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# One-loop matching of improved four-fermion staggered operators with an improved gluon action

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We present results for one-loop matching factors of four-fermion operators composed of HYP-smearred staggered fermions. We generalize previous calculations by using the tree-level improved Symanzik gauge action. These results are needed for our companion numerical calculation of  $B_K$  and related matrix elements. We find that the impact on one-loop matching factors of using the improved gluon action is much smaller than that from the use of either HYP smearing or mean-field improvement. The one-loop coefficients for mean-field improved, HYP-smearred operators with the Symanzik gauge action have a maximum magnitude of  $O(1) \times \alpha_s$ , indicating that perturbation theory is reasonably convergent.

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## I. INTRODUCTION

Numerical simulations of lattice QCD are now able to calculate a range of phenomenologically interesting non-perturbative quantities with high precision. Of particular interest are hadronic matrix elements of operators that appear in the electroweak Hamiltonian, or in extensions of the standard model. For such quantities it is necessary (in order to make use of Wilson coefficients calculated in continuum perturbation theory) to determine the matching factors which relate operators regularized on the lattice with those regularized in the continuum. For the latter one typically uses naive dimensional regularization (NDR) with  $\overline{\text{MS}}$  subtraction.

In this paper we calculate matching factors (which are, in general, matrices) for four-fermion operators composed of light staggered quarks. These arise, for example, in the calculation of the  $K^0 - \bar{K}^0$  mixing parameter  $B_K$ . Their flavor structure forbids mixing with lower-dimensional operators, so the matching is only between operators of dimension 6. In the electroweak theory, the operator that arises has a “left-left” structure, due to the left-handed coupling of the W-bosons. In extensions of the standard model, however,  $\Delta S = 2$  operators can arise with other Dirac structures. For this reason we calculate matching factors for all possible Dirac structures.

In recent years, it has become increasingly common to determine matching factors non-perturbatively, either us-

ing the Rome-Southampton non-perturbative renormalization method [1], or using approaches based on the Schrödinger functional [2]. These methods replace hard-to-estimate truncation errors by controllable statistical and systematic errors. We are implementing such calculations for improved staggered fermions, but have so far only obtained results for bilinear operators [3]. We expect that the implementation for four-fermion operators, which involves mixing with a long list of lattice operators, will be more challenging. The use of one-loop matching is a useful intermediate step, and, as we will describe, the necessary calculations are relatively simple generalizations of previous work. We also note that, since our present numerical calculations involve very small lattices spacings ( $a \approx 0.045$  fm), the truncation errors are quite small, since they are proportional to  $\alpha_s^2$  with  $\alpha_s$  evaluated at a scale  $\approx 1/a$  [4].

Our companion numerical calculations use valence staggered fermions which have been improved by the use of HYP-smearred links (links replaced with hypercubic blocked links [5]). The ensembles are those generated by the MILC collaboration [6], in which the gauge action is Symanzik-improved, and the quark action is the asqtad staggered action. Previous calculations have obtained the matching factors for four-fermion operators composed of HYP-smearred staggered fermions [7], but only using the Wilson gauge action. Here we generalize these results to the case of an improved gauge action. This extends our earlier work in which we calculated matching factors for bilinear operators using improved gauge actions and HYP-smearred staggered fermions [8].

At first sight, the generalization from the Wilson gauge action (for which, in Feynman gauge, the gluon propagator is diagonal in Euclidean indices) to an improved gauge action (in which the propagator is not diagonal) appears non-trivial. In particular, one-loop calculations using unimproved staggered fermions [9, 10] were simplified us-

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ing the diagonal nature of the gluon propagator. The inclusion of smeared links, however, leads to the natural introduction of a “composite gluon propagator” which represents both the smearing and the gluon propagator itself. This propagator contains non-zero off-diagonal elements, and so calculations of one-loop matching factors for HYP-smeared staggered fermions must already deal with the presence of such elements [7, 11]. This means that the generalization to an improved gluon propagator requires no change to the analytic expressions—all one needs to change is the composite gluon propagator before numerical evaluation of the loop integral. As we discuss here, this simplification holds not only for bilinear operators [8], but also for four-fermion operators.

The paper is organized as follows. In Sec. II, we recall our notation and conventions for actions and operators. In Sec. III, we present the Feynman diagrams and describe their evaluation. Because we are building on the work of Refs. [7] and [8], we provide only a minimal discussion of technical details. In Sec. IV, we present our numerical results for matching factors, providing the complete matching matrix for the operators relevant to  $B_K$ , and a partial matrix (the part that will likely be used in practice) for other four-fermion operators. We conclude briefly in Sec. V.

## II. ACTIONS AND OPERATORS

The HYP-smeared staggered action has the same form as the unimproved staggered fermion action,

$$S_{\text{HYP}} = \sum_n \bar{\chi}(n) \left[ \sum_\mu \eta_\mu(n) \nabla_\mu^{\text{H}} + m \right] \chi(n), \quad (1)$$

where  $\eta_\mu(n) = (-1)^{n_1 + \dots + n_{\mu-1}}$ , and the covariant difference operator is

$$\nabla_\mu^{\text{H}} \chi(n) = \frac{1}{2} [V_\mu(n) \chi(n + \hat{\mu}) - V_\mu^\dagger(n - \hat{\mu}) \chi(n - \hat{\mu})]. \quad (2)$$

Here and in the following we set the lattice spacing  $a$  to unity, except where confusion could arise. HYP improvement consists of using HYP-smeared links,  $V_\mu$ , instead of the original “thin” links,  $U_\mu$ . We set the HYP-smearing parameters to the values that remove the tree-level coupling of quarks to gluons having one or more components of momenta equal to  $\pi$ . These values are  $\alpha_1 = 0.875$ ,  $\alpha_2 = 4/7$  and  $\alpha_3 = 0.25$ , in the notation of Ref. [5]. These are the values used in our numerical simulations.

After gauge-fixing, we expand both the thin and smeared links in the usual way,

$$U_\mu(n) = \exp[i g_0 A_\mu(n + \hat{\mu}/2)], \quad (3)$$

$$V_\mu(n) = \exp[i g_0 B_\mu(n + \hat{\mu}/2)]. \quad (4)$$

where  $g_0$  is the bare gauge coupling. The relation between the fluctuations of smeared and thin links can be

written

$$B_\mu(n + \hat{\mu}/2) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \sum_\nu h_{\mu\nu}(k) A_\nu(k) e^{ik \cdot (n + \hat{\mu}/2)} + \mathcal{O}(A^2). \quad (5)$$

Here,  $h_{\mu\nu}(k)$  is the smearing kernel, which depends on the smearing parameters and the details of the HYP construction. It contains non-zero off-diagonal components because a smeared link in one direction contains contributions from thin links in all four directions. It turns out that we need only the linear term in Eq. (5) in a one-loop calculation. The contribution of the quadratic term (which gives rise to tadpole diagrams) turns out to vanish due to the projection back into the SU(3) group that is part of the definition of HYP-smearing [9, 12, 13].

The smearing kernel  $h_{\mu\nu}$  can be conveniently decomposed into diagonal and off-diagonal parts:

$$h_{\mu\nu}(k) = \delta_{\mu\nu} D_\mu(k) + (1 - \delta_{\mu\nu}) \bar{s}_\mu \bar{s}_\nu \tilde{G}_{\nu,\mu}(k), \quad (6)$$

with  $\bar{s}_\mu = \sin(k_\mu/2)$ , and

$$D_\mu(k) = 1 - \sum_{\nu \neq \mu} \bar{s}_\nu^2 + \sum_{\substack{\nu < \rho \\ \nu, \rho \neq \mu}} \bar{s}_\nu^2 \bar{s}_\rho^2 - \bar{s}_\nu^2 \bar{s}_\rho^2 \bar{s}_\sigma^2, \quad (7)$$

$$\tilde{G}_{\nu,\mu}(k) = 1 - \frac{(\bar{s}_\rho^2 + \bar{s}_\sigma^2)}{2} + \frac{\bar{s}_\rho^2 \bar{s}_\sigma^2}{3}. \quad (8)$$

Here  $\mu, \nu, \rho$ , and  $\sigma$  all differ from each other. By contrast, the smearing kernel for an action containing the original thin links is simply  $h_{\mu\nu} = \delta_{\mu\nu}$ .

