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Phys. Rev. D **83**, 086001 — Published 5 April 2011

DOI: 10.1103/PhysRevD.83.086001

LR12569D

NSF-KITP-10-018

String scattering in flat space and a scaling limit of Yang-Mills correlators

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Abstract

We use the flat space limit of the AdS/CFT correspondence to derive a simple relation between the $2 \to 2$ scattering amplitude of massless string states in type IIB superstring theory on ten-dimensional Minkowski space and a scaling limit of the $\mathcal{N} = 4$ super Yang-Mills four point functions. We conjecture that this relation holds non-perturbatively and at arbitrarily high energy.

1 Introduction and summary

Originating as a phenomenological description of hadron scattering, string theory is defined at the basic level by a prescription for perturbative scattering amplitudes. Using the AdS/CFT correspondence [1], one can go beyond a perturbative formulation and define string theory in asymptotically AdS backgrounds in terms of the dual gauge theory. Immediately after the discovery of AdS/CFT, it was argued [2, 3] that the flat space limit of the correspondence should lead to a holographic formulation of string theory in Minkowski space. This idea was explored in [4] to obtain an explicit relation between scattering amplitudes of the bulk gravitational theory and certain singularities of the CFT correlation functions at large 't Hooft coupling. The aim of this paper is to extend this relation to string scattering amplitudes.

Relegating the derivation and the more general case to the later sections, here we state our main results in the set-up of the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills (SYM). The string coupling g_s, the AdS radius R and the string length l_s are related to the Yang-Mills coupling $g_{\rm YM}$ and the number of colors N via

$$4\pi g_s = g_{\rm YM}^2 \ , \qquad\qquad \left(\frac{R}{l_s}\right)^4 = g_{\rm YM}^2 N \ .$$

On the one hand, the natural observables for type IIB string theory on 10D Minkowski space are scattering amplitudes of stable string states. For simplicity, we consider the $2 \to 2$ scattering amplitude of (massless) dilaton particles,

$$T = l_s^6 \, \mathcal{T}\left(g_s, -t/s, l_s^2 s\right) \ ,$$

where s and t are the usual Mandelstam invariants. We recall that the scattering angle θ is given by $\sin^2(\theta/2) = -t/s$. On the other hand, the natural observables for SYM on 4D Minkowski space are correlation functions of gauge-invariant operators. In particular, we are interested in the connected [1] four point function of the lagrangian density $\mathcal{O}$, which is the dual operator to the bulk dilaton and is half BPS. Conformal invariance implies that the four-point function on Minkowski space $\mathbb{M}^4$ takes the form

$$\frac{\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle}{\langle \mathcal{O}(x_1)\mathcal{O}(x_3)\rangle \, \langle \mathcal{O}(x_2)\mathcal{O}(x_4)\rangle} = \mathcal{A}(g_{\rm YM}^2, N, \sigma, \rho^2) \ ,$$

[1]The connected four-point function $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle_{\rm c}$ is given by subtracting disconnected contributions from the full four-point function: $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle_{\rm c} \equiv \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle - \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \langle \mathcal{O}(x_3)\mathcal{O}(x_4)\rangle - \langle \mathcal{O}(x_1)\mathcal{O}(x_3)\rangle \langle \mathcal{O}(x_2)\mathcal{O}(x_4)\rangle - \langle \mathcal{O}(x_1)\mathcal{O}(x_4)\rangle \langle \mathcal{O}(x_2)\mathcal{O}(x_3)\rangle$. In the text we simply denote $\langle \cdots \rangle_{\rm c}$ by $\langle \cdots \rangle$. The two-point functions of BPS operators are not renormalized [5].

