Non-Abelian duality and confinement: From N=2 to N=1 supersymmetric QCD

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Non-Abelian Duality and Confinement: from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ Supersymmetric QCD

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Abstract

Recently we discovered and discussed non-Abelian duality in the *quark vacua* of $\mathcal{N} = 2$ super-Yang–Mills theory with the U(N) gauge group and N_f flavors ($N_f > N$). Both theories from the dual pair support non-Abelian strings which confine monopoles. Now we introduce an $\mathcal{N} = 2$ -breaking deformation, a mass term $\mu \mathcal{A}^2$ for the adjoint fields. Starting from a small deformation we eventually make it large which enforces complete decoupling of the adjoint fields. We show that the above non-Abelian duality fully survives in the limit of $\mathcal{N} = 1$ SQCD, albeit some technicalities change. For instance, non-Abelian strings which used to be BPS-saturated in the $\mathcal{N} = 2$ limit, cease to be saturated in $\mathcal{N} = 1$ SQCD. Our duality is a distant relative of Seiberg's duality in $\mathcal{N} = 1$ SQCD. Both share some common features but have many drastic distinctions. This is due to the fact that Seiberg's duality apply to the monopole rather than quark vacua.

More specifically, in our theory we deal with $N < N_f < \frac{3}{2}N$ *massive* quark flavors. We consider the vacuum in which N squarks condense. Then we identify a crossover transition from weak to strong coupling. At strong coupling we find a dual theory, U($N_f - N$) SQCD, with N_f light dyon flavors. Dyons condense triggering the formation of non-Abelian strings which confine monopoles. Screened quarks and gauge bosons of the original theory decay into confined monopole-antimonopole pairs and form stringy mesons.

1 Introduction and setting the goal: from $\mathcal{N} = 2$ to $\mathcal{N} = 1$

The dual Meissner effect as the confinement mechanism [1] in Yang–Mills theories remains obscure despite a remarkable breakthrough in $\mathcal{N} = 2$ supersymmetric theories, where the exact Seiberg–Witten solution was found [2, 3]. Seiberg and Witten demonstrated [2, 3] that magnetic monopoles do condense in the so-called monopole vacua of the $\mathcal{N} = 2$ theory after one switches on a small $\mathcal{N} = 2$ -breaking deformation of the $\mu \mathcal{A}^2$ type. Upon condensation of the monopoles, chromoelectric flux tubes (strings) of the Abrikosov–Nielsen–Olesen (ANO) type [4] are formed. This leads to confinement of (probe) quarks attached to the endpoints of confining strings.

The Seiberg–Witten mechanism of confinement is essentially Abelian [1] [5, 6, 7, 8, 9]. This is due to the fact that in the Seiberg–Witten solution the non-Abelian gauge group of the underlying $\mathcal{N} = 2$ theory (say, SU(N)) is broken down to the Abelian subgroup U(1)$^{N-1}$ by condensation (in the strongly coupled monopole vacua) of the *adjoint scalars* inherent to $\mathcal{N} = 2$. The subsequent monopole condensation occurs essentially in the Abelian U(1)$^{N-1}$ theory. This feature makes the $\mathcal{N} = 2$ theory dissimilar from pure Yang–Mills, in which there is no dynamical Abelianization. Hence, to get closer to reality, a natural desire arises to eliminate the adjoint scalars, passing if not to $\mathcal{N} = 0$, at least to $\mathcal{N} = 1$. That's what we will eventually do.

However, $\mathcal{N} = 1$ theories do not support monopoles (dyons), at least at the quasiclassical level, and the very meaning of the dual Meissner effect gets obscure. In search of a non-Abelian confinement mechanism similar in spirit to the Meissner mechanism of Nambu, 't Hooft, and Mandelstam we recently explored a different, albeit related, scenario [10, 11]. To begin with, we focused on the quark (rather than monopole) vacuum of $\mathcal{N} = 2$ supersymmetric QCD (SQCD) with the U(N) gauge group (rather than SU(N)) and N_f flavors of fundamental quark hypermultiplets, with N_f in the interval $N < N_f < 2N$. Then, there is no confinement of the chromoelectric charges; on the contrary, they are Higgs-screened. Instead, the

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[1] By non-Abelian confinement we mean such dynamical regime in which at distances of the flux tube formation all gauge bosons are equally important, while the Abelian confinement occurs when the relevant gauge dynamics at such distances is Abelian. Note that Abelian confinement can take place in non-Abelian theories, the Seiberg–Witten solution is just one example.
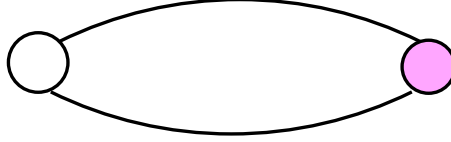
Figure 1: Meson formed by monopole-antimonopole pair connected by two strings. Open and closed circles denote the monopole and antimonopole, respectively.

chromomagnetic charges are confined by *non-Abelian* strings. They — the chromomagnetic charges — manifest themselves in a clear-cut manner as junctions of two nonidentical, albeit degenerate, strings. Moreover, at strong coupling (where, as we will see, a dual description is applicable) the quarks and gauge bosons of the original theory decay into monopole-antimonopole pairs on the curves of marginal stability (CMS). The (anti)monopoles forming the pair are confined. In other words, the original quarks and gauge bosons evolve in the strong coupling domain into "stringy mesons" with two constituents being connected by two strings as shown in Fig. 1. These mesons are expected to lie on Regge trajectories.

All these phenomena take place in the quark vacua of the $\mathcal{N} = 2$ theory [10, 11]. This theory is slightly different from the Seiberg–Witten model. Namely, as was mentioned, the U(1) gauge factor is added, and, then, the Fayet–Iliopoulos (FI) [12] D-term ξ is introduced. Then, we single out the vacuum in which $r = N$ (s)quarks condense. A global color-flavor locked symmetry survives in the limit of equal quark mass terms. At large ξ this theory is at weak coupling and supports non-Abelian flux tubes (strings) [13, 14, 15, 16] (see also [17, 18, 19, 20] for reviews). It is the formation of these strings that ensures confinement of monopoles. Upon reducing the FI parameter ξ, the theory goes through a crossover transition [10, 21, 22] into a strongly coupled regime which can be described in terms of weakly coupled *dual* $\mathcal{N} = 2$ SQCD, with the U($\tilde{N}$)$\times$U(1)$^{N-\tilde{N}}$ gauge group and N_f flavors of light *dyons*.[2] Here

$$\tilde{N} = N_f - N \,, \tag{1.1}$$

as in Seiberg's duality in $\mathcal{N} = 1$ theories [24, 25] where the emergence of the dual SU($\tilde{N}$) group was first observed.

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[2]This is in perfect agreement with the results obtained in [23] where the SU($\tilde{N}$) dual gauge group was identified at the root of the baryonic Higgs branch in the SU(N) gauge theory with massless (s)quarks.

The dual theory supports non-Abelian strings due to condensation of light dyons much in the same way as the string formation in the original theory is due to condensation of quarks. Moreover, the number of distinct strings is, of course, the same in the original and dual theories. The strings of the dual theory confine monopoles too, rather than quarks [10]. This is due to the fact that the light dyons condensing in the dual theory carry weight-like chromoelectric charges (in addition to chromomagnetic charges). In other words, they carry the quark charges. The strings formed through condensation of these dyons can confine only states with the root-like magnetic charges, i.e. the monopoles [10]. Thus, our $\mathcal{N} = 2$ non-Abelian duality is *not* electromagnetic.

The chromoelectric charges of quarks (or gauge bosons) are Higgs-screened a large ξ. As was mentioned above, in the domain of small ξ (where the dual description is applicable) these states decay into the monopole-antimonopole pairs on CMS, see [11] for the proof of this fact. The monopoles and antimonopoles forming the pair cannot abandon each other because they are confined. Therefore, the original quarks and gauge bosons, with the passage to the strong coupling domain of small ξ, evolve into "stringy mesons" shown in Fig. 1. A detailed discussion of these stringy mesons can be found in [19]. The same picture takes place when we move in the opposite direction, with the interchange of two theories from the dual pair.

Deep in the non-Abelian quantum regime the confined monopoles were demonstrated [11] to belong to the *fundamental representation* of the global (color-flavor locked) group. Therefore, the monopole-antimonopole mesons can be both, in the adjoint and singlet representation of this group. This pattern of confinement seems to be similar to what we have in actuality, except that the role of the "constituent quarks" inside mesons is played by the monopoles.

Low-energy dynamics on the world sheet of the non-Abelian strings under discussion are described by two-dimensional CP models [13, 14, 15, 16]. From the world-sheet standpoint different (degenerate) non-Abelian strings are different supersymmetric vacua of the CP models. Confined monopoles are in fact kinks interpolating between these vacua. Nonperturbative generation of the dynamical scale Λ_{CP} stabilizes the kink inverse sizes and masses at $O(\Lambda_{\text{CP}})$ [15, 19]. This is in contradistinction with the absence of *classical* stabilization of monopoles in the non-Abelian regime (see e.g the discussion

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of the so-called "monopole clouds" in [26]).[3]

In this paper we report on the second stage of the program, namely the study of non-Abelian duality in the *absence* of the adjoint fields, in $\mathcal{N} = 1$ SQCD. To pass from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ we add a deformation term $\mu \mathcal{A}^2$ in the superpotential. We show that the picture of the non-Abelian monopole confinement outlined above for $\mathcal{N} = 2$ survives this deformation all the way up to large μ where the adjoint fields decouple leaving us with $\mathcal{N} = 1$ SQCD.

We start our work from $\mathcal{N} = 2$ SQCD with the U(N) gauge group and N_f *massive* quark flavors where

$$N < N_f < \frac{3}{2}N \tag{1.2}$$

to ensure infrared freedom in the dual theory at large μ. Since the deformation superpotential (2.2) plays the role of an effective FI term (being combined with the nonvanishing quark mass terms), there is no need to introduce the FI term through the D term. Although it is certainly doable, this would be nothing but an unnecessary complication.

At small μ the deformation superpotential (2.2) reduces to the Fayet–Iliopoulos F-term with the effective FI parameter ξ determined by $\xi \sim \sqrt{\mu m}$, where m presents a typical scale of the quark masses. We focus on the so-called r vacuum in which $r = N$ quarks condense, with the subsequent formation of the non-Abelian strings which confine monopoles. Much in the same way as in our previous $\mathcal{N} = 2$ studies [10, 11] with the Fayet–Iliopoulos D-term, there is a crossover transition in ξ. It takes place at the boundary of the weak and strong coupling domains. At strong coupling occurring as one reduces $\sqrt{m\mu}$, a dual description applies, in terms of a weakly coupled non-Abelian infrared free SQCD with the dual gauge group U($\tilde{N}$)$\times$U(1)$^{N-\tilde{N}}$ and N_f light dyon flavors. The dual gauge group is Higgsed too (with a global color-flavor locked symmetry preserved) and supports non-Abelian strings. These strings still confine *monopoles* rather than quarks.

Next, we increase the deformation parameter μ decoupling the adjoint fields and sending the original theory to the limit of $\mathcal{N} = 1$ SQCD. At large μ the dual theory is demonstrated to be weakly coupled and infrared free,

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[3]To better explain this statement we should point out that, say, in the monopole vacua of $\mathcal{N} = 2$ SQCD, the Seiberg–Witten solution tells us [2, 3] that the theory dynamically Abelianizes. That's why it is so difficult to apply the standard confinement scenario, based on monopole condensation, to theories without adjoint scalars. We just do not know what does that mean, non-Abelian monopole in the Higgs/Coulomb phase.

with the U($\tilde{N}$) gauge group and N_f light dyons D^{lA}, ($l = 1, ...\tilde{N}$ is the color index in the dual gauge group, while $A = 1, ..., N_f$ is the flavor index). Our proof is valid provided the dyon condensate $\sim \xi \sim \sqrt{\mu m}$ is small enough. Non-Abelian strings (albeit this time non-BPS saturated) are formed which confine monopoles — qualitatively the same type of confinement as in the $\mathcal{N} = 2$ duality [10, 11]. In the domain of small $\sqrt{\mu m}$ quark and gauge bosons of original $\mathcal{N} = 1$ SQCD are presented by stringy mesons built from the monopole-antimonopoles pairs connected by two non-Abelian strings, see Fig. 1.