We use the tree-level Symanzik-improved gluon action [14, 15];

$$S_g = \frac{6}{g_0^2} \left[ \frac{5}{3} \sum_{\text{pl}} \frac{\text{ReTr}(1 - U_{\text{pl}})}{3} - \frac{1}{12} \sum_{\text{rt}} \frac{\text{ReTr}(1 - U_{\text{rt}})}{3} \right], \quad (9)$$

where “pl” and “rt” represent plaquette and rectangle, respectively. In fact, the MILC collaboration use the (partial) one-loop Symanzik-improved action determined in Refs. [16, 17]. However, the one-loop contributions to this action contribute to matching factors of valence fermionic operators only at two-loop level, so the consistent choice for our one-loop calculation is the tree-level action (9). For purposes of comparison, we also use the Wilson gluon action, which is obtained from Eq. (9) by dropping the rectangle term and setting the coefficient of the plaquette to unity instead of 5/3.

Since we use MILC asqtad ensembles in our numerical studies, the sea quarks are asqtad staggered fermions rather than HYP-smeared. We do not display the sea-quark action, however, since sea-quarks only enter at two-loop order in the matching of valence fermionic operators. Our one-loop matching factors are thus valid for any choice of sea quarks.

We now turn to the definitions of our lattice four-fermion operators, which are the same as those used in Ref. [7]. Our construction follows the hypercube method

of Ref. [18]. The operators come in two classes, differing in the contractions of their color indices. First we have *one color-trace* operators, labeled with a subscript  $I$ :

$$\begin{aligned} [S \times F][S' \times F']_I(y) = & \\ \frac{1}{4^4} \sum_{A,B,A',B'} & [\bar{\chi}_a^{(1)}(2y+A) \overline{(\gamma_S \otimes \xi_F)}_{AB} \chi_b^{(2)}(2y+B)] \\ \times [\bar{\chi}_{a'}^{(3)}(2y+A') \overline{(\gamma_{S'} \otimes \xi_{F'})}_{A'B'} \chi_{b'}^{(4)}(2y+B')] & \\ \times \mathcal{V}^{ab'}(2y+A, 2y+B') \mathcal{V}^{a'b}(2y+A', 2y+B). & \quad (10) \end{aligned}$$

Here,  $y \in \mathbf{Z}^4$  is the coordinate of  $2^4$  hypercubes. Hypercube vectors<sup>1</sup>  $S$  and  $S'$  denote the spins of the component bilinears, while  $F$  and  $F'$  denote the tastes. Indices  $a$ ,  $b$ ,  $a'$ , and  $b'$  denote colors, while superscripts  $(i)$  for  $i = 1, 2, 3, 4$  label different flavors (not tastes). Using four different flavors forbids penguin diagrams, which would lead to mixing with lower-dimension operators.<sup>2</sup> Two “fat” Wilson lines  $\mathcal{V}^{ab'}(2y+A, 2y+B')$  and  $\mathcal{V}^{a'b}(2y+A', 2y+B)$  ensure the gauge invariance of the four-fermion operators. A fat Wilson line  $\mathcal{V}^{ab'}(2y+A, 2y+B')$ , for example, is constructed by averaging over all the shortest paths connecting  $2y+A$  and  $2y+B'$ , with each path formed by products of HYP-smear links  $V_\mu$ . When we use the unimproved staggered action the Wilson lines are composed of unsmeared thin links,  $U_\mu$ .

The second class are the *two color-trace* operators, for which we use the subscript  $II$ :

$$\begin{aligned} [S \times F][S' \times F']_{II}(y) = & \\ \frac{1}{4^4} \sum_{A,B,A',B'} & [\bar{\chi}_a^{(1)}(2y+A) \overline{(\gamma_S \otimes \xi_F)}_{AB} \chi_b^{(2)}(2y+B)] \\ \times [\bar{\chi}_{a'}^{(3)}(2y+A') \overline{(\gamma_{S'} \otimes \xi_{F'})}_{A'B'} \chi_{b'}^{(4)}(2y+B')] & \\ \times \mathcal{V}^{ab}(2y+A, 2y+B) \mathcal{V}^{a'b'}(2y+A', 2y+B'). & \quad (11) \end{aligned}$$

These operators differ from those with one color-trace only by the choice of fat Wilson lines—here they connect within each bilinear, whereas for the one color-trace operators they connect between bilinears.

We also consider mean-field improvement of the staggered action and operators following Refs. [9, 19–21]. This is also referred to as tadpole improvement. Mean-field improvement is achieved by rescaling the staggered fields and the links. In the case of HYP-smear staggered fermions the rescaling is

$$\chi \rightarrow \psi = \sqrt{v_0} \chi, \quad (12)$$

$$\bar{\chi} \rightarrow \bar{\psi} = \sqrt{v_0} \bar{\chi}, \quad (13)$$

$$V_\mu \rightarrow \tilde{V}_\mu = V_\mu / v_0, \quad (14)$$

$$v_0 \equiv \left[ \frac{1}{3} \text{ReTr} \langle V_{\text{pl}} \rangle \right]^{1/4}, \quad (15)$$

with  $V_{\text{pl}}$  the plaquette composed of HYP-smear links. One then constructs the operators described above out of  $\psi$ ,  $\bar{\psi}$ , and  $\tilde{V}_\mu$ . The resulting operators are expected to be closer to their continuum counterparts because the rescaled links fluctuate around unity. Mean-field improvement can be implemented in simulations after the data has been collected, as long as the contributions to the four-fermion operators having different numbers of links are stored separately.

### III. FEYNMAN DIAGRAMS AND THEIR EVALUATION

Feynman rules for the gauge and staggered-fermion actions, and for insertions of the four-fermion operators, can be found in literature and we do not reproduce them here. The rules for unimproved staggered fermions are given in Refs. [9, 19, 22], and the generalization to HYP-smear staggered fermions can be found in Ref. [7, 11]. The gluon propagator for the Symanzik action was determined in Ref. [14]; we use the simpler form presented in our earlier work [12].<sup>3</sup>

We show the Feynman diagrams contributing to one-loop matching factors to the two types of four-fermion operators in Figs. 1 and 2. Analytic formulae for these diagrams for HYP-smear staggered fermions with the Wilson gluon action are given in Ref. [7]. We do not repeat these results here, since it turns out, as already mentioned in the Introduction, that the generalization to the improved gluon action is relatively simple. Instead we explain the recipe by which the results of Ref. [7] can be generalized.

The key point is that, since all gauge links are HYP-smear (whether in the action or the operators), the gluon propagator always comes with a smearing kernel on each end. Thus what appears is the *composite gluon propagator* (called the “smeared-smeared propagator” in Ref. [12]):

$$\mathcal{T}_{\mu\nu}(k) \equiv \sum_{\alpha\beta} h_{\mu\alpha}(k) h_{\nu\beta}(k) \mathcal{D}_{\alpha\beta}(k). \quad (16)$$

Here  $\mu$  and  $\nu$  are the directions of the initial and final smeared gauge links,  $h$  is given in Eq. (6), and  $\mathcal{D}_{\alpha\beta}(k)$  is the gluon propagator in Feynman gauge. Even with the Wilson gauge action, where  $\mathcal{D}$  is diagonal, the fact that  $h$  has non-vanishing off-diagonal elements implies that  $\mathcal{T}$  does too. Thus the generalization to the Symanzik gauge action, for which  $\mathcal{D}$  itself has non-vanishing off-diagonal elements, does not introduce any fundamentally new types of contribution to  $\mathcal{T}$ . Of course, the expression for  $\mathcal{T}$  is much more involved, but this does not present

<sup>1</sup> These are vectors whose entries are 0 or 1.

<sup>2</sup> The relation of these four-flavor operators to the  $\Delta S = 2$  continuum operators is discussed below.

<sup>3</sup> To be precise, we use the formulae of Appendix A of Ref. [12] with  $\omega = 1$ ,  $c = -1/12$  and  $c' = 0$ . The result for the Wilson gauge action is obtained by further setting  $c = 0$ .

