Regularization schemes and higher order corrections

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Regularization Schemes and Higher Order Corrections

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I apply commonly used regularization schemes to a multi-loop calculation to examine the properties of the schemes at higher orders. I find complete consistency between the conventional dimensional regularization scheme and dimensional reduction, but I find that the four dimensional helicity scheme produces incorrect results at next-to-next-to-leading order and singular results at next-to-next-to-next-to-leading order. It is not, therefore, a unitary regularization scheme.

I. INTRODUCTION

Dimensional regularization [1] is an elegant and efficient means of handling the divergences that arise in perturbation theory beyond the tree level. Among its many favorable qualities it respects gauge and Lorentz invariance and allows one to handle both ultraviolet and infrared divergences in the same manner. The application of dimensional regularization to different kinds of problems has led to the development of a variety of regularization schemes, which share the dimensional regularization of momentum integrals, but differ in their handling of external (or observed) states and of spin degrees of freedom.

The original formulation of dimensional regularization [1], known as the 't Hooft-Veltman (HV) scheme, specifies that observed states are to be treated as four dimensional, while internal states are to be treated as $D_m = 4-2\varepsilon$ dimensional. That is, both their momenta and spin degrees of freedom were to be continued from four to D_m dimensions. It turns out that one has the freedom to choose the value of the trace of the Dirac unit matrix to take its canonical value of four, so fermions continue to have two spin degrees of freedom, even though their momenta are continued to D_m dimensions. Internal gauge bosons, however, have D_m-2 spin degrees of freedom (internal massive gauge bosons have D_m-1 degrees of freedom).

A slight variation on the HV scheme has come to be called Conventional Dimensional Regularization (CDR) [2]. In this variation, all particles and momenta are taken to be D_m dimensional. This often turns out to be computationally more convenient, since one set of rules governs all interactions. This is particularly so when computing higher order corrections to theories subject to infrared sensitivities, like QCD. In the HV scheme, if two external states have infrared sensitive overlaps, they must be treated as internal, or D_m dimensional states. In the CDR scheme, all states are already treated as D_m dimensional, so there is no possibility of failing to properly account for infrared overlaps.

A third variation, called Dimensional Reduction (DRED) [3], was devised for application to supersymmetric theories. In supersymmetry, it is essential that the number of bosonic degrees of freedom is exactly equal to the number of fermionic degrees of freedom. This requirement is violated in the HV and CDR schemes. In the DRED scheme, the continuation to D_m dimensions is taken as a compactification from four dimensions. Thus, while space-time is taken to be four dimensional and particles have the standard number of degrees of freedom, momenta span a D_m dimensional vector space and momentum integrals are regularized dimensionally.

A fourth variation, the Four Dimensional Helicity (FDH) scheme [4, 5], was developed primarily for use in constructing one loop amplitudes from unitarity cuts. The most efficient building blocks for such calculations are tree-level helicity amplitudes, which necessarily have two spin degrees of freedom for both fermions and gauge bosons. The FDH scheme resembles the DRED scheme in that it regularizes momentum integrals dimensionally while maintaining the spin degrees of freedom of a four dimensional theory (and therefore appears to be a valid supersymmetric regularization scheme [5]), but there are crucial differences, which I will discuss in detail.

The fact that the HV scheme respects the unitarity of the S-matrix was proven at its introduction [1]. The arguments which establish the validity of the HV scheme carry over to the CDR scheme and establish that it too is a valid regularization scheme. After some initial confusion over the proper renormalization procedure [6–8] for the DRED scheme, it was established that it too is a proper, unitary regularization scheme [8] and that it is indeed equivalent to the CDR scheme [9]. The FDH scheme has never been subjected to such stringent examination. It has been used successfully in a number of landmark next-to-leading order (NLO) calculations, but it has never been established whether it is a proper, unitary regularization scheme, or merely a set of short-cuts that allow expert users to obtain correct results.

In this paper, I will perform a well-known multi-loop calculation in the various regularization schemes. I will show that while the HV and CDR scheme calculations yield the correct result and the DRED scheme calculation, while far more complicated is completely equivalent, the FDH scheme calculation yields incorrect results which inevitably violate unitarity at sufficiently high order. A detailed comparison of the various calculations identifies the source of the unitarity violations in the FDH scheme.

The plan of this paper is as follows: in section two, I will describe the test calculation to be performed and present the result to be obtained. In sections three, four and five, I will describe in detail the calculation to next-to-next-to-leading order (NNLO) as it is performed in the CDR, DRED and FDH schemes, respectively. In section six, I present partial results at N^3LO which solidify the conclusion that the CDR and DRED schemes are equivalent and correct, but that the FDH scheme violates unitarity. In section seven, I will discuss my results and draw my conclusions.

II. THE TEST ENVIRONMENT

To test the regularization schemes, I will calculate two quantities: the massless non-singlet contributions to

1. the hadronic decay width of a fictitious neutral vector boson V, of mass M_V;

2. the single photon approximation to the total hadronic annihilation cross section for an electron – positron pair.

I will perform these calculations by means of the optical theorem, taking the imaginary part of the forward scattering amplitudes. In both cases, this means taking the imaginary part of the vacuum polarization tensor sandwiched between external states. Since the optical theorem is a direct consequence of the unitarity of the S-matrix, any unitary regularization scheme must give the same result, once one expands in terms of a standard coupling. To avoid complications involving prescriptions for handling γ_5 and the Levi-Civita tensor, I will take V to have only vector-like couplings. In this way, the vacuum polarization tensor for the V boson will be identical to that of the off-shell photon, up to coupling constants and so the QCD expansion of the two results will differ only by constant numerical factors.

Each regularization scheme will start from the same four dimensional Lagrangian,

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2}A^a_\mu \left(\partial^\mu\partial^\nu(1-\xi^{-1}) - g^{\mu\nu}\Box\right)A^a_\nu - g\,f^{abc}(\partial^\mu A^{a\nu})A^b_\mu A^c_\nu - \frac{g^2}{4}f^{abc}f^{ade}A^{b\mu}A^{c\nu}A^d_\mu A^e_\nu \\ &+ i\sum_f \overline{\psi}^i_f\left(\delta_{ij}\not{\partial} - igt^a_{ij}\not{A}^a - ig_V Q_f \not{V}\right)\psi^j_f - \overline{c}^a\Box c^a + g\,f^{abc}\left(\partial_\mu \overline{c}^a\right)A^{b\mu}c^c\,,\end{aligned} \tag{1}$$

where $A^{a\mu}$ is the QCD gauge field, V^μ is the massive vector boson, ψ_f is the quark field of flavor f, $\overline{c}^a$ and c^a are the Faddeev-Popov ghost fields, g is the QCD coupling, g_V is the V gauge coupling and Q_f represents the charge of the quark flavor f under the V symmetry. I will not be computing non-trivial corrections in g_V, so there is no need to specify the V-self interaction parts of the Lagrangian.

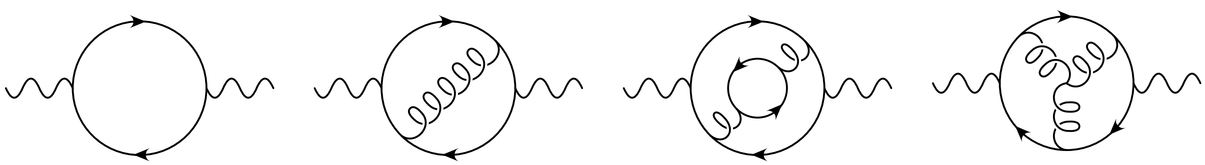

FIG. 1: Sample diagrams of one-, two- and three-loop contributions to the vacuum polarization of V.

The result to N^{3}LO is well known [10–14],

$$\Gamma^V_{had} = \Gamma^V_{0,\mathrm{had}}\,\mathscr{F}(\alpha_s^{\overline{\mathrm{MS}}}, Q^2 = M_V^2) \qquad \Gamma^V_{0,\mathrm{had}} = \frac{\alpha_V\, M_V}{3} N_c \sum_f Q_f^2$$
$$\sigma^{e^+e^-\to\,\mathrm{had}}(Q^2) = \sigma_0^{e^+e^-\to\,\mathrm{had}}(Q^2)\,\mathscr{F}(\alpha_s^{\overline{\mathrm{MS}}}, Q^2) \qquad \sigma_0^{e^+e^-\to\,\mathrm{had}}(Q^2) = \frac{4\pi\,\alpha^2}{3\,Q^2} N_c \sum_f Q_f^2 \tag{2}$$

and

$$\begin{aligned}
\mathscr{F}(\alpha_s^{\overline{\mathrm{MS}}}, Q^2) = \Bigg\{ & 1 + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right) C_F \frac{3}{4}\left[1 + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)\beta_0^{\overline{\mathrm{MS}}} \ln\frac{\mu^2}{Q^2} + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2 \left(\beta_1^{\overline{\mathrm{MS}}} \ln\frac{\mu^2}{Q^2} + \beta_0^{\overline{\mathrm{MS}}\,2} \ln^2\frac{\mu^2}{Q^2}\right)\right] \\
& + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2 \left[\left(-C_F^2\frac{3}{32} + C_F C_A\left(\frac{123}{32} - \frac{11}{4}\zeta_3\right) + C_F N_f\left(-\frac{11}{16} + \frac{1}{2}\zeta_3\right)\right)\right. \\
& \qquad \left.\times\left(1 + 2\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)\beta_0^{\overline{\mathrm{MS}}} \ln\frac{\mu^2}{Q^2}\right)\right] \\
& + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^3 \left[-C_F^3\frac{69}{128} + C_F^2 C_A\left(-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\right)\right. \\
& \qquad + C_F C_A^2\left(\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right) \\
& \qquad + C_F^2 N_f\left(-\frac{29}{128} + \frac{19}{8}\zeta_3 - \frac{5}{2}\zeta_5\right) + C_F C_A N_f\left(-\frac{485}{54} + \frac{56}{9}\zeta_3 + \frac{5}{12}\zeta_5\right) \\
& \qquad \left. + C_F N_f^2\left(\frac{151}{216} - \frac{19}{36}\zeta_3\right) - \frac{1}{4}\pi^2 C_F \beta_0^{\overline{\mathrm{MS}}\,2}\right] + \mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^4\right)\Bigg\}\,.
\end{aligned} \tag{3}$$

To obtain the hadronic decay width at LO, NLO and NNLO, I need to compute the QCDcorrections to the vacuum polarization of the V (photon) at 1, 2 and 3 loops respectively. Sample diagrams are shown in Fig. (1).

A. Methods

In each scheme, I will need to compute the vacuum polarization of V and the necessary coupling renormalization constants. As a cross-check on the reliability of my calculational framework, I reproduce known results on the QCD β-functions and mass anomalous dimensions to three-loop order, as well as the three-loop QCD contributions to the β-function of V (where needed).

In all calculations, I generate the contributing diagrams using QGRAF [15]. The symbolic algebra program FORM [16] is used to implement the Feynman rules and perform algebraic manipulations to reduce the result to a set of Feynman integrals to be performed and their coefficients. The set of Feynman integrals are then reduced to Master Integrals using the program REDUZE [17]. Using the method of Ref. [18], the vertex corrections can be expressed in terms of the same propagator integrals used to compute the vacuum polarization and wave-function renormalizations. The complete set of Master Integrals at one, two and three loops are shown in Fig. (2). Most of the Master Integrals are trivial iterated-bubble diagrams and the others

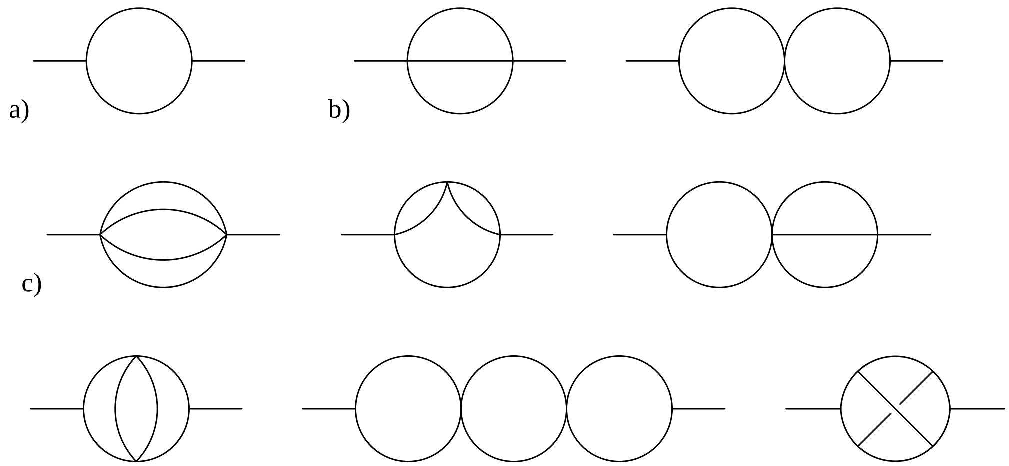

FIG. 2: Master integrals for the evaluation of vacuum polarization at a) one loop, b) two loops and c) three loops.

were evaluated long ago [19, 20]. As an additional cross check, the integral reduction and evaluation is also performed using the program MINCER[21, 22].

B. Notation

The various schemes that I will consider span a variety of vector spaces, each with their own metric tensor. To establish some level of consistency, I will denote the metric tensor of classical four-dimensional space-time as $\eta^{\mu\nu}$; the metric tensor of the D_m dimensional vector space in which momentum integrals are regularized will be denoted as $\hat{g}^{\mu\nu}$; and the metric tensor of the largest vector space will be denoted $g^{\mu\nu}$. Where it does not vanish, the complement of $\hat{g}^{\mu\nu}$ will be denoted as $\delta^{\mu\nu} = g^{\mu\nu} - \hat{g}^{\mu\nu}$. Similarly, the Dirac matrices γ^{μ}, will be denoted $\gamma^{\mu}_{(4)}$ when they are strictly four-dimensional, $\hat{\gamma}^{\mu}$ when they span the D_m dimensional space and $\overline{\gamma}^{\mu}$ in the space spanned by $\delta^{\mu\nu}$.

I will now present the details of the calculation in the CDR, DRED and FDH schemes.

III. CONVENTIONAL DIMENSIONAL REGULARIZATION

In the CDR scheme, the calculation is quite straightforward. The Lagrangian and Feynman rules are just the same as for a four-dimensional calculation, except that the Dirac matrices γ^μ and the metric tensor $g^{\mu\nu}$ have been extended to span a D_m dimensional vector space. That is,

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\,, \qquad g^{\mu\nu}g_{\mu\nu} = D_m\,, \qquad \gamma^\mu\gamma_\mu = D_m\,, \qquad g^{\mu\nu} \equiv \hat{g}^{\mu\nu}\,. \tag{4}$$

The Dirac trace, $\mathrm{Tr}\,[1] = 4$, retains its standard normalization.