An interesting aspect, to be discussed in the bulk of the paper, is the relationship of our duality with that of Seiberg. To make ourselves clear in this point we should undertake a small digression in the issue of vacua.

$\mathcal{N} = 1$ SQCD with N_f flavors ($N + 1 \leq N_f < \frac{3}{2}N$) has a large number of distinct vacua. We need to classify them. To this end we can invoke our knowledge of the vacuum structure in related theories, such as $\mathcal{N} = 2$ SQCD, which is controlled by the exact Seiberg–Witten solution [2, 3].

Let us turn to the latter. Among others, it has N supersymmetric vacua which are generically referred to as the "monopole vacua." The gauge symmetry is spontaneously broken down to U(1)$^{N-1}$ in these vacua,[4] and the subsequent switch-on of a small-μ deformation leads to the monopole condensation (in fact, in some of these vacua it is dyons that condense), the (Abelian) flux tube formation and confinement of (probe) quarks. As μ grows and eventually becomes large, the adjoint fields of the $\mathcal{N} = 2$ theory decouple, and we are left with $\mathcal{N} = 1$ SQCD. The N monopole vacua go though a crossover transition into a non-Abelian phase. We will say that the above vacua evolve and become the monopole vacua of $\mathcal{N} = 1$ SQCD. The name "monopole" is symbolic. We just continue to refer in this way to the vacua which used to be the monopole vacua of the $\mathcal{N} = 2$ Seiberg–Witten theory at small μ, into the domain of large μ where the Seiberg–Witten control over dynamics is lost.

At large μ we recover $\mathcal{N} = 1$ SQCD. The N monopole vacua are those in which Seiberg's duality was established [24, 25]. If the quark fields of the electric theory are endowed with small masses to lift the continuous vacuum manifold, in the Seiberg magnetic dual theories the meson field M condenses. Since it is singlet with respect to the dual color gauge groups SU($\tilde{N}$), this

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[4]In our theory we will have U(1)^N, since instead of the Seiberg–Witten SU(N) group we work with U(N).

gauge group remains unbroken. We stress that the Seiberg M condensation occurs in the vacua of the dual theory if in the original electric theory we stay in the monopole vacua. There is no obvious connection between M and the monopole fields which — the monopole fields — are not defined at all in this set up.[5]

Our task is to explore dualities in $\mathcal{N} = 1$ SQCD in the vacua other than the N monopole vacua. For a deeper understanding of the problem we start, however, from the *quark vacua* of the $\mathcal{N} = 2$ Seiberg–Witten theory (with addition of the the U(1) gauge group and the corresponding Fayet–Iliopoulos term [12]). This was the beginning of our program of duality explorations, the $\mathcal{N} = 2$ limit [10, 11, 21, 22, 19]. In this paper we report the study of the μ-deformation leading us away from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ SQCD, remaining in the quark vacua.

Now, turning to the relation between our duality (plus monopole confinement) and that of Seiberg [24, 25], we observe that the light dyons D^{lA} of our U($\tilde{N}$) dual theory are simultaneously similar to and dissimilar from Seiberg's "dual quarks." They have the same quantum numbers, but dynamics are different. One can conjecture that, in fact, Seiberg's dual quarks is a different-phase implementation of the dyons D^{lA}. If so everything else becomes clear. Indeed, in quantum numbers, the stringy mesons formed from the monopole-antimonopole pairs correspond to Seiberg's neutral mesons M_A^B $(A, B = 1, ..., N_f)$. Both incorporate the singlet and adjoint representations of the global flavor group. The difference is that in our dual theory these stringy mesons are nonperturbative objects which are rather heavy in the weak coupling regime of the dual theory. This is in sharp contrast with the fact that, in Seiberg's dual theory, the M_A^B mesons appear as fundamental fields at the Lagrangian level and are light.

The explanation for this dynamical differences was, in fact, given above: Seiberg's duality refers to the monopole vacua while ours to the quark vacua

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[5]A side remark which will not be elaborated below is in order here. In $\mathcal{N} = 1$ SQCD with nonvanishing quark masses the Intriligator–Seiberg–Shih (ISS) vacuum was detected in 2006 [27, 28]. With a generic set of the mass terms the ISS vacuum is non-supersymmetric (i.e. its energy is lifted from zero). Given a special choice of the mass parameters it can be made supersymmetric at the *classical* level. Then, quantum corrections will lift it from zero, albeit the breaking can be small. In [29] we considered non-Abelian strings and their junctions in the ISS-like vacuum of $\mathcal{N} = 1$ SQCD. Finally, once we started speaking of Seiberg's duality beyond the original Seiberg's duality, we cannot help mentioning an inspiring paper of Komargodski [30].

(of the $r = N$ type). Dyons D^{lA} do not condense in the monopole vacua, and the (infrared free) dual theory is in the Coulomb phase. In our vacua, at strong coupling (weak coupling in the dual theory), the light dyons condense, triggering formation of the non-Abelian strings and, as a result, the confinement of monopoles. The dyon condensate is proportional to $\sqrt{\mu m}$ and represents a would-be run-away vacuum not seen in the Seiberg dual description, where μ is considered to be strictly infinite (see Fig. 3 in Section 6).

Concluding the introductory section we reiterate that the overall picture of duality we obtained in previous works [10, 11] in $\mathcal{N} = 2$ theories survives the passage to $\mathcal{N} = 1$. Some details change, for instance, the strings cease to be BPS-saturated (correspondingly, supersymmetry on the string world sheet is lost at the classical level). Nevertheless, the general pattern of the phenomenon stays *intact*.

The paper is organized as follows. In Sec. 2 we outline our basic setup, $\mathcal{N} = 2$ SQCD. Then we introduce a small deformation parameter μ and briefly review non-Abelian duality observed in [10, 11]. In Sec. 3 we describe how the quarks and gauge bosons of the original theory pass into the monopole-antimonopole pairs, stringy mesons, in the crossover domain. In Sec. 4 we increase μ eventually decoupling gauge singlets and adjoint scalars of the dual theory. This is the limit of $\mathcal{N} = 1$ SQCD. Section 5 is devoted to formation of non-Abelian strings and monopole-antimonopole mesons in the $\mathcal{N} = 1$ theory. Then we proceed to duality in the quark vacua of the $\mathcal{N} = 1$ theory. In Sec. 6 we compare our picture to that of Seiberg. Finally, Sec. 7 summarizes our results and conclusions. In Appendix we treat technical details of the U(3) model with $N_f = 5$ at small μ.

2   Basic theory at small μ

This section presents our basic setup at small μ, i.e near the $\mathcal{N} = 2$ limit.

The gauge symmetry of the basic bulk model is U(N)=SU(N)×U(1). In the absence of deformation the model under consideration is $\mathcal{N} = 2$ SQCD with N_f massive quark hypermultiplets. We assume that $N_f > N$ but $N_f < \frac{3}{2}N$, see Eq. (1.2). The latter inequality ensures that the dual theory is *not* asymptotically free.

In addition, we will introduce the mass term μ for the adjoint matter breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$.

The field content is as follows. The $\mathcal{N} = 2$ vector multiplet consists of the U(1) gauge field A_μ and the SU(N) gauge field A_μ^a, where $a = 1, ..., N^2 - 1$, and their Weyl fermion superpartners plus complex scalar fields a, and a^a and their Weyl superpartners, respectively. The N_f quark multiplets of the U(N) theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners — all in the fundamental representation of the SU(N) gauge group. Here $k = 1, ..., N$ is the color index while A is the flavor index, $A = 1, ..., N_f$. We will treat q^{kA} and $\tilde{q}_{Ak}$ as rectangular matrices with N rows and N_f columns.

Let us first discuss the undeformed $\mathcal{N} = 2$ theory. The superpotential has the form

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2} \sum_{A=1}^{N_f} \left(\frac{1}{2} \tilde{q}_A \mathcal{A} q^A + \tilde{q}_A \mathcal{A}^a T^a q^A \right) , \tag{2.1}$$

where $\mathcal{A}$ and $\mathcal{A}^a$ are chiral superfields, the $\mathcal{N} = 2$ superpartners of the gauge bosons of U(1) and SU(N), respectively.

Next, we add the mass term for the adjoint fields which breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$,

$$\mathcal{W}_{[\mu]} = \sqrt{\frac{N}{2}} \frac{\mu_1}{2} \mathcal{A}^2 + \frac{\mu_2}{2} (\mathcal{A}^a)^2, \tag{2.2}$$

where μ_1 and μ_2 is are mass parameters for the chiral superfields in $\mathcal{N} = 2$ gauge supermultiplets, U(1) and SU(N), respectively. Generally speaking μ_1 need not coincide with μ_2, but we will assume these parameters to be of the same order of magnitude and will generically denote them as μ. Clearly, the mass term (2.2) splits the $\mathcal{N} = 2$ supermultiplets, breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. First we assume that μ is small, much smaller than the quark masses m_A,

$$\mu \ll m_A, \qquad A = 1, ..., N_f . \tag{2.3}$$

The bosonic part of the action of our basic theory has the form (for details see [19])

$$
\begin{aligned}
S = & \int d^4x \left[\frac{1}{4g_2^2} \left(F_{\mu\nu}^a \right)^2 + \frac{1}{4g_1^2} \left(F_{\mu\nu} \right)^2 + \frac{1}{g_2^2} |D_\mu a^a|^2 + \frac{1}{g_1^2} |\partial_\mu a|^2 \right. \\
& \left. + \left| \nabla_\mu q^A \right|^2 + \left| \nabla_\mu \bar{\tilde{q}}^A \right|^2 + V(q^A, \tilde{q}_A, a^a, a) \right] .
\end{aligned}
\tag{2.4}
$$

Here D_μ is the covariant derivative in the adjoint representation of SU(N), while

$$\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu - i A_\mu^a T^a \qquad (2.5)$$

acts in the fundamental representation. We suppress the color SU(N) indices of the matter fields. The normalization of the SU(N) generators T^a is as follows

$$\mathrm{Tr}\,(T^a T^b) = \tfrac{1}{2}\,\delta^{ab}\,.$$

The coupling constants g_1 and g_2 correspond to the U(1) and SU(N) sectors, respectively. With our conventions, the U(1) charges of the fundamental matter fields are $\pm 1/2$, see Eq. (2).

The scalar potential $V(q^A, \tilde{q}_A, a^a, a)$ in the action (2.4) is the sum of D and F terms,

$$
\begin{aligned}
V(q^A, \tilde{q}_A, a^a, a) &= \frac{g_2^2}{2}\left(\frac{1}{g_2^2} f^{abc}\bar{a}^b a^c + \bar{q}_A\, T^a q^A - \tilde{q}_A T^a\, \bar{\tilde{q}}^A\right)^2 \\[2mm]
&+ \frac{g_1^2}{8}\left(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A\right)^2 \\[2mm]
&+ 2g_2^2\left|\tilde{q}_A T^a q^A + \frac{1}{\sqrt{2}}\frac{\partial\mathcal{W}_\mu}{\partial a^a}\right|^2 + \frac{g_1^2}{2}\left|\tilde{q}_A q^A + \sqrt{2}\,\frac{\partial\mathcal{W}_\mu}{\partial a}\right|^2 \\[2mm]
&+ \frac{1}{2}\sum_{A=1}^{N_f}\left\{\left|(a + \sqrt{2}m_A + 2T^a a^a)q^A\right|^2 \right. \\[2mm]
&+ \left.\left|(a + \sqrt{2}m_A + 2T^a a^a)\bar{\tilde{q}}^A\right|^2\right\}\,. \qquad (2.6)
\end{aligned}
$$

Here f^{abc} denote the structure constants of the SU(N) group, m_A is the mass term for the A-th flavor, and the sum over the repeated flavor indices A is implied.