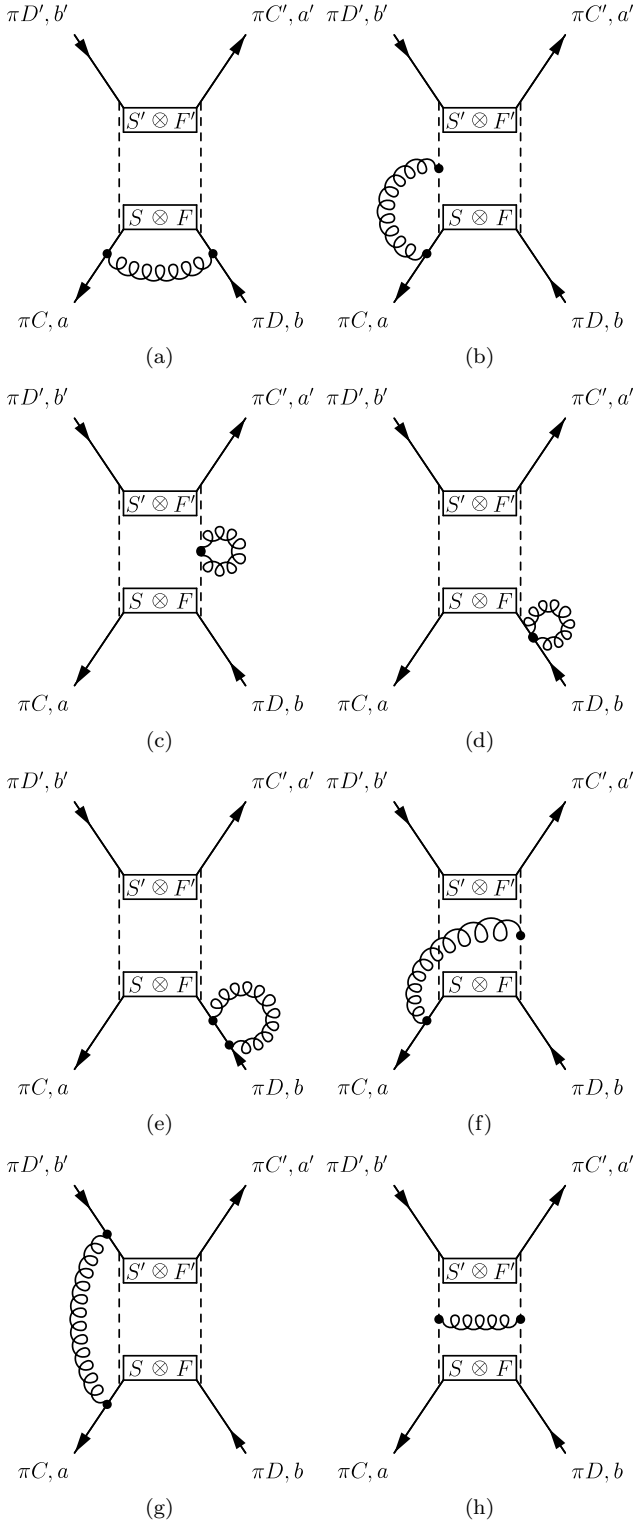


FIG. 1. Feynman diagrams contributing to the one-loop matrix elements of one color-trace operators. We show only one diagram of each type. Hypercube vectors ( $C$ ,  $D$ ,  $C'$ , and  $D'$ ) multiplied by  $\pi$  denote external quark momenta.  $a$ ,  $b$ ,  $a'$ , and  $b'$  are color indices. Dashed lines indicate the Wilson lines which make the four-fermion operator gauge invariant. Boxes indicate the hypercube bilinears of which the four-fermion operator is composed.

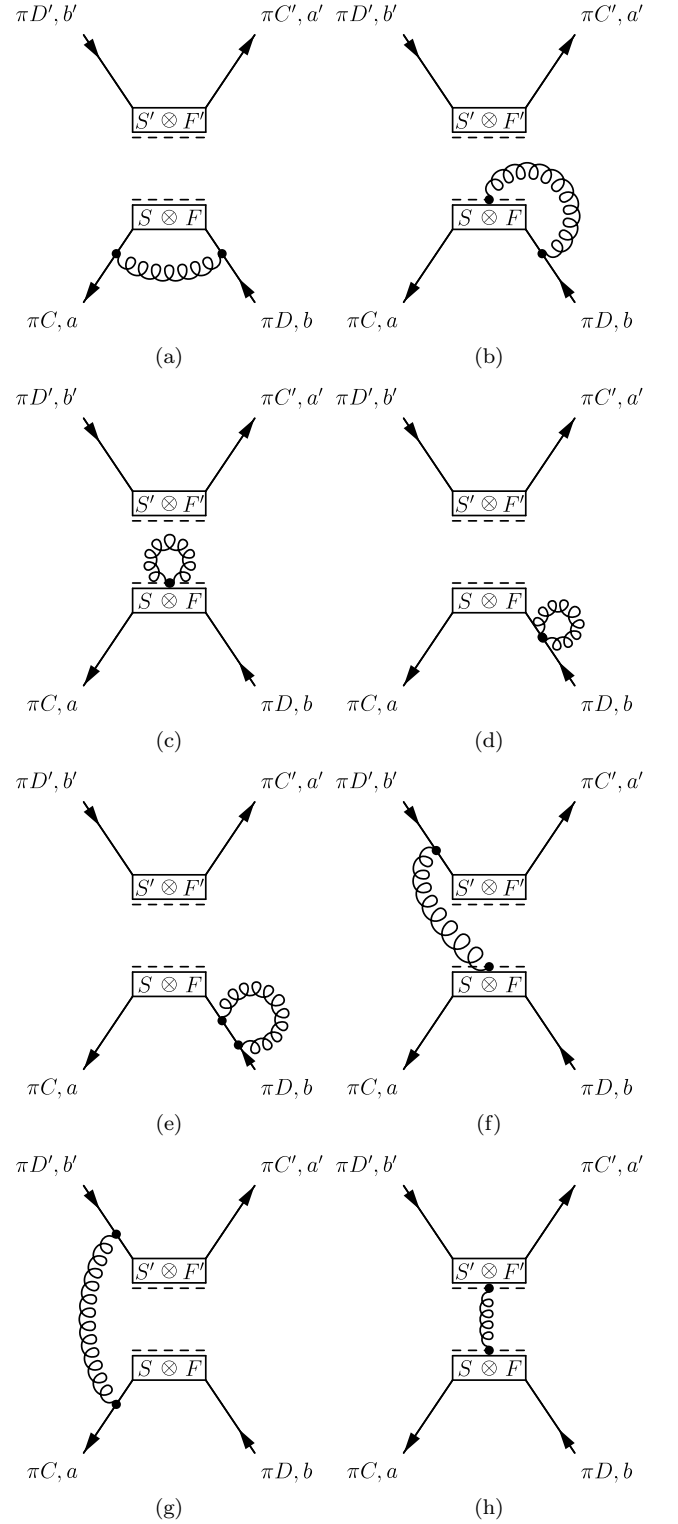


FIG. 2. Feynman diagrams contributing to the one-loop matrix elements of two color-trace operators. Notation is as in Fig. 1.

problems since the expression is evaluated numerically when doing the loop integral. This situation is in contrast to what happens if the action and operators are composed of thin links, for then  $\mathcal{T}$  is diagonal, which greatly simplifies the resulting expressions.

In order to simplify the expressions for Feynman diagrams, Ref. [7] used the following properties of  $\mathcal{T}_{\mu\nu}$ : it is symmetric, and its off-diagonal elements are proportional to  $\bar{s}_\mu \bar{s}_\nu$  multiplied by a function even in each of the components of  $k_\mu$ . For the Wilson gauge action, with diagonal  $\mathcal{D}$ , these properties follow from the fact that  $h_{\mu\nu}$  has the same properties. For the Symanzik gauge action, it turns out that  $\mathcal{D}$  also has these properties, from which it is simple to show that  $\mathcal{T}$  does too. Thus the simplifications used in Ref. [7] apply for both gauge actions.