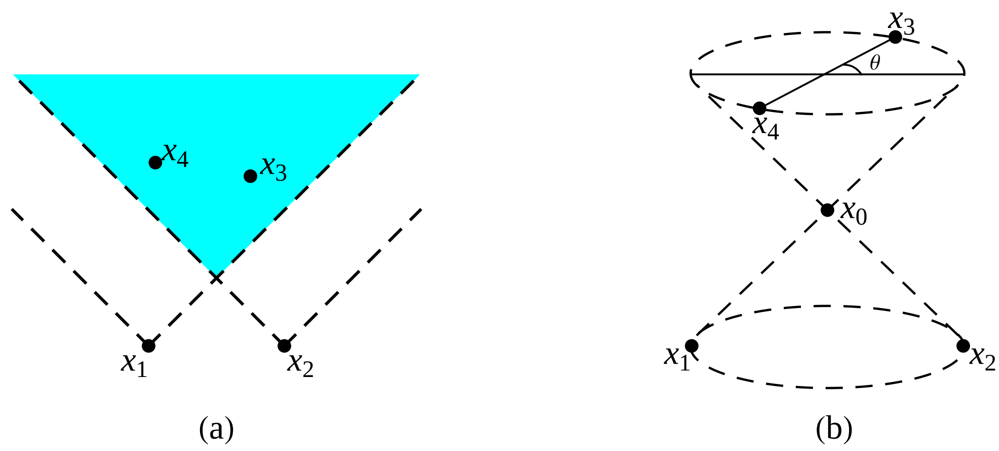

Figure 1: (a) Causal relations between the operator insertion points used in the flat space limit: x_{12} and x_{34} are spacelike and x_3 and x_4 are in the future of both x_1 and x_2. (b) Example of configuration providing the flat space limit of the correlator. The auxiliary point x_0 is null related to all the insertion points. The conformal invariants for this configuration are $\rho = 0$ and $\sigma = \sin^2(\theta/2)$ where θ is shown in the figure.

where σ and ρ are the conformal invariant combinations

$$\sigma^2 = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2}\,, \qquad \sinh^2\rho = \frac{\det x_{ij}^2}{4\, x_{13}^2 x_{24}^2 x_{12}^2 x_{34}^2}\,.$$

The determinant is taken over i and j. We consider $x_{12} = x_1 - x_2$ and $x_{34} = x_3 - x_4$ spacelike and x_3 and x_4 in the future of both x_1 and x_2, as depicted in figure 1(a). We shall see that the flat space limit of the AdS/CFT correspondence gives [2] a precise relation between the dimensionless functions $\mathcal{T}$ and $\mathcal{A}$. More precisely, the scaling limit of the four point function,

$$\mathcal{F}(g_{\rm YM}^2, \sigma, \xi) = \lim_{N\to\infty} \frac{\mathcal{A}\left(g_{\rm YM}^2, N, \sigma, -\frac{(1-\sigma)\xi^2}{\sigma g_{\rm YM}\sqrt{N}}\right)}{\sigma^8\left(g_{\rm YM}^2 N\right)^{7/4}} \tag{1.1}$$

with

$$\xi \equiv \left(-\frac{\sigma}{1-\sigma}\sqrt{g_{\rm YM}^2 N}\rho^2\right)^{1/2} \tag{1.2}$$

fixed, is related to the flat space scattering amplitude through the relation

$$\mathcal{F}(g_{\rm YM}^2, \sigma, \xi) = \frac{1}{2^{17}3^2\pi^3\xi\sqrt{\sigma(1-\sigma)}} \int_0^\infty d\nu \nu^{11} e^{-\xi\nu} i\mathcal{T}\left(\frac{g_{\rm YM}^2}{4\pi}, \sigma, \nu^2 + i\epsilon\right)\,. \tag{1.3}$$

[2]Our conjectural relations are obtained under the assumption that the wave packets are localized well enough so that their interactions are confined to the small flat region in AdS. A major progress in [4] was the construction of wave packets that are much better localized than previously considered.

One can also invert this relation and obtain

$$i\mathcal{T}\left(g_s, -t/s, l_s^2 s\right) = \frac{2^{17}3^2\pi^3\sqrt{stu}}{l_s^{11}s^7}\int_{-i\infty}^{i\infty}\frac{d\xi}{2\pi i}\xi\mathcal{F}(4\pi g_s, -t/s, \xi)e^{\xi l_s\sqrt{s}}\,. \tag{1.4}$$

Crossing symmetry of the scattering amplitude,

$$\mathcal{T}\left(g_s, \sin^2\frac{\theta}{2}, l_s^2 s\right) = \mathcal{T}\left(g_s, \cos^2\frac{\theta}{2}, l_s^2 s\right)\ ,$$

implies that

$$\mathcal{F}(g_{\rm YM}^2, \sigma, \xi) = \mathcal{F}(g_{\rm YM}^2, 1-\sigma, \xi)\ .$$

Formulas (1.1 - 1.4), as well as their generalizations (2.8, 2.10, 2.11, 2.13) to other dimensions, are the main results of this paper. Notice that the dual flat space limit of SYM is a large N limit very different from the planar limit. In the former, one keeps $g_{\rm YM}$ fixed (instead of the 't Hooft coupling $g_{\rm YM}^2 N$) and scales the kinematical variable $\rho \simeq \det(x_{ij}^2)^{1/2}$ to zero as $N^{-1/4}$. One way to approach the limit $\rho \to 0$ is to make the four points x_i approach the lightcone of a point x_0 in $\mathbb{M}^4$, as in figure 1(b).