Although D_m is given the representation $D_m = 4-2\varepsilon$, the sign of ε is not determined. If it is taken to be positive, so that $D_m < 4$, then the Feynman integrals that one encounters are convergent under the rules of ultraviolet power counting. On the other hand, infrared power counting would prefer $\varepsilon < 0 \Rightarrow D_m > 4$. In practice, the sign of ε does not matter and it can be used to regularize both infrared and ultraviolet divergences. Regardless of the sign of ε, it is important that the vector space in which momenta take values is larger than the standard $3+1$ dimensional space-time. This means that the standard four dimensional metric tensor $\eta^{\mu\nu}$ spans a smaller space than the D_m dimensional metric tensor, and the four dimensional Dirac matrices $\gamma^{0,1,2,3}$ form a subset of the full γ^μ,

$$g^{\mu\nu}g_\mu^\rho = g^{\nu\rho}\,, \qquad g^{\mu\nu}\eta_\mu^\rho = \eta^{\nu\rho}\,, \qquad \eta^{\mu\nu}\eta_\mu^\rho = \eta^{\nu\rho}\,. \tag{5}$$

These considerations are of particular importance when considering chiral objects involving γ_5 and the Levi-Civita tensor, but will play a role in our discussion below.

Because the Dirac trace is unchanged, fermions still have exactly two degrees of freedom in the CDR scheme. Gauge bosons, however, acquire extra spin degrees of freedom in the D_m dimensional vector space. The spin sum over polarization vectors in a physical (axial) gauge takes the form

$$-g_{\mu\nu}\sum_\lambda \varepsilon^{*\mu}(k,\lambda)\,\varepsilon^\nu(k,\lambda) = g_{\mu\nu}\left(g^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n}\right) = D_m - 2 = 2-2\varepsilon\,, \tag{6}$$

where n is the axial gauge reference vector. For massive vector bosons, the spin sum becomes

$$-g_{\mu\nu}\sum_\lambda \varepsilon^{*\mu}(k,\lambda)\,\varepsilon^\nu(k,\lambda) = g_{\mu\nu}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}\right) = D_m - 1 = 3-2\varepsilon\,, \tag{7}$$

A. Renormalization

The renormalization constants in the CDR scheme are defined as

$$\begin{aligned}
&\Gamma^{(B)}_{AAA} = Z_1\Gamma_{AAA}\,, \qquad \psi_f^{(B)\,i} = Z_2^{\frac{1}{2}}\,\psi_f^i\,, \qquad A_\mu^{(B)\,a} = Z_3^{\frac{1}{2}}\,A_\mu^a \\
&\Gamma^{(B)}_{c\bar{c}A} = \widetilde{Z}_1\Gamma_{q\bar{q}A}\,, \qquad c^{(B)\,a} = \widetilde{Z}_3^{\frac{1}{2}}\,c^a\,, \qquad \overline{c}^{(B)\,a} = \widetilde{Z}_3^{\frac{1}{2}}\,\overline{c}^a\,, \\
&\Gamma^{(B)}_{q\bar{q}A} = Z_{1F}\Gamma_{q\bar{q}A}\,, \qquad \xi^{(B)} = \xi\,Z_3\,,
\end{aligned} \tag{8}$$

where Γ_{abc} represents the vertex function involving fields a, b and c.

Although we treat the quark fields as massless, we can compute the mass anomalous dimension by introducing a fictitious scalar particle ϕ and computing the β-function of its Yukawa coupling to the quarks. The equivalence is clear from the Standard Model, where the Higgs Yukawa coupling and the fermion mass are proportional at leading electroweak order and must behave the same under QCD renormalization. For this purpose, I introduce one more renormalization constant, $\Gamma^{(B)}_{q\bar{q}\phi} = Z_{1\phi}\Gamma_{q\bar{q}\phi}$. One can introduce a wave-function renormalization for ϕ, $Z_{3\phi}$, but it will not contribute because $Z_{3\phi} = 1 + \mathscr{O}(\alpha_\phi)$. Note also that I do not need to compute the QCD corrections to the β-function for α_V, which will start at order α_V^2 because of the Ward Identity.

In the $\overline{\text{MS}}$ scheme, the couplings renormalize as

$$\begin{aligned} \alpha_s^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_s^{\overline{\text{MS}}}}\,\alpha_s^{\overline{\text{MS}}}, &\qquad Z_{\alpha_s^{\overline{\text{MS}}}} &= \frac{Z_1^2}{Z_3^3} = \frac{Z_{1F}^2}{Z_2^2 Z_3} = \frac{\widetilde{Z}_1^2}{\widetilde{Z}_3^2 Z_3} \\ \alpha_\phi^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_\phi^{\overline{\text{MS}}}}\,\alpha_\phi^{\overline{\text{MS}}}, &\qquad Z_{\alpha_\phi^{\overline{\text{MS}}}} &= \frac{Z_{1\phi}^2}{Z_2^2 Z_{3\phi}} \end{aligned} \tag{9}$$

The structure of the renormalization constants $Z_{\alpha_s^{\overline{\text{MS}}}}$ and $Z_{\alpha_\phi^{\overline{\text{MS}}}}$ is determined entirely by their lowest order $(1/\varepsilon)$ poles, which in turn define the β-functions.

$$\begin{aligned} \beta^{\overline{\text{MS}}}(\alpha_s^{\overline{\text{MS}}}) &= \mu^2 \frac{d}{d\mu^2}\frac{\alpha_s^{\overline{\text{MS}}}}{\pi} = -\varepsilon\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\left(1 + \frac{\alpha_s^{\overline{\text{MS}}}}{Z_{\alpha_s^{\overline{\text{MS}}}}}\frac{\partial Z_{\alpha_s^{\overline{\text{MS}}}}}{\partial \alpha_s^{\overline{\text{MS}}}}\right)^{-1} \\ &= -\varepsilon\frac{\alpha_s^{\overline{\text{MS}}}}{\pi} - \sum_{n=0}^{\infty}\beta_n^{\overline{\text{MS}}}\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^{n+2} \\ \beta_\phi^{\overline{\text{MS}}}(\alpha_s^{\overline{\text{MS}}}) &= \mu^2 \frac{d}{d\mu^2}\frac{\alpha_\phi^{\overline{\text{MS}}}}{\pi} = -\left(\varepsilon\frac{\alpha_\phi^{\overline{\text{MS}}}}{\pi} + \frac{\alpha_\phi^{\overline{\text{MS}}}}{Z_{\alpha_\phi^{\overline{\text{MS}}}}}\frac{\partial Z_{\alpha_\phi^{\overline{\text{MS}}}}}{\partial \alpha_s^{\overline{\text{MS}}}}\beta^{\overline{\text{MS}}}(\alpha_s^{\overline{\text{MS}}})\right)\left(1 + \frac{\alpha_\phi^{\overline{\text{MS}}}}{Z_{\alpha_\phi^{\overline{\text{MS}}}}}\frac{\partial Z_{\alpha_\phi^{\overline{\text{MS}}}}}{\partial \alpha_\phi^{\overline{\text{MS}}}}\right)^{-1} \\ &= -\frac{\alpha_\phi^{\overline{\text{MS}}}}{\pi}\left(\varepsilon + \sum_{n=0}^{\infty}\beta_{\phi,n}^{\overline{\text{MS}}}\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^{n+1}\right) \end{aligned} \tag{10}$$

The mass anomalous dimension,

$$\gamma^{\overline{\text{MS}}}(\alpha_s^{\overline{\text{MS}}}) = \frac{\mu^2}{m^{\overline{\text{MS}}}}\frac{d}{d\mu^2}m^{\overline{\text{MS}}} = \sum_{n=0}^{\infty} -\gamma_n^{\overline{\text{MS}}}\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^{n+1} \tag{11}$$

is defined in terms of m, rather than m^2, with the result that $\gamma_n^{\overline{\text{MS}}} = \frac{1}{2}\beta_{\phi,n}^{\overline{\text{MS}}}$. The results for $\beta_n^{\overline{\text{MS}}}$ and $\gamma_n^{\overline{\text{MS}}}$ through three loops are given in Appendix A.

B. Vacuum polarization in the CDR scheme

The imaginary part of the unrenormalized vacuum polarization tensor in the CDR scheme is

$$\begin{aligned}\Im\left[\Pi^{(B)}_{\mu\nu}(Q)\Big|_{CDR}\right] &= \frac{-Q^2 g_{\mu\nu}+Q_\mu Q_\nu}{3}\alpha_V^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^\epsilon \Bigg\{ \\ & 1+\left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^\epsilon C_F\left[\frac{3}{4}+\epsilon\left(\frac{55}{8}-6\zeta_3\right)+\epsilon^2\left(\frac{1711}{48}-\frac{15}{4}\zeta_2-19\zeta_3-9\zeta_4\right)+\mathcal{O}(\epsilon^3)\right] \\ & +\left(\frac{\alpha_s^B}{\pi}\right)^2\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\epsilon}\left[\frac{1}{\epsilon}\left(\frac{11}{16}C_F C_A-\frac{1}{8}C_F N_f\right)\right. \\ & -\frac{3}{32}C_F^2+C_F C_A\left(\frac{487}{48}-\frac{33}{4}\zeta_3\right)+C_F N_f\left(-\frac{11}{6}+\frac{3}{2}\zeta_3\right) \\ & +\epsilon\left(C_F^2\left(-\frac{143}{32}-\frac{111}{8}\zeta_3+\frac{45}{2}\zeta_5\right)+C_F C_A\left(\frac{50339}{576}-\frac{231}{32}\zeta_2-\frac{109}{2}\zeta_3-\frac{99}{8}\zeta_4-\frac{15}{4}\zeta_5\right)\right. \\ & \left.\left. +C_F N_f\left(-\frac{4417}{288}+\frac{21}{16}\zeta_2+\frac{19}{2}\zeta_3+\frac{9}{4}\zeta_4\right)\right)+\mathcal{O}(\epsilon^2)\right]+\mathcal{O}\left(\left(\frac{\alpha_s^B}{\pi}\right)^3\right)\Bigg\}. \end{aligned} \tag{12}$$

Upon renormalizing the QCD coupling according to Eq. (9), setting $\alpha_V^B \to \alpha_V\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon$, and dropping terms of order (ϵ), I obtain

$$\begin{aligned}\Im\left[\Pi_{\mu\nu}(Q)\big|_{CDR}\right] &= \frac{-Q^2 g_{\mu\nu}+Q_\mu Q_\nu}{3}\alpha_V N_c\sum_f Q_f^2\left\{1+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)C_F\frac{3}{4}\left[1+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)\beta_0^{\overline{\rm MS}}\ln\frac{\mu^2}{Q^2}\right]\right. \\ & \left.+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^2\left[-C_F^2\frac{3}{32}+C_F C_A\left(\frac{123}{32}-\frac{11}{4}\zeta_3\right)+C_F N_f\left(-\frac{11}{16}+\frac{1}{2}\zeta_3\right)\right]+\mathcal{O}\left(\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^3\right)\right\}. \end{aligned} \tag{13}$$

In this way of performing the calculation, all of the QCD states that appear are internal states, so the HV scheme gives exactly the same result.

C. Total Decay rate and annihilation cross section in the CDR scheme

The decay rate and the annihilation cross section are determined by computing the imaginary part of the forward scattering amplitude. For the decay rate, this means attaching the polarization vector $\epsilon^\mu(Q,\lambda)$ and its conjugate $\epsilon^\nu(Q,\lambda)^*$ ($Q^2=M_V^2$) and averaging over the spins,

$$\Gamma^{CDR}_{V\to\,\text{hadrons}} = \frac{1}{M_V}\frac{1}{N_{\text{spins}}}\sum_\lambda \epsilon^\mu(Q,\lambda)\,\Im\left[\Pi_{\mu\nu}(Q)\big|_{CDR}\right]\epsilon^\nu(Q,\lambda)^*, \tag{14}$$

where

$$\frac{1}{N_{\text{spins}}}\sum_\lambda \epsilon^\mu(Q,\lambda)\,\epsilon^\nu(Q,\lambda)^* = \frac{1}{N_{\text{spins}}}\left(-g^{\mu\nu}+\frac{Q^\mu Q^\nu}{M_V^2}\right). \tag{15}$$

Notice that because the imaginary part of the vacuum polarization tensor is finite, it does not matter whether the spin sum is taken in $D_m = 4-2\varepsilon$ dimensions as in the CDR scheme or in four dimensions as in the HV scheme as the difference is of order ε. The result is

$$\Gamma^{CDR}_{V\to\,\text{hadrons}} = \frac{\alpha_V\,M_V}{3}\,N_c\sum_f Q_f^2\left\{1+\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)C_F\,\frac{3}{4}\left[1+\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)\beta_0^{\overline{\text{MS}}}\ln\frac{\mu^2}{Q^2}\right]\right.$$
$$\left.+\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^2\left[-C_F^2\,\frac{3}{32}+C_F\,C_A\left(\frac{123}{32}-\frac{11}{4}\zeta_3\right)+C_F\,N_f\left(-\frac{11}{16}+\frac{1}{2}\zeta_3\right)\right]+\mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^3\right)\right\}, \tag{16}$$

in agreement with Eqs. (2-3).

For the annihilation cross section $\sigma_{e^+\,e^-\to\,\text{hadrons}}$, one attaches fermion bilinears to each end of the vacuum polarization tensor and averages over the spins.

$$\sigma^{CDR}_{e^+\,e^-\to\,\text{hadrons}} = \frac{2}{Q^2}\frac{e^2}{4}\sum_{\lambda\,\lambda'}\frac{\left\langle\overline{v}(p_{e^+},\lambda)\,|\gamma^\mu|\,u(p_{e^-},\lambda')\right\rangle}{Q^2}\,\Im\left[\Pi_{\mu\,\nu}(Q)\big|_{CDR,\,\alpha_V\to\alpha}\right]\frac{\left\langle\overline{u}(p_{e^-},\lambda')\,|\gamma^\nu|\,v(p_{e^+},\lambda)\right\rangle}{Q^2}\,. \tag{17}$$

Because this is a forward scattering amplitude, the spinor bilinears can be combined into a trace,

$$\frac{1}{2}\sum_{\lambda\,\lambda'}\left\langle\overline{v}(p_{e^+},\lambda)\,|\gamma^\mu|\,u(p_{e^-},\lambda')\right\rangle\left\langle\overline{u}(p_{e^-},\lambda')\,|\gamma^\nu|\,v(p_{e^+},\lambda)\right\rangle = \frac{1}{2}\,\text{Tr}\left[\not{p}_{\text{e}^+}\,\gamma^\mu\,\not{p}_{\text{e}^-}\,\gamma^\nu\right] = \left(-Q^2\,g^{\mu\,\nu}+Q^\mu\,Q^\nu\right), \tag{18}$$

where the last identification results from the fact that $Q^\mu = p^\mu_{e^-}+p^\mu_{e^+}$, $\quad p_{e^-}\cdot Q = p_{e^+}\cdot Q = Q^2/2$. The result is

$$\sigma^{CDR}_{e^+\,e^-\to\,\text{hadrons}} = \frac{4\pi\,\alpha^2}{3\,Q^2}\,N_c\sum_f Q_f^2\left\{1+\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)C_F\,\frac{3}{4}\left[1+\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)\beta_0^{\overline{\text{MS}}}\ln\frac{\mu^2}{Q^2}\right]\right.$$
$$\left.+\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^2\left[-C_F^2\,\frac{3}{32}+C_F\,C_A\left(\frac{123}{32}-\frac{11}{4}\zeta_3\right)+C_F\,N_f\left(-\frac{11}{16}+\frac{1}{2}\zeta_3\right)\right]+\mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^3\right)\right\}, \tag{19}$$

again in agreement with Eqs. (2-3).