We now describe how the analytic formulae of Ref. [7] must be changed when using the Symanzik action.<sup>4</sup> Two independent approaches to the matching calculation were used in that work. In the first, explicit expressions were given for all diagrams (Appendices A, B and C of Ref. [7]). To obtain the expressions for the Symanzik gauge action one must make the following replacement:

$$\sum_\lambda h_{\mu\lambda} h_{\nu\lambda} \rightarrow (4 \sum_\rho \bar{s}_\rho^2) \sum_{\alpha\beta} h_{\mu\alpha} h_{\nu\beta} \mathcal{D}_{\alpha\beta}^{\text{Imp}}. \quad (17)$$

In the second method (Appendix D of Ref. [7]), maximal use was made of the matching calculation for bilinear operators. For this part of the calculation, one can simply use our results for matching factors of bilinear operators with the Symanzik gauge action [8]. For two classes of diagrams [those of Figs. 2(f) and (h)], bilinear results are not sufficient, and for these Ref. [7] gives explicit expressions. These are written in terms of the diagonal and off-diagonal parts of  $\mathcal{T}$ , and in particular in terms of  $P_\mu$  and  $O_{\mu\nu}$  defined through

$$\mathcal{T}_{\mu\nu} = \frac{\delta_{\mu\nu} P_\mu + (1 - \delta_{\mu\nu}) 4 \bar{s}_\mu \bar{s}_\nu O_{\mu\nu}}{4 \sum_\mu \bar{s}_\mu^2}, \quad (18)$$

(where repeated indices are not summed). Here the recipe is to construct  $\mathcal{T}_{\mu\nu}$ , Eq. (16), using the Symanzik gluon propagator, use this in Eq. (18) to obtain new expressions for  $P_\mu$  and  $O_{\mu\nu}$ , and use the latter in the results of Ref. [7].

As in Ref. [7], we evaluate matching factors using both methods described above and find agreement. This is a non-trivial check on the numerical implementation of the analytic expressions. We have also checked the relations which follow from Fierz identities and from the  $U(1)_\epsilon$  symmetry of staggered fermions.

<sup>4</sup> As discussed in Ref. [8], the simple recipe described here does not work if one uses the asqtad action because not every diagram can be expressed in terms of the composite gluon propagator (due to the presence of the Naik term). Some diagrams would need to be calculated anew.

## IV. MATCHING FACTORS

We calculate the matching factors in the usual way by evaluating the  $qq\bar{q}\bar{q}$  matrix elements of the operators both on the lattice and in the continuum, and projecting onto the different color and spin-taste contributions. We do so at one-loop order, requiring the evaluation of the diagrams of Figs. 1 and 2 on the lattice. On the continuum side, only diagrams of types (a) and (g) contribute, since the continuum four-fermion operators do not contain gauge fields. In the continuum calculation one must choose the continuation to  $4 + \epsilon$  dimensions of the operators and the projectors onto different spin structures. We follow the conventions described in Refs. [7] and [23].

The matching formula between continuum and lattice-regularized operators then takes the general form

$$\mathcal{O}_i^{\text{Cont}}(\mu) = \sum_j Z_{ij}(\mu, a) \mathcal{O}_j^{\text{Lat}}(1/a), \quad (19)$$

with  $\mu$  the continuum regularization scale, and the lattice spacing now made explicit. At one-loop order, and with a suitable choice of lattice operators, the matching factor has the form

$$Z_{ij} = \delta_{ij} + \frac{g^2}{(4\pi)^2} \left[ -\gamma_{ij} \log(\mu a) + c_{ij} \right] + \mathcal{O}(a). \quad (20)$$

where  $\gamma_{ij}$  and  $c_{ij}$  are, respectively, the one-loop anomalous dimension matrix and the finite coefficients. The latter are given by the difference of finite terms in the continuum and lattice one-loop calculations,

$$c_{ij} = C_{ij}^{\text{Cont}} - C_{ij}^{\text{Lat}}. \quad (21)$$

The general expressions for  $\gamma_{ij}$  and  $C_{ij}^{\text{Cont}}$  are given in Ref. [7] and we do not reproduce them here.<sup>5</sup> We only note that the mixing structure in the continuum is much simpler than that on the lattice because taste is conserved.

Mean-field improvement of the action and operators leads to a change in  $C_{ij}^{\text{Lat}}$  and thus in the finite part of the matching factors:

$$c_{ij} \xrightarrow{\text{MF}} c_{ij} - C_F I_{MF} T_{ij}, \quad (22)$$

where  $C_F = 4/3$ , and

$$I_{MF} = (4\pi)^2 \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \left( (\bar{s}_2)^2 \mathcal{T}_{11} - \bar{s}_1 \bar{s}_2 \mathcal{T}_{12} \right), \quad (23)$$

<sup>5</sup> In Ref. [7], a more elaborate notation is used in which  $\gamma$  and  $C$  become matrices in “color-trace” space. We do not use this notation here. We take this opportunity to correct two typographical errors in Table XIV of Ref. [7]: the entries in the  $\hat{\gamma}_{ij}$  column which are  $-4$  and  $4/3$  should be changed to  $+4$  and  $-4/3$ , respectively.

with  $T_{\mu\nu}$  the composite gluon propagator defined in Eq. (16). General results for the mean-field-improvement coefficients,  $T_{ij}$ , can be found in Ref. [20], and are quoted below for the operators considered here.

For each of the indices  $i$  and  $j$ , there are  $16^4$  choices of the  $S, F, S', F'$ . Although lattice symmetries reduce the number of independent entries,  $c_{ij}$  remains a large matrix. We have obtained expressions for all its entries, but present here only the most interesting subset.

### A. Matching Factors for $B_K$

The continuum  $\Delta S = 2$  four-fermion operator whose matrix element enters into the kaon mixing parameter  $B_K$  is

$$\mathcal{O}_{B_K}^{\text{Cont}} = [\bar{s}^a \gamma_\mu (1 - \gamma_5) d^a] [\bar{s}^b \gamma_\mu (1 - \gamma_5) d^b]. \quad (24)$$

In order to calculate  $\langle \bar{K}_0 | \mathcal{O}_{B_K}^{\text{Cont}} | K_0 \rangle$  using staggered fermions, one must first relate it, in the continuum, to a matrix element in an augmented theory in which there are four tastes for each continuum flavor. In fact, as explained in Ref. [24], one also needs to choose the quarks in each bilinear to have different flavors [as has been done in the lattice operators defined in Eqs. (10) and (11)]. This is necessary so that the Wick contractions in the original and augmented continuum theories agree. Thus one ends up with an eightfold increase in the number of valence flavors. To maintain the equivalence of the sea-quark sectors (and thus the dynamics) of these two theories one must take the 8th root of the fermion determinant. This rooting is not controversial in the formal continuum limit, and the equality of the corresponding matrix elements in the two continuum theories is valid non-perturbatively. We do note, however, that the need for rooting in the augmented theory implies that this theory is partially quenched, as has been stressed in Ref. [25].

The end result, in the “two spin-trace” formulation of Refs. [24, 26], is that the relevant operator in the augmented continuum theory is (keeping only the positive parity part)

$$\mathcal{O}_{B_K}^{\text{Cont}'} = \mathcal{O}_{V1}^{\text{Cont}'} + \mathcal{O}_{V2}^{\text{Cont}'} + \mathcal{O}_{A1}^{\text{Cont}'} + \mathcal{O}_{A2}^{\text{Cont}'}, \quad (25)$$

where

$$\mathcal{O}_{V1}^{\text{Cont}'} \equiv [\bar{S}_a(\gamma_\mu \otimes \xi_5) D_b] [\bar{S}'_b(\gamma_\mu \otimes \xi_5) D'_a], \quad (26)$$

$$\mathcal{O}_{V2}^{\text{Cont}'} \equiv [\bar{S}_a(\gamma_\mu \otimes \xi_5) D_a] [\bar{S}'_b(\gamma_\mu \otimes \xi_5) D'_b], \quad (27)$$

$$\mathcal{O}_{A1}^{\text{Cont}'} \equiv [\bar{S}_a(\gamma_\mu \gamma_5 \otimes \xi_5) D_b] [\bar{S}'_b(\gamma_\mu \gamma_5 \otimes \xi_5) D'_a], \quad (28)$$

$$\mathcal{O}_{A2}^{\text{Cont}'} \equiv [\bar{S}_a(\gamma_\mu \gamma_5 \otimes \xi_5) D_a] [\bar{S}'_b(\gamma_\mu \gamma_5 \otimes \xi_5) D'_b]. \quad (29)$$

Here  $S, D, S'$  and  $D'$  are Dirac fields having an implicit taste index running over four values. This index is contracted with the taste matrix  $\xi_5$ . The overall normalization of this operator is unimportant as it cancels in the ratio which defines  $B_K$ . Note also that the choice

of taste matrix is arbitrary in the continuum theory—we use  $\xi_5$  since that is what is used in lattice calculations.