The paper is organized as follows. In Section 2, we derive the relations (1.1 - 1.4) between the flat space string theory and a scaling limit of gauge theory. We present two derivations, one of which is in a new approach while the other is a direct generalization of the methods in [4]. Since the relations (1.1 - 1.4) are rather abstract and involve an unfamiliar limit, in Section 3 we consider special cases to gain better intuition. These limits correspond to the tree-level and hard scattering approximations on the string theory side. Finally in Section 4 we discuss the primary obstacles in testing or applying the relations, and also comment on possible ways to overcome such difficulties.

2 Derivations of the relations

In this section we derive the relations (1.1 - 1.4). In fact we present below two derivations, which are related but independent. The first one directly leads to (the generalization of) the relation (1.4) that expresses the scaled CFT correlator in terms of a string scattering amplitude. On the other hand the relation (1.3) gives the scattering amplitude as an integral transform of the CFT correlator. Since (1.3) mathematically provides the inverse map of (1.4), logically it suffices to derive only the latter. In our second derivation, however, we are able to directly obtain (1.3) by generalizing the analysis performed in [4].

2.1 From strings to Yang-Mills

We shall follow the notation of [4] and use the embedding space formalism, which is useful in relating the analysis in global coordinates in this section to the expressions (1.1) given in Poincaré coordinates of AdS. A point X in AdS_{d+1} of radius R, is described by $X \in \mathbb{R}^{2,d}$ with

$$X^2 \equiv -(X^{-1})^2 - (X^0)^2 + (X^1)^2 + \ldots + (X^d)^2 = -R^2 \,. \tag{2.1}$$

A point in the boundary of AdS is a null ray ($P \sim \lambda P$ and $P^2 = 0$) in $\mathbb{R}^{2,d}$ [6]. Correlation functions

$$A(P_1, \ldots, P_n) = \langle \mathcal{O}_1(P_1) \ldots \mathcal{O}_n(P_n) \rangle$$

of scalar primary operators, are Lorentz invariant and homogeneous functions on the light-cone of the embedding space $\mathbb{R}^{2,d}$,

$$A(\ldots, \lambda P_i, \ldots) = \lambda^{-\Delta_i} A(\ldots, P_i, \ldots),$$

where Δ_i is the dimension of the operator at position P_i. Using the AdS/CFT correspondence we can write

$$A = \int_{\text{AdS}} \prod_{i=1}^{4} dX_i G_{B\partial}(X_i, P_i)\, G(X_1, \ldots, X_4) \tag{2.2}$$

where the bulk-boundary propagator is [3]

$$G_{B\partial}(X, P) = \frac{C_\Delta}{R^{(d-1)/2}} \frac{1}{(-2P \cdot X/R + i\epsilon)^\Delta} \,, \tag{2.3}$$

and $G(X_i)$ is the amputated bulk Green's function [4]. The normalization constant reads

$$C_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}} \Gamma\left(\Delta - \frac{d}{2} + 1\right)} \,,$$

and dX_i is the short hand for the volume form on AdS.

Let us now consider a bulk theory with an intrinsic length scale l_s independent of the AdS radius R. We are interested in the limit $l_s \ll R$ so that flat space physics dominates. In this limit, the important integration region in (2.2) is $|X_i - X_j| \ll R$ [5], where the AdS

[3] Since supergravity fields like the dilaton are dual to BPS operators whose two and three point functions are not renormalized, it is very plausible that their propagators do not receive quantum corrections even when the wave length is the string scale.

[4] At intermediate stages of the derivation, we use the Green's function G, which is an off-shell field theory quantity hard to define in string theory. However, the final result only depends on the on-shell string scattering amplitude.