Thus, I have established that I can reproduce the known results in the CDR scheme through three loop order, which is a strong check on my computational framework.

IV. DIMENSIONAL REDUCTION

In Dimensional Reduction, one starts from standard four-dimensional space-time and compactifies to a *smaller* vector space of dimension $D_m = 4-2\varepsilon < 4$ in which momenta take values. The particles in the spectrum, however, retain the spin degrees of freedom of four dimensions. That is, both fermions and gauge bosons still have two degrees of freedom. This is by design, of course, since it is required by supersymmetry. All Dirac algebra can be treated as four dimensional. However, now the four dimensional metric tensor $\eta^{\mu\nu}$ spans a larger space than the D_m dimensional metric $\hat{g}^{\mu\nu}$ that might arise from tensor momentum integrals,

$$\hat{g}^{\mu\nu}\,\eta^{\rho}_{\mu} = \hat{g}^{\nu\rho}\,. \tag{20}$$

There is also a very serious consequence of the fact that the D_m dimensional vector space is smaller than four dimensional space-time. The Ward Identity only applies to the D_m dimensional vector space! This means that the 2ε spin degrees of freedom that are not protected by the Ward Identity must renormalize differently than the $2-2\varepsilon$ degrees of freedom that are protected. In supersymmetric theories, the supersymmetry provides the missing Ward Identity which demands that the 2ε spin degrees of freedom be treated as gauge bosons. In non-supersymmetric theories, however, they must be considered to be distinct particles, with distinct couplings and renormalization properties. It is common to refer to these extra degrees of freedom as "ε-scalars" or as "evanescent" degrees of freedom.

Once the evanescent degrees of freedom (which I will label $A_e^{a\tilde{\mu}}$, to distinguish them from the gluons, $A^{a\mu}$) are recognized as independent particles, it is apparent that their couplings are also independent, not only of the QCD coupling, but of one another. That is, the coupling g_e of the evanescent gluons to the quarks is not only distinct from g, the coupling of QCD, but is also distinct from λ_i, the quartic couplings of the evanescent gluons to themselves. (The quartic gauge coupling of QCD splits into three independent quartic couplings of the evanescent gluons.) Note that the massive vector boson V^μ also has evanescent degrees of freedom, $V_e^{\tilde{\mu}}$, which couple to quarks with strength g_{Ve}.

Thus, the Lagrangian in the DRED scheme becomes:

$$\begin{aligned}
\mathscr{L} = &-\frac{1}{2}A^a_\mu\left(\partial^\mu\partial^\nu(1-\xi^{-1})-\hat{g}^{\mu\nu}\Box\right)A^a_\nu - g\,f^{abc}(\partial^\mu A^{a\nu})A^b_\mu A^c_\nu - \frac{g^2}{4}f^{abc}f^{ade}A^{b\mu}A^{c\nu}A^d_\mu A^e_\nu \\
&+ i\sum_f \overline{\psi}^i_f\left(\delta_{ij}\partial\!\!\!/ - i g t^a_{ij}A\!\!\!/^a - i g_V Q_f V\!\!\!/\right)\psi^j_f - \overline{c}^a\Box c^a + g\,f^{abc}\left(\partial_\mu\overline{c}^a\right)A^{b\mu}c^c \\
&+ \frac{1}{2}A^a_{e\tilde{\mu}}\Box A^{a\tilde{\mu}}_e - g\,f^{abc}(\partial^\mu A^{a\tilde{\nu}}_e)A^b_\mu A^c_{e\tilde{\nu}} + \frac{g^2}{2}f^{abc}f^{adf}A^{b\mu}A^{c\tilde{\nu}}_e A^d_\mu A^f_{e\tilde{\nu}} - \frac{1}{4}\sum_i \lambda_i H^{bcdf}_i A^{b\tilde{\mu}}_e A^{c\tilde{\nu}}_e A^d_{e\tilde{\mu}} A^f_{e\tilde{\nu}} \\
&+ \sum_f \overline{\psi}^i_f\left(g_e t^a_{ij}A\!\!\!/^a_e + g_{Ve}Q_f V\!\!\!/_e\right)\psi^j_f\,.
\end{aligned} \tag{21}$$

As mentioned above, the quartic coupling of the evanescent gluons splits into three terms, which mix under renormalization. One can choose the tensors H_i^{bcde} to be [23]

$$
\begin{aligned}
H_1^{bcde} =& \frac{1}{2}\left(f^{abc}\,f^{ade} + f^{abe}\,f^{adc}\right) \\
H_2^{bcde} =& \delta^{bc}\delta^{de} + \delta^{bd}\delta^{ce} + \delta^{be}\delta^{cd} \\
H_3^{bcde} =& \frac{1}{2}\left(\delta^{bc}\delta^{de} + \delta^{be}\delta^{cd}\right) - \delta^{bd}\delta^{ce}\,,
\end{aligned} \tag{22}
$$

Although the quartic couplings enter the β-functions and anomalous dimension at three loops and are essential to the renormalization program, they do not explicitly contribute to the calculation at hand.

Now that the correct spectrum has been identified, one must carefully consider the renormalization program. The naïve application of the principle of minimal subtraction leads to the violation of unitarity [6]. Because the contributions of evanescent states and couplings to scattering amplitudes are weighted by a factor ε, the leading one-loop contribution is finite and therefore not subtracted. As one proceeds to higher orders, there is a mismatch among the counterterms such that the renormalization program fails to remove all of the ultraviolet singularities.

A successful renormalization program for the DRED scheme [8, 9] applies the principle of minimal subtraction to the evanescent Green functions (that is, Green functions with external evanescent states) themselves. At each order, the renormalization scheme renders the evanescent Green functions finite. Since evanescent Green functions enter into the scattering amplitudes of physical particles at order ε and they are rendered finite by renormalization, they never contribute to physical scattering amplitudes.

The evanescent coupling still contributes to Green functions with only physical external states, but the contribution is rendered finite by the prescribed renormalization program [8, 9, 23, 24]. Because the evanescent coupling, α_e renormalizes differently than the gauge coupling α_s, the two cannot be identified, even at the end of the calculation. One can choose a renormalization point where the two coincide, but they evolve differently under renormalization group transformations and their values will diverge as one moves away from the renormalization point.

Still, the evanescent coupling is essentially a fictitious quantity and one finds that if one computes a physical quantity in the DRED scheme and then converts the running couplings of the DRED scheme to those of a scheme such as CDR that has no evanescent couplings, the factors of α_e drop out [23, 24].

A. Renormalization

The renormalization constants in the DRED scheme are defined as

$$\begin{aligned}
&\Gamma^{(B)}_{AAA} = Z_1 \Gamma_{AAA}\,, \qquad \psi^{(B)\,i}_f = Z_2^{\frac{1}{2}}\,\psi^i_f\,, \qquad A^{(B)\,a}_\mu = Z_3^{\frac{1}{2}}\,A^a_\mu \\
&\Gamma^{(B)}_{c\bar{c}A} = \widetilde{Z}_1 \Gamma_{q\bar{q}A}\,, \qquad c^{(B)\,a} = \widetilde{Z}_3^{\frac{1}{2}}\,c^a\,, \qquad \bar{c}^{(B)\,a} = \widetilde{Z}_3^{\frac{1}{2}}\,\bar{c}^a\,, \\
&\Gamma^{(B)}_{q\bar{q}A} = Z_{1F}\,\Gamma_{q\bar{q}A}\,, \qquad \xi^{(B)} = \xi\, Z_3\,, \\
&\Gamma^{(B)}_{q\bar{q}e} = Z_{1e} \Gamma_{q\bar{q}e}\,, \qquad A^{(B)\,a}_{e\,\mu} = Z_{3e}^{\frac{1}{2}} A^a_{e\,\mu}\,, \qquad \Gamma^{(B)\,i}_{eeee} = Z^i_{1eeee}\,\Gamma^i_{eeee}\,, \\
&\Gamma^{(B)}_{q\bar{q}V_e} = Z_{1Ve} \Gamma_{q\bar{q}V_e}\,, \qquad V^{(B)}_{e\mu} = Z_{3Ve}^{\frac{1}{2}} V_{e\mu}\,.
\end{aligned} \tag{23}$$

In addition, I again introduce the fictitious scalar that allows me to compute the mass anomalous dimension for massless quarks. Note that while the Ward Identity protects α_V from leading QCD corrections, it does not protect α_{Ve}. That is why I need to introduce renormalization constants for vertex and wave-function and why I need to compute the β-function of α_{Ve}.

In the $\overline{\text{DR}}$ scheme (modified minimal subtraction in the DRED scheme), the couplings renormalize as

$$\begin{aligned}
\alpha_s^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_s^{\overline{\text{DR}}}}\,\alpha_s^{\overline{\text{DR}}}\,, \qquad Z_{\alpha_s^{\overline{\text{DR}}}} = \frac{Z_1^2}{Z_3^3} = \frac{Z_{1F}^2}{Z_2^2 Z_3} = \frac{\widetilde{Z}_1^2}{\widetilde{Z}_3^2 Z_3}\,, \\
\alpha_e^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_e^{\overline{\text{DR}}}}\,\alpha_e^{\overline{\text{DR}}}\,, \qquad Z_{\alpha_e^{\overline{\text{DR}}}} = \frac{Z_{1e}^2}{Z_2^2 Z_{3e}}\,, \\
\alpha_{Ve}^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_{Ve}^{\overline{\text{DR}}}}\,\alpha_{Ve}^{\overline{\text{DR}}}\,, \qquad Z_{\alpha_{Ve}^{\overline{\text{DR}}}} = \frac{Z_{1Ve}^2}{Z_2^2 Z_{3Ve}}\,, \\
\alpha_\phi^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_\phi^{\overline{\text{DR}}}}\,\alpha_\phi^{\overline{\text{DR}}}\,, \qquad Z_{\alpha_\phi^{\overline{\text{DR}}}} = \frac{Z_{1\phi}^2}{Z_2^2 Z_{3\phi}}\,.
\end{aligned} \tag{24}$$

and the β-functions are given by

$$\begin{aligned}
\beta^{\overline{\rm DR}} = \mu^2 \frac{d}{d\mu^2}\frac{\alpha_s^{\overline{\rm DR}}}{\pi} &= -\left(\varepsilon\frac{\alpha_s^{\overline{\rm DR}}}{\pi} + \frac{\alpha_s^{\overline{\rm DR}}}{Z_{\alpha_s^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_s^{\overline{\rm DR}}}}{\partial\alpha_e^{\overline{\rm DR}}}\beta_e^{\overline{\rm DR}} + \frac{\alpha_s^{\overline{\rm DR}}}{Z_{\alpha_s^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_s^{\overline{\rm DR}}}}{\partial\eta_i^{\overline{\rm DR}}}\beta_{\eta_i}^{\overline{\rm DR}}\right)\left(1+\frac{\alpha_s^{\overline{\rm DR}}}{Z_{\alpha_s^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_s^{\overline{\rm DR}}}}{\partial\alpha_s^{\overline{\rm DR}}}\right)^{-1} \\
&= -\varepsilon\frac{\alpha_s^{\overline{\rm DR}}}{\pi} - \sum_{i,j,k,l,m}\beta_{ijklm}^{\overline{\rm DR}}\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i\left(\frac{\alpha_e^{\overline{\rm DR}}}{\pi}\right)^j\left(\frac{\eta_1^{\overline{\rm DR}}}{\pi}\right)^k\left(\frac{\eta_2^{\overline{\rm DR}}}{\pi}\right)^l\left(\frac{\eta_3^{\overline{\rm DR}}}{\pi}\right)^m \\
\beta_e^{\overline{\rm DR}} = \mu^2 \frac{d}{d\mu^2}\frac{\alpha_e^{\overline{\rm DR}}}{\pi} &= -\left(\varepsilon\frac{\alpha_e^{\overline{\rm DR}}}{\pi} + \frac{\alpha_e^{\overline{\rm DR}}}{Z_{\alpha_e^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_e^{\overline{\rm DR}}}}{\partial\alpha_s^{\overline{\rm DR}}}\beta^{\overline{\rm DR}} + \frac{\alpha_e^{\overline{\rm DR}}}{Z_{\alpha_e^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_e^{\overline{\rm DR}}}}{\partial\eta_i^{\overline{\rm DR}}}\beta_{\eta_i}^{\overline{\rm DR}}\right)\left(1+\frac{\alpha_e^{\overline{\rm DR}}}{Z_{\alpha_e^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_e^{\overline{\rm DR}}}}{\partial\alpha_e^{\overline{\rm DR}}}\right)^{-1} \\
&= -\varepsilon\frac{\alpha_e^{\overline{\rm DR}}}{\pi} - \sum_{i,j,k,l,m}\beta_{e,ijklm}^{\overline{\rm DR}}\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i\left(\frac{\alpha_e^{\overline{\rm DR}}}{\pi}\right)^j\left(\frac{\eta_1^{\overline{\rm DR}}}{\pi}\right)^k\left(\frac{\eta_2^{\overline{\rm DR}}}{\pi}\right)^l\left(\frac{\eta_3^{\overline{\rm DR}}}{\pi}\right)^m \\
\beta_{Ve}^{\overline{\rm DR}} = \mu^2 \frac{d}{d\mu^2}\frac{\alpha_{Ve}^{\overline{\rm DR}}}{\pi} &= -\left(\varepsilon\frac{\alpha_{Ve}^{\overline{\rm DR}}}{\pi} + \frac{\alpha_{Ve}^{\overline{\rm DR}}}{Z_{\alpha_{Ve}^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_{Ve}^{\overline{\rm DR}}}}{\partial\alpha_s^{\overline{\rm DR}}}\beta^{\overline{\rm DR}} + \frac{\alpha_{Ve}^{\overline{\rm DR}}}{Z_{\alpha_{Ve}^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_{Ve}^{\overline{\rm DR}}}}{\partial\alpha_e^{\overline{\rm DR}}}\beta_e^{\overline{\rm DR}} + \frac{\alpha_{Ve}^{\overline{\rm DR}}}{Z_{\alpha_{Ve}^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_{Ve}^{\overline{\rm DR}}}}{\partial\eta_i^{\overline{\rm DR}}}\beta_{\eta_i}^{\overline{\rm DR}}\right) \\
&\qquad\times\left(1+\frac{\alpha_{Ve}^{\overline{\rm DR}}}{Z_{\alpha_{Ve}^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_{Ve}^{\overline{\rm DR}}}}{\partial\alpha_{Ve}^{\overline{\rm DR}}}\right)^{-1} \\
&= -\frac{\alpha_{Ve}^{\overline{\rm DR}}}{\pi}\left(\varepsilon + \sum_{i,j,k,l,m}\beta_{Ve,ijklm}^{\overline{\rm DR}}\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i\left(\frac{\alpha_e^{\overline{\rm DR}}}{\pi}\right)^j\left(\frac{\eta_1^{\overline{\rm DR}}}{\pi}\right)^k\left(\frac{\eta_2^{\overline{\rm DR}}}{\pi}\right)^l\left(\frac{\eta_3^{\overline{\rm DR}}}{\pi}\right)^m\right) \\
\beta_{\phi}^{\overline{\rm DR}} = \mu^2 \frac{d}{d\mu^2}\frac{\alpha_{\phi}^{\overline{\rm DR}}}{\pi} &= -\left(\varepsilon\frac{\alpha_{\phi}^{\overline{\rm DR}}}{\pi} + \frac{\alpha_{\phi}^{\overline{\rm DR}}}{Z_{\alpha_{\phi}^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_{\phi}^{\overline{\rm DR}}}}{\partial\alpha_s^{\overline{\rm DR}}}\beta^{\overline{\rm DR}} + \frac{\alpha_{\phi}^{\overline{\rm DR}}}{Z_{\alpha_{\phi}^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_{\phi}^{\overline{\rm DR}}}}{\partial\alpha_e^{\overline{\rm DR}}}\beta_e^{\overline{\rm DR}} + \frac{\alpha_{\phi}^{\overline{\rm DR}}}{Z_{\alpha_{\phi}^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_{\phi}^{\overline{\rm DR}}}}{\partial\eta_i^{\overline{\rm DR}}}\beta_{\eta_i}^{\overline{\rm DR}}\right) \\
&\qquad\times\left(1+\frac{\alpha_{\phi}^{\overline{\rm DR}}}{Z_{\alpha_{\phi}^{\overline{\rm DR}}}}\frac{\partial Z_{\alpha_{\phi}^{\overline{\rm DR}}}}{\partial\alpha_{\phi}^{\overline{\rm DR}}}\right)^{-1} \\
&= -\frac{\alpha_{\phi}^{\overline{\rm DR}}}{\pi}\left(\varepsilon + \sum_{i,j,k,l,m}\beta_{\phi,ijklm}^{\overline{\rm DR}}\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i\left(\frac{\alpha_e^{\overline{\rm DR}}}{\pi}\right)^j\left(\frac{\eta_1^{\overline{\rm DR}}}{\pi}\right)^k\left(\frac{\eta_2^{\overline{\rm DR}}}{\pi}\right)^l\left(\frac{\eta_3^{\overline{\rm DR}}}{\pi}\right)^m\right)
\end{aligned} \tag{25}$$