The augmented continuum theory has been chosen to be the continuum limit of the lattice staggered theory.<sup>6</sup> In particular, at tree level, the operators  $\mathcal{O}_j^{\text{Cont}'}$  match onto lattice operators [defined in Eqs. (10) and (11)] as follows:

$$\mathcal{O}_{V1}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{V1}^{\text{Lat}} = [V_\mu \times P][V_\mu \times P]_I, \quad (30)$$

$$\mathcal{O}_{V2}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{V2}^{\text{Lat}} = [V_\mu \times P][V_\mu \times P]_{II}, \quad (31)$$

$$\mathcal{O}_{A1}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{A1}^{\text{Lat}} = [A_\mu \times P][A_\mu \times P]_I, \quad (32)$$

$$\mathcal{O}_{A2}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{A2}^{\text{Lat}} = [A_\mu \times P][A_\mu \times P]_{II}. \quad (33)$$

Thus the tree-level matching relation for the  $B_K$  operator is

$$\mathcal{O}_{B_K}^{\text{Cont}'} = \mathcal{O}_{V1}^{\text{Lat}} + \mathcal{O}_{V2}^{\text{Lat}} + \mathcal{O}_{A1}^{\text{Lat}} + \mathcal{O}_{A2}^{\text{Lat}} + O(g^2) + O(a^2). \quad (34)$$

At one-loop order, many lattice operators contribute to the matching formula. It is convenient to divide them into the two classes: (A) the four operators which arise at tree-level, which have the  $\xi_5$  taste matrices in both bilinears, and (B) the remaining operators, which all turn out to have taste matrices other than  $\xi_5$  in the bilinears. This division is useful for two reasons. First, in present numerical calculations only operators from class (A) are kept, so these are the matching coefficients that are needed.<sup>7</sup> Second, these are the only operators for which anomalous-dimension matrix elements and finite continuum coefficients are non-zero. Thus we write the one-loop matching formula as

$$\mathcal{O}_{B_K}^{\text{Cont}'} = \sum_{i \in (A)} z_i \mathcal{O}_i^{\text{Lat}} - \frac{g^2}{(4\pi)^2} \sum_{j \in (B)} d_j^{\text{Lat}} \mathcal{O}_j^{\text{Lat}}, \quad (35)$$

$$z_i = 1 + \frac{g^2}{(4\pi)^2} \left( -4 \log(\mu a) - \frac{11}{3} - d_i^{\text{Lat}} \right) \quad (36)$$

where the subscripts to the sums indicate that  $i$  runs over the four operators in class (A) while  $j$  runs over all operators in class (B). We have put in the values of the anomalous dimensions and finite coefficients. The constants  $d_j^{\text{Lat}}$  are obtained by summing elements of the matrices  $C_{ij}^{\text{Lat}}$  introduced above.

Numerical values for the  $d_{1-4}^{\text{Lat}}$  are given in table I. We compare the naive staggered action (with operators having thin links) to the HYP-smear staggered action (with operators having smeared links). In the former case, we implement mean-field improvement (since,

<sup>6</sup> Here we assume that rooting introduces no problems with the continuum limit, following the discussion in Refs. [25, 27–29].

<sup>7</sup> The rationale for this is that we use external kaons with taste  $\xi_5$ . As shown in Ref. [30], however, leaving out the operators with other tastes leads to an error of  $\mathcal{O}(\alpha_s m_K^2 / \Lambda_\chi^2)$ , which is of next-to-leading order in staggered chiral perturbation theory. This error must be accounted for when fitting.

TABLE I. Results for  $d_i^{\text{Lat}}$  for various choices of gauge and fermion action and of the four-fermion operators.

	(a)	(b)	(c)	(d)	(e)	(f)
Gluon action	Wilson	Sym	Wilson	Sym	Wilson	Sym
Quark action	Naive	Naive	HYP	HYP	HYP	HYP
Mean-field imp.	Y	Y	N	N	Y	Y
$d_{V1}^{\text{Lat}}$	-2.349(1)	-2.487(1)	-4.984(1)	-3.649(1)	-2.174(1)	-1.722(1)
$d_{V2}^{\text{Lat}}$	-12.915(2)	-11.537(2)	-11.108(2)	-8.584(2)	-5.487(2)	-4.729(2)
$d_{A1}^{\text{Lat}}$	-2.951(1)	-3.077(1)	-5.496(1)	-4.119(1)	-2.686(1)	-2.192(1)
$d_{A2}^{\text{Lat}}$	-3.725(1)	-2.895(1)	1.012(1)	1.087(1)	1.012(1)	1.087(1)
Range	10.57	9.04	12.18	9.67	6.50	5.82

in general, perturbation theory is very poorly convergent without this improvement), while for the HYP-smearred case we show results both with and without mean-field improvement. For each choice of fermion, we compare the results obtained using the Wilson and tree-level Symanzik gauge actions, with the former being from Ref. [7]. We see that improving the gauge action has a small effect, which, in most cases, reduces the size of the coefficients. A much more significant reduction is obtained by HYP smearing, as can be seen by comparing, for example, columns (b) and (f). (This is the appropriate comparison because both columns show results with mean-field improvement implemented.) The results needed for our companion numerical calculation [31] are those of column (f).

As can be seen from Eq. (36), the full one-loop correction includes the anomalous dimension and finite continuum contributions as well as  $d_i^{\text{Lat}}$ . Thus the total size of the one loop correction depends on the renormalization scale and lattice spacing through the combination  $\mu a$ . A better measure of the size of the correction is the range of the coefficients  $d_{1-4}^{\text{Lat}}$ , from which anomalous dimension and continuum contributions cancel. The ranges are given in table I, and show a small reduction with the use of the improved gauge action.

To give a sense of the numerical size of the matching coefficients themselves, we show in table II results for the  $z_{1-4}$  for the “ultrafine” MILC lattices ( $a \approx 0.045$  fm), setting  $\mu = 1/a$  (“horizontal matching”), and using  $\alpha_s = g^2/(4\pi) = 0.2096$  (the value in the  $\overline{\text{MS}}$  scheme at  $\mu = 1/a$ ). For the actions we use in practice [column (f)] the one-loop corrections range between +2% and -8%.

TABLE II. Values of  $z_i$  for the MILC ultrafine ensembles. The notation for actions is as in table I.

	(a)	(b)	(c)	(d)	(e)	(f)
$z_{V1}$	0.978	0.980	1.022	1.000	0.975	0.968
$z_{V2}$	1.154	1.131	1.124	1.082	1.030	1.018
$z_{A1}$	0.988	0.990	1.031	1.008	0.984	0.975
$z_{A2}$	1.001	0.987	0.922	0.921	0.922	0.921

To give a complete view of the one-loop matching, we present, in Tables III and IV, the coefficients  $d_j^{\text{Lat}}$  for all other operators which appear at this order. We see that

improving the gauge action leads, as above, to a small reduction in the magnitude of all the matching coefficients. Note that, since these mixings are pure lattice artifacts, with no anomalous dimension or other continuum contributions, reducing the size of the coefficients is an unambiguous improvement.

As noted above, in present numerical calculations the mixing with these operators is not being incorporated in the lattice operators, but rather is a source of systematic error that must be estimated by fitting. An alternative approach would be to include the dominant operators from the Tables in the one-loop matched operator. As one can see, there are relatively few operators having  $O(1)$  coefficients:

1.  $[S \times V_\mu][S \times V_\mu]$ ,
2.  $[P \times V_\mu][P \times V_\mu]$  and  $[P \times V_\mu][P \times V_\nu]$ ,
3.  $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ ,  $[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\rho]$   
 $[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\nu]$  and  $[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\eta]$ .