2.1 Vacuum structure

Now, let us discuss the vacuum structure of this theory [31]. The vacua of the theory (2.4) are determined by the zeros of the potential (2.6). In general, the theory has a number of the so called r-vacua, in which r quarks

condense. The range of variation of r is $r = 0, ..., N$. Say, $r = 0$ vacua (there are N such vacua) are always at strong coupling. We have already explained that they are called the monopole vacua [2, 3]. In this paper we will focus on a particular set of vacua with the maximal number of condensed quarks, $r = N$. The reason for this choice is that all U(1) factors of the gauge group are spontaneously broken in these vacua, and, as a result, they support non-Abelian strings [13, 14, 15, 16]. The occurrence of strings ensures the monopole confinement in these vacua.

Let us first assume that our theory is at weak coupling, so that we can analyze it quasiclassically. Below we will explicitly formulate conditions on the quark mass terms and μ which will enforce such a regime.

With generic values of the quark masses we have

$$C^N_{N_f} = \frac{N_f!}{N!(N_f - N)!} \tag{2.7}$$

isolated r-vacua in which $r = N$ quarks (out of N_f) develop vacuum expectation values (VEVs). Consider, say, the vacuum in which the first N flavors develop VEVs, to be denoted as $(1, 2 ..., N)$. In this vacuum the adjoint fields develop VEVs too, namely,

$$\left\langle \left(\frac{1}{2}\, a + T^a\, a^a \right) \right\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & m_N \end{pmatrix}, \tag{2.8}$$

For generic values of the quark masses, the SU(N) subgroup of the gauge group is broken down to U(1)$^{N-1}$. However, in the *special limit*

$$m_1 = m_2 = ... = m_{N_f}, \tag{2.9}$$

the adjoint field VEVs do not break the SU(N)×U(1) gauge group. In this limit the theory acquires a global flavor SU(N_f) symmetry.

With all quark masses equal and to the leading order in μ, the mass term for the adjoint matter (2.2) reduces to the Fayet–Iliopoulos F-term of the U(1) factor of the SU(N)×U(1) gauge group, which does *not* break $\mathcal{N} = 2$ supersymmetry [6, 8]. In this limit the Fayet–Iliopoulos F-term can be transformed into the Fayet–Iliopoulos D-term by an SU(2)$_R$ rotation; the theory reduces to $\mathcal{N} = 2$ SQCD described in detail, say, in [19]. Higher orders in the parameter μ break $\mathcal{N} = 2$ supersymmetry by splitting all $\mathcal{N} = 2$ multiplets.

If the quark masses are unequal the U(N) gauge group is broken down to U(1)^N by the adjoint field VEV's (2.8). To the leading order in μ, the superpotential (2.2) reduces to N distinct FI terms, one in each U(1) gauge factor. $\mathcal{N} = 2$ supersymmetry in each individual low-energy U(1) theory remains unbroken [31]. It is broken, however, being considered in the full U(N) gauge theory.

Using (2.2) and (2.8) it is not difficult to obtain the quark field VEVs from Eq. (2.6). By virtue of a gauge rotation they can be written as

$$
\langle q^{kA} \rangle = \langle \bar{\tilde{q}}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \sqrt{\xi_N} & 0 & \dots & 0 \end{pmatrix} ,
$$

$$
k = 1, ..., N , \qquad A = 1, ..., N_f , \tag{2.10}
$$

where we present the quark fields as matrices in the color (k) and flavor (A) indices. The Fayet–Iliopoulos F-term parameters for each U(1) gauge factor are given (in the quasiclassical approximation) by the following expressions:

$$
\xi_P = 2 \left\{ \sqrt{\frac{2}{N}} \, \mu_1 \hat{m} + \mu_2 (m_P - \hat{m}) \right\} , \qquad P = 1, ..., N \tag{2.11}
$$

and $\hat{m}$ is the average value of the first N quark masses,

$$
\hat{m} = \frac{1}{N} \sum_{P=1}^{N} m_P . \tag{2.12}
$$

While the adjoint VEVs do not break the SU(N)×U(1) gauge group in the limit (2.9), the quark condensate (2.10) does result in the spontaneous breaking of both gauge and flavor symmetries. A diagonal global SU(N) combining the gauge SU(N) and an SU(N) subgroup of the flavor SU(N_f) group survives, however. This is color-flavor locking. Below we will refer to this diagonal global symmetry as to SU(N)$_{C+F}$.

Thus, the pattern of the color and flavor symmetry breaking is as follows:

$$
\text{U}(N)_{\text{gauge}} \times \text{SU}(N_f)_{\text{flavor}} \to \text{SU}(N)_{C+F} \times \text{SU}(\tilde{N})_F \times \text{U}(1) , \tag{2.13}
$$

where $\tilde{N} = N_f - N$. Here SU(N)$_{C+F}$ is a global unbroken color-flavor rotation, which involves the first N flavors, while the SU($\tilde{N}$)$_F$ factor stands for the

flavor rotation of the $\tilde{N}$ quarks. The presence of the global $SU(N)_{C+F}$ group is instrumental for formation of the non-Abelian strings [13, 14, 15, 16, 31]. As we will see shortly, the global symmetry of the dual theory is, of course, the same, albeit the physical origin is different.

With unequal quark masses, the global symmetry (2.13) is broken down to $U(1)^{N_f-1}$ both by the adjoint and squark VEVs. This should be contrasted with the theory with the Fayet–Iliopoulos term introduced through the D-term, in which the quark VEVs are all equal and do not break the color-flavor symmetry.

Since the global (flavor) $SU(N_f)$ group is broken by the quark VEVs anyway, it will be helpful for our purposes to consider the following mass splitting:

$$m_P = m_{P'}, \qquad m_K = m_{K'}, \qquad m_P - m_K = \Delta m \qquad (2.14)$$

where

$$P, P' = 1, ..., N \quad \text{and} \quad K, K' = N+1, ..., N_f. \qquad (2.15)$$

This mass splitting respects the global group (2.13) in the $(1, 2, ..., N)$ vacuum. Moreover, this vacuum becomes isolated. No Higgs branch develops. We will often use this limit below.

2.2 Perturbative excitations

Now let us discuss the perturbative excitation spectrum. To the leading order in μ, in the limit (2.14), the superpotential (2.2) reduces to the Fayet–Iliopoulos F-term of the $U(1)$ factor of the gauge group. Since both $U(1)$ and $SU(N)$ gauge groups are broken by the squark condensation, all gauge bosons become massive. In fact, with nonvanishing ξ_P's (see Eq. (2.11)), both the quarks and adjoint scalars combine with the gauge bosons to form long $\mathcal{N} = 2$ supermultiplets [8], for a review see [19]. In the limit (2.14)

$$\xi_P \equiv \xi,$$

and all states come in representations of the unbroken global group (2.13), namely, in the singlet and adjoint representations of $SU(N)_{C+F}$,

$$(1, 1), \quad (N^2 - 1, 1), \qquad (2.16)$$

and in the bifundamental representations

$$(\bar{N}, \tilde{N}), \quad (N, \bar{\tilde{N}}). \qquad (2.17)$$

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We mark representations in (2.16) and (2.17) with respect to two non-Abelian factors in (2.13). The singlet and adjoint fields are (i) the gauge bosons, and (ii) the first N flavors of the squarks q^{kP} $(P = 1, ..., N)$, together with their fermion superpartners. The bifundamental fields are the quarks q^{kK} with $K = N + 1, ..., N_f$. These quarks transform in the two-index representations of the global group (2.13) due to the color-flavor locking.

In the limit (2.14) the mass of the $(N^2 - 1, 1)$ adjoint fields is

$$m_{(N^2-1, 1)} = g_2 \sqrt{\xi}, \tag{2.18}$$

while the singlet field mass is

$$m_{(1, 1)} = g_1 \sqrt{\frac{N}{2}} \sqrt{\xi}. \tag{2.19}$$

The bifundamental field masses are given by

$$m_{(\bar{N}, \tilde{N})} = \Delta m. \tag{2.20}$$

The above quasiclassical analysis is valid if the theory is at weak coupling. This is the case if the quark VEVs are sufficiently large so that the gauge coupling constant is frozen at a large scale. From (2.10) we see that the quark condensates are of the order of $\sqrt{\mu m}$ (see also [2, 3, 23, 32]). As was mentioned, we assume that $\mu_1 \sim \mu_2 \sim \mu$. In this case the weak coupling condition reduces to

$$\sqrt{\mu m} \gg \Lambda_{\mathcal{N}=2}, \tag{2.21}$$

where $\Lambda_{\mathcal{N}=2}$ is the scale of the $\mathcal{N} = 2$ theory, and we assume that all quark masses are of the same order $m_A \sim m$. In particular, the condition (2.21), combined with the condition (2.3) of smallness of μ, implies that the average quark mass m is very large.

3 Duality at small μ in the quark vacua

3.1 Dual theory

Now we will relax the condition (2.21) and pass to the strong coupling domain at

$$|\sqrt{\xi_P}| \ll \Lambda_{\mathcal{N}=2}, \qquad |m_A| \ll \Lambda_{\mathcal{N}=2}. \tag{3.1}$$

$\mathcal{N} = 2$ SQCD with the Fayet–Iliopoulos term (introduced as the D-term) was shown [10, 11] to undergo a crossover transition upon reduction of the FI parameter. The results obtained in [10] are based on studying the Seiberg–Witten curve [2, 3, 23] in $\mathcal{N} = 2$ SQCD on the Coulomb branch, and, therefore, do not depend on the type of the FI deformation. We briefly review these results here adjusting our consideration [10, 11] to fit the case of the Fayet–Iliopoulos F-term induced by the adjoint mass μ.

The domain (3.1) can be described in terms of weakly coupled (infrared free) dual theory with with the gauge group

$$U(\tilde{N}) \times U(1)^{N-\tilde{N}},\qquad (3.2)$$

and N_f flavors of light *dyons*.[6]

Light dyons D^{lA}

$$l = 1, ..., \tilde{N}, \quad A = 1, ..., N_f \qquad (3.3)$$

are in the fundamental representation of the gauge group $SU(\tilde{N})$ and are charged under the Abelian factors indicated in Eq. (3.2). In addition, there are $(N - \tilde{N})$ light dyons D^J $(J = \tilde{N} + 1, ..., N)$, neutral under the $SU(\tilde{N})$ group, but charged under the U(1) factors.

In Appendix A we present the low-energy effective action for the dual theory in a specific example: $N = 3$, $N_f = 5$, and $\tilde{N} = 2$. In particular, starting from this action, we find the dyon condensates in the quasiclassical approximation. Generalization of these results to arbitrary N and $\tilde{N}$ has the following form

$$\langle D^{lA}\rangle = \langle \bar{\tilde{D}}^{lA}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \dots & 0 & \sqrt{\xi_1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & \sqrt{\xi_{\tilde{N}}} \end{pmatrix},$$

$$\langle D^J\rangle = \langle \bar{\tilde{D}}^J\rangle = \sqrt{\frac{\xi_J}{2}}, \qquad J = \tilde{N} + 1, ..., N. \qquad (3.4)$$

The most important feature apparent in (3.4), as compared to the squark VEVs of the original theory (2.10), is a "vacuum jump" [10],

$$(1, ..., N)_{\sqrt{\xi}\gg\Lambda_{\mathcal{N}=2}} \to (N + 1, ..., N_f, \tilde{N} + 1, ..., N)_{\sqrt{\xi}\ll\Lambda_{\mathcal{N}=2}}. \qquad (3.5)$$

---

[6]Previously the $SU(\tilde{N})$ gauge group was identified as dual [23] on the Coulomb branch at the root of the baryonic Higgs branch in the $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang–Mills theory with massless quarks.