Through three-loop order, the η_i do not contribute to the QCD β-function, $\beta^{\overline{\rm DR}}$, nor to the vacuum polarization of V (or V_e). To three-loop order, I find agreement with known results [23, 24] and derive new results for the β-function of α_{Ve}. The coefficients of the β-functions and anomalous dimensions are given in Appendix B.

By comparing $\beta_{Ve,}^{\overline{\rm DR}}$ and $\gamma^{\overline{\rm DR}}$ in Eqs. (B4-B5), we see that the term "ε-scalar" is a misnomer. If the evanescent part of V were a true scalar, its β-function would coincide (but for a factor of 2) with the mass anomalous dimension. The pure $\alpha_s^{\overline{\rm DR}}$ terms do coincide, because there is no non-vanishing contraction of the Lorentz indices of the evanescent V and those of the gluons. Because there are contractions between the Lorentz indices of the evanescent V and those of the evanescent gluons, however, terms involving $\alpha_e^{\overline{\rm DR}}$ do not agree.

Calculations in the DRED scheme naturally produce results in terms of $\alpha_s^{\overline{\mathrm{DR}}}$ while the standard result has been expressed in terms of $\alpha_s^{\overline{\mathrm{MS}}}$. One can always convert one renormalized coupling to another. The rule for converting $\alpha_s^{\overline{\mathrm{DR}}} \to \alpha_s^{\overline{\mathrm{MS}}}$, derived in Refs. [24, 25], is

$$\alpha_s^{\overline{\mathrm{DR}}} = \alpha_s^{\overline{\mathrm{MS}}} \left[1 + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)\frac{C_A}{12} + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2 \frac{11}{72}C_A^2 - \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)\left(\frac{\alpha_e^{\overline{\mathrm{DR}}}}{\pi}\right)\frac{C_F\,N_f}{16} + \ldots \right] \tag{26}$$

When the result is expressed in terms of $\alpha_s^{\overline{\mathrm{MS}}}$, all $\alpha_e^{\overline{\mathrm{DR}}}$ terms drop out.

B. Vacuum polarization in the DRED scheme

In the DRED scheme, there are two independent transverse vacuum polarization tensors,

$$\Im\left[\Pi_{\mu\nu}^{(B)}(Q)\Big|_{DRED}\right] = \frac{-Q^2\,\hat{g}_{\mu\nu} + Q_\mu Q_\nu}{3}\,\Im\left[\Pi_A^{(B)}(Q)\Big|_{DRED}\right] - Q^2\,\frac{\delta_{\mu\nu}}{2\,\varepsilon}\,\Im\left[\Pi_B^{(B)}(Q)\Big|_{DRED}\right], \tag{27}$$

where

$$\begin{aligned}
\Im\left[\Pi_A^{(B)}(Q)\Big|_{DRED}\right] &= \alpha_V^B\,N_c\sum_f Q_f^2\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{\varepsilon}\Bigg\{ \\
&1 + \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{\varepsilon} C_F\left[\frac{3}{4} + \varepsilon\left(\frac{51}{8} - 6\,\zeta_3\right) + \varepsilon^2\left(\frac{497}{16} - \frac{15}{4}\zeta_2 - 15\,\zeta_3 - 9\,\zeta_4\right) + \mathscr{O}(\varepsilon^3)\right] \\
&+ \left(\frac{\alpha_e^B}{\pi}\right)\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{\varepsilon} C_F\left[-\varepsilon\,\frac{3}{4} - \varepsilon^2\,\frac{29}{8} + \mathscr{O}(\varepsilon^3)\right] \\
&+ \left(\frac{\alpha_s^B}{\pi}\right)^2\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{2\varepsilon}\left[\frac{1}{\varepsilon}\left(\frac{11}{16}C_F\,C_A - \frac{1}{8}C_F\,N_f\right) - \frac{3}{32}C_F^2 + \left(\frac{77}{8} - \frac{33}{4}\zeta_3\right)C_F\,C_A - \left(\frac{7}{4} - \frac{3}{2}\zeta_3\right)C_F\,N_f\right. \\
&\quad + \varepsilon\left(C_F^2\left(-\frac{141}{32} - \frac{111}{8}\,\zeta_3 + \frac{45}{2}\,\zeta_5\right) + C_F\,C_A\left(\frac{15301}{192} - \frac{231}{32}\,\zeta_2 - \frac{193}{4}\,\zeta_3 - \frac{99}{8}\,\zeta_4 - \frac{15}{4}\,\zeta_5\right)\right. \\
&\quad\quad \left.\left. + C_F\,N_f\left(-\frac{1355}{96} + \frac{21}{16}\,\zeta_2 + \frac{17}{2}\,\zeta_3 + \frac{9}{4}\,\zeta_4\right)\right) + \mathscr{O}(\varepsilon^2)\right] \\
&+ \left(\frac{\alpha_e^B}{\pi}\right)^2\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{2\varepsilon}\left[\frac{3}{4}C_F^2 - \frac{3}{8}C_F\,C_A + \frac{3}{16}C_F\,N_f - \varepsilon\left(\frac{47}{8}C_F^2 - \frac{11}{4}C_F\,C_A + \frac{7}{4}C_F\,N_f\right) + \mathscr{O}(\varepsilon^2)\right] \\
&+ \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{\alpha_e^B}{\pi}\right)\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{2\varepsilon}\left[-\frac{9}{8}C_F^2 - \varepsilon\left(\frac{141}{16}C_F^2 + \frac{21}{16}C_F\,C_A\right) + \mathscr{O}(\varepsilon^2)\right] + \mathscr{O}\left(\left(\frac{\alpha_s^B}{\pi},\frac{\alpha_e^B}{\pi}\right)^3\right)\Bigg\},
\end{aligned} \tag{28}$$

and

$$
\begin{aligned}
\Im\left[\Pi_B^{(B)}(Q)\Big|_{DRED}\right] = \alpha_{Ve}^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} \Bigg\{ & \varepsilon + 2\varepsilon^2 + \left(4 - \frac{3}{2}\zeta_2\right)\varepsilon^3 + \mathscr{O}(\varepsilon^4) \\
& \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} C_F \left[\frac{3}{2} + \varepsilon\frac{29}{4} + \varepsilon^2\left(\frac{227}{8} - \frac{15}{2}\zeta_2 - 6\zeta_3\right) + \mathscr{O}(\varepsilon^3)\right] \\
& + \left(\frac{\alpha_e^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} C_F \left[-1 - 4\varepsilon - \varepsilon^2\left(\frac{27}{2} - 5\zeta_2\right) + \mathscr{O}(\varepsilon^3)\right] \\
& + \left(\frac{\alpha_s^B}{\pi}\right)^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\varepsilon} \left[\frac{1}{\varepsilon}\left(\frac{9}{8}C_F^2 + \frac{11}{16}C_F C_A - \frac{1}{8}C_F N_f\right) + \frac{279}{32}C_F^2 + \frac{199}{32}C_F C_A - \frac{17}{16}C_F N_f\right. \\
& \quad + \varepsilon\left(C_F^2\left(\frac{3139}{64} - \frac{189}{16}\zeta_2 - \frac{45}{4}\zeta_3\right) + C_F C_A\left(\frac{2473}{64} - \frac{231}{32}\zeta_2 - \frac{75}{8}\zeta_3\right)\right. \\
& \quad\quad \left.\left. + C_F N_f\left(-\frac{207}{32} + \frac{21}{16}\zeta_2 + \frac{3}{2}\zeta_3\right)\right) + \mathscr{O}(\varepsilon^2)\right] \\
& + \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{\alpha_e^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\varepsilon}\left[-\frac{1}{\varepsilon}\frac{9}{4}C_F^2 - \frac{129}{8}C_F^2 - \frac{3}{8}C_F C_A\right. \\
& \quad\quad \left. - \varepsilon\left(\left(\frac{671}{8} - \frac{189}{8}\zeta_2 - 9\zeta_3\right)C_F^2 + \frac{53}{16}C_F C_A\right) + \mathscr{O}(\varepsilon^2)\right] \\
& + \left(\frac{\alpha_e^B}{\pi}\right)^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\varepsilon}\left[\frac{1}{\varepsilon}\left(C_F^2 - \frac{1}{4}C_F C_A + \frac{1}{8}C_F N_f\right) + \frac{13}{2}C_F^2 - \frac{3}{2}C_F C_A + \frac{15}{16}C_F N_f\right. \\
& + \varepsilon\left(\left(31 - \frac{21}{2}\zeta_2 - \frac{3}{4}\zeta_3\right)C_F^2 - \left(\frac{53}{8} - \frac{21}{8}\zeta_2 - \frac{3}{8}\zeta_3\right)C_F C_A + \left(\frac{157}{32} - \frac{21}{16}\zeta_2\right)C_F N_f\right) + \left.\mathscr{O}(\varepsilon^2)\right] \\
& + \mathscr{O}\left(\left(\frac{\alpha_s^B}{\pi}, \frac{\alpha_e^B}{\pi}\right)^3\right)\Bigg\},
\end{aligned}
\tag{29}
$$

where $\mathscr{O}\left(\left(\frac{\alpha_s^B}{\pi}, \frac{\alpha_e^B}{\pi}\right)^3\right)$ denotes terms for which the sum of the powers of $\left(\frac{\alpha_s^B}{\pi}\right)$ and $\left(\frac{\alpha_e^B}{\pi}\right)$ is at least three.

Upon renormalization according to Eq. (24) and expanding in terms of $\alpha_s^{\overline{\rm MS}}$ according to Eq. (26), I find that

$$\Im\left[\Pi_A(Q)|_{DRED}\right] = \alpha_V\, N_c \sum_f Q_f^2 \left\{1+\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)\frac{3}{4}C_F\left[1+\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)\beta_{20}^{\overline{\rm DR}}\ln\frac{\mu^2}{Q^2}\right]\right.$$
$$\left.+\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^2\left[-C_F^2\frac{3}{32}+C_F\,C_A\left(\frac{121}{32}-\frac{11}{4}\zeta_3\right)+C_F\,N_f\left(-\frac{11}{16}+\frac{1}{2}\zeta_3\right)\right]+\mathscr{O}\left(\left(\frac{\alpha_s^B}{\pi},\frac{\alpha_e^B}{\pi}\right)^3\right)\right\}$$
$$= \alpha_V\, N_c \sum_f Q_f^2 \left\{1+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)C_F\,\frac{3}{4}\left[1+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)\beta_0^{\overline{\rm MS}}\ln\frac{\mu^2}{Q^2}\right]\right.$$
$$\left.+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^2\left[-C_F^2\,\frac{3}{32}+C_F\,C_A\left(\frac{123}{32}-\frac{11}{4}\zeta_3\right)+C_F\,N_f\left(-\frac{11}{16}+\frac{1}{2}\zeta_3\right)\right]+\mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^3\right)\right\},$$
$$\Im\left[\Pi_B(Q)|_{DRED}\right] = \mathscr{O}(\varepsilon)\,. \tag{30}$$

C. Total Decay rate and annihilation cross section in the DRED scheme

As in the CDR scheme, the decay rate and annihilation cross section are determined from the imaginary part of the forward scattering amplitude.

$$\Gamma^{DRED}_{V\to\text{ hadrons}} = \frac{1}{M_V}\frac{1}{N_{\rm spins}}\sum_\lambda \varepsilon^\mu(Q,\lambda)\,\Im\left[\Pi_{\mu\nu}(Q)\big|_{DRED}\right]\,\varepsilon^\nu(Q,\lambda)^*\,, \tag{31}$$

where

$$\frac{1}{N_{\rm spins}}\sum_\lambda \varepsilon^\mu(Q,\lambda)\,\varepsilon^\nu(Q,\lambda)^* = \frac{1}{3}\left(-\hat{g}^{\mu\nu}+\frac{Q^\mu\,Q^\nu}{M_V^2}-\delta^{\mu\nu}\right)\,. \tag{32}$$

The evanescent part of the spin average contracts only with the $\Pi_B(Q)$ term, which has been renormalized to be of order (ε), so that the result is:

$$\Gamma^{DRED}_{V\to\text{ hadrons}} = \frac{\alpha_V\,M_V}{3}\,N_c\sum_f Q_f^2\left\{1+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)C_F\,\frac{3}{4}\left[1+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)\beta_0^{\overline{\rm MS}}\ln\frac{\mu^2}{Q^2}\right]\right.$$
$$\left.+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^2\left[-C_F^2\,\frac{3}{32}+C_F\,C_A\left(\frac{123}{32}-\frac{11}{4}\zeta_3\right)+C_F\,N_f\left(-\frac{11}{16}+\frac{1}{2}\zeta_3\right)\right]+\mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^3\right)\right\}, \tag{33}$$

just like in the CDR calculation.