The remainder of the coefficients are an order of magnitude or more smaller (i.e.  $|d_i^{\text{Lat}}| < 0.2$ ). Since these coefficients are multiplied by  $g^2/(4\pi)^2 \approx 0.017 - 0.025$  for  $a \approx 0.045 - 0.12$  fm, we expect the contributions to  $B_K$  from the remaining operators to be very small.

## B. Matching Factors for other four-fermion operators

Models of new physics can lead, after integrating out heavy particles, to  $\Delta S = 2$  operators with different Dirac structure from that in  $O_{B_K}^{\text{Cont}}$ . To constrain these models one needs to know the matrix elements of these new operators. A standard basis is [32]

$$O_2^{\text{Cont}} = [\bar{s}^a(1 - \gamma_5)d^a][\bar{s}^b(1 - \gamma_5)d^b], \quad (37)$$

$$O_3^{\text{Cont}} = [\bar{s}^a(1 - \gamma_5)d^b][\bar{s}^b(1 - \gamma_5)d^a], \quad (38)$$

$$O_4^{\text{Cont}} = [\bar{s}^a(1 - \gamma_5)d^a][\bar{s}^b(1 + \gamma_5)d^b], \quad (39)$$

$$O_5^{\text{Cont}} = [\bar{s}^a(1 - \gamma_5)d^b][\bar{s}^b(1 + \gamma_5)d^a]. \quad (40)$$



TABLE III. Matching coefficients  $d_j^{\text{Lat}}$  [defined in Eq. (35)] for the operator required for calculating  $B_K$ , and for operators  $j$  which have different taste than the continuum operator (25). Lattice operators and fermion action are HYP-smearred and the gauge action is either Wilson—column (c)—or Symanzik—column (d). Results in column (c) are obtained from Tables I-IV of Ref. [7]. The coefficients  $T_j$  give the impact of mean-field improvement:  $d_j^{\text{Lat}} \rightarrow d_j^{\text{Lat}} + C_F I_{MF} T_j$ , with  $I_{MF} = 1.053786$  for the Wilson gauge action and  $I_{MF} = 0.722795$  for the Symanzik gauge action. Greek indices are implicitly summed, with the condition that they are unequal, and the further constraint that for the operator  $[V_\mu \times T_{\nu\rho}][V_\mu \times T_{\nu\rho}]$ ,  $\nu < \rho$ , while for the operators  $[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\rho]$  and  $[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\eta]$ ,  $\mu < \nu$ . Results are accurate to at least  $\pm 2$  in the last digit quoted.

$\mathcal{O}_j^{\text{Lat}}$	color trace	(c)	(d)	$T_j$
$[S \times V_\mu][S \times V_\mu]$	I	-3.450	-2.805	1
$[S \times V_\mu][S \times V_\mu]$	II	-0.263	-0.249	0
$[S \times V_\mu][S \times V_\nu]$	I	0.118	0.108	0
$[S \times V_\mu][S \times V_\nu]$	II	-0.104	-0.097	0
$[S \times A_\mu][S \times A_\mu]$	I	0.043	0.028	0
$[S \times A_\mu][S \times A_\mu]$	II	-0.052	-0.035	0
$[S \times A_\mu][S \times A_\nu]$	I	-0.015	-0.010	0
$[S \times A_\mu][S \times A_\nu]$	II	0.002	0.002	0
$[V_\mu \times S][V_\mu \times S]$	I	-0.044	-0.029	0
$[V_\mu \times S][V_\mu \times S]$	II	-0.008	-0.005	0
$[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$	I	-0.124	-0.086	0
$[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$	II	-0.114	-0.084	0
$[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\rho}]$	I	0.029	0.023	0
$[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\rho}]$	II	-0.023	-0.019	0
$[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\rho}]$	I	0.016	0.014	0
$[V_\mu \times T_{\mu\nu}][V_\mu \times T_{\nu\rho}]$	II	0.002	0.002	0
$[V_\mu \times T_{\nu\rho}][V_\mu \times T_{\mu\nu}]$	I	0.016	0.014	0
$[V_\mu \times T_{\nu\rho}][V_\mu \times T_{\mu\nu}]$	II	0.002	0.002	0
$[V_\mu \times T_{\nu\rho}][V_\mu \times T_{\nu\rho}]$	I	-0.118	-0.091	0
$[V_\mu \times T_{\nu\rho}][V_\mu \times T_{\nu\rho}]$	II	-0.037	-0.030	0
$[V_\mu \times T_{\nu\rho}][V_\mu \times T_{\nu\eta}]$	I	0.027	0.022	0
$[V_\mu \times T_{\nu\rho}][V_\mu \times T_{\nu\eta}]$	II	-0.020	-0.016	0
$[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$	I	2.071	1.547	-1
$[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$	II	-0.538	-0.485	0
$[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\nu]$	I	-0.410	-0.383	0
$[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\nu]$	II	0.452	0.417	0
$[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\rho]$	I	0.129	0.120	0
$[T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\rho]$	II	0.126	0.118	0
$[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\mu]$	I	0.129	0.120	0
$[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\mu]$	II	0.126	0.118	0
$[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\rho]$	I	-2.930	-2.331	1
$[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\rho]$	II	-0.346	-0.316	0
$[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\eta]$	I	0.652	0.610	0
$[T_{\mu\nu} \times V_\rho][T_{\mu\nu} \times V_\eta]$	II	-0.153	-0.143	0

In this section we present one-loop matching coefficients for these operators.<sup>8</sup>

<sup>8</sup> Linear combinations of these operators are also needed for calculating the  $K \rightarrow \pi$  matrix elements of the  $I = 3/2$  part of the electromagnetic penguin contribution to  $\epsilon'/\epsilon$ . In the context of staggered fermions this is explained in Ref. [24].

TABLE IV. Matching coefficients  $d_j^{\text{Lat}}$  (continued from Table III). Again, Greek indices are implicitly summed, with the condition that they are unequal, with the further constraint that for the operator  $[A_\mu \times T_{\nu\rho}][A_\mu \times T_{\nu\rho}]$ ,  $\nu < \rho$ , and for the operators  $[T_{\mu\nu} \times A_\rho][T_{\mu\nu} \times A_\rho]$  and  $[T_{\mu\nu} \times A_\rho][T_{\mu\nu} \times A_\eta]$ ,  $\mu < \nu$ .

