^i\, S_1(P_T) \tag{D4}$$

$$T^{ij} \;=\; P_T^i\, P_T^j\, S_2(P_T) + \delta^{ij}\, S_3(P_T) \tag{D5}$$

$$T^{ijl} \;=\; P_T^i\, P_T^j\, P_T^k\, S_4(P_T) + (P_T^i\,\delta^{jl} + P_T^j\,\delta^{il} + P_T^l\,\delta^{ij})\, S_5(P_T)\,, \tag{D6}$$

where the $\boldsymbol{P}_T$ components ($P_T^X = P_T\cos\phi_h$, $P_T^Y = P_T \sin\phi_h$) give the proper tensorial structure, while $S_1$–$S_5$ are five scalar functions which can only depend on $P_T$ (modulus), and can easily be determined by contracting Eqs. (D1)–(D3) with some symmetric tensorial structures ($P_T^i$, $\delta^{ij}$, etc..., as appropriate) to obtain simple scalar relations. Finally, one finds

$$S_1(P_T) = \frac{1}{P_T}\int d^2\boldsymbol{k}_\perp\; (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)\, \Delta f(x,k_\perp)\, \Delta D(z,p_\perp) \tag{D7}$$

$$S_2(P_T) = \frac{1}{P_T^2}\int d^2\boldsymbol{k}_\perp\; [2(\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)^2 - k_\perp^2]\, \Delta f(x,k_\perp)\, \Delta D(z,p_\perp) \tag{D8}$$

$$S_3(P_T) = \int d^2\boldsymbol{k}_\perp\,[k_\perp^2 - (\boldsymbol{k}_\perp\cdot\hat{\boldsymbol{P}}_T)^2]\,\Delta f(x,k_\perp)\,\Delta D(z,p_\perp) \tag{D9}$$

$$S_4(P_T) = \frac{1}{P_T^3}\int d^2\boldsymbol{k}_\perp\,[4(\boldsymbol{k}_\perp\cdot\hat{\boldsymbol{P}}_T)^3 - 3k_\perp^2(\boldsymbol{k}_\perp\cdot\hat{\boldsymbol{P}}_T)]\,\Delta f(x,k_\perp)\,\Delta D(z,p_\perp) \tag{D10}$$

$$S_5(P_T) = \frac{1}{P_T}\int d^2\boldsymbol{k}_\perp\,[k_\perp^2(\boldsymbol{k}_\perp\cdot\hat{\boldsymbol{P}}_T) - (\boldsymbol{k}_\perp\cdot\hat{\boldsymbol{P}}_T)^3]\,\Delta f(x,k_\perp)\,\Delta D(z,p_\perp)\,. \tag{D11}$$

As a consequence, we have

$$\int d^2\boldsymbol{k}_\perp\;\cos\phi_\perp\,\Delta f\,\Delta D = \cos\phi_h\;\int d^2\boldsymbol{k}_\perp\,(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)\,\Delta f\,\Delta D \tag{D12}$$

$$\int d^2\boldsymbol{k}_\perp\;\sin\phi_\perp\,\Delta f\,\Delta D = \sin\phi_h\;\int d^2\boldsymbol{k}_\perp\,(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)\,\Delta f\,\Delta D \tag{D13}$$

$$\int d^2\boldsymbol{k}_\perp\;\cos^2\phi_\perp\,\Delta f\,\Delta D = \frac{1}{2}\int d^2\boldsymbol{k}_\perp\,\Big\{1+\cos 2\phi_h\,[2(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^2-1]\Big\}\,\Delta f\,\Delta D \tag{D14}$$

$$\int d^2\boldsymbol{k}_\perp\;\sin^2\phi_\perp\,\Delta f\,\Delta D = \frac{1}{2}\int d^2\boldsymbol{k}_\perp\,\Big\{1-\cos 2\phi_h\,[2(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^2+1]\Big\}\,\Delta f\,\Delta D \tag{D15}$$

$$\int d^2\boldsymbol{k}_\perp\;\cos\phi_\perp\;\sin\phi_\perp\,\Delta f\,\Delta D = \cos\phi_h\;\sin\phi_h\int d^2\boldsymbol{k}_\perp\,[2(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^2-1]\,\Delta f\,\Delta D \tag{D16}$$