For the annihilation cross section $\sigma_{e^+e^-\to\text{ hadrons}}$, one attaches fermion bilinears to each end of the vacuum polarization tensor and averages over the spins.

$$\sigma^{DRED}_{e^+e^-\to\text{ hadrons}} = \frac{2}{Q^2}\frac{e^2}{4}\sum_{\lambda\,\lambda'} \frac{\left\langle \overline{v}(p_{e^+},\lambda)\,|\hat{\gamma}^\mu|\,u(p_{e^-},\lambda')\right\rangle}{Q^2} \Im\left[\Pi_{\mu\nu}(Q)\big|_{DRED,\,\alpha_V\to\alpha}\right] \frac{\left\langle \overline{u}(p_{e^-},\lambda')\,|\hat{\gamma}^\nu|\,v(p_{e^+},\lambda)\right\rangle}{Q^2}$$
$$+ \frac{2}{Q^2}\frac{e^2_{\ell e}}{4}\sum_{\lambda\,\lambda'} \frac{\left\langle \overline{v}(p_{e^+},\lambda)\,|\overline{\gamma}^\mu|\,u(p_{e^-},\lambda')\right\rangle}{Q^2} \Im\left[\Pi_{\mu\nu}(Q)\big|_{DRED,\,\alpha_V\to\alpha}\right] \frac{\left\langle \overline{u}(p_{e^-},\lambda')\,|\overline{\gamma}^\nu|\,v(p_{e^+},\lambda)\right\rangle}{Q^2}\,, \tag{34}$$

where $e_{\ell e}$ represents the coupling of the evanescent photon to the electron. Combining the spinor bilinears into traces,

$$\frac{1}{2}\sum_{\lambda\,\lambda'}\left\langle \overline{v}(p_{e^+},\lambda)\,|\hat{\gamma}^\mu|\,u(p_{e^-},\lambda')\right\rangle\left\langle \overline{u}(p_{e^-},\lambda')\,|\hat{\gamma}^\nu|\,v(p_{e^+},\lambda)\right\rangle = \frac{1}{2}\,\mathrm{Tr}\left[\not{p}_{e^+}\,\gamma^\mu\,\not{p}_{e^-}\,\gamma^\nu\right] = \left(-Q^2\,\hat{g}^{\mu\nu} + Q^\mu\,Q^\nu\right)$$
$$\frac{1}{2}\sum_{\lambda\,\lambda'}\left\langle \overline{v}(p_{e^+},\lambda)\,|\overline{\gamma}^\mu|\,u(p_{e^-},\lambda')\right\rangle\left\langle \overline{u}(p_{e^-},\lambda')\,|\overline{\gamma}^\nu|\,v(p_{e^+},\lambda)\right\rangle = \frac{1}{2}\,\mathrm{Tr}\left[\not{p}_{e^+}\,\overline{\gamma}^\mu\,\not{p}_{e^-}\,\overline{\gamma}^\nu\right] = \left(-Q^2\,\delta^{\mu\nu}\right) \tag{35}$$

The final result is

$$\sigma^{DRED}_{e^+e^-\to\text{ hadrons}} = \frac{4\pi\,\alpha^2}{3\,Q^2}\,N_c\sum_f Q_f^2\left\{1+\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)C_F\,\frac{3}{4}\left[1+\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)\beta_0^{\overline{\mathrm{MS}}}\ln\frac{\mu^2}{Q^2}\right]\right.$$
$$\left.+\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2\left[-C_F^2\,\frac{3}{32}+C_F\,C_A\left(\frac{123}{32}-\frac{11}{4}\zeta_3\right)+C_F\,N_f\left(-\frac{11}{16}+\frac{1}{2}\,\zeta_3\right)\right]+\mathcal{O}\left(\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^3\right)\right\}\,, \tag{36}$$

again in agreement with Eqs. (2-3). As promised, under the DRED scheme renormalization program, evanescent Green functions are rendered finite by renormalization and contribute to scattering amplitudes at order (ε). Also as promised, the results are completely equivalent to those of the CDR scheme.

V. THE FOUR DIMENSIONAL HELICITY SCHEME

In the Four Dimensional Helicity scheme, one defines an enlarged vector space of dimensionality $D_m = 4-2\,\varepsilon$, in which loop momenta take values, as in the CDR scheme. In addition, one defines a still larger vector space, of dimensionality $D_s = 4$, in which internal spin degrees of freedom take values. The precise rules for the FDH scheme are given in Ref. [5]. They are:

1. As in ordinary dimensional regularization, all momentum integrals are integrated over D_m dimensional momenta. Metric tensors resulting from tensor integrals are D_m dimensional.

2. All “observed” external states are taken to be four dimensional, as are their momenta and polarization vectors. This facilitates the use of helicity states for observed particles.

3. All “unobserved” or internal states are treated as D_s dimensional, and the D_s dimensional vector space is taken to be larger than the D_m dimensional vector space. Unobserved states include virtual states inside of loops, virtual states inside of trees as well as external states that have infrared sensitive overlaps with other external states.

4. Both the D_s and D_m dimensional vector spaces are larger than the standard four dimensional space-time, so that contraction of four dimensional objects with D_m or D_s dimensional objects yields only four dimensional components.

To keep track of the many vector spaces and their overlapping domains, I give the result of the contractions of the various metric tensors with one another,

$$\begin{array}{llll} g^{\mu\nu}\, g_{\mu\nu} = D_s\,, & \hat{g}^{\mu\nu}\, \hat{g}_{\mu\nu} = D_m\,, & \eta^{\mu\nu}\, \eta_{\mu\nu} = 4\,, & \delta^{\mu\nu}\, \delta_{\mu\nu} = D_x = D_s - D_m \\ g^{\mu\nu} \hat{g}^{\rho}_{\nu} = \hat{g}^{\mu\rho}\,, & g^{\mu\nu} \eta^{\rho}_{\nu} = \eta^{\mu\rho}\,, & \hat{g}^{\mu\nu} \eta^{\rho}_{\nu} = \eta^{\mu\rho}\,, & \\ g^{\mu\nu} \delta^{\rho}_{\nu} = \delta^{\mu\rho}\,, & \hat{g}^{\mu\nu} \delta^{\rho}_{\nu} = 0\,, & \eta^{\mu\nu} \delta^{\rho}_{\nu} = 0\,. & \end{array} \tag{37}$$

Like the HV scheme, the FDH scheme treats observed states as four dimensional. In inclusive calculations, however, where there are infrared overlaps among external states, the external states are taken to be D_s dimensional in the infrared regions.

As in the DRED scheme, spin degrees of freedom take values in a vector space that is larger than that in which momenta take values. It would seem, therefore, that the same remarks regarding the Ward Identity and the conclusion that the $D_x = D_s - D_m$ dimensional components of the gauge fields and their couplings must be considered as distinct from the D_m dimensional gauge fields and couplings would apply. That is not, however, how the FDH scheme is used. All field components in the D_s dimensional space are treated as gauge fields and no distinction is made between the couplings. It is common, however, to define an interpolating scheme, the “δ_R” scheme, in which $D_s = 4 - 2\,\varepsilon\,\delta_R$. The parameter δ_R interpolates between the HV scheme ($\delta_R = 1$) and the FDH scheme ($\delta_R = 0$). Using this scheme gives one a handle on the impact of the evanescent degrees of freedom on the result, but not on the impact of a distinct evanescent coupling.

It is claimed [5] that the essential difference between the FDH and DRED schemes is that in the former $D_m > 4$, while in the latter $D_m < 4$. It must be this difference, then, that allows for the very different handling of the evanescent couplings and degrees of freedom. We shall see what impact this choice has in the calculation and discussion below.

A. Renormalization

I will not give detailed results for the renormalization parameters of the FDH scheme. There is no point in doing so because, as I will show, the rules of the FDH scheme enumerated in the previous section are not consistent with a successful renormalization program. The first sign that there is a problem with the renormalization program comes in the computation of the one-loop renormalization constants. In particular, the gluon vacuum polarization tensor splits into two independent components, $\Pi_A^{\mu\nu} = \Pi_A(Q^2)\left((-Q^2\hat{g}^{\mu\nu} + Q^\mu Q^\nu\right)$ and $\Pi_B^{\mu\nu} = \Pi_B(Q^2)\,\delta^{\mu\nu}$, both of which are singular. This is a clear warning that what the FDH scheme calls the gluon is in fact two distinct sets of degrees of freedom. If I ignore Π_B and just renormalize Π_A, I find the usual result that

$$\beta_0^{\overline{\rm FDH}} = \frac{11}{12}C_A - \frac{1}{6}N_f\,. \tag{38}$$

Note that I also get this result if I take the spin average (trace) of the full vacuum polarization tensor. Because Π_B is weighted by a factor of 2ε, its contribution to the spin average is not singular. Because the leading order term in the quantities being calculated is of order one, and the NLO term of order α_s, this result for the one-loop β-function is all that is needed to compute the renormalized cross section at NNLO. Furthermore, the many NLO results that have been obtained using the FDH scheme have all renormalized using the above result for $\beta_0^{\overline{\rm FDH}}$.

When I try to proceed to the two-loop beta function, I find that both Π_A and Π_B contribute singular terms to the spin-averaged vacuum polarization, while if I again ignore Π_B and renormalize Π_A, I obtain the usual value for β_1,

$$\beta_1^{\overline{\rm FDH}} = \frac{17}{24}C_A^2 - \frac{5}{24}C_A N_f - \frac{1}{8}C_F N_f\,. \tag{39}$$

This seems to be the choice made in Ref. [5] as they quote only the result for terms proportional to $Q^\mu Q^\nu$, which would be part of my Π_A. Since the standard lore has been that $\alpha_s^{\overline{\rm FDH}}$ and $\alpha_s^{\overline{\rm DR}}$ coincide, at least through second order corrections, this seems to be the most reasonable choice. Furthermore, it means that the conversion to $\alpha_s^{\overline{\rm MS}}$ will be [5, 25]

$$\alpha_s^{\overline{\rm FDH}} = \alpha_s^{\overline{\rm MS}}\left[1 + \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)\frac{C_A}{12} + \ldots\right] \tag{40}$$

As it turns out, it does not matter what choice one makes as even the one-loop result for $\beta_0^{\overline{\rm FDH}}$, which seems safe if only because it is familiar, leads to the violation of unitarity.

B. Vacuum polarization in the FDH scheme

Leaving aside the question of renormalization beyond one-loop, I will proceed with the calculation of the V-boson vacuum polarization. In performing calculations in the FDH scheme, it becomes apparent that the results are identical, term-by-term. to the calculation in the DRED scheme, except that the evanescent gluons are identified as gluons and the coupling α_e is set to α_s. Therefore I find that

$$\Im\left[\Pi^{(B)}_{\mu\nu}(Q)\Big|_{FDH}\right] = \frac{-Q^2\,\hat{g}_{\mu\nu}+Q_\mu Q_\nu}{3}\,\Im\left[\Pi^{(B)}_{A}(Q)\Big|_{FDH}\right] - Q^2\,\frac{\delta_{\mu\nu}}{2\,\varepsilon}\,\Im\left[\Pi^{(B)}_{B}(Q)\Big|_{FDH}\right], \tag{41}$$

where

$$\begin{aligned}\Im\left[\Pi^{(B)}_{A}(Q)\Big|_{FDH}\right] &= \alpha_V^B\,N_c\sum_f Q_f^2\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{\varepsilon}\Bigg\{\\ &1+\left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{\varepsilon} C_F\left[\frac{3}{4}+\varepsilon\left(\frac{45}{8}-6\,\zeta_3\right)+\varepsilon^2\left(\frac{439}{16}-\frac{15}{4}\zeta_2-15\,\zeta_3-9\,\zeta_4\right)+\mathscr{O}(\varepsilon^3)\right]\\ &+\left(\frac{\alpha_s^B}{\pi}\right)^2\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{2\varepsilon}\Bigg[\frac{1}{\varepsilon}\left(\frac{11}{16}C_F\,C_A-\frac{1}{8}C_F\,N_f\right)-\frac{15}{32}C_F^2+\left(\frac{37}{4}-\frac{33}{4}\zeta_3\right)C_F\,C_A-\left(\frac{25}{16}-\frac{3}{2}\zeta_3\right)C_F\,N_f\\ &+\varepsilon\Bigg(C_F^2\left(-\frac{235}{32}-\frac{111}{8}\,\zeta_3+\frac{45}{2}\,\zeta_5\right)+C_F\,C_A\left(\frac{14521}{192}-\frac{231}{32}\,\zeta_2-\frac{193}{4}\,\zeta_3-\frac{99}{8}\,\zeta_4-\frac{15}{4}\,\zeta_5\right)\\ &+C_F\,N_f\left(-\frac{1187}{96}+\frac{21}{16}\,\zeta_2+\frac{17}{2}\,\zeta_3+\frac{9}{4}\,\zeta_4\right)\Bigg)+\mathscr{O}(\varepsilon^2)\Bigg]+\mathscr{O}\left(\left(\frac{\alpha_s^B}{\pi}\right)^3\right)\Bigg\},\end{aligned} \tag{42}$$

and

$$\begin{aligned}\Im\left[\Pi^{(B)}_{B}(Q)\Big|_{FDH}\right] &= \alpha_V^B\,N_c\sum_f Q_f^2\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{\varepsilon}\Bigg\{\\ &\varepsilon+\left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{\varepsilon} C_F\left[\frac{1}{2}+\varepsilon\frac{13}{4}+\varepsilon^2\left(\frac{119}{8}-\frac{5}{2}\zeta_2-6\,\zeta_3\right)+\mathscr{O}(\varepsilon^3)\right]\\ &+\left(\frac{\alpha_s^B}{\pi}\right)^2\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{2\varepsilon}\Bigg[\frac{1}{\varepsilon}\left(-\frac{1}{8}C_F^2+\frac{7}{16}C_F\,C_A\right)-\frac{29}{32}C_F^2+\frac{139}{32}C_F\,C_A-\frac{1}{8}C_F\,N_f\\ &+\varepsilon\Bigg(C_F^2\left(-\frac{245}{64}+\frac{21}{16}\,\zeta_2-3\,\zeta_3\right)+C_F\,C_A\left(\frac{1837}{64}-\frac{147}{32}\,\zeta_2-9\,\zeta_3\right)\\ &+C_F\,N_f\left(-\frac{25}{16}+\frac{3}{2}\,\zeta_3\right)\Bigg)+\mathscr{O}(\varepsilon^2)\Bigg]+\mathscr{O}\left(\left(\frac{\alpha_s^B}{\pi}\right)^3\right)\Bigg\}.\end{aligned} \tag{43}$$