$\mathcal{O}_j^{\text{Lat}}$	color trace	(c)	(d)	$T_j$
$[T_{\mu\nu} \times A_\mu][T_{\mu\nu} \times A_\mu]$	I	-0.026	-0.017	0
$[T_{\mu\nu} \times A_\mu][T_{\mu\nu} \times A_\mu]$	II	-0.045	-0.030	0
$[T_{\mu\nu} \times A_\mu][T_{\mu\nu} \times A_\nu]$	I	-0.010	-0.007	0
$[T_{\mu\nu} \times A_\mu][T_{\mu\nu} \times A_\nu]$	II	-0.003	-0.002	0
$[T_{\mu\nu} \times A_\mu][T_{\mu\nu} \times A_\rho]$	I	0.001	0.000	0
$[T_{\mu\nu} \times A_\mu][T_{\mu\nu} \times A_\rho]$	II	-0.002	-0.001	0
$[T_{\mu\nu} \times A_\rho][T_{\mu\nu} \times A_\mu]$	I	0.001	0.000	0
$[T_{\mu\nu} \times A_\rho][T_{\mu\nu} \times A_\mu]$	II	-0.002	-0.001	0
$[T_{\mu\nu} \times A_\rho][T_{\mu\nu} \times A_\rho]$	I	-0.068	-0.046	0
$[T_{\mu\nu} \times A_\rho][T_{\mu\nu} \times A_\rho]$	II	-0.024	-0.016	0
$[T_{\mu\nu} \times A_\rho][T_{\mu\nu} \times A_\eta]$	I	0.000	-0.000	0
$[T_{\mu\nu} \times A_\rho][T_{\mu\nu} \times A_\eta]$	II	0.003	0.002	0
$[A_\mu \times S][A_\mu \times S]$	I	-0.003	-0.002	0
$[A_\mu \times S][A_\mu \times S]$	II	-0.022	-0.014	0
$[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$	I	-0.124	-0.086	0
$[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$	II	-0.114	-0.084	0
$[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\rho}]$	I	0.022	0.017	0
$[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\rho}]$	II	-0.002	-0.002	0
$[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\nu\rho}]$	I	-0.004	-0.003	0
$[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\nu\rho}]$	II	0.011	0.009	0
$[A_\mu \times T_{\nu\rho}][A_\mu \times T_{\mu\nu}]$	I	-0.004	-0.003	0
$[A_\mu \times T_{\nu\rho}][A_\mu \times T_{\mu\nu}]$	II	0.011	0.009	0
$[A_\mu \times T_{\nu\rho}][A_\mu \times T_{\nu\rho}]$	I	-0.106	-0.083	0
$[A_\mu \times T_{\nu\rho}][A_\mu \times T_{\nu\rho}]$	II	-0.074	-0.055	0
$[A_\mu \times T_{\nu\rho}][A_\mu \times T_{\nu\eta}]$	I	-0.017	-0.015	0
$[A_\mu \times T_{\nu\rho}][A_\mu \times T_{\nu\eta}]$	II	0.011	0.009	0
$[P \times V_\mu][P \times V_\mu]$	I	2.566	2.004	-1
$[P \times V_\mu][P \times V_\mu]$	II	-0.547	-0.503	0
$[P \times V_\mu][P \times V_\nu]$	I	0.151	0.141	0
$[P \times V_\mu][P \times V_\nu]$	II	0.326	0.307	0
$[P \times A_\mu][P \times A_\mu]$	I	-0.063	-0.041	0
$[P \times A_\mu][P \times A_\mu]$	II	-0.042	-0.028	0
$[P \times A_\mu][P \times A_\nu]$	I	0.012	0.008	0
$[P \times A_\mu][P \times A_\nu]$	II	-0.005	-0.003	0

As for  $\mathcal{O}_{BK}^{\text{Cont}}$ , the first step is to match the operators into the augmented continuum theory. The result is

$$\mathcal{O}_2^{\text{Cont}'} = \mathcal{O}_{S2}^{\text{Cont}'} + \mathcal{O}_{P2}^{\text{Cont}'} + \frac{1}{2} \left( \mathcal{O}_{S1}^{\text{Cont}'} + \mathcal{O}_{P1}^{\text{Cont}'} - \mathcal{O}_{T1}^{\text{Cont}'} \right), \quad (41)$$

$$\mathcal{O}_3^{\text{Cont}'} = \mathcal{O}_{S1}^{\text{Cont}'} + \mathcal{O}_{P1}^{\text{Cont}'} + \frac{1}{2} \left( \mathcal{O}_{S2}^{\text{Cont}'} + \mathcal{O}_{P2}^{\text{Cont}'} - \mathcal{O}_{T2}^{\text{Cont}'} \right), \quad (42)$$

$$\mathcal{O}_4^{\text{Cont}'} = \mathcal{O}_{S2}^{\text{Cont}'} - \mathcal{O}_{P2}^{\text{Cont}'} + \frac{1}{2} \left( \mathcal{O}_{V1}^{\text{Cont}'} - \mathcal{O}_{A1}^{\text{Cont}'} \right), \quad (43)$$

$$\mathcal{O}_5^{\text{Cont}'} = \mathcal{O}_{S1}^{\text{Cont}'} - \mathcal{O}_{P1}^{\text{Cont}'} + \frac{1}{2} \left( \mathcal{O}_{V2}^{\text{Cont}'} - \mathcal{O}_{A2}^{\text{Cont}'} \right), \quad (44)$$

where some of the operators are defined in Eqs. (26-29), and the others are

$$\mathcal{O}_{S_1}^{\text{Cont}'} \equiv [\bar{S}_a(\mathbf{1} \otimes \xi_5) D_b][\bar{S}'_b(\mathbf{1} \otimes \xi_5) D'_a], \quad (45)$$

$$\mathcal{O}_{S_2}^{\text{Cont}'} \equiv [\bar{S}_a(\mathbf{1} \otimes \xi_5) D_a][\bar{S}'_b(\mathbf{1} \otimes \xi_5) D'_b], \quad (46)$$

$$\mathcal{O}_{P_1}^{\text{Cont}'} \equiv [\bar{S}_a(\gamma_5 \otimes \xi_5) D_b][\bar{S}'_b(\gamma_5 \otimes \xi_5) D'_a], \quad (47)$$

$$\mathcal{O}_{P_2}^{\text{Cont}'} \equiv [\bar{S}_a(\gamma_5 \otimes \xi_5) D_a][\bar{S}'_b(\gamma_5 \otimes \xi_5) D'_b], \quad (48)$$

$$\mathcal{O}_{T_1}^{\text{Cont}'} \equiv \sum_{\mu < \nu} [\bar{S}_a(\gamma_\mu \gamma_\nu \otimes \xi_5) D_b][\bar{S}'_b(\gamma_\mu \gamma_\nu \otimes \xi_5) D'_a], \quad (49)$$

$$\mathcal{O}_{T_2}^{\text{Cont}'} \equiv \sum_{\mu < \nu} [\bar{S}_a(\gamma_\mu \gamma_\nu \otimes \xi_5) D_a][\bar{S}'_b(\gamma_\mu \gamma_\nu \otimes \xi_5) D'_b]. \quad (50)$$

At tree-level, the new operators in the augmented continuum theory match onto lattice operators in the obvious way:

$$\mathcal{O}_{S_1}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{S_1}^{\text{Lat}} = [S \times P][S \times P]_I, \quad (51)$$

$$\mathcal{O}_{S_2}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{S_2}^{\text{Lat}} = [S \times P][S \times P]_{II}, \quad (52)$$

$$\mathcal{O}_{P_1}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{P_1}^{\text{Lat}} = [P \times P][P \times P]_I, \quad (53)$$

$$\mathcal{O}_{P_2}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{P_2}^{\text{Lat}} = [P \times P][P \times P]_{II}, \quad (54)$$

$$\mathcal{O}_{T_1}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{T_1}^{\text{Lat}} = \sum_{\mu < \nu} [T_{\mu\nu} \times P][T_{\mu\nu} \times P]_I, \quad (55)$$

$$\mathcal{O}_{T_2}^{\text{Cont}'} \stackrel{\text{tree}}{=} \mathcal{O}_{T_2}^{\text{Lat}} = \sum_{\mu < \nu} [T_{\mu\nu} \times P][T_{\mu\nu} \times P]_{II}. \quad (56)$$

As for  $\mathcal{O}_{B_K}^{\text{Cont}'}$ , each of the operators  $\mathcal{O}_{2-5}^{\text{Cont}'}$  matches at one loop order onto class (A) lattice operators composed of bilinears with taste  $\xi_5$ , and class (B) operators having other tastes. Continuum mixing involves only class (A) operators—mixing with operators of class (B) is a lattice effect. We display here only results for mixing with class (A) operators, for several reasons. First, these are likely to be the subset of operators used in numerical simulations. Second, the contribution of class (B) operators is of next-to-next-to-leading order in staggered chiral perturbation theory, using the power counting of Ref. [30]. This is because the contribution is suppressed both by  $\alpha_s$ , and by the need to have a chiral loop due to the mismatch between the taste of the operator and external states. Unlike for  $\mathcal{O}_{B_K}^{\text{Cont}'}$ , there is no chiral enhancement to raise the contribution to next-to-leading order. Finally, we do not show results for class (B) mixing for the sake of brevity.