$$\begin{aligned}\int d^2\boldsymbol{k}_\perp\;\cos^3\phi_\perp\,\Delta f\,\Delta D &= \cos^3\phi_h\int d^2\boldsymbol{k}_\perp\,[4(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3-3(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)]\,\Delta f\,\Delta D\\ &+3\cos\phi_h\int d^2\boldsymbol{k}_\perp\,[(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)-(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3]\,\Delta f\,\Delta D\end{aligned} \tag{D17}$$

$$\begin{aligned}\int d^2\boldsymbol{k}_\perp\;\sin^3\phi_\perp\,\Delta f\,\Delta D &= \sin^3\phi_h\int d^2\boldsymbol{k}_\perp\,[4(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3-3(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)]\,\Delta f\,\Delta D\\ &+3\sin\phi_h\int d^2\boldsymbol{k}_\perp\,[(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)-(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3]\,\Delta f\,\Delta D\end{aligned} \tag{D18}$$

$$\begin{aligned}\int d^2\boldsymbol{k}_\perp\;\cos^2\phi_\perp\sin\phi_\perp\,\Delta f\,\Delta D &= \cos^2\phi_h\sin\phi_h\int d^2\boldsymbol{k}_\perp\,[4(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3-3(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)]\,\Delta f\,\Delta D\\ &+\;\sin\phi_h\int d^2\boldsymbol{k}_\perp\,[(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)-(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3]\,\Delta f\,\Delta D\end{aligned} \tag{D19}$$

$$\begin{aligned}\int d^2\boldsymbol{k}_\perp\;\cos\phi_\perp\sin^2\phi_\perp\,\Delta f\,\Delta D &= \cos\phi_h\sin^2\phi_h\int d^2\boldsymbol{k}_\perp\,[4(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3-3(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)]\,\Delta f\,\Delta D\\ &+\;\cos\phi_h\int d^2\boldsymbol{k}_\perp\,[(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)-(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3]\,\Delta f\,\Delta D\,.\end{aligned} \tag{D20}$$

From these equations one can easily reconstruct

$$\int d^2\boldsymbol{k}_\perp\;\cos 2\phi_\perp\,\Delta f\,\Delta D = \cos 2\phi_h\;\int d^2\boldsymbol{k}_\perp\,[2(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^2-1]\,\Delta f\,\Delta D \tag{D21}$$

$$\int d^2\boldsymbol{k}_\perp\;\sin 2\phi_\perp\,\Delta f\,\Delta D = \sin 2\phi_h\;\int d^2\boldsymbol{k}_\perp\,[2(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^2-1]\,\Delta f\,\Delta D \tag{D22}$$

$$\int d^2\boldsymbol{k}_\perp\;\cos 3\phi_\perp\,\Delta f\,\Delta D = \cos 3\phi_h\int d^2\boldsymbol{k}_\perp\,[4(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3-3(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)]\,\Delta f\,\Delta D \tag{D23}$$

$$\int d^2\boldsymbol{k}_\perp\;\sin 3\phi_\perp\,\Delta f\,\Delta D = \sin 3\phi_h\int d^2\boldsymbol{k}_\perp\,[4(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)^3-3(\hat{\boldsymbol{k}}_\perp\cdot\hat{\boldsymbol{P}}_T)]\,\Delta f\,\Delta D\,. \tag{D24}$$

All of these terms are easily recognizable in Eqs. (63)-(78).

## Appendix E: Integration by rotation in the hadronic plane

Eqs. (63)-(78) can also be obtained in a simple way looking to a slightly different reference frame. Let us define the production plane as the plane containing the virtual photon $\gamma^*$, the proton momentum and the produced hadron $h$. We can define a new $\gamma^* - p$ c.m. frame where the $X' - Z'$ plane is the production plane. This new frame is rotated by an angle $\phi_h$ with respect to the c.m. frame $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ depicted in Fig. 1 (we drop for simplicity the subscript $cm$):

$$\hat{\boldsymbol{X}} = \hat{\boldsymbol{X}}' \cos\phi_h - \hat{\boldsymbol{Y}}' \sin\phi_h \tag{E1}$$

$$\hat{\boldsymbol{Y}} = \hat{\boldsymbol{X}}' \sin\phi_h + \hat{\boldsymbol{Y}}' \cos\phi_h \,. \tag{E2}$$