Upon renormalizing such that

$$\left(\frac{\alpha_s^B}{\pi}\right)\to\left(\frac{\alpha_s^{\overline{\text{FDH}}}}{\pi}\right)\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{-\varepsilon}\left(1-\frac{\beta_0^{\overline{\text{FDH}}}}{\varepsilon}\left(\frac{\alpha_s^{\overline{\text{FDH}}}}{\pi}\right)\right), \qquad \alpha_V^B\to\alpha_V\left(\frac{4\pi}{Q^2\,e^{\gamma_E}}\right)^{-\varepsilon}, \tag{44}$$

I find that

$$\Im\left[\Pi_A(Q)|_{FDH}\right] = \alpha_V\, N_c \sum_f Q_f^2 \left\{ 1 + \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)\frac{3}{4}C_F \left[1 + \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)\beta_0^{\overline{\rm FDH}} \ln\frac{\mu^2}{Q^2}\right] \right.$$
$$\left. + \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)^2 \left[-C_F^2\frac{15}{32} + C_F\, C_A\left(\frac{131}{32} - \frac{11}{4}\zeta_3\right) + C_F\, N_f\left(-\frac{5}{8} + \frac{1}{2}\zeta_3\right)\right] + \mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)^3\right)\right\}$$
$$= \alpha_V\, N_c \sum_f Q_f^2 \left\{ 1 + \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right) C_F\, \frac{3}{4}\left[1 + \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)\beta_0^{\overline{\rm MS}} \ln\frac{\mu^2}{Q^2}\right] \right.$$
$$\left. + \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^2 \left[-C_F^2\frac{15}{32} + C_F\, C_A\left(\frac{133}{32} - \frac{11}{4}\zeta_3\right) + C_F\, N_f\left(-\frac{5}{8} + \frac{1}{2}\zeta_3\right)\right] + \mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^3\right)\right\},$$
$$\Im\left[\Pi_B(Q)|_{FDH}\right] = \alpha_V\, N_c \sum_f Q_f^2 \left\{ \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)\frac{1}{2}C_F \left[1 + \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)\beta_0^{\overline{\rm FDH}} \ln\frac{\mu^2}{Q^2}\right] \right.$$
$$+ \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)^2 \left[\frac{1}{\varepsilon}\left(-C_F^2\frac{1}{8} - C_F\, C_A\frac{1}{48} + C_F\, N_f\frac{1}{12}\right)\left(1 + 3\varepsilon\ln\frac{\mu^2}{Q^2}\right)\right. \tag{45}$$
$$\left.\left. -C_F^2\frac{29}{32} + C_F\, C_A\frac{131}{96} - C_F\, N_f\frac{5}{12}\right] + \mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)^3\right)\right\}$$
$$= \alpha_V\, N_c \sum_f Q_f^2 \left\{ \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)\frac{1}{2}C_F \left[1 + \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)\beta_0^{\overline{\rm FDH}} \ln\frac{\mu^2}{Q^2}\right] \right.$$
$$+ \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^2 \left[\frac{1}{\varepsilon}\left(-C_F^2\frac{1}{8} - C_F\, C_A\frac{1}{48} + C_F\, N_f\frac{1}{12}\right)\left(1 + 3\varepsilon\ln\frac{\mu^2}{Q^2}\right)\right.$$
$$\left.\left. -C_F^2\frac{29}{32} + C_F\, C_A\frac{45}{32} - C_F\, N_f\frac{5}{12}\right] + \mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^3\right)\right\}.$$

C. Total Decay rate and annihilation cross section in the FDH scheme

The results of the vacuum polarization calculation look to be disastrous as Π_B is singular at order α_s^2. However, the rules of the FDH scheme, enumerated above, specify that external states are taken to be four dimensional. This means that the spin average of the vector polarizations is

$$\frac{1}{N_{\rm spins}}\sum_\lambda \varepsilon^\mu(Q,\lambda)\,\varepsilon^\nu(Q,\lambda)^* = \frac{1}{3}\left(-\eta^{\mu\nu} + \frac{Q^\mu\, Q^\nu}{M_V^2}\right), \tag{46}$$

which annihilates $\Pi_B^{\mu\nu}\big|_{FDH}$. For the annihilation rate, the rules are a bit ambiguous, as they could be read to mean that the lepton spinors are four dimensional but the vertex (γ^μ) connecting them to the loop part of the amplitude is D_s dimensional. This would bring $\Pi_B^{\mu\nu}\big|_{FDH}$ into the calculation and lead to a singular result at order α_s^2. However, Rule 4 could also be read to mean that the vertex sandwiched between four dimensional states is also reduced to being four dimensional.

Assuming this interpretation, I find that

$$\Gamma^{FDH}_{V\to \text{ hadrons}} = \frac{\alpha_V M_V}{3} N_c \sum_f Q_f^2 \left\{ 1 + \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right) C_F \frac{3}{4} \left[1 + \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right) \beta_0^{\overline{\text{MS}}} \ln\frac{\mu^2}{Q^2}\right] \right. \\ \left. + \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^2 \left[-C_F^2 \frac{15}{32} + C_F C_A \left(\frac{133}{32} - \frac{11}{4}\zeta_3\right) + C_F N_f \left(-\frac{5}{8} + \frac{1}{2}\zeta_3\right)\right] + \mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^3\right) \right\}, \tag{47}$$

and

$$\sigma^{FDH}_{e^+e^-\to \text{ hadrons}} = \frac{4\pi\alpha^2}{3Q^2} N_c \sum_f Q_f^2 \left\{ 1 + \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right) C_F \frac{3}{4} \left[1 + \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right) \beta_0^{\overline{\text{MS}}} \ln\frac{\mu^2}{Q^2}\right] \right. \\ \left. + \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^2 \left[-C_F^2 \frac{15}{32} + C_F C_A \left(\frac{133}{32} - \frac{11}{4}\zeta_3\right) + C_F N_f \left(-\frac{5}{8} + \frac{1}{2}\zeta_3\right)\right] + \mathscr{O}\left(\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^3\right) \right\}. \tag{48}$$

The results agree with one another, are correct through NLO and are finite through NNLO. Unfortunately, the NNLO terms are not correct! Because the discrepancy is finite, there remains the possibility that the conversion from $\alpha_s^{\overline{\text{FDH}}}$ to $\alpha_s^{\overline{\text{MS}}}$ given in Eq. (40) is incorrect, although this would contradict previous results [5, 25]. If this were the case, then one would expect that the N^{3}LO result would also be finite but incorrect. If, instead, the finite discrepancy at NNLO is the result of a failure of the renormalization program, the N^{3}LO result should be singular.

VI. PARTIAL RESULTS AT N^{3}LO

Although first computed some time ago, the vacuum polarization at four loops [13, 14] remains a formidable calculation. It is only necessary, however, to look at a small part of the calculation: the terms proportional to the square of the number of fermion flavors, N_f^2. This is fortunate for a couple of reasons: 1) there are only three four-loop diagrams to be computed, see Fig. (3), (plus three more in the DRED scheme, where the gluons are replaced by evanescent gluons); and 2) the contributions from renormalization in the CDR and FDH schemes come only from the leading term in the QCD β-function (β_0 and β_0^2). Thus, my result will not depend on how the higher order terms of the β-function are chosen in the FDH scheme.

A. The CDR scheme

In the CDR scheme, there are only three four-loop diagrams that need to be calculated. The first two are simply iterated bubble diagrams and are essentially trivial. The third is slightly non-trivial, so I again

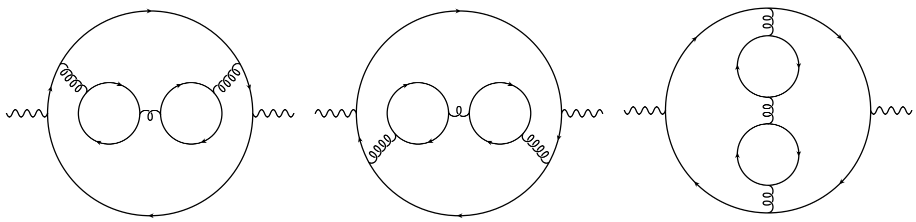

FIG. 3: Four loop diagrams that contribute to the N_f^2 term at N^{3}LO.

use my QGRAF-FORM-REDUZE suite of programs to address the problem. All of the four-loop master integrals can be found in Ref. [26]. I find the result of the four-loop calculation to be

$$\Im\left[\Pi^{(B)}_{\mu\nu}(Q)\Big|_{CDR}\right]_{\alpha_s^3 N_f^2} = \frac{-Q^2 g_{\mu\nu} + Q_\mu Q_\nu}{3}\alpha_V^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{4\varepsilon} \times \left(\frac{\alpha_s^B}{\pi}\right)^3 C_F N_f^2 \left[\frac{1}{48\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{121}{288} - \frac{1}{3}\zeta_3\right) + \frac{2777}{576} - \frac{3}{8}\zeta_2 - \frac{19}{6}\zeta_3 - \frac{1}{2}\zeta_4\right] \tag{49}$$

Renormalizing, I find

$$\Im\left[\Pi_{\mu\nu}(Q)\Big|_{CDR}\right]_{\alpha_s^3 N_f^2} = \frac{-Q^2 g_{\mu\nu} + Q_\mu Q_\nu}{3}\alpha_V N_c \sum_f Q_f^2 \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^3 C_F N_f^2 \times \left[\frac{151}{216} - \frac{1}{24}\zeta_2 - \frac{19}{36}\zeta_3 + \left(\frac{11}{48} - \frac{1}{6}\zeta_3\right)\ln\left(\frac{\mu^2}{Q^2}\right) + \frac{1}{48}\ln^2\left(\frac{\mu^2}{Q^2}\right)\right] \tag{50}$$

Using this term to compute the $\alpha_s^3 N_f^2$ contribution to the decay rate and annihilation cross section as in Eqs. (14,17), I find the result expected from Eqs. (2-3).

B. The DRED scheme

In the DRED scheme, there are three extra four-loop diagrams to compute, obtained by replacing gluon propagators with evanescent gluon propagators. I find

$$\begin{aligned}
\Im\left[\Pi_A^{(B)}(Q)\Big|_{DRED}\right]_{\alpha_s^3 N_f^2} &= \alpha_V^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{4\varepsilon} C_F N_f^2 \Bigg\{ \\
&\left(\frac{\alpha_s^B}{\pi}\right)^3 \left[\frac{1}{48\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{13}{32} - \frac{1}{3}\zeta_3\right) + \frac{7847}{1728} - \frac{3}{8}\zeta_2 - \frac{53}{18}\zeta_3 - \frac{1}{2}\zeta_4\right] \\
&+ \left(\frac{\alpha_e^B}{\pi}\right)^3 \left[-\frac{1}{\varepsilon}\frac{3}{64} - \frac{83}{128}\right]\Bigg\} \\
\Im\left[\Pi_B^{(B)}(Q)\Big|_{DRED}\right]_{\alpha_s^3 N_f^2} &= \alpha_{Ve}^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{4\varepsilon} C_F N_f^2 \Bigg\{ \\
&\left(\frac{\alpha_s^B}{\pi}\right)^3 \left[\frac{1}{72\varepsilon^2} + \frac{1}{\varepsilon}\frac{73}{432} + \frac{3595}{2592} - \frac{1}{4}\zeta_2 - \frac{1}{3}\zeta_3\right] \\
&+ \left(\frac{\alpha_e^B}{\pi}\right)^3 \left[-\frac{1}{48\varepsilon^2} - \frac{1}{\varepsilon}\frac{11}{48} - \frac{155}{96} + \frac{3}{8}\zeta_2\right]\Bigg\}
\end{aligned} \tag{51}$$

Upon renormalizing according to Eq. (24) and converting the coupling to $\alpha_s^{\overline{\text{MS}}}$, I obtain

$$\begin{aligned}
&\Im\left[\Pi_A(Q)|_{DRED}\right]_{\alpha_s^3 N_f^2} \\
&\quad = \alpha_V N_c \sum_f Q_f^2 C_F N_f^2 \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^3 \left[\frac{151}{216} - \frac{1}{24}\zeta_2 - \frac{19}{36}\zeta_3 + \left(\frac{11}{48} - \frac{1}{6}\zeta_3\right)\ln\left(\frac{\mu^2}{Q^2}\right) + \frac{1}{48}\ln^2\left(\frac{\mu^2}{Q^2}\right)\right], \\
&\Im\left[\Pi_B(Q)|_{DRED}\right]_{\alpha_s^3 N_f^2} = \mathscr{O}(\varepsilon).
\end{aligned} \tag{52}$$

As for the CDR scheme, this leads to the expected result for the decay rate and annihilation cross section.