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In other words, if we pick up the vacuum with nonvanishing VEVs of the first N quark flavors in the original theory at large ξ, Eq. (2.4), and then reduce ξ below $\Lambda_{\mathcal{N}=2}$, the system goes through a crossover transition and ends up in the vacuum of the *dual* theory with the nonvanishing VEVs of $\tilde{N}$ last dyons (plus VEVs of $(N - \tilde{N})$ SU($\tilde{N}$) singlets).

The Fayet–Iliopoulos parameters ξ_P in (3.4) are determined by the quantum version of the classical expressions (2.11). They are expressible via the roots of the Seiberg–Witten curve in the given $r = N$ vacuum [31]. Namely,

$$\xi_P = 2 \left\{ \sqrt{\frac{2}{N}} \, \mu_1 \, \hat{m} - \mu_2 (\sqrt{2} e_P + \hat{m}) \right\}, \qquad (3.6)$$

where e_P are the double roots of the Seiberg–Witten curve [23],

$$y^2 = \prod_{P=1}^{N} (x - \phi_P)^2 - 4 \left(\frac{\Lambda}{\sqrt{2}} \right)^{N - \tilde{N}} \prod_{A=1}^{N_f} \left(x + \frac{m_A}{\sqrt{2}} \right), \qquad (3.7)$$

while ϕ_P are gauge invariant parameters on the Coulomb branch. We recall that $\hat{m}$ in Eq. (3.6) is still the average of the first N quark masses (2.12). The curve (3.7) describes the Coulomb branch of the theory for $\tilde{N} < N - 1$. The case $\tilde{N} = N - 1$ is special. In this case we must make a shift in Eq. (3.7) [23],

$$m_A \to \tilde{m}_A = m_A + \frac{\Lambda_{\mathcal{N}=2}}{N}, \qquad \tilde{N} = N - 1 . \qquad (3.8)$$

We will not consider this special case at large μ since it is incompatible with the condition $N_f < 3/2 \, N$ or $\tilde{N} < N/2$.

In the $r = N$ vacuum the curve (3.7) has N double roots and reduces to

$$y^2 = \prod_{P=1}^{N} (x - e_P)^2, \qquad (3.9)$$

where quasiclassically (at large masses) e_P's are given by the mass parameters, $\sqrt{2} e_P \approx -m_P$, $P = 1, ..., N$.

As long as we keep ξ_P and masses small enough (i.e. in the domain (3.1)) the coupling constants of the infrared free dual theory (frozen at the scale of the dyon VEVs) are small: the dual theory is at weak coupling.

At small masses, in the region (3.1), the double roots of the Seiberg–Witten curve are

$$\sqrt{2}e_I = -m_{I+N}, \qquad \sqrt{2}e_J = \Lambda_{\mathcal{N}=2} \exp\left(\frac{2\pi i}{N - \tilde{N}}J\right) \qquad (3.10)$$

for $\tilde{N} < N - 1$, where

$$I = 1, ..., \tilde{N} \quad \text{and} \quad J = \tilde{N} + 1, ..., N. \qquad (3.11)$$

In particular, the $\tilde{N}$ first roots are determined by the masses of the last $\tilde{N}$ quarks — a reflection of the fact that the non-Abelian sector of the dual theory is not asymptotically free and is at weak coupling in the domain (3.1).

From Eqs. (3.4), (3.6) and (3.10) we see that the VEVs of the non-Abelian dyons D^{lA} are determined by $\sqrt{\mu m}$ and are much smaller than the VEVs of the Abelian dyons D^J in the domain (3.1). The latter are of the order of $\sqrt{\mu \Lambda_{\mathcal{N}=2}}$. This circumstance is most crucial for our analysis in this paper. It will allow us to eventually increase μ and decouple the adjoint fields without spoiling the weak coupling condition in the dual theory, see Sec. 4.

In the special case $\tilde{N} = N - 1$ masses in (3.10) should be shifted according to (3.8).

Now, let us consider either equal quark masses or the mass choice (2.14). Both, the gauge group and the global flavor $SU(N_f)$ group, are broken in the vacuum. However, the color-flavor locked form inherent to (3.4) under the given mass choice guarantees that the diagonal global $SU(\tilde{N})_{C+F}$ symmetry survives. More exactly, the unbroken *global* group of the dual theory is

$$SU(N)_F \times SU(\tilde{N})_{C+F} \times U(1). \qquad (3.12)$$

The $SU(\tilde{N})_{C+F}$ factor in (3.12) is a global unbroken color-flavor rotation, which involves the last $\tilde{N}$ flavors, while the $SU(N)_F$ factor stands for the flavor rotation of the first N dyons.

Thus, a color-flavor locking takes place in the dual theory too. Much in the same way as in the original theory, the presence of the global $SU(\tilde{N})_{C+F}$ group is the reason behind formation of the non-Abelian strings. For generic quark masses the global symmetry (2.13) is broken down to $U(1)^{N_f - 1}$.

In the equal mass limit, or given the mass choice (2.14), the global unbroken symmetry (3.12) of the dual theory at small ξ coincides with the global group (2.13) which manifest in the $r = N$ vacuum of the original theory at large ξ. This has been already announced previously.
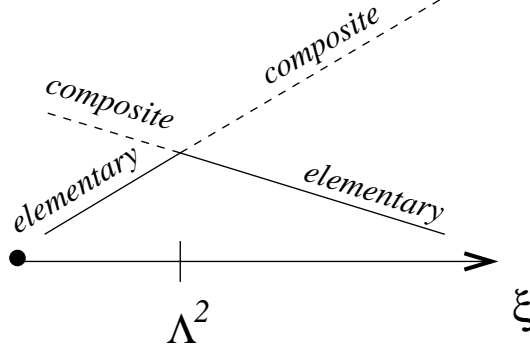
Figure 2: Evolution of the SU(N) and SU($\tilde{N}$) gauge bosons and light quarks (dyons) vs. ξ.

Note, however, that this global symmetry is realized in two very distinct ways in the dual pair at hand. As was already mentioned, the quarks and U(N) gauge bosons of the original theory at large ξ come in the $(1,1)$, $(N^2 - 1, 1)$, $(\bar{N}, \tilde{N})$, and $(N, \bar{\tilde{N}})$ representations of the global group (2.13), while the dyons and U($\tilde{N}$) gauge bosons form $(1,1)$, $(1, \tilde{N}^2 - 1)$, $(N, \bar{\tilde{N}})$, and $(\bar{N}, \tilde{N})$ representations of (3.12). We see that the adjoint representations of the $(C + F)$ subgroup are different in two theories. How can this happen?

The quarks and gauge bosons which form the adjoint $(N^2 - 1)$ representation of SU(N) at large ξ and the dyons and gauge bosons which form the adjoint $(\tilde{N}^2 - 1)$ representation of SU($\tilde{N}$) at small ξ are, in fact, *distinct* states. The $(N^2 - 1)$ adjoints of SU(N) become heavy and decouple as we pass from large to small ξ along the line $\xi \sim \Lambda_{\mathcal{N}=2}$. Moreover, some composite $(\tilde{N}^2 - 1)$ adjoints of SU($\tilde{N}$), which are heavy and invisible in the low-energy description at large ξ become light at small ξ and form the D^{lK} dyons $(K = N + 1, ..., N_f)$ and gauge bosons of U($\tilde{N}$). The phenomenon of the level crossing takes place (Fig. 2). Although this crossover is smooth in the full theory, from the standpoint of the low-energy description the passage from large to small ξ means a dramatic change: the low-energy theories in these domains are completely different; in particular, the degrees of freedom in these theories are different.

This logic leads us to the following conclusion. In addition to light dyons and gauge bosons included in the low-energy theory at small ξ we must have heavy fields which form the adjoint representation $(N^2 - 1, 1)$ of the global symmetry (3.12). These are screened quarks and gauge bosons from the

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large-ξ domain. Let us denote them as $M_P^{P'}$ (here $P, P' = 1, ..., N$).

As has been already noted in Sec. 1, at small ξ they decay into the monopole-antimonopole pairs on the curves of marginal stability (CMS).[7] This is in accordance with results obtained for $\mathcal{N} = 2$ SU(2) gauge theories [2, 3, 33] on the Coulomb branch at zero ξ, while for the theory at hand it is proven in [11]. The general rule is that the only states which exist at strong coupling inside CMS are those which can become massless on the Coulomb branch [2, 3, 33]. For our theory these are light dyons shown in Eq. (3.4), gauge bosons of the dual gauge group and monopoles.

At small nonvanishing ξ the monopoles and antimonopoles produced in the decay process of the adjoint $(N^2 - 1, 1)$ states cannot escape from each other and fly off to asymptotically large separations because they are confined. Therefore, the (screened) quarks or gauge bosons evolve into stringy mesons $M_P^{P'}$ $(P, P' = 1, ..., N)$ in the strong coupling domain of small ξ – the monopole-antimonopole pairs connected by two strings [10, 11], as shown in Fig. 1.

By the same token, at large ξ, in addition to the light quarks and gauge bosons, we have heavy fields $M_K^{K'}$ (here $K, K' = N + 1, ..., N_f$), which form the adjoint $(\tilde{N}^2 - 1)$ representation of SU($\tilde{N}$). This is schematically depicted in Fig. 2.

The $M_K^{K'}$ states are (screened) light dyons and gauge bosons of the dual theory. In the large-ξ domain they decay into the monopole-antimonopole pairs and form stringy mesons [10] shown in Fig. 1.

3.2 More on the non-Abelian strings and confined monopoles

Since dyons develop VEVs in the $r = N$ vacuum which break the gauge group, see (3.4), our dual theory supports strings. In fact, the minimal stings in our theory are the Z_N strings, progenitors of the non-Abelian strings [13, 14, 15, 16].

---

[7]An explanatory remark regarding our terminology is in order. Strictly speaking, such pairs can be formed by monopole-antidyons and dyon-antidyons as well, the dyons carrying root-like electric charges. In this paper we refer to all such states collectively as to "monopoles." This is to avoid confusion with dyons which appear in Eq. (3.4). The latter dyons carry weight-like electric charges and, roughly speaking, behave as quarks, see [10] for further details.

At generic m_A the dual gauge group (3.2) reduces to $U(1)^N$; the low-energy theory is $U(1)^N$ gauge theory with the Fayet–Iliopoulos F-term for each $U(1)$ factor. The Z_N strings for this theory are thoroughly studied in [31]. In the low-energy approximation the Z_N strings are BPS saturated. Tensions of all N elementary Z_N strings are given by the FI parameters [31],

$$T_P^{\text{BPS}} = 2\pi|\xi_P|, \qquad P = 1, ..., N. \tag{3.13}$$

In the limit (2.14) the color-flavor locking takes place and the global group of the dual theory becomes that of Eq. (3.12). In this case $\tilde{N}$ of the set of N Z_N strings (associated with windings of non-Abelian D^{lA} dyons) acquire orientational zero modes and become non-Abelian. They can be analyzed within the general framework developed in [13, 14, 15, 16], see [19] for a review. The internal dynamics of the orientational zero modes are described by two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric CP model living on the string world sheet [13, 14, 15, 16]. For the original theory (2.4) it is $CP(N-1)$ model for $\tilde{N} = 0$. For nonzero $\tilde{N}$ the string becomes semilocal. Semilocal strings do not have fixed transverse radius, they acquire size moduli, see [34] for a review of the Abelian semilocal strings. The non-Abelian semilocal strings in $\mathcal{N} = 2$ SQCD with $N_f > N$ were studied in [13, 16, 35, 36]. The internal dynamics of these strings is *qualitatively* described by a weighted $CP(N_f - 1)$ model with N positive and $\tilde{N}$ negative charges associated with N orientational moduli and $\tilde{N}$ size moduli. (Aspects of a quantitative description and its interrelation with the weighted $CP(N_f-1)$ model will be discussed in [37].) In the dual theory N and $\tilde{N}$ interchange; it is governed by the weighted $CP(N_f-1)$ model with $\tilde{N}$ positive (orientations) and N negative (size) charges n_K, $K = (N + 1), ..., N_f$ and ρ_P, $P = 1, ..., N$, respectively [11]. The above moduli are subject to the constraint

$$|n_K|^2 - |\rho_P|^2 = 2\tilde{\beta}, \tag{3.14}$$

where $\tilde{\beta}$ is a coupling constant of the dual world-sheet theory. It is determined by the gauge coupling constant of the dual bulk theory at the scale $\sim \sqrt{\xi}$ [14, 15, 11],

$$\tilde{\beta} = \frac{2\pi}{\tilde{g}_2^2}. \tag{3.15}$$

Distinct elementary non-Abelian strings correspond to different vacua of the CP model under consideration. Confined monopoles of the bulk theory are identified with the junctions of two degenerate elementary non-Abelian

strings [38, 15, 16]. These are seen as kinks interpolating between different vacua of the CP model. Non-perturbative generation of the dynamical scale Λ_{CP} in the CP model stabilizes these kinks in the non-Abelian regime, making their inverse sizes and masses of the order of Λ_{CP} [15, 19]. Thus, the notion of the confined monopole becomes well-defined in the non-Abelian limit.