[5] This condition is clear in the case of massive particles since their propagators decay exponentially with a finite correlation length equal to $1/\text{mass} \sim l_s \ll R$. For massless particles the situation is more subtle. However, if the diagrams under consideration have no IR divergences in flat space then as we take $R \to \infty$ the integrals become dominated by the flat space region $|X_i - X_j| \ll R$.

curvature effects are negligible. Therefore, we can approximate

$$A \approx \int_{\mathrm{AdS}} dX_1 G_{B\partial}(X_1,P_1) \int_{\mathbb{M}} \prod_{i=2}^{4} dY_i G_{B\partial}(X_1+Y_i,P_i)\, G_{\text{flat}}(0,Y_2,Y_3,Y_4) \tag{2.4}$$

where $Y_i = X_i - X_1$ parametrize the neighborhood of X_1 and G_{flat} is the flat space amputated Green's function. Rewriting the bulk-boundary propagator as

$$G_{B\partial}(X,P) = \frac{(-i)^{\Delta} C_{\Delta} R^{\Delta}}{\Gamma(\Delta) R^{(d-1)/2} l_s^{\Delta}} \int_0^{\infty} \frac{d\beta}{\beta} \beta^{\Delta} e^{-2i\beta P\cdot X/l_s} \ ,$$

we can easily perform the Y-integrals and obtain the off-shell flat space scattering amplitude[6]

$$\int_{\mathbb{M}} \prod_{i=2}^{4} dY_i e^{-2i\beta_i P_i\cdot Y_i/l_s}\, G_{\text{flat}}(0,Y_2,Y_3,Y_4) = iT^{(d+1)}\left(k_1 = -\sum_{i=2}^{4} k_i, k_i = -2\beta_i P_i/l_s\right) \ .$$

The integral over X_1 has the form

$$\int_{\mathrm{AdS}} dX_1 e^{-2iQ\cdot X_1/l_s}$$

with $Q = \sum_{i=1}^{4} \beta_i P_i$. Using Poincaré coordinates, this integral can be simplified to

$$-i\pi^{\frac{d}{2}} R^{d+1} \int_0^{\infty} \frac{dy}{y} (-iy)^{-\frac{d}{2}} e^{iy - i(R/l_s)^2 Q^2/y} \ .$$

The four point function is then given by

$$\begin{aligned} A \approx \pi^{\frac{d}{2}} R^{3-d} \prod_{i=1}^{4} \frac{(-i)^{\Delta_i} C_{\Delta_i} R^{\Delta_i}}{\Gamma(\Delta_i) l_s^{\Delta_i}} \int_0^{\infty} \frac{dy}{y} (-iy)^{-\frac{d}{2}} e^{iy} \\ \int_0^{\infty} \prod_{i=1}^{4} \frac{d\beta_i}{\beta_i} \beta_i^{\Delta_i} e^{-i(R/l_s)^2 Q^2/y}\, T^{(d+1)}\left(k_1 = -\sum_{i=2}^{4} k_i, k_i = -2\beta_i P_i/l_s\right) \ , \end{aligned} \tag{2.5}$$

with $Q^2 = -\frac{1}{2}\sum_{i,j} \beta_i \beta_j P_{ij}$ and $P_{ij} = -2P_i \cdot P_j$. Since the exponent contains the large factor $(R/l_s)^2$ we shall attempt to perform the integral over $\vec{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$ by saddle point. This requires diagonalizing the real symmetric matrix P_{ij}. In general, this matrix has 4 eigenvalues of order 1 and the integral is small. However, when one eigenvalue is very small, of order $(l_s/R)^2$, the integral gets enhanced. This is the kinematical regime of interest which we now describe. Writing $P = (\cos\tau, \sin\tau, \mathbf{e})$ we describe the AdS boundary in global coordinates with $\mathbf{e}$ a d-dimensional unit vector parametrizing S^{d-1} and τ being global time. We then take

$$\begin{aligned} P_1 &= (0,-1,-1,0,\mathbf{0}) + \ldots \\ P_2 &= (0,-1,1,0,\mathbf{0}) + \ldots \\ P_3 &= (0,1,\cos\theta,\sin\theta,\mathbf{0}) + \ldots \\ P_4 &= (0,1,-\cos\theta,-\sin\theta,\mathbf{0}) + \ldots \end{aligned} \tag{2.6}$$

[6] To be precise, only the part of P_i orthogonal to X_1 contributes. However, this is a subdominant effect.

where $\mathbf{0}$ is the origin of $\mathbb{R}^{d-2}$ and the dots stand for small deviations whose explicit form will not be important. With this choice the matrix P_{ij} has the following eigenvalues and eigenvectors

$$\begin{array}{ll} \lambda_0 = 0 + \dots & \vec{\beta}_0 = (1,1,1,1) + \dots \\ \lambda_s = 8 + \dots & \vec{\beta}_s = (-1,-1,1,1) + \dots \\ \lambda_t = -8\sigma + \dots & \vec{\beta}_t = (-1,1,-1,1) + \dots \\ \lambda_u = -8(1-\sigma) + \dots & \vec{\beta}_u = (-1,1,1,-1) + \dots \end{array}$$

where $\sigma = \sin^2 \frac{\theta}{2}$. As we shall see, the only important effect of the small deviations from (2.6) is to produce a non-vanishing eigenvalue $\lambda_0 \neq 0$. In particular, $\det P_{ij} \approx 2^9 \sigma(1-\sigma)\lambda_0$.