C. The FDH scheme

In the FDH scheme, however, I find that

$$\begin{aligned}
\Im\left[\Pi_A^{(B)}(Q)\Big|_{FDH}\right]_{\alpha_s^3 N_f^2} &= \alpha_V^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{4\varepsilon} C_F N_f^2 \\
&\times \left(\frac{\alpha_s^B}{\pi}\right)^3 \left[\frac{1}{48\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{23}{64} - \frac{1}{3}\zeta_3\right) + \frac{13453}{3456} - \frac{3}{8}\zeta_2 - \frac{53}{18}\zeta_3 - \frac{1}{2}\zeta_4\right], \\
\Im\left[\Pi_B^{(B)}(Q)\Big|_{FDH}\right]_{\alpha_s^3 N_f^2} &= \alpha_V^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{4\varepsilon} C_F N_f^2 \\
&\times \left(\frac{\alpha_s^B}{\pi}\right)^3 \left[-\frac{1}{144\varepsilon^2} - \frac{1}{\varepsilon}\frac{13}{216} - \frac{295}{1296} + \frac{1}{8}\zeta_2 - \frac{1}{3}\zeta_3\right].
\end{aligned} \tag{53}$$

I renormalize according to

$$\alpha_s^B = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \alpha_s^{\overline{\rm FDH}} \left[1 - \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)\frac{\beta_0^{\overline{\rm FDH}}}{\epsilon} + \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)^2 \left(\frac{\beta_0^{\overline{\rm FDH}\,2}}{\epsilon^2} - \frac{1}{2}\frac{\beta_1^{\overline{\rm FDH}}}{\epsilon}\right)\right], \tag{54}$$

keeping only terms proportional to $\alpha_s^{\overline{\rm FDH}\,3} N_f^2$. Such terms can only come from the $\beta_0^{\overline{\rm FDH}}$ and $\beta_0^{\overline{\rm FDH}\,2}$ terms, so any uncertainty about $\beta_1^{\overline{\rm FDH}}$ has no effect here. The renormalized result is

$$\begin{aligned}
&\Im\left[\Pi_A(Q)|_{FDH}\right]_{\alpha_s^3 N_f^2} \\
&\quad = \alpha_V N_c \sum_f Q_f^2 C_F N_f^2 \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)^3 \left[-\frac{1}{192\,\epsilon} + \frac{1843}{3456} - \frac{1}{24}\zeta_2 - \frac{19}{36}\zeta_3 + \left(\frac{3}{16} - \frac{1}{6}\zeta_3\right)\ln\left(\frac{\mu^2}{Q^2}\right) + \frac{1}{48}\ln^2\left(\frac{\mu^2}{Q^2}\right)\right], \\
&\Im\left[\Pi_B(Q)|_{FDH}\right]_{\alpha_s^3 N_f^2} \\
&\quad = \alpha_V N_c \sum_f Q_f^2 C_F N_f^2 \left(\frac{\alpha_s^{\overline{\rm FDH}}}{\pi}\right)^3 \left[\frac{1}{144\,\epsilon^2} - \frac{5}{432\,\epsilon} - \frac{869}{2592} + \frac{1}{18}\zeta_2 - \frac{5}{27}\ln\left(\frac{\mu^2}{Q^2}\right) - \frac{1}{36}\ln^2\left(\frac{\mu^2}{Q^2}\right)\right].
\end{aligned} \tag{55}$$

The demand that external states be four dimensional removes the Π_B term, but there is also a pole in Π_A and no finite renormalization to put the result in terms of $\alpha_s^{\overline{\rm MS}}$ can remove it. I must therefore conclude that the FDH scheme is not consistent with unitarity.

VII. DISCUSSION

In this paper, I have performed a high-order calculation in each of three regularization schemes: the conventional dimensional regularization (CDR) scheme; the dimensional reduction (DRED) scheme; and the four dimensional helicity (FDH) scheme. Of these, the CDR scheme is by far the most widely used, and was, in fact, used to compute the original results that I use as my test basis. The FDH scheme has primarily been used to produce one-loop helicity amplitudes, although it has been used in a few cases in two-loop calculations and also as a supersymmetric regulator. The primary purpose of this paper was to put the FDH scheme to a stringent test and determine its reliability in a high order calculation. The DRED scheme is primarily used as a supersymmetric regulator and is quite cumbersome for non-supersymmetric calculations. It is, however, closely related to the FDH scheme and has been demonstrated [8, 9, 23, 24] to be equivalent to the CDR scheme through four loops. A close comparison of the details of the calculations in the FDH and DRED schemes helps to identify where and when things go wrong with the former.

In the cases of the CDR and DRED schemes, I have reproduced the known result for the hadronic decay width of a massive vector boson (or equivalently, the e^+e^- annihilation rate to hadrons) through NNLO, and a few terms at N^{3}LO. This represents computing the QCD corrections to the vacuum polarization of the photon (V boson) through three loops, with partial results at four loops. In addition, I have reproduced the renormalization parameters of QCD (β-function(s), mass anomalous dimension) through three loop order. This establishes that I have theoretical control over all of the needed calculations through three loop order. In order to obtain the partial N^{3}LO result in the DRED scheme, I also needed the three-loop QCD corrections to the β-function of the evanescent photon (V boson).

The calculation of the V boson decay rate provides another instance of the equivalence the CDR and DRED schemes at the four-loop level [23]. The ability to obtain the correct result using the DRED scheme required a delicate balance of the many extra couplings and their renormalization effects upon one another. Indeed, given the complexity needed to make the DRED scheme work, it seems that there should be little surprise that the FDH scheme, with its greater simplicity, should fail.

Perhaps, it is worth considering how it is that the FDH scheme has been used successfully in so many calculations. Its most common use has been in the construction of one-loop scattering amplitudes via unitarity cuts, using four-dimensional helicity amplitudes as the primary building blocks. Thus, it is natural that it restricts observed (external) states to be four-dimensional. Because the FDH scheme defines that $D_s > D_m > 4$, this restriction excludes evanescent fields from appearing as external states. This is very important because, as one can see from comparing Eqs. (30) and (55), terms involving external evanescent states are the most dangerous. Even though it does not renormalize evanescent states and couplings properly the FDH is able to get the non-evanescent part of the vacuum polarization tensor correct at NLO, while the evanescent part is ready to contribute a finite error at NLO. Because the DRED scheme defines $4 > D_m$, the evanescent states are *parts* of the classical four dimensional states. It would not seem natural to exclude them from appearing as external states. Instead, they are handled through the renormalization program so that their effects are removed from physical scattering amplitudes. In the FDH scheme, the evanescent states are instead *additions* to the four dimensional states (as are the extra degrees of freedom that come from regularizing momentum integrals) and there is no barrier to excluding them as observed states.

In an FDH scheme calculation, a tree-level term is strictly four-dimensional and is free from evanescent contributions. (Depending on interpretation, this may be a stronger condition than is given in the rules of Ref [5], but it is the actual condition imposed if one defines the tree-level amplitude as being a four-dimensional helicity amplitude.) Because evanescent terms are absent at tree-level, they cannot generate ultraviolet poles at one loop. Even if one were to renormalize them properly, as in the DRED scheme, there would be nowhere to make the counter-term insertion! In fact, the one-loop contributions are not even finite, as the counting over the number of states (2ε) makes the result of order ε. This is clearly illustrated in Eq. (28). Neither α_s, nor α_e appear at LO. Therefore, the contributions at NLO are finite for α_s and of order ε (because of the counting over the number of states) for α_e. In more complicated QCD calculations,

α_s will appear at LO and will therefore contribute an ultraviolet pole at one-loop, which will be removed by renormalization. α_e, however, will still make its first appearance at NLO and that contribution will be of order ε. Thus, one can expect that the FDH scheme, used as above, should be reliable for computing NLO corrections through finite order (ε^0). The error from improperly identifying evanescent quantities should be of order ε. At NNLO and beyond however, the failure to properly identify and renormalize the evanescent parameters leads to incorrect results and the violation of unitarity.

So, as suggested [5], one of the FDH scheme's most important assets is that it defines $D_s > D_m > 4$. This feature is also the scheme's undoing, though not of necessity. Because the effects of external evanescent states can be removed (or indeed never seen) by imposing a four-dimensionality restriction, and because the effects of internal evanescent states therefore contribute at order ε at one loop, it appears that one can simply ignore the distinction between gauge and evanescent terms. In contrast, because the DRED scheme must deal with external evanescent terms from the beginning, its advocates were forced to develop a successful renormalization program [8, 9]. Extensive testing [8, 9, 23, 24] has shown that this program works to at least the fourth order and that it handles the effects of both internal and external evanescent contributions. As I remarked earlier, calculations in the DRED and FDH schemes are term-by-term identical, except for the identification of the couplings and propagating states. Thus, one could make the FDH scheme a unitary regularization scheme for non-supersymmetric calculations by recognizing the distinction between gauge and evanescent terms and adopting the DRED scheme's renormalization program. This would, of course, do away with any notion of the FDH scheme being simple, but it would at least be correct. The FDH scheme would still be distinguished from the DRED scheme by the fact that $D_s > D_m > 4$, which facilitates helicity amplitude calculations and, in chiral theories, improves its situation with regard to γ_5 and the Levi-Civita tensor [27, 28]. Furthermore, with a valid renormalization program, the requirement of four-dimensional observed states could be made optional. This would lead to two linked, slightly different, schemes, just like the HV and CDR schemes. This suggestion has already been made by Signer and Stöckinger [29] who in fact define their version of the DRED scheme to have precisely the $D_s > D_m > 4$ hierarchy of the FDH scheme.

Thus, in conclusion, the CDR and DRED schemes are correct and equivalent ways of performing QCD calculations through N^{3}LO. The FDH scheme, however, has been shown to be incorrect and to violate unitarity beyond NLO when applied to non-supersymmetric theories. It must therefore be viewed as a shortcut for performing NLO calculations and should only be used for such calculations with great caution.

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Appendix A: Renormalization parameters for the CDR scheme

To three-loop order, I find the coefficients of the β-function to be

$$\beta_0^{\overline{\rm MS}} = \frac{11}{12}C_A - \frac{1}{6}N_f\,, \qquad \beta_1^{\overline{\rm MS}} = \frac{17}{24}C_A^2 - \frac{5}{24}C_A N_f - \frac{1}{8}C_F N_f\,,$$
$$\beta_2^{\overline{\rm MS}} = \frac{2857}{3456}C_A^3 - \frac{1415}{3456}C_A^2 N_f - \frac{205}{1152}C_A C_F N_f + \frac{1}{64}C_F^2 N_f + \frac{79}{3456}C_A N_f^2 + \frac{11}{576}C_F N_f^2\,, \tag{A1}$$

while the coefficients of the mass anomalous dimension are

$$\gamma_0^{\overline{\rm MS}} = \frac{3}{4}C_F\,, \qquad \gamma_1^{\overline{\rm MS}} = \frac{3}{32}C_F^2 + \frac{97}{96}C_F C_A - \frac{5}{48}C_F N_f\,,$$
$$\gamma_2^{\overline{\rm MS}} = \frac{129}{128}C_F^3 - \frac{129}{256}C_F^2 C_A + \frac{11413}{6912}C_F C_A^2 - \left(\frac{23}{64} - \frac{3}{8}\zeta_3\right) C_F^2 N_f - \left(\frac{139}{864} + \frac{3}{8}\right) C_F C_A N_f - \frac{35}{1728}C_F N_f^2\,, \tag{A2}$$

in agreement with known results [30–33].

Appendix B: Renormalization parameters for the DRED scheme

The coefficients of the QCD β-function, $\beta^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}})$ through three loops are:

$$\beta_{20}^{\overline{\rm DR}} = \frac{11}{12}C_A - \frac{1}{6}N_f\,, \qquad \beta_{30}^{\overline{\rm DR}} = \frac{17}{24}C_A^2 - \frac{5}{24}C_A N_f - \frac{1}{8}C_F N_f\,,$$
$$\beta_{40}^{\overline{\rm DR}} = \frac{3115}{3456}C_A^3 - \frac{1439}{3456}C_A^2 N_f - \frac{193}{1152}C_A C_F N_f + \frac{1}{64}C_F^2 N_f + \frac{79}{3456}, C_A N_f^2 + \frac{11}{576}C_F N_f^2\,, \tag{B1}$$
$$\beta_{31}^{\overline{\rm DR}} = -\frac{1}{16}C_F N_f \left(\frac{3}{2}C_F\right)\,, \qquad \beta_{22}^{\overline{\rm DR}} = -\frac{1}{16}C_F N_f \left(\frac{1}{2}C_A - C_F - \frac{1}{4}N_f\right)\,,$$

where the notation is that

$$\beta^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}}) = -\varepsilon\frac{\alpha_s^{\overline{\rm DR}}}{\pi} - \sum_{i,j,k,l,m} \beta_{ijklm}^{\overline{\rm DR}} \left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i \left(\frac{\alpha_e^{\overline{\rm DR}}}{\pi}\right)^j \left(\frac{\eta_1^{\overline{\rm DR}}}{\pi}\right)^k \left(\frac{\eta_2^{\overline{\rm DR}}}{\pi}\right)^l \left(\frac{\eta_3^{\overline{\rm DR}}}{\pi}\right)^m. \tag{B2}$$

The last three indices of $\beta_{ijklm}^{\overline{\rm DR}}$ are omitted when they are all equal to 0.

The β-function of evanescent QCD coupling, $\beta_{e,}^{\overline{\rm DR}}(\alpha_e^{\overline{\rm DR}})$ is