The identification of confined monopoles with the CP-model kinks reveals the global quantum numbers of the monopoles. Say, it was known for a long time that the kinks in the quantum limit form a fundamental representation of the global SU(N) group in the $\mathcal{N} = (2,2)$ supersymmetric CP($N-1$) models [39, 40]. In [11] we generalized this result to the case of the $\mathcal{N} = (2,2)$ supersymmetric weighted CP models. We showed that the kinks (confined monopoles) are in the fundamental representation of the global group (3.12).

More exactly, in the limit (2.14) they form the $(N,1) + (1,\tilde{N})$ representations of the global group (3.12). This means that the total number of stringy mesons M_A^B formed by the monopole-antimonopole pairs connected by two different elementary non-Abelian strings (Fig. 1) is N_f^2. The mesons $M_P^{P'}$ form the singlet and $(N^2-1, 1)$ adjoint representations of the global group (3.12), the mesons M_P^K and M_K^P form bifundamental representations $(N, \tilde{\bar{N}})$ and $(\bar{N}, \tilde{N})$, while the mesons $M_K^{K'}$ form the singlet and $(1, \tilde{N}^2 - 1)$ adjoint representations. (Here, as usual, $P = 1, ..., N$ and $K = (N+1), ..., N_f$.)

All these mesons with not too high spins have masses

$$m_{M_P^{P'}} \sim \sqrt{\xi}, \tag{3.16}$$

as determined by the string tensions (3.13) . They are heavier than the elementary states, namely, dyons and dual gauge bosons which form the (1,1), $(N, \tilde{\bar{N}})$, $(\bar{N}, \tilde{N})$, and $(1, \tilde{N}^2 - 1)$ representations and have masses $\sim \tilde{g}_2 \sqrt{\xi}$.

Therefore, the (1,1), $(N, \tilde{\bar{N}})$, $(\bar{N}, \tilde{N})$, and $(1, \tilde{N}^2 - 1)$ stringy mesons decay into elementary states, and we are left with $M_P^{P'}$ stringy mesons in the representation $(N^2 - 1, 1)$. Thus our confinement picture in the bulk theory outlined above is confirmed by the world-sheet analysis.

This concludes our extended introduction and adjustments necessary to pass to the study of the $\mathcal{N} = 1$ theories.

4   Flowing to $\mathcal{N} = 1$ QCD

With all preparatory work done, we begin our journey in the $\mathcal{N} = 1$ theories. In this section we increase the adjoint mass μ and decouple the adjoint matter. In the course of this process the theory at hand flows to $\mathcal{N} = 1$ SQCD. So, now we assume that

$$|\mu| \gg |m_A|, \qquad A = 1, ..., N_f. \tag{4.1}$$

Then, the $\mathcal{N} = 2$ multiplets are split. We consider the quark masses to be small enough to guarantee that the original theory (2.4) is at strong coupling, while the dual theory is at weak coupling.

4.1   Decoupling the $\mathrm{U}(1)^{(N-\tilde{N})}$ sector

At first, we will impose the condition

$$|\mu| \ll \Lambda_{\mathcal{N}=2}, \tag{4.2}$$

implying (in conjunction with (4.1)) that all parameters $\sqrt{\xi_P}$ are much smaller than $\Lambda_{\mathcal{N}=2}$. Then our dual theory is at weak coupling, see Eqs. (3.6) and (3.10). From (3.10) we see that VEVs of non-Abelian dyons D^{lA} are much smaller those of the Abelian dyons D^J. Consider the low-energy limit of the dual theory, i.e. energies much lower than $\sqrt{\mu \Lambda_{\mathcal{N}=2}}$. In this scale the Abelian dyons D^J

$$J = (\tilde{N} + 1), ..., N$$

are heavy and decouple. These dyons interact with $(N - \tilde{N} + 1)$ U(1) gauge fields, see Eq. (3.2). In this set of gauge bosons, $(N - \tilde{N})$ U(1) fields also become heavy (with masses $g\sqrt{\mu \Lambda_{\mathcal{N}=2}}$). Only one remains. As a result, in the low-energy limit we are left with the dual theory with the gauge group

$$\mathrm{U}(\tilde{N}) \tag{4.3}$$

and N_f flavors of dyons

$$D^{lA}, \quad l = 1, ..., \tilde{N}, \quad A = 1, ..., N_f.$$

The superpotential in this theory is

$$\mathcal{W} = \sqrt{2} \sum_{A=1}^{N_f} \left(\frac{1}{2} \tilde{D}_A b_{U(1)} D^A + \tilde{D}_A b^p \, T^p D^A \right) + \mathcal{W}_{[\mu]}(b_{U(1)}, b^p), \tag{4.4}$$

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Here $b_{U(1)}$ is a chiral superfield, the $\mathcal{N} = 2$ superpartner of $B_\mu^{U(1)}$, where $B_\mu^{U(1)}$ is a particular linear combination of the dual gauge fields not interacting with the D^J dyons. We renormalized $b_{U(1)}$ so that charges of the D^{lA} dyons with respect to this field are $\frac{1}{2}$. This amounts to redefining its coupling constant $\tilde{g}_{U(1)}^2$. Moreover, b^p is an SU($\tilde{N}$) adjoint chiral field, the $\mathcal{N} = 2$ superpartner of the dual SU($\tilde{N}$) gauge field. Finally, $\mathcal{W}_{[\mu]}$ is a μ dependent part of the superpotential, cf. (2.2).

The deformation superpotential $\mathcal{W}_{[\mu]}$ given in Eq. (2.2) can be expressed in terms of invariants u_k, see Eq. (A.7). Namely,

$$
\mathcal{W}_{[\mu]} = \mu_2 \, u_2 - \frac{\mu_2}{N} \left(1 - \sqrt{\frac{2}{N}} \frac{\mu_1}{\mu_2} \right) u_1^2 , \tag{4.5}
$$

where u_2 and u_1 should be understood as functions of $b_{U(1)}$ and b^p. These functions are determined by the exact Seiberg–Witten solution. We will treat them in Sec. 4.2. Note, that with the singlet dyons decoupled, the VEVs of the non-Abelian dyons are

$$
\langle D^{lA} \rangle = \langle \tilde{\bar{D}}^{lA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \dots & 0 & \sqrt{\xi_1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & \sqrt{\xi_{\tilde{N}}} \end{pmatrix} , \tag{4.6}
$$

where the first $\tilde{N}$ ξ's are of the order of μm, see (3.10).

4.2   Decoupling the adjoint matter

As will be shown in Sec. 4.3, the masses of the gauge fields and dyons D^{lA} in the U($\tilde{N}$) gauge theory, with the superpotential (4.4), do not exceed $\sqrt{\mu m}$, while the adjoint matter mass of is of the order of μ. Therefore, in the limit (4.1) the adjoint matter decouples. Below scale μ our theory becomes dual to $\mathcal{N} = 1$ SQCD with the scale

$$
\tilde{\Lambda}^{N-2\tilde{N}} = \frac{\Lambda_{\mathcal{N}=2}^{N-\tilde{N}}}{\mu^{\tilde{N}}} . \tag{4.7}
$$

The only condition we impose to keep this infrared free theory in the weak coupling regime is

$$
\sqrt{\mu m} \ll \tilde{\Lambda} . \tag{4.8}
$$

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This means that at large μ we must keep the quark masses small enough. The larger the value of μ the smaller the quark masses, so that the product μm is constrained from above by $\tilde{\Lambda}^2$. This is always doable.

We would like to stress that, although this procedure is perfectly justified in the $r = N$ vacuum we work in, it does not work, say, in the monopole vacua. In these vacua VEVs of the light matter (the Abelian monopoles) are of the order of $\sqrt{\mu \Lambda_{\mathcal{N}=2}}$, which, in turn, sets the mass scale in the dual Abelian $U(1)^N$ gauge theory [2]. Therefore, we cannot decouple the adjoint matter keeping the dual theory at weak coupling. As soon as we increase μ well above the above scale, we break the weak coupling condition in the dual $U(1)^N$ gauge theory.

In contrast, in the $r = N$ vacuum we can take μ much larger than the quark masses and decouple the adjoint matter. If the condition (4.8) is fulfilled, the dual theory stays at weak coupling. The reason is that it is the quark masses rather than $\Lambda_{\mathcal{N}=2}$ that determine the "non-Abelian" roots of the Seiberg–Witten curve and VEVs of the non-Abelian dyons, see (3.10).

Given the superpotential (4.4) we can explicitly integrate out the adjoint matter. To this end we expand $\mathcal{W}_{[\mu]}$ in powers of $b_{U(1)}$ and b^p,

$$
\begin{aligned}
\mathcal{W}_{[\mu]}(b_{U(1)}, b^p) \;=\;& c_1 \, \mu_2 \, b_{U(1)}^2 + c_2 \, \mu_2 \, (b^p)^2 \\[1mm]
&+\; c_3 \, \mu_2 m \, b_{U(1)} + c_4 \, \mu_2 \, \Lambda_{\mathcal{N}=2} \, b_{U(1)} \\[2mm]
&+\; O\left(\frac{\mu_2 \, (b^p)^4}{\Lambda_{\mathcal{N}=2}^2}\right) + O\left(\frac{\mu_2 \, b_{U(1)}^3}{\Lambda_{\mathcal{N}=2}}\right),
\end{aligned}
\tag{4.9}
$$

where

$$
m = \frac{1}{N_f} \sum_{A=1}^{N_f} m_A .
\tag{4.10}
$$

We then note that

$$
c_4 = 0 .
\tag{4.11}
$$

Indeed, a nonvanishing c_4 would produce a VEV of $b_{U(1)}$ of the order of $\Lambda_{\mathcal{N}=2}$ which, in turn, would imply VEVs of certain dyons D^{lA} to be of the order of $\sqrt{\mu \Lambda_{\mathcal{N}=2}}$, in direct contradiction with Eqs. (4.6) and (3.10).

Moreover, since VEVs of $b_{U(1)}$ and b^p are of the order of the quark masses (rather than $\Lambda_{\mathcal{N}=2}$) we can neglect higher-order terms in the expansion (4.9)

and keep only linear and quadratic terms in the b fields. Higher-order terms are suppressed by powers of $m/\Lambda_{\mathcal{N}=2}$.