In order to perform the $\vec{\beta}$-integral in (2.5) we change coordinates as follows,

$$4\vec{\beta} = \nu \vec{\beta}_0 + \nu_s \vec{\beta}_s + \nu_t \vec{\beta}_t + \nu_u \vec{\beta}_u \ .$$

The integrals over ν_s, ν_t and ν_u can be readily performed by saddle point, turning the second line of (2.5) into

$$\pi^{\frac{3}{2}} 4^{2-\sum \Delta_i} (l_s/R)^3 \frac{(-iy)^{\frac{3}{2}}}{\sqrt{\sigma(1-\sigma)}} \int_0^\infty \frac{d\nu}{\nu} \nu^{\sum \Delta_i - 3} e^{-i\frac{\xi^2 \nu^2}{4y}} iT^{(d+1)}(-t/s = \sigma, l_s^2 s = \nu^2) \qquad (2.7)$$

where we have introduced the scaling variable

$$-\xi^2 = \lim_{\substack{\lambda_0 \to 0 \\ R/l_s \to \infty}} \frac{R^2 \lambda_0}{2 l_s^2} = \lim_{\substack{\det P_{ij} \to 0 \\ R/l_s \to \infty}} \frac{R^2}{l_s^2} \frac{\det P_{ij}}{4 P_{12} P_{34} \sqrt{P_{13} P_{24} P_{14} P_{23}}} \ . \qquad (2.8)$$

To match the definitions in the Introduction, recall that in the usual Poincaré coordinates $P_i = (P^+, P^-, P^\mu) = (1, x_i^2, x_i^\mu)$, one has $P_{ij} = x_{ij}^2$.

We can now return to (2.5) and do the y-integral,

$$A \approx \frac{4(2\pi)^{\frac{3+d}{2}} l_s^3}{R^d \sqrt{\sigma(1-\sigma)}} \prod_{i=1}^{4} \frac{(-i)^{\Delta_i} C_{\Delta_i} R^{\Delta_i}}{\Gamma(\Delta_i)(4 l_s)^{\Delta_i}} \int_0^\infty \frac{d\nu}{\nu} \nu^{\sum \Delta_i - 3} (\xi\nu)^{\frac{3-d}{2}} K_{\frac{3-d}{2}}(\xi\nu) iT^{(d+1)} \qquad (2.9)$$

where K is the modified Bessel function.

Let us now specialize to the case of elastic scattering, where $\Delta_1 = \Delta_3$ and $\Delta_2 = \Delta_4$. In this case, it is convenient to define a reduced four point function $\mathcal{A}$ by dividing A by the disconnected correlator,

$$A(P_i) = \frac{C_{\Delta_1} C_{\Delta_2} \mathcal{A}(P_i)}{(-2P_1 \cdot P_3 + i\epsilon)^{\Delta_1} (-2P_2 \cdot P_4 + i\epsilon)^{\Delta_2}} \ .$$

The reduced function $\mathcal{A}$ here coincides with $\mathcal{A}$ that defined in (1.1). Let us also specialize to the case where the bulk theory is critical string theory on $\mathrm{AdS}_{d+1} \times M_{9-d}$. Then, the scattering amplitude $T^{(d+1)}$ is equal to the ten-dimensional string scattering amplitude $T = l_s^6 \mathcal{T}$ divided by the volume vR^{9-d} of the compact space M_{9-d}. Using (2.9), we conclude that the scaling function

$$\mathcal{F}(\xi) = \lim_{R/l_s \to \infty} \left(\frac{R}{l_s}\right)^{9-2\Delta_1-2\Delta_2} \frac{\mathcal{A}(P_i)}{\sigma^{\Delta_1+\Delta_2}} \tag{2.10}$$

is directly related to the string scattering amplitude via

$$\mathcal{F}(\xi) = \frac{\mathcal{N}}{\sqrt{\sigma(1-\sigma)}} \xi^{\frac{3-d}{2}} \int_0^\infty d\nu\, \nu^{2\Delta_1+2\Delta_2-\frac{5+d}{2}} K_{\frac{3-d}{2}}(\xi\nu) i\mathcal{T}(l_s^2 s = \nu^2 + i\epsilon) \tag{2.11}$$

where we have suppressed the $\sigma = -t/s$ and g_s dependence and defined

$$\mathcal{N} = \frac{(2\pi)^{\frac{3-d}{2}} 2^{d-2\Delta_1-2\Delta_2}}{v\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma\left(\Delta_1 - \frac{d}{2}+1\right)\Gamma\left(\Delta_2 - \frac{d}{2}+1\right)}\ .$$