Notice that $\hat{\boldsymbol{X}}' = \hat{\boldsymbol{P}}_T = \hat{\boldsymbol{h}}$. Any integration in Eqs. (63)-(78), at fixed values of the external variables, can be recast as the sum of one or more contributions of this kind:

$$\int d^2\boldsymbol{k}_\perp\, k_\perp \cos\phi_\perp\, f(k_\perp, p_\perp) \qquad \int d^2\boldsymbol{k}_\perp\, k_\perp \sin\phi_\perp\, f(k_\perp, p_\perp) \tag{E3}$$

$$\int d^2\boldsymbol{k}_\perp\, k_\perp^2 \cos 2\phi_\perp\, f(k_\perp, p_\perp) \qquad \int d^2\boldsymbol{k}_\perp\, k_\perp^2 \sin 2\phi_\perp\, f(k_\perp, p_\perp) \tag{E4}$$

$$\int d^2\boldsymbol{k}_\perp\, k_\perp^3 \cos 3\phi_\perp\, f(k_\perp, p_\perp) \qquad \int d^2\boldsymbol{k}_\perp\, k_\perp^3 \sin 3\phi_\perp\, f(k_\perp, p_\perp)\,, \tag{E5}$$

where

$$p_\perp^2 = P_T^2 + z_h^2\, k_\perp^2 - 2\, z_h (\boldsymbol{k}_\perp \cdot \boldsymbol{P}_T)\,. \tag{E6}$$

Let us consider, for instance, Eq. (E3); using Eq. (E1), we have

$$\begin{aligned}\int d^2\boldsymbol{k}_\perp\, k_\perp \cos\phi_\perp\, f(k_\perp, p_\perp) &= \int d^2\boldsymbol{k}_\perp\, k_\perp^X\, f(k_\perp, p_\perp) = \int d^2\boldsymbol{k}_\perp\, (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{X}})\, f(k_\perp, p_\perp)\\ &= \int d^2\boldsymbol{k}_\perp \left[(\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{X}}') \cos\phi_h - (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{Y}}') \sin\phi_h\right] f(k_\perp, \boldsymbol{k}_\perp \cdot \hat{\boldsymbol{X}}') \qquad \text{(E7)}\\ &= \cos\phi_h \int d^2\boldsymbol{k}_\perp\, (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)\, f(k_\perp, p_\perp)\,, \qquad \text{(E8)}\end{aligned}$$

where in the step (E7) we have underlined that $f$ is a function of $(\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T) \equiv (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{X}}')$ by means of Eq. (E6). With similar arguments we have, for all integrals of the kind (E3)–(E5):

$$\int d^2\boldsymbol{k}_\perp\, k_\perp \cos\phi_\perp \Rightarrow \cos\phi_h \int d^2\boldsymbol{k}_\perp\, (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T) \tag{E9}$$

$$\int d^2\boldsymbol{k}_\perp\, k_\perp \sin\phi_\perp \Rightarrow \sin\phi_h \int d^2\boldsymbol{k}_\perp\, (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T) \tag{E10}$$

$$\int d^2\boldsymbol{k}_\perp\, k_\perp^2 \cos 2\phi_\perp \Rightarrow \cos 2\phi_h \int d^2\boldsymbol{k}_\perp\, [2(\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)^2 - k_\perp^2] \tag{E11}$$

$$\int d^2\boldsymbol{k}_\perp\, k_\perp^2 \sin 2\phi_\perp \Rightarrow \sin 2\phi_h \int d^2\boldsymbol{k}_\perp\, [2(\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)^2 - k_\perp^2] \tag{E12}$$

$$\int d^2\boldsymbol{k}_\perp\, k_\perp^3 \cos 3\phi_\perp \Rightarrow \cos 3\phi_h \int d^2\boldsymbol{k}_\perp\, (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)\, [4(\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)^2 - 3k_\perp^2] \tag{E13}$$

$$\int d^2\boldsymbol{k}_\perp\, k_\perp^3 \sin 3\phi_\perp \Rightarrow \sin 3\phi_h \int d^2\boldsymbol{k}_\perp\, (\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)\, [4(\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{P}}_T)^2 - 3k_\perp^2]\,, \tag{E14}$$

which coincide with Eqs. (D12), (D13) and (D21)–(D24).

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