Now, substituting (4.9) into (4.4) and integrating over $b_{U(1)}$ and b^p we get the superpotential which depends only on D^{lA}. Minimizing it and requiring VEVs of D^{lA} to be given by (4.6) (see also (3.10)) we fix the coefficients c_1, c_2 and c_3,

$$c_1 = \frac{\tilde{N}}{4}\left(1 + \gamma\frac{\tilde{N}}{N}\right), \qquad c_2 = \frac{1}{2}, \qquad c_3 = \frac{\tilde{N}}{\sqrt{2}}\gamma\left(1 + \frac{\tilde{N}}{N}\right), \qquad (4.12)$$

where

$$\gamma = 1 - \sqrt{\frac{2}{N}}\frac{\mu_1}{\mu_2}. \qquad (4.13)$$

After eliminating the adjoint matter the superpotential takes the form

$$\begin{aligned}
\mathcal{W} &=& -\frac{1}{2\mu_2}\left[(\tilde{D}_A D^B)(\tilde{D}_B D^A) - \frac{\alpha_D}{\tilde{N}}(\tilde{D}_A D^A)^2\right] \\
&& + \left[m_A - \frac{\gamma\left(1 + \frac{\tilde{N}}{N}\right)}{1 + \gamma\frac{\tilde{N}}{N}}m\right](\tilde{D}_A D^A),
\end{aligned} \qquad (4.14)$$

where the color indices are contracted inside each parentheses, while

$$\alpha_D = \frac{\gamma\frac{\tilde{N}}{N}}{1 + \gamma\frac{\tilde{N}}{N}}. \qquad (4.15)$$

This equation presents our final large-μ answer for the superpotential of the theory dual to $\mathcal{N} = 1$ SQCD in the $(1, \ldots, N)$ vacuum. The second term is the dyon mass term while the first one describes the dyon interaction.

One can check that minimization of this superpotential leads to correct dyon VEVs, cf. Eq. (4.6). Of course, the theory with the superpotential (4.14) possesses many other vacua in which different dyons (and different number of dyons) develop VEVs. We consider only one particular vacuum here. As was explained in Sec. 3, if we choose the $(1, \ldots, N)$ vacuum in the original theory above the crossover, then we end up in the $(0, \ldots, 0, N + 1, \ldots, N_f)$ vacuum in the dual theory below the crossover, see (3.5). Vacua with different number of condensed D's seen in (4.14) are spurious. The reason is that if we start from an $r < N$ vacuum in the original theory

the dual gauge group (below the crossover) would be different from $U(\tilde{N})$. Thus, the dual theory would not be the $U(\tilde{N})$ gauge theory of dyons D^{lA} ($l = 1, ..., \tilde{N}$), with the superpotential (4.14).

Summarizing this section, we pass to the limit of large μ decoupling the adjoint matter in the dual theory. This leaves us with the dual $U(\tilde{N})$ gauge theory with the superpotential (4.14). At this point one should ask: Are we sure that μ is large enough to decouple the adjoint matter in the original theory (2.4), as well as in the dual theory, so that the original theory becomes $\mathcal{N} = 1$ SQCD?

Strictly speaking, it is not easy to directly answer this question since in the domain (4.8) the original theory is at strong coupling, and our control over its dynamics is limited. Nevertheless, one can give the following argument. Let us denote the low-energy scale of the original $\mathcal{N} = 1$ SQCD as Λ. In terms of the scale of the original theory (2.4) at large μ we have

$$\Lambda^{2N-\tilde{N}} = \mu^N \Lambda_{\mathcal{N}=2}^{N-\tilde{N}} . \tag{4.16}$$

The (s)quark masses are small, and the scale of excitations in $\mathcal{N} = 1$ SQCD is determined by the parameter (4.16). The nonvanishing masses just lift the Higgs branch making all vacua isolated. Therefore, if we require that

$$\mu \gg \Lambda \tag{4.17}$$

we can be sure that the adjoint mater is decoupled in the original theory. Now, the weak coupling condition for the dual theory (4.8) can be rewritten in terms of Λ as follows:

$$m \ll \Lambda \left(\frac{\Lambda}{\mu}\right)^{\frac{3N}{N-2\tilde{N}}} . \tag{4.18}$$

Since the quark mass scale m is at our disposal, we can always choose it to be sufficiently small. Below we assume that both conditions (4.17) and (4.18) are met.

If we further increase μ (keeping the quark masses fixed) we hit the upper bound in (4.18) and the dual theory (4.14) goes through a crossover into strong coupling.[8] Still further increase of the parameter μ,

$$\sqrt{\mu m} \gg \Lambda ,$$

_____
[8]To avoid this, one can simultaneously decrease m.

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brings us in the weak coupling regime in the original $\mathcal{N} = 1$ SQCD. In this regime the (s)quark fields condense thus completely Higgsing the U(N) gauge group. Non-Abelian strings are formed which confine monopoles. This regime is quite similar to that studied in [41, 42, 43] in the massless version of the theory (2.4), with a large Fayet–Iliopoulos D-term.

We stress that in this domain (large m) the (s)quark fields condense, while in our present setup (small m) the quarks and gauge bosons decay into the monopole-antimonopole stringy mesons on CMS.

4.3 Perturbative mass spectrum

In this section we briefly discuss the perturbative mass spectrum of the dual U($\tilde{N}$) gauge theory, with the superpotential (4.14), at large μ. At first we assume the limit (2.14) for the quark masses.

The U($\tilde{N}$) gauge group is completely Higgsed, and the masses of the gauge bosons are

$$m_{SU(\tilde{N})} = \tilde{g}_2 \sqrt{\xi} \tag{4.19}$$

for the SU($\tilde{N}$) gauge bosons, and

$$m_{U(1)} = \tilde{g}_1 \sqrt{\frac{N}{2}} \sqrt{\xi}. \tag{4.20}$$

for the U(1) gauge boson. Here $\tilde{g}_1$ and $\tilde{g}_2$ are dual gauge couplings for U(1) and SU($\tilde{N}$) gauge bosons respectively, while ξ is a common value of the first $\tilde{N}$ ξ_P's (see Eqs. (3.6) and (3.10)),

$$\xi = 2 \left\{ \sqrt{\frac{2}{N}} \mu_1 \hat{m} + \mu_2(m_{\text{last}} - \hat{m}) \right\}, \qquad m_{\text{last}} = m_K,$$

$$K = (N+1), ..., N_f. \tag{4.21}$$

The dyon masses are determined by the D-term potential

$$V_D^{\text{dual}} = \frac{\tilde{g}_2^2}{2} \left(\bar{D}_A T^p D_A - \tilde{D}_A T^p \bar{\tilde{D}}^A \right)^2 + \frac{\tilde{g}_1^2}{8} \left(|D^A|^2 - |\tilde{D}_A|^2 \right)^2 \tag{4.22}$$

and the F-term potential following from the superpotential (4.14). Diagonalizing the quadratic form given by these two potentials we find that, out of $4\tilde{N}N_F$ real degrees of freedom of the scalar dyons, $\tilde{N}^2$ are eaten by the

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Higgs mechanism, $(\tilde{N}^2 - 1)$ real scalar dyons have the same mass as the non-Abelian gauge fields, Eq. (4.19), while one scalar dyon has mass (4.20). These dyons are scalar superpartners of the SU($\tilde{N}$) and U(1) gauge bosons in $\mathcal{N} = 1$ massive vector supermultiplets, respectively.

Another $2(\tilde{N}^2 - 1)$ dyons form a $(1, \tilde{N}^2 - 1)$ representation of the global group (3.12). Their mass is as follows:

$$m_{(1, \tilde{N}^2 - 1)} = \frac{\xi}{\mu_2} = 2 \left(m_{\text{last}} - \gamma \hat{m} \right) , \tag{4.23}$$

where ξ is given in Eq. (4.21), while two real singlet dyons have mass

$$m_{(1,1)} = \sqrt{\frac{N}{2}} \frac{\xi}{\mu_1} = 2 \left(\hat{m} - \sqrt{\frac{N}{2}} \frac{\mu_2}{\mu_1} \Delta m \right) . \tag{4.24}$$

Masses of $4N\tilde{N}$ bifundamental fields are given by the mass split of N first and $\tilde{N}$ last quark masses, see (2.14),

$$m_{(\bar{N}, \tilde{N})} = \Delta m . \tag{4.25}$$

All these dyons are the scalar components of the $\mathcal{N} = 1$ chiral multiplets.

We see that the masses of the gauge multiplets and those of chiral matter get a large split in the limit of large μ and small m_A. Chiral matter become much lighter than the gauge multiplets cf. [44, 19]. Most important is the fact that in the theory (4.14) vacuum expectation values are developed by the light dyons, with masses given by (4.24) in the limit (2.14). Thus, we have an extreme type-I superconductivity in the vacuum of the dual theory.

For generic quark masses the perturbative excitation spectrum is rather complicated. We summarize it here for a particular case

$$\sqrt{\frac{\tilde{N}}{2}} \tilde{g}_1 = \tilde{g}_2 , \qquad \gamma = 0 . \tag{4.26}$$

The first condition means that (with our normalizations) the gauge couplings in the SU($\tilde{N}$) and U(1) sectors are the same, while the last condition implies that we consider a single-trace deformation superpotential in (2.2). Under these conditions the masses of the gauge bosons $(A_\mu)_l^k$ are

$$m_{\text{gauge}} = \tilde{g}_2 \sqrt{\frac{\xi_k + \xi_l}{2}} . \tag{4.27}$$

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Moreover, $\tilde{N}^2$ real dyons have the same masses. They are the $\mathcal{N} = 1$ superpartners of the massive gauge bosons. Another $2\tilde{N}^2$ real dyons form a $\tilde{N} \times \tilde{N}$ complex matrix. The masses of the elements of this matrix are

$$m_{KK'} = m_K + m_{K'}, \qquad K, K' = (N+1), ..., N_f. \qquad (4.28)$$

The remaining $4N\tilde{N}$ of dyons (which become bifundamentals in the limit (2.14)) have masses

$$m_{PK} = m_P - m_K, \qquad P = 1, ..., N, \qquad K = (N+1), ..., N_f. \qquad (4.29)$$

Again, we see that the dyons with masses (4.28) and (4.29) are much lighter than the gauge bosons and their scalar superpartners. It is the diagonal elements of the dyon matrix with the masses (4.28) that develop vacuum expectation values.

5 Strings and confined monopoles at large μ

Since in the dual theory (4.14) the dyons develop vacuum expectation values, see Eq. (3.4), this theory support strings. Consider the limit (2.14) in which the global color-flavor group (3.12) is restored and these strings become non-Abelian. As was discussed in Sec. 4.3, the mass terms of those dyons that develop VEVs are much smaller than the gauge boson masses in the dual gauge group $U(\tilde{N})$. Therefore, we deal with the type-I superconductor. A detailed discussion of the non-Abelian string solutions for this case will be presented elsewhere. Here we briefly mention certain peculiar features of such strings.

These strings are not BPS-saturated; their profile functions satisfy second-order equations of motion. These profile functions have logarithmic long-range tails formed by light dyonic scalars with masses (4.23) and (4.24), see [45] where Abelian strings in the extreme type-I superconducting vacuum were studied. The string tension in this regime is

$$T = \frac{4\pi|\xi|}{\log(\tilde{g}\mu/m)}, \qquad (5.1)$$

while their transverse sizes scale as

$$R \sim \frac{\log(\tilde{g}\mu/m)}{\tilde{g}\sqrt{\xi}}, \qquad (5.2)$$

with the logarithmic accuracy Here ξ is given in Eq. (4.21), and we assume that $\tilde{g}_2 \sim \tilde{g}_1 \sim \tilde{g}$.

As was mentioned in Sec. 3.2, the internal dynamics of the non-Abelian strings in the $\mathcal{N} = 2$ limit at small μ is qualitatively described by an $\mathcal{N} = (2,2)$ supersymmetric weighted CP model [13, 14, 15, 16], see also [37]. In the dual bulk theory, the string world-sheet model is $CP(N_f - 1)$ with $\tilde{N}$ positive charges associated with the orientational modes and N negative charges associated with string's size moduli (the latter are specific for semilocal string) [13, 16, 35, 36, 10, 11].