We write the one-loop matching formula as

$$\mathcal{O}_i^{\text{Cont}'} = \sum_{j \in (A)} z_{ij} \mathcal{O}_j^{\text{Lat}} - \frac{g^2}{(4\pi)^2} \sum_{k \in (B)} d_{ik}^{\text{Lat}} \mathcal{O}_k^{\text{Lat}}, \quad (57)$$

with  $i = 2 - 4$  and

$$z_{ij} = b_{ij} + \frac{g^2}{(4\pi)^2} \left( -\gamma_{ij} \log(\mu a) + d_{ij}^{\text{Cont}} - d_{ij}^{\text{Lat}} - C_F I_{MF} T_{ij} \right). \quad (58)$$

We stress that we are using a different basis for the continuum and lattice operators, so that the tree-level contribution is no longer  $\delta_{ij}$ . Because of this, we denote the finite parts as  $d_{ij}$  rather than  $C_{ij}$ . The  $d_{ij}^{\text{Cont}}$  matrix comes from Ref. [7]. The elements of  $d_{ij}^{\text{Lat}}$ , which are calculated numerically, are new results. The  $T_{ij}$  term is present only if the operator is mean-field improved. Numerical values for  $I_{MF}$  are given in the caption of Table III.

Results for the four operators are presented in Tables V-VIII. We show results for HYP-smear operators without mean-field improvement, but include the values of  $T_{ij}$  so that mean-field improvement can be easily implemented.

TABLE V. Matching coefficients entering Eq. (58) for  $i = 2$ , i.e. for  $\mathcal{O}_2^{\text{Cont}'}$ . The finite lattice coefficients are for the HYP-smear fermion action and operators, and either (c) the Wilson gauge action or (d) the Symanzik gauge action (following the labeling used in Table I. The  $d_{ij}^{\text{Cont}}$  matrix comes from Ref. [7]. Results are accurate to the number of digits quoted.

Operator $j$	$b_{2j}$	$\gamma_{2j}$	$d_{2j}^{\text{Cont}}$	$d_{2j}^{\text{Lat}}(c)$	$d_{2j}^{\text{Lat}}(d)$	$T_{2j}$
$\mathcal{O}_{S_1}^{\text{Lat}}$	-1/2	6	-59/12	2.70	2.34	-1
$\mathcal{O}_{S_2}^{\text{Lat}}$	1	-10	+73/12	-17.80	-14.53	6
$\mathcal{O}_{P_1}^{\text{Lat}}$	-1/2	6	-59/12	3.64	3.17	-1
$\mathcal{O}_{P_2}^{\text{Lat}}$	1	-10	+73/12	5.42	4.06	-2
$\mathcal{O}_{T_1}^{\text{Lat}}$	1/2	-14/3	+29/12	-2.94	-2.52	1
$\mathcal{O}_{T_2}^{\text{Lat}}$	0	2/3	-5/4	0.03	0.01	0

TABLE VI. Matching coefficients entering Eq. (58) for  $i = 3$ . Notation as in Table V.

Operator $j$	$b_{3j}$	$\gamma_{3j}$	$d_{3j}^{\text{Cont}}$	$d_{3j}^{\text{Lat}}(c)$	$d_{3j}^{\text{Lat}}(d)$	$T_{3j}$
$\mathcal{O}_{S_1}^{\text{Lat}}$	1	8	1/3	-4.35	-2.88	2
$\mathcal{O}_{S_2}^{\text{Lat}}$	-1/2	0	-5/3	7.13	5.38	-3
$\mathcal{O}_{P_1}^{\text{Lat}}$	1	8	1/3	-6.23	-4.56	2
$\mathcal{O}_{P_2}^{\text{Lat}}$	-1/2	0	-5/3	-0.25	-0.14	1
$\mathcal{O}_{T_1}^{\text{Lat}}$	0	8/3	-1	0.14	0.24	0
$\mathcal{O}_{T_2}^{\text{Lat}}$	1/2	16/3	-1/3	-2.31	-1.55	1

TABLE VII. Matching coefficients entering Eq. (58) for  $i = 4$ . Notation as in Table V.

Operator $j$	$b_{4j}$	$\gamma_{4j}$	$d_{4j}^{\text{Cont}}$	$d_{4j}^{\text{Lat}}(c)$	$d_{4j}^{\text{Lat}}(d)$	$T_{4j}$
$\mathcal{O}_{S_1}^{\text{Lat}}$	0	0	-3	0	0	0
$\mathcal{O}_{S_2}^{\text{Lat}}$	1	-16	+23/3	-19.12	-16.02	6
$\mathcal{O}_{P_1}^{\text{Lat}}$	0	0	+3	0	0	0
$\mathcal{O}_{P_2}^{\text{Lat}}$	-1	16	-23/3	-6.93	-5.09	2
$\mathcal{O}_{V_1}^{\text{Lat}}$	-1/2	8	-23/6	3.02	2.72	-1
$\mathcal{O}_{V_2}^{\text{Lat}}$	0	0	+3/2	0.37	0.34	0
$\mathcal{O}_{A_1}^{\text{Lat}}$	1/2	-8	+23/6	-3.27	-2.95	1
$\mathcal{O}_{A_2}^{\text{Lat}}$	0	0	-3/2	0.40	0.37	0

TABLE VIII. Matching coefficients entering Eq. (58) for  $i = 5$ . Notation as in Table V.

Operator $j$	$b_{5j}$	$\gamma_{5j}$	$d_{5j}^{\text{Cont}}$	$d_{5j}^{\text{Lat}}(c)$	$d_{5j}^{\text{Lat}}(d)$	$T_{5j}$
$\mathcal{O}_{S1}^{\text{Lat}}$	1	2	1/6	-4.67	-3.42	2
$\mathcal{O}_{S2}^{\text{Lat}}$	0	-6	-1/2	-2.32	-2.45	0
$\mathcal{O}_{P1}^{\text{Lat}}$	-1	-2	-1/6	6.55	5.10	-2
$\mathcal{O}_{P2}^{\text{Lat}}$	0	6	1/2	-3.32	-2.59	0
$\mathcal{O}_{V1}^{\text{Lat}}$	0	3	1/4	0.16	0.27	0
$\mathcal{O}_{V2}^{\text{Lat}}$	-1/2	-1	-1/12	5.24	4.04	-2
$\mathcal{O}_{A1}^{\text{Lat}}$	0	-3	-1/4	-0.16	-0.27	0
$\mathcal{O}_{A2}^{\text{Lat}}$	1/2	1	1/12	0.05	0.09	0

The tables show that, for coefficients with magnitudes larger than about 5, improvement of the gauge action leads to a small reduction in the size of the finite lattice coefficients.<sup>9</sup> Even with this improvement, we see that the largest lattice coefficients have a magnitude as large as 16, although most have a magnitude smaller than 10. These numbers are larger than those we found for the  $B_K$  operator (Table I). On a coarse lattice with  $a \approx 0.12$  fm, and  $g^2/(4\pi)^2 \approx 0.025$ , the largest one-loop term could give a 40% correction. This is not a precise statement because we have not included the contribution from finite continuum terms and from the anomalous dimensions. Nevertheless, it is a warning to expect perturbation theory to be less convergent for the operators  $\mathcal{O}_{2-5}$  than for  $\mathcal{O}_{B_K}$ .

The situation is better if one implements mean-field improvement. As can be seen from the Tables, this reduces all the larger coefficients, so that the largest magnitude is now  $\approx 10$ . Since mean-field improvement is relatively straightforward to implement, our results sug-

gest that this would be worth the required investment.

Finally, we note that, unlike for  $\mathcal{O}_{B_K}$ , one finds very large finite coefficients if one uses mean-field improved naive fermions (with either gauge action). The largest magnitudes are  $\approx 50$ , indicating a complete breakdown in the convergence of perturbation theory. Because of this, we do not include the results in the tables.

## V. CONCLUSION

We have calculated the matching factors for four-fermion operators using various fermion and gauge actions. Most useful are our results for the fermion action and operators constructed using HYP-smear links with the Symanzik improved gluon action. These are needed for our ongoing calculation of  $B_K$  and related matrix elements [4, 31]. For these operators, the one-loop corrections are of moderate size for the  $B_K$  operator, with the range of corrections being  $\approx 10 \times \alpha_s/(4\pi) \approx \alpha_s$ , which is the naively expected size. The same holds true for the operators induced by new physics, as long as one implements mean-field improvement. For all operators, we find that improving the gauge action generically leads to a reduction in the size of one-loop matching coefficients, but that the effect is relatively small.

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<sup>9</sup> The range of the coefficients is not useful for these operators as the contributions from anomalous dimensions differ.

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