$$\beta_{e,02}^{\overline{\rm DR}} = \frac{1}{2}C_A - C_F - \frac{1}{4}N_f\,, \qquad \beta_{e,11}^{\overline{\rm DR}} = \frac{3}{2}C_F\,,$$
$$\beta_{e,03}^{\overline{\rm DR}} = \frac{3}{8}C_A^2 - \frac{5}{4}C_AC_F + C_F^2 - \frac{3}{16}C_AN_f + \frac{3}{8}C_FN_f\,, \qquad \beta_{e,12}^{\overline{\rm DR}} = -\frac{3}{8}C_A^2 + \frac{5}{2}C_AC_F - \frac{11}{4}C_F^2 - \frac{5}{16}C_FN_f\,,$$
$$\beta_{e,21}^{\overline{\rm DR}} = -\frac{7}{64}C_A^2 + \frac{55}{48}C_AC_F + \frac{3}{16}C_F^2 + \frac{1}{16}C_AN_f - \frac{5}{24}C_FN_f$$
$$\beta_{e,02100}^{\overline{\rm DR}} = -\frac{9}{8} \qquad \beta_{e,02010}^{\overline{\rm DR}} = \frac{5}{4} \qquad \beta_{e,02001}^{\overline{\rm DR}} = \frac{3}{4}$$
$$\beta_{e,01200}^{\overline{\rm DR}} = \frac{27}{64} \qquad \beta_{e,01020}^{\overline{\rm DR}} = -\frac{15}{4} \qquad \beta_{e,01002}^{\overline{\rm DR}} = -\frac{21}{32} \qquad \beta_{e,01101}^{\overline{\rm DR}} = -\frac{9}{16}$$
$$\beta_{e,04}^{\overline{\rm DR}} = -\left(\frac{7}{4} + \frac{9}{4}\zeta_3\right)C_F^3 + \left(\frac{17}{8} + \frac{15}{2}\zeta_3\right)C_F^2C_A - \left(\frac{3}{4} + \frac{69}{16}\zeta_3\right)C_FC_A^2 + \left(\frac{1}{16} + \frac{9}{16}\zeta_3\right)C_A^3$$
$$+ \left(\frac{13}{32} - \frac{33}{16}\zeta_3\right)C_F^2N_f + \left(\frac{1}{32} + \frac{51}{32}\zeta_3\right)C_FC_AN_f - \left(\frac{21}{128} + \frac{9}{32}\zeta_3\right)C_A^2N_f - \left(\frac{1}{128}C_F - \frac{7}{256}C_A\right)N_f^2$$
$$\beta_{e,13}^{\overline{\rm DR}} = \left(\frac{13}{2} - 3\,\zeta_3\right)C_F^3 - (10 - 6\,\zeta_3)\,C_F^2C_A + \left(\frac{133}{32} - \frac{15}{4}\zeta_3\right)C_FC_A^2 - \left(\frac{25}{64} - \frac{3}{4}\zeta_3\right)C_A^3$$
$$+ \left(\frac{13}{16} - \frac{3}{4}\zeta_3\right)C_F^2N_f - \frac{9}{8}(1 - \zeta_3)\,C_FC_AN_f + \left(\frac{7}{32} - \frac{3}{8}\zeta_3\right)C_A^2N_f + \frac{3}{64}C_AN_f^2$$
$$\beta_{e,22}^{\overline{\rm DR}} = -\left(\frac{139}{64} - \frac{27}{4}\zeta_3\right)C_F^3 - \left(\frac{793}{128} + 18\,\zeta_3\right)C_F^2C_A + \left(\frac{1587}{256} + \frac{207}{16}\zeta_3\right)C_FC_A^2 - \left(\frac{427}{512} + \frac{45}{16}\zeta_3\right)C_A^3$$
$$- \left(\frac{569}{256} - \frac{99}{16}\zeta_3\right)C_F^2N_f + \left(\frac{31}{16} - \frac{171}{32}\zeta_3\right)C_FC_AN_f - \left(\frac{871}{1024} - \frac{45}{32}\zeta_3\right)C_A^2N_f + \left(\frac{1}{16}C_F - \frac{1}{256}C_A\right)N_f^2$$
$$\beta_{e,31}^{\overline{\rm DR}} = \frac{129}{64}C_F^3 - \frac{457}{128}C_F^2C_A + \frac{11875}{3456}C_FC_A^2 - \frac{3073}{4608}C_A^3$$
$$- \left(\frac{23}{32} - \frac{3}{4}\zeta_3\right)C_F^2N_f - \left(\frac{157}{1728} + \frac{3}{4}\zeta_3\right)C_FC_AN_f + \frac{463}{2304}C_A^2N_f - \left(\frac{35}{864}C_F + \frac{5}{576}C_A\right)N_f^2$$

(B3)

$$\beta_{e,03100}^{\overline{\rm DR}} = -\frac{9}{64} + \frac{243}{128}N_f \qquad \beta_{e,03010}^{\overline{\rm DR}} = \frac{5}{8} - \frac{45}{64}N_f \qquad \beta_{e,03001}^{\overline{\rm DR}} = \frac{3}{32} - \frac{81}{64}N_f$$
$$\beta_{e,12100}^{\overline{\rm DR}} = -\frac{219}{16} \qquad \beta_{e,12010}^{\overline{\rm DR}} = \frac{145}{48} \qquad \beta_{e,12001}^{\overline{\rm DR}} = \frac{73}{8}$$
$$\beta_{e,21100}^{\overline{\rm DR}} = -\frac{1125}{1024} \qquad \beta_{e,21010}^{\overline{\rm DR}} = \frac{105}{128} \qquad \beta_{e,21001}^{\overline{\rm DR}} = \frac{615}{512}$$
$$\beta_{e,02200}^{\overline{\rm DR}} = \frac{1413}{512} - \frac{729}{1024}N_f \qquad \beta_{e,02020}^{\overline{\rm DR}} = -\frac{115}{32} + \frac{135}{64}N_f \qquad \beta_{e,02002}^{\overline{\rm DR}} = -\frac{161}{256} - \frac{567}{512}N_f$$
$$\beta_{e,02110}^{\overline{\rm DR}} = \frac{75}{8} \qquad \beta_{e,02101}^{\overline{\rm DR}} = -\frac{471}{128} + \frac{243}{256}N_f \qquad \beta_{e,02011}^{\overline{\rm DR}} = -\frac{85}{8}$$
$$\beta_{e,01300}^{\overline{\rm DR}} = -\frac{1701}{1024} \qquad \beta_{e,01210}^{\overline{\rm DR}} = -\frac{405}{128} \qquad \beta_{e,01201}^{\overline{\rm DR}} = \frac{1701}{512}$$
$$\beta_{e,01120}^{\overline{\rm DR}} = \frac{135}{32} \qquad \beta_{e,01111}^{\overline{\rm DR}} = \frac{135}{16} \qquad \beta_{e,01102}^{\overline{\rm DR}} = -\frac{81}{128}$$
$$\beta_{e,01021}^{\overline{\rm DR}} = -\frac{315}{32} \qquad \beta_{e,01012}^{\overline{\rm DR}} = -\frac{315}{32} \qquad \beta_{e,01003}^{\overline{\rm DR}} = \frac{63}{128}$$

The mass anomalous dimension in the DRED scheme is

$$\gamma^{\overline{\rm DR}}_{10} = \frac{3}{4}C_F$$

$$\gamma^{\overline{\rm DR}}_{20} = \frac{3}{32}C_F^2 + \frac{91}{96}C_A C_F - \frac{5}{48}C_F N_f \qquad \gamma^{\overline{\rm DR}}_{11} = -\frac{3}{8}C_F^2 \qquad \gamma^{\overline{\rm DR}}_{02} = \frac{1}{4}C_F^2 - \frac{1}{8}C_A C_F + \frac{1}{16}C_F N_f$$

$$\gamma^{\overline{\rm DR}}_{30} = \frac{129}{128}C_F^3 - \frac{133}{256}C_F^2 C_A + \frac{10255}{6912}C_F C_A^2 - \left(\frac{23}{64} - \frac{3}{8}\zeta_3\right) C_F^2 N_f - \left(\frac{281}{1728} - \frac{3}{8}\zeta_3\right) C_A C_F N_f - \frac{35}{1728}C_F N_f^2$$

$$\gamma^{\overline{\rm DR}}_{21} = -\frac{27}{64}C_F^3 - \frac{21}{32}C_F^2 C_A - \frac{15}{256}C_F C_A^2 + \frac{9}{64}C_F^2 N_f$$

$$\gamma^{\overline{\rm DR}}_{12} = \frac{9}{8}C_F^3 - \frac{21}{32}C_F^2 C_A + \frac{3}{64}C_F C_A^2 + \frac{3}{128}C_F C_A N_f + \frac{3}{16}C_F^2 N_f$$

$$\gamma^{\overline{\rm DR}}_{03} = -\frac{3}{8}C_F^3 + \frac{3}{8}C_F^2 C_A - \frac{3}{32}C_F C_A^2 + \frac{1}{16}C_F C_A N_f - \frac{5}{32}C_F^2 N_f - \frac{1}{128}C_F N_f^2$$

$$\gamma^{\overline{\rm DR}}_{02100} = \frac{3}{8} \qquad \gamma^{\overline{\rm DR}}_{02010} = -\frac{5}{12} \qquad \gamma^{\overline{\rm DR}}_{02001} = -\frac{1}{4}$$

$$\gamma^{\overline{\rm DR}}_{01200} = -\frac{9}{64} \qquad \gamma^{\overline{\rm DR}}_{01101} = \frac{3}{16} \qquad \gamma^{\overline{\rm DR}}_{01020} = \frac{5}{4} \qquad \gamma^{\overline{\rm DR}}_{01002} = -\frac{7}{32} \tag{B4}$$

The above results for $\beta^{\overline{\rm DR}}$, $\beta^{\overline{\rm DR}}_{e,}$ and $\gamma^{\overline{\rm DR}}$ all agree with the results of Refs. [23, 24]

The QCD contributions to the β-function of the evanescent part of a non-QCD gauge coupling is a new result. I find

$$\beta^{\overline{\rm DR}}_{Ve,10} = \frac{3}{2}C_F \qquad \beta^{\overline{\rm DR}}_{Ve,01} = -C_F$$

$$\beta^{\overline{\rm DR}}_{Ve,20} = \frac{3}{16}C_F^2 + \frac{91}{48}C_F C_A - \frac{5}{24}C_F N_f \qquad \beta^{\overline{\rm DR}}_{Ve,11} = -\frac{11}{4}C_F^2 - \frac{3}{4}C_F C_A \qquad \beta^{\overline{\rm DR}}_{Ve,02} = C_F^2 + \frac{3}{8}C_F N_f$$

$$\beta^{\overline{\rm DR}}_{Ve,30} = \frac{129}{64}C_F^3 - \frac{133}{128}C_F^2 C_A - \left(\frac{23}{32} - \frac{3}{4}\zeta_3\right) C_F^2 N_f + \frac{10255}{3456}C_F C_A^2 - \left(\frac{281}{864} + \frac{3}{4}\zeta_3\right) C_F C_A N_f - \frac{35}{864}C_F N_f^2$$

$$\beta^{\overline{\rm DR}}_{Ve,21} = -\left(\frac{139}{64} - \frac{27}{4}\zeta_3\right) C_F^3 - \left(\frac{331}{64} + \frac{81}{8}\zeta_3\right) C_F^2 C_A + \frac{11}{16}C_F^2 N_f - \left(\frac{195}{256} - \frac{27}{8}\zeta_3\right) C_F C_A^2 + \frac{5}{64}C_F C_A N_f$$

$$\beta^{\overline{\rm DR}}_{Ve,12} = \left(\frac{13}{2} - 3\zeta_3\right) C_F^3 - \left(\frac{7}{8} - \frac{9}{2}\zeta_3\right) C_F^2 C_A + \left(\frac{63}{64} - \frac{3}{4}\zeta_3\right) C_F^2 N_f + \left(\frac{7}{16} - \frac{3}{2}\zeta_3\right) C_F C_A^2 - \left(\frac{3}{64} - \frac{3}{4}\zeta_3\right) C_F C_A N_f$$

$$\beta^{\overline{\rm DR}}_{Ve,03} = -\left(\frac{7}{4} + \frac{9}{4}\zeta_3\right) C_F^3 + \left(\frac{1}{8} + \frac{27}{8}\zeta_3\right) C_F^2 C_A - \frac{27}{32}C_F^2 N_f + \left(\frac{1}{16} - \frac{9}{8}\zeta_3\right) C_F C_A^2 + \frac{3}{64}C_F C_A N_f + \frac{3}{64}C_F N_f^2$$

$$\beta^{\overline{\rm DR}}_{Ve,02100} = \frac{3}{8} \qquad \beta^{\overline{\rm DR}}_{Ve,02010} = -\frac{25}{6} \qquad \beta^{\overline{\rm DR}}_{Ve,02001} = -\frac{1}{4}$$

$$\beta^{\overline{\rm DR}}_{Ve,01200} = -\frac{63}{64} \qquad \beta^{\overline{\rm DR}}_{Ve,01101} = \frac{21}{16} \qquad \beta^{\overline{\rm DR}}_{Ve,01020} = \frac{65}{4} \qquad \beta^{\overline{\rm DR}}_{Ve,01002} = -\frac{49}{32} \tag{B5}$$

[1] G. ’t Hooft and M. J. G. Veltman, Nucl. Phys. **B44**, 189 (1972).

[2] J. Collins, *Renormalization* (Cambridge University Press, Cambridge, England, 1984).

[3] W. Siegel, Phys. Lett. **B84**, 193 (1979).

[4] Z. Bern and D. A. Kosower, Nucl. Phys. **B379**, 451 (1992).

[5] Z. Bern, A. De Freitas, L. J. Dixon, and H. L. Wong, Phys. Rev. **D66**, 085002 (2002), hep-ph/0202271.

[6] R. van Damme and G. 't Hooft, Phys. Lett. **B150**, 133 (1985).

[7] D. M. Capper, D. R. T. Jones, and P. van Nieuwenhuizen, Nucl. Phys. **B167**, 479 (1980).

[8] I. Jack, D. R. T. Jones, and K. L. Roberts, Z. Phys. **C62**, 161 (1994), hep-ph/9310301.

[9] I. Jack, D. R. T. Jones, and K. L. Roberts, Z. Phys. **C63**, 151 (1994), hep-ph/9401349.

[10] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **B85**, 277 (1979).

[11] M. Dine and J. R. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979).

[12] W. Celmaster and R. J. Gonsalves, Phys. Rev. **D21**, 3112 (1980).

[13] S. G. Gorishnii, A. L. Kataev, and S. A. Larin, Phys. Lett. **B212**, 238 (1988).

[14] S. G. Gorishnii, A. L. Kataev, and S. A. Larin, Phys. Lett. **B259**, 144 (1991).

[15] P. Nogueira, J. Comput. Phys. **105**, 279 (1993).

[16] J. A. M. Vermaseren (2000), Report No. NIKHEF-00-0032, math-ph/0010025.

[17] C. Studerus, Comput. Phys. Commun. **181**, 1293 (2010), 0912.2546.

[18] A. I. Davydychev, P. Osland, and O. Tarasov, Phys.Rev. **D58**, 036007 (1998), hep-ph/9801380.

[19] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Nucl. Phys. **B174**, 345 (1980).

[20] D. I. Kazakov, Theor. Math. Phys. **58**, 223 (1984).

[21] S. G. Gorishnii, S. A. Larin, L. R. Surguladze, and F. V. Tkachov, Comput. Phys. Commun. **55**, 381 (1989).

[22] S. A. Larin, F. V. Tkachov, and J. A. M. Vermaseren (1991), Report No. NIKHEF-H-91-18.

[23] R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila, and M. Steinhauser, JHEP **12**, 024 (2006), hep-ph/0610206.

[24] R. Harlander, P. Kant, L. Mihaila, and M. Steinhauser, JHEP **09**, 053 (2006), hep-ph/0607240.

[25] Z. Kunszt, A. Signer, and Z. Trocsanyi, Nucl.Phys. **B411**, 397 (1994), hep-ph/9305239.

[26] P. Baikov and K. Chetyrkin, Nucl.Phys. **B837**, 186 (2010), in memoriam Sergei Grigorievich Gorishny, 1958-1988, 1004.1153.

[27] W. Siegel, Phys.Lett. **B94**, 37 (1980).

[28] D. Stockinger, JHEP **0503**, 076 (2005), hep-ph/0503129.

[29] A. Signer and D. Stockinger, Nucl. Phys. **B808**, 88 (2009), 0807.4424.

[30] O. Tarasov, A. Vladimirov, and A. Zharkov, Phys.Lett. **B93**, 429 (1980).

[31] S. Larin and J. Vermaseren, Phys.Lett. **B303**, 334 (1993), hep-ph/9302208.

[32] K. G. Chetyrkin, Phys. Lett. **B404**, 161 (1997), hep-ph/9703278.

[33] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. **B405**, 327 (1997), hep-ph/9703284.