At large μ the semilocal strings at hand are no longer BPS-saturated. Their size moduli ρ_P are lifted, and the string tends to shrink in type-I superconductors and to expand in type-II superconductors [46, 34]. Remember, we deal with type I. Thus, the shrinkage of the semilocal strings results in conventional local strings. They are stable. The size moduli of the semilocal strings acquire masses of the order of

$$m_\rho \sim \frac{1}{\tilde{g}\sqrt{\xi}\,R^2} \sim \frac{\tilde{g}\sqrt{\xi}}{\log\left(\tilde{g}\mu/m\right)}\,, \tag{5.3}$$

cf. [46]. Then, the world-sheet theory effectively reduces to $CP(\tilde{N}-1)$ model which describes the orientational mode dynamics. In particular, as a matter of fact, the constraint (3.14) is replaced by

$$|n_K|^2 = 2\tilde{\beta}\,. \tag{5.4}$$

Another feature of the non-Abelian strings in the extreme type-I superconductors is that the coupling constant $\tilde{\beta}$ of the $CP(\tilde{N}-1)$ model becomes very large,

$$\tilde{\beta} \sim \frac{\tilde{g}^2 \mu}{m}\,. \tag{5.5}$$

This effect is due to the presence of a long-range tail in the string in the type-I superconductor. Using the one-loop renormalization equation in the asymptotically free $CP(\tilde{N}-1)$ model

$$4\pi\tilde{\beta}(\xi) = \tilde{N}\ln\frac{\sqrt{\xi}}{\Lambda_{CP}} \tag{5.6}$$

we find that the $CP(\tilde{N}-1)$ model scale becomes exponentially small,

$$\Lambda_{CP} \sim \sqrt{\xi}\exp\left(-\mathrm{const}\,\frac{\tilde{g}^2\mu}{m}\right)\,. \tag{5.7}$$

Now, it is time discuss confined monopoles of the bulk theory correspond-ing to kinks in the world-sheet CP model. At large μ the non-Abelian strings are no longer BPS-saturated, and, consequently, the $\mathcal{N} = (2,2)$ supersym-metry in the world-sheet CP model is lost. Non-supersymmetric $CP(\tilde{N}-1)$ model no longer has $\tilde{N}$ degenerate vacua, the true vacuum is unique, but the model has a family of quasi-vacua [47, 48]. The splittings are of the order of Λ_{CP}. Thus, $\tilde{N}$ different non-Abelian strings are split in their tensions. This implies two-dimensional confinement of monopoles, along the string [48], in addition to their permanent attachment to strings. The monopoles cannot move freely along the string. They are combined into monopole-antimonopole pairs, the attraction is due to the fact that the string between the monopole and antimonopole at hand has a slightly higher tension than the strings out-side.

However, this effect is tiny (because of the small value of the parameter Λ_{CP}) and does *not* determine the distance between the monopole and an-timonopole in the stringy meson in Fig. 1. This distance is determined by the classical string tension (5.1) itself (and the kink masses), rather than the tiny quantum differences between the tensions of different non-Abelian strings. Therefore, we will ignore this effect, the tension splitting.

Another effect which affects the formation of monopole-antimonopole stringy mesons at large μ is the lifting of the size moduli of the semilo-cal string, see (5.3). Although the kinks that are in the $(1, \tilde{N})$ representation of the global group (3.12) are still light (their masses are of the order of Λ_{CP}), the kinks in the $(N, 1)$ representation become heavier. We expect them to have masses of the order of the masses of the ρ-excitations (see (5.3)),

$$m_{(N,1)}^{\text{kink}} \sim \frac{\tilde{g}\sqrt{\xi}}{\log{(\tilde{g}\mu/m)}} \,. \tag{5.8}$$

These kinks (confined monopoles) form stringy mesons in the adjoint representation of the $SU(N)$ subgroup of the global group. We recall that the $(N^2-1, 1)$ stringy mesons are former (screened) quarks and gauge bosons of the original $\mathcal{N} = 1$ SQCD. As was already explained, below the crossover (at small $\sqrt{\mu m}$, see (4.8)) the quarks and gauge bosons decay into the monopole-antimonopole pairs and form stringy mesons $M_P^{P'}$, $P = 1, ..., N$, shown in Fig. 1.

From the kink mass formulas (5.3) and (5.1) and the string tension we

expect the mass of the $M_P^{P'}$ mesons to be

$$m_{M_P^{P'}} \sim \sqrt{T}\,, \tag{5.9}$$

provided the meson spins are of the order of unity. The masses of these stringy mesons are determined by the string tension, much in the same way as in the $\mathcal{N} = 2$ limit, see (3.16).

6   Relation to Seiberg's duality

The last but not the least topic to discuss is the relation between our duality (and the monopole confinement mechanism) at large μ and Seiberg's duality in $\mathcal{N} = 1$ SQCD [24, 25]. The light dyons D^{lA} of our $\mathrm{U}(\tilde{N})$ dual theory could be identified with Seiberg's "dual quarks". This is natural since they carry the same quantum numbers: both are in fundamental representations of the dual gauge group $\mathrm{U}(\tilde{N})$ and the global flavor group $\mathrm{SU}(N_f)$.

Moreover, the stringy mesons formed by the monopole-antimonopole pairs correspond to Seiberg's neutral mesons M_A^B, $A, B = 1, ..., N_f$, which are in the singlet or adjoint representations of global flavor group both in our and Seiberg's dual descriptions of $\mathcal{N} = 1$ SQCD. This conceptual similarity does not extend further, however. There is a crucial distinction: in our dual theory the stringy mesons are non-perturbative objects and are rather heavy, with masses determined by the string tension, (5.9). The dual gauge bosons and in particular, dyons D^{lA}, are much lighter, see (4.19), (4.20) and (4.23), (4.24) respectively.

At the same time, in Seiberg's dual theory, the M_A^B mesons appear as fundamental fields at the Lagrangian level and are light. As was already mentioned in Sec. 1, our understanding of these dramatic differences is that Seiberg's duality refers to N monopole vacua in which the meson fields M_A^B condense, making the dyons ("dual quarks") heavy [25, 28]. Let us briefly review how this happens. Consider the $\mathrm{U}(\tilde{N})$ version of Seiberg's dual theory with the superpotential

$$\mathcal{W}_S = \sqrt{2}\,(\tilde{D}_A D^B) M_B^A + \Lambda m_A\, M_A^A\,, \tag{6.1}$$

where we conjectured that Seiberg's "dual quarks" can be identified with our dyons D^{lA}. Following [25, 28], we assume that the M_A^B fields develop VEVs

making dyons heavy and integrate dyons out. The gluino condensation in the U($\tilde N$) gauge theory with no matter induces the superpotential

$$\mathcal{W}_S^{\text{eff}} = \tilde N\, \Lambda^{\frac{2\tilde N - N}{\tilde N}} \, (\det M)^{\frac{1}{\tilde N}} + \Lambda m_A\, M_A^A\,. \tag{6.2}$$

Strictly speaking the scale of Seiberg's dual theory should appear in the first term here. However, this scale is estimated to be of the order of the scale of the original $\mathcal{N} = 1$ QCD Λ [28], and in this estimate we do not distinguish between the two.

Minimizing this superpotential with respect to M_A^B we find

$$\langle M \rangle \sim \Lambda^{\frac{N - \tilde N}{N}} m^{\frac{\tilde N}{N}}\,. \tag{6.3}$$

The presence of N vacua in $\mathcal{N} = 1$ SQCD is well-known and follows e.g. from Witten's index. It is also known that these vacua are continuously connected to N monopole vacua of $\mathcal{N} = 2$ SQCD through the μ deformation [49, 50, 51, 32]. Since the "dual quarks" do not condense in these vacua the non-asymptotically free Seiberg's dual theory is in the Coulomb phase ("free dyonic phase"). This is true for energies above the scale of the M-field VEVs (6.3). Below this scale, all dyons decouple and Seiberg's dual theory becomes pure Yang-Milles theory with the U($\tilde N$) gauge group. It flows into the strong coupling, and the SU($\tilde N$) sector becomes confining. The U(1) gauge factor remains unbroken.

In Fig. 3 we show schematically the evolution of different vacua versus μ at small m. The vertical axis in this Figure corresponds to μ, while the horizontal axis schematically represents VEVs of various fields in the given vacuum. At small μ, near the Coulomb branch of $\mathcal{N} = 2$ SQCD, we have the U(1)^N Abelian gauge theories in the N monopole vacua. Condensation of the monopoles leads to formation of the electric ANO strings and Abelian confinement of quarks [2, 3]. One U(1) factor remains unbroken. At $\mu \sim \Lambda$ these vacua go through a crossover into the non-Abelian phase. In the limit of infinite μ they are described via Seiberg's dual theory. It is the U($\tilde N$) infrared-free non-Abelian gauge theory with neutral mesonic fields described by the superpotential (6.1) [24, 25, 28]. As was reviewed above, the M fields condense, and the theory is in the Coulomb phase for dyons D^{lA}.

Our dual theory applies to the $r = N$ quark vacuum of the original $\mathcal{N} = 1$ SQCD, rather than the monopole vacua. In the strong coupling regime at small m (described by a weakly coupled dual theory in the domain
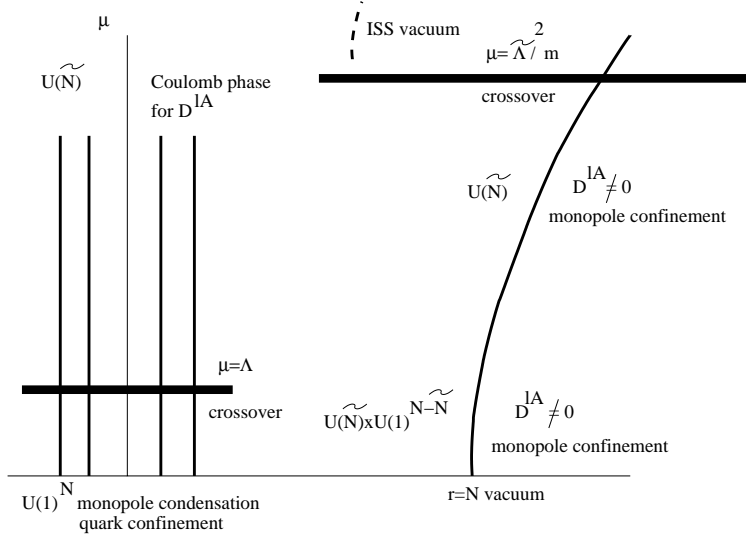
Figure 3: μ-evolution of different vacua at small m. N monopole vacua are shown by thick solid lines on the left, while $r = N$ vacuum is on the right. The ISS vacuum is shown by thick dashed line. Gauge groups in different regimes are indicated as well as condensed or confined states.

(4.8)) the light dyons D^{lA} condense in this vacuum, triggering formation of the non-Abelian strings with confinement of monopoles ensuing automatically. This vacuum has dyon condensate proportional to $\sqrt{\mu m}$, see (4.6), and represents a run-away vacuum not seen in Seiberg's dual description, where μ is considered to be strictly infinite.

There exists a bunch of other "hybrid" vacua in the theory, in which at small μ we deal with $r < N$ quarks and some monopoles condensing. All of them have unbroken U(1) gauge group [52]. We do not study them in this paper.

A few words about the relation of our $r = N$ vacuum to the Intriligator–Seiberg–Shih vacuum [27]. This vacuum looks rather similar to ours. The dyons D^{lA} (Seiberg's "dual quarks") condense in both of these vacua. However, clearly these vacua are different. In particular, for a generic choice of the quark masses, supersymmetry is broken in the ISS vacuum, while the $r = N$ vacuum is supersymmetric. Also, the ISS vacuum has dyon VEVs of the order of $\sqrt{m\Lambda}$ while in the $r = N$ vacuum they are much larger, proportional to $\sqrt{\mu m}$. Still, the presence of the D^{lA} condensate indicates that the ISS vacuum could have physics similar to that in our $r = N$ vacuum.