When $\Delta_1 = \Delta_2 = d = 4$ this Bessel transform reduces to the Laplace transform of equation (1.3).

2.2 From Yang-Mills to strings

The inverse transform of (2.11), and hence (1.4) as a special case, can be found by using the wave packet construction of [2, 4]. Since the set-up is exactly the same as [4] and this is the second derivation of the main results, we will be brief. The basic idea was to use sources localized around the boundary points (2.6) to produce wave packets that scatter in a small flat region of the bulk. The operators smeared around P_1 and P_2 represent incoming particles while those around P_3 and P_4 correspond to outgoing particles. As before, the AdS radius R is taken to be large compared with the string length and the wave length of the massless particles. The $(d+1)$-dimensional spacetime momenta of each wave packet is $k_i = (\omega_i, \mathbf{k}_i)$. We work in the center-of-mass frame $\mathbf{k}_1 + \mathbf{k}_2 = 0$. Repeating the argument leading to (3.37) of [4], one can relate the $(d+1)$-dimensional scattering amplitude $T^{(d+1)}$ to $\mathcal{A}$ as

$$\begin{aligned} & i(2\pi)^{d+1}\delta^{d+1}(\textstyle\sum k_i)T^{(d+1)} \\ &= \mathcal{L}\,\delta(\textstyle\sum \omega_i)\delta^2(\sum \mathbf{k}_{i||}) \displaystyle\int d\tau e^{-il_s\omega_1\tau/\sin\theta} \\ &\times \mathcal{A}\left(\sigma, \rho^2 = \frac{(\tau+i\epsilon)^2/(R/l_s)^2 - (\mathbf{k}_{4\perp}/\omega_1)^2\sin^2\theta}{16\sin^4(\theta/2)}\right) \end{aligned} \tag{2.12}$$

up to lower order terms in the limit $R/l_s \to \infty$. We have grouped prefactors into

$$\mathcal{L} = \frac{(2\pi)^{\frac{d+1}{2}} l_s^{2d-5}(l_s^2 s)^{2-\Delta_1-\Delta_2}}{2v\mathcal{N}(\sin(\theta/2))^{2\Delta_1+2\Delta_2}(R/l_s)^{2\Delta_1+2\Delta_2-2d+2}}.$$

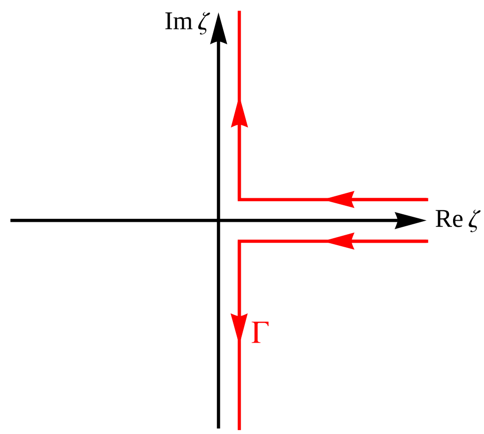

Figure 2: Integration contour used in equation 2.13.

The delta function $\delta^2(\sum \mathbf{k}_{i||})$ enforces momentum conservation in the plane spanned by $\mathbf{k}_1$ and $\mathbf{k}_3$, and $\mathbf{k}_{4\perp}$ is the projection of $\mathbf{k}_4$ perpendicular to that plane. On the right hand side we need an extra delta function $\delta^{d-2}(\mathbf{k}_{4\perp})$, which arise in the limit $R/l_s \to \infty$ if $\mathcal{A}$ scales as (2.10). The coefficient of the delta function can be computed by integrating over $\mathbf{k}_{4\perp}$. We then obtain the relation