In particular, it could exhibit confinement of monopoles and a phenomenon similar to our decay of quarks and gauge bosons of the original $\mathcal{N} = 1$ SQCD into the monopole-antimonopole stringy mesons. Since the ISS vacuum is not supersymmetric, it may not exist at all μ. This would explain why we do not see this vacuum in our dual theory (4.14) in the domain (4.18). We show this vacuum by the dashed line in Fig. 3. The fate of the ISS vacuum in the framework of our construction calls for further studies.

7   Conclusions

Let us summarize our findings. We started from our recent development of the non-Abelian duality in the *quark vacua* of $\mathcal{N} = 2$ super-Yang–Mills theory with the U(N) gauge group and N_f flavors ($N_f > N$). The fact that $N_f > N$ is very crucial, as will be emphasized below. The quark mass terms are introduced in a judiciously chosen way. Instead of the Fayet–Iliopoulos term of the D type, as previously, we introduce it through a superpotential (i.e. F type). We construct dual pairs. Both theories from the dual pair support non-Abelian strings which confine monopoles.

Next we undertake a next step, basically in the uncharted waters. we introduce an $\mathcal{N} = 2$ -breaking deformation, a mass term $\mu \mathcal{A}^2$ for the adjoint fields. Our final goal is to make the adjoint fields heavy and thus pass to $\mathcal{N} = 1$ SQCD.

Starting from a small deformation we eventually make it large which enforces complete decoupling of the adjoint fields. We show that the $\mathcal{N} = 2$ non-Abelian duality fully survives in the limit of $\mathcal{N} = 1$ SQCD, albeit some technicalities change. For instance, non-Abelian strings which used to be BPS-saturated in the $\mathcal{N} = 2$ limit, cease to be saturated in $\mathcal{N} = 1$ SQCD. They become strings typical of the extreme type-I superconducting regime.

Our duality is a distant relative of Seiberg's duality in $\mathcal{N} = 1$ SQCD. Both share common features but have many drastic distinctions. This is due to the fact that Seiberg's duality apply to the monopole rather than quark vacua.

More specifically, in our theory we deal with $N < N_f < \frac{3}{2}N$ massive quark flavors. We consider the vacuum in which N squarks condense. Then we identify a crossover transition from weak to strong coupling. At strong coupling we find a dual theory, U($N_f - N$) SQCD, with N_f light dyon flavors. The dual theory is at weak coupling provided μm is small enough (at large μ

this requires taking m to be rather small). Condensation of light dyons D^{lA} in this theory triggers the formation of non-Abelian strings and confinement of monopoles. Quarks and gauge bosons of the original $\mathcal{N} = 1$ SQCD decay into the monopole-antimonopole pairs on CMS and form stringy mesons shown in Fig. 1.

We would like to stress that the condition $\tilde{N} > 1$ is crucial for our construction. As was explained in Sec. 4, the presence of the dual non-Abelian group allows us to increase μ, eventually decoupling the adjoint field and, simultaneously, keeping the dual theory at weak coupling. The reason is that we can take quark masses rather small to satisfy the weak coupling condition (4.8). If the dual gauge group were Abelian, the light matter VEV's would be of the order of $\sqrt{\mu \Lambda_{\mathcal{N}=2}}$, hence the theory would go into the strong coupling regime once we increase μ above $\Lambda_{\mathcal{N}=2}$.

Acknowledgments

Appendix:
U(3) theory with $N_f = 5$ at small μ

In this Appendix following [10] we consider specific example of U(3) gauge theory with $N_f = 5$ quark flavors (so that $N = 3$, $\tilde{N} = 2$) and present the low-energy dual theory at small values of FI parameter, see (3.1). The gauge group (3.2) in this case has the form

$$\mathrm{U}(2) \times \mathrm{U}(1)_8 \times \mathrm{U}(1) \,, \tag{A.1}$$

where $\mathrm{U}(1)_8$ denotes a U(1) factor of the gauge group which is associated with T_8 generator of the U(3) gauge group of the original theory.

The bosonic part of the effective low-energy action of the theory in the

domain (3.1) has the form

$$S_{\text{dual}} = \int d^4x \left[\frac{1}{4\tilde{g}_2^2} \left(F_{\mu\nu}^p\right)^2 + \frac{1}{4g_1^2} \left(F_{\mu\nu}\right)^2 + \frac{1}{4\tilde{g}_8^2} \left(F_{\mu\nu}^8\right)^2 + \frac{1}{\tilde{g}_2^2} |\partial_\mu b^p|^2 \right.$$

$$+ \frac{1}{g_1^2} |\partial_\mu a|^2 + \frac{1}{\tilde{g}_8^2} |\partial_\mu b^8|^2 + \left|\nabla_\mu^1 D^A\right|^2 + \left|\nabla_\mu^1 \tilde{D}_A\right|^2 + \left|\nabla_\mu^2 D^3\right|^2 + \left|\nabla_\mu^2 \tilde{D}_3\right|^2$$

$$\left. + V(D, \tilde{D}, b^p, b^8, a) \right], \tag{A.2}$$

Here B_μ^p ($p = 1, 2, 3$), B_μ^8 and A_μ are gauge fields of (A.1), while $F_{\mu\nu}^p$, $F_{\mu\nu}^8$ and $F_{\mu\nu}$ are their field strengths. Their scalar $\mathcal{N} = 2$ superpartners b^p and b^8 in terms of the fields of the original theory (2.4) have the form

$$b^3 = \frac{1}{\sqrt{2}} (a^3 + a_D^3) \quad \text{for} \quad p = 3, \qquad b^8 = \frac{1}{\sqrt{10}} (a^8 + 3a_D^8), \tag{A.3}$$

where subscript D means dual scalar fields [2, 3], while field a is the same as in (2.4). Covariant derivatives are defined in accordance with the charges of the D^{lA} and D^3 dyons, see [10] for more details. Namely,

$$\nabla_\mu^1 = = \partial_\mu - i \left(\frac{1}{2} A_\mu + \sqrt{2} B_\mu^p \frac{\tau^p}{2} + \frac{1}{2}\sqrt{\frac{10}{3}} B_\mu^8 \right),$$

$$\nabla_\mu^2 = = \partial_\mu - i \left(\frac{1}{2} A_\mu - \sqrt{\frac{10}{3}} B_\mu^8 \right). \tag{A.4}$$

The coupling constants g_1, $\tilde{g}_8$ and $\tilde{g}_2$ correspond to two U(1) and the SU(2) gauge groups, respectively. The scalar potential $V(D, \tilde{D}, b^p, b^8, a)$ in the ac-

tion (A.2) is

$$
V(D, \tilde{D}, b^p, b^8, a) = \frac{\tilde{g}_2^2}{4} \left(\bar{D}_A \tau^p D_A - \tilde{D}_A \tau^p \bar{\tilde{D}}^A \right)^2
$$

$$
+ \quad \frac{10}{3} \frac{\tilde{g}_8^2}{8} \left(|D^A|^2 - |\tilde{D}_A|^2 - 2|D^3|^2 + 2|\tilde{D}_3|^2 \right)^2
$$

$$
+ \quad \frac{\tilde{g}_1^2}{8} \left(|D^A|^2 - |\tilde{D}_A|^2 + |D^3|^2 - |\tilde{D}_3|^2 \right)^2
$$

$$
+ \quad \frac{\tilde{g}_2^2}{2} \left| \sqrt{2} \tilde{D}_A \tau^p D_A + \sqrt{2} \, \frac{\partial \mathcal{W}_\mu}{\partial b^p} \right|^2 + \frac{\tilde{g}_1^2}{2} \left| \tilde{D}_A D^A + \tilde{D}_3 D_3 + \sqrt{2} \, \frac{\partial \mathcal{W}_\mu}{\partial a} \right|^2
$$

$$
+ \quad \frac{\tilde{g}_8^2}{2} \left| \sqrt{\frac{10}{3}} \tilde{D}_A D^A - 2\sqrt{\frac{10}{3}} \tilde{D}_3 D^3 + \sqrt{2} \, \frac{\partial \mathcal{W}_\mu}{\partial b^8} \right|^2
$$

$$
+ \quad \frac{1}{2} \left\{ \left| (a + \tau^p \sqrt{2} \, b^p + \sqrt{\frac{10}{3}} b^8 + \sqrt{2} m_A) D^A \right|^2 \right.
$$

$$
+ \quad \left| (a + \tau^p \sqrt{2} \, b^p + \sqrt{\frac{10}{3}} b^8 + \sqrt{2} m_A) \bar{\tilde{D}}_A \right|^2
$$

$$
+ \quad \left| a - 2\sqrt{\frac{10}{3}} b^8 + \sqrt{2} m_3 \right|^2 \left(|D^3|^2 + |\tilde{D}_3|^2 \right) \left. \right\} . \tag{A.5}
$$

The theory (A.2) is at weak coupling in the domain (3.1) but the derivatives of the superpotential (2.2) entering in (A.5) (which determine VEVs of dyons) are rather complicated functions of fields a, b^8 and b^p. In [31] we used the exact Seiberg-Witten solution of our theory to determine these derivatives in $r = N$ vacuum. Here we briefly review this calculation.

First we make a quantum generalization

$$
\frac{\partial \mathcal{W}_\mu}{\partial b^p} \to \mu_2 \frac{\partial u_2}{\partial b^p}, \qquad \frac{\partial \mathcal{W}_\mu}{\partial b^8} \to \mu_2 \frac{\partial u_2}{\partial b^8}, \qquad \frac{\partial \mathcal{W}_\mu}{\partial a} \to \mu_1 \sqrt{\frac{2}{N}} \frac{\partial u_2}{\partial a}, \quad \text{(A.6)}
$$

where

$$u_k = \langle \mathrm{Tr} \left(\frac{1}{2} a + T^a a^a \right)^k \rangle, \qquad k = 1, ..., N,$$ (A.7)

are gauge invariant parameters which describe the Coulomb branch.

To select the desired vacuum (1,2,3) (which transforms into (4,5,3) vacuum below crossover) among all other vacua in the Seiberg-Witten curve we require that the curve has the factorized form (3.9), while the double roots e_P are semiclassically (at large masses) are given by mass parameters, $\sqrt{2} e_P \approx -m_P$, $P = 1, ..., N$.

Using explicit expressions from [53, 54, 55, 56] which express derivatives of u_k with respect to scalar fields a^a ($a = 1, 2, 3$) of the original theory (2.4) and taking into account monodromies which convert these derivatives into derivatives with respect to b^p, b^8 and a [31, 10] we obtain

$$\frac{\partial u_2}{\partial a} = e_1 + e_2 + e_3, \qquad \frac{1}{\sqrt{2}} \frac{\partial u_2}{\partial b^3} = e_1 - e_2,$$

$$\frac{1}{\sqrt{10}} \frac{\partial u_2}{\partial b^8} = \frac{1}{\sqrt{3}} (e_1 + e_2 - 2 e_3),$$ (A.8)

where e_P are double roots of the Seiberg-Witten curve (3.7) with shifted masses (3.8) for the case $N = 3$, $\tilde{N} = 2$.

Vacua of the theory (A.2) are determined by zeros of all D and F-terms in (A.5). Using the derivatives of the superpotential (2.2) obtained above we get the VEV's of dyons in the form

$$\langle D^{lA} \rangle = \langle \bar{\tilde{D}}^{lA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & \sqrt{\xi_1} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\xi_2} \end{pmatrix},$$

$$\langle D^3 \rangle = \langle \bar{\tilde{D}}^3 \rangle = \sqrt{\frac{\xi_3}{2}},$$ (A.9)

where FI parameters ξ_P are determined by (3.6). The obvious generalization of this formula to an arbitrary N and $\tilde{N}$ gives Eq. (3.4) quoted in the main text.

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