$$i\mathcal{T}(l_s^2 s) = \frac{2l_s^3\sqrt{stu}}{(2\pi)^{\frac{d+1}{2}}\mathcal{N}}(l_s^2 s)^{\frac{d-1}{2}-\Delta_1-\Delta_2}\int d^{d-2}y \int d\tau e^{-i\tau l_s\sqrt{s}}\mathcal{F}\left(\xi = \sqrt{y^2-(\tau+i\epsilon)^2}\right) ,$$

which can be simplified to [7]

$$i\mathcal{T}(l_s^2 s) = -\frac{l_s^3\sqrt{stu}}{\pi^2\mathcal{N}}(l_s^2 s)^{\frac{d+1}{4}-\Delta_1-\Delta_2}\int_\Gamma d\xi\, \xi^{\frac{d-1}{2}} K_{\frac{d-3}{2}}\left(-\xi l_s\sqrt{s}\right)\mathcal{F}(\xi) , \tag{2.13}$$

where Γ is the contour shown in the figure 2. Specializing to $d = \Delta_1 = \Delta_2 = 4$ we obtain one of our main formulas (1.4).

3 Analysis of special cases

Now that we have derived the relations (1.1 - 1.4) between the string scattring amplitude and the CFT correlator, we wish to understand their implications better. They are highly abstract and given in terms of an unusual scaling limit. To gain intuition we study two special limits, namely the tree-level and hard scattering approximations of the string amplitude.

[7]The consistency of (2.13) and (2.11) can be checked using

$$\int_\Gamma d\xi\, \xi K_\alpha(a\xi)K_\alpha(-b\xi) = -\frac{\pi^2}{a}\delta(a-b) ,$$

with $a, b > 0$.

3.1 Tree-level string theory

Let us now focus on the first term in the g_s expansion. The tree-level four dilaton scattering amplitude in type IIB string theory is given by [8]

$$T_{\rm tree} = \kappa^2 \left(\frac{tu}{s} + \frac{su}{t} + \frac{st}{u} \right) \mathcal{B} \left(\frac{\alpha' s}{4}, \frac{\alpha' t}{4}, \frac{\alpha' u}{4} \right) \tag{3.1}$$

with $\mathcal{B}(\alpha_1, \alpha_2, \alpha_3) = \prod_{i=1}^{3} [\Gamma(1-\alpha_i)/\Gamma(1+\alpha_i)]$. Equation (1.3) then gives the planar contribution to the scaling function $\mathcal{F}$

$$\mathcal{F}_{\rm planar} = i g_{\rm YM}^4 \frac{\pi^2}{18\xi} \frac{(1-\sigma+\sigma^2)^2}{(\sigma(1-\sigma))^{3/2}} \int_0^\infty d\nu \nu^{13} \mathcal{B}\left(\nu^2 + i\epsilon, -\sigma\nu^2, (\sigma-1)\nu^2\right) e^{-2\xi\nu} \ . \tag{3.2}$$

The small α' expansion of the string scattering amplitude (3.1) corresponds to the large ξ expansion,

$$\mathcal{F}_{\rm planar} = i g_{\rm YM}^4 \frac{\pi^2}{9} \frac{(1-\sigma+\sigma^2)^2}{(\sigma(1-\sigma))^{3/2}} \left[\frac{\Gamma(14)}{(2\xi)^{15}} + 2\zeta(3)\sigma(1-\sigma) \frac{\Gamma(20)}{(2\xi)^{21}} + O\left(\xi^{-25}\right) \right] \ , \tag{3.3}$$

for Re $\xi > 0$. The first term in this expansion can be confirmed by taking the scaling limit of the SYM four point function computed in the supergravity approximation [7]. The other terms are predictions for the contributions of higher derivative corrections to the spacetime effective action.

The real part of $\mathcal{F}$ is directly related to the imaginary part of the scattering amplitude and therefore signals the presence of production thresholds. At tree level, it can be written as

$$\text{Re } \mathcal{F}_{\rm planar} = -g_{\rm YM}^4 \frac{\pi^3}{36\xi} \frac{(1-\sigma+\sigma^2)^2}{(\sigma(1-\sigma))^{3/2}} \sum_{k=1}^{\infty} \frac{(-1)^k k^7}{(k!)^2} \frac{\Gamma(1+\sigma k)\Gamma(1+(1-\sigma)k)}{\Gamma(1-\sigma k)\Gamma(1-(1-\sigma)k)} e^{-2\xi\sqrt{k}} \ .$$

The exponential decay at large positive ξ means that the real part of $\mathcal{F}$ is undetectable in the small α' expansion (3.3).

3.2 Hard scattering

It is well-known that string amplitudes are soft at high energy and fixed angle [8, 9, 10]. For instance the amplitude (3.1) decays as

$$T_{\rm tree} \approx i\kappa^2 s \frac{(1-\sigma+\sigma^2)^2}{\sigma(1-\sigma)} e^{-\alpha' s\, q(\sigma)} \ , \tag{3.4}$$

[8]Recall that $2\kappa^2 = (2\pi)^7 g_s^2 l_s^8$ and that the polarization tensor for dilaton with momentum k is $e_{\mu\nu} = (\eta_{\mu\nu} - k_\mu \bar{k}_\nu - k_\nu \bar{k}_\mu)/\sqrt{8}$ with $\bar{k}\cdot k = 1, \bar{k}^2 = 0$, so that $e_{\mu\nu}e^{\mu\nu} = 1$.

for large s and fixed $\sigma = -t/s$, with

$$-2q(\sigma) = \sigma \log \sigma + (1-\sigma)\log(1-\sigma) \ .$$

Since $q(\sigma) > 0$ for $0 < \sigma < 1$, the integral in (3.2) converges for all values of $\xi \in \mathbb{C}$, making $\xi \mathcal{F}_{\rm planar}$ an entire function of ξ. Indeed, we can define

$$\xi \mathcal{F}_{\rm planar} = i g_{\rm YM}^4 \frac{\pi^2}{18} \frac{(1-\sigma+\sigma^2)^2}{(\sigma(1-\sigma))^{3/2}} \sum_{l=0}^{\infty} a_l(\sigma)\, \xi^l \ ,$$

where the series has an infinite radius of convergence. The high energy asymptotics (3.4) gives the large order behavior

$$a_l(\sigma) \approx i(-1)^l \frac{\Gamma(7+l/2)}{2^{15} l!\, q(\sigma)^{7+l/2}} \ , \qquad l \to \infty \ .$$

At any order in the string perturbative expansion the string amplitude shows a similar soft behavior in the hard scattering regime. More precisely, the genus G amplitude decays as $e^{-\alpha' s\, q(\sigma)/(1+G)}$ at large s. This implies that $\xi\mathcal{F}$ is an entire function of ξ at any order in the g_s perturbative expansion. The definition (1.1) suggests that this is true even non-perturbatively. Furthermore, one expects the contribution of intermediate black hole states to this $2 \to 2$ exclusive process to be suppressed by $e^{-S_{BH}/2}$ [11, 12]. This fast decay with energy also makes $\xi\mathcal{F}$ an entire function of ξ.

4 Discussion

We derived the conjectural relations (1.1 - 1.4) and their generalizations (2.8, 2.10, 2.11, 2.13) using two methods, both on the gravity side of the AdS/CFT duality. Our relations involve a novel limit of correlation functions, which is quite natural in the gravity picture and corresponds to the flat space limit. Our derivations make use of the well-localized wave packets constructed in [4], and we believe that the interactions outside the flat space region do not contribute to the correlation functions.

This limit is however unusual and hard to analyze on the field theory side, making it hard to test our relations. Obtaining the string scattering amplitude from (1.4) requires computing the CFT four point function in a Lorentzian and strongly coupled regime. This is surely a daunting task. On the other hand, we can use the relation (1.3) to translate what is known about the string amplitude, like the all-genus calculations of the high-energy fixed-angle [8, 9, 10] and small-angle scattering [13, 14], into predictions for the gauge theory correlator.

It should also be possible to generalize the relation found here to more general S-matrix elements. In particular, a similar relation should exist between n-point functions and n-particle scattering amplitudes. We believe this will involve a scaling limit where $(R/l_s)^2 \det x_{ij}^2$ is kept fixed while $R/l_s \to \infty$. Another important issue is the limitation to external massless particles. Overcoming this would allow us, for instance, to relate the three point function of primary operators to the decay rate of a massive string state into two lighter ones.

Clearly the more interesting goal is to find a reduction of SYM that computes directly $\mathcal{F}$ or $\mathcal{T}$ without passing through the four point function of the full theory, in a way rather similar to the BMN limit [9]. This would be the long sought holographic dual of flat space.

Acknowledgments

We thank A. Buchel, L. Cornalba, M. Costa, J. Gomis, A. Sever, J. Polchinski, R. Porto and P. Vieira for useful discussions. Research at the Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. JP is funded by FCT grant SFRH/BPD/34052/2006. This research was supported in part by the National Science Foundation under Grant No. NSF PHY05-51164.

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