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Abstract

We consider a relativistic plasma containing charged chiral fermions in an external magnetic field, e.g a chirally symmetric quark-gluon plasma created in relativistic heavy ion collisions. We show that triangle anomalies imply the existence of a new type of collective gapless excitation in this system that stems from the coupling between the density waves of the electric and chiral charges; we call it "the Chiral Magnetic Wave" (CMW). The CMW exists even in a neutral plasma, i.e. in the absence of the axial and vector chemical potentials. We demonstrate the existence of CMW and study its properties using three different approaches: i) relativistic magnetohydrodynamics; ii) dimensional reduction to $(1+1)$ Sine-Gordon model, appropriate in a strong magnetic field; and iii) holographic QCD (Sakai-Sugimoto model), appropriate at strong coupling. We also briefly discuss the phenomenological implications of the CMW for heavy ion collisions.

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1 Introduction

Recently, the rôle of triangle anomalies in the dynamics of relativistic plasmas in magnetic field and/or at finite angular momentum has excited considerable attention. Such plasmas are created for example in relativistic heavy ion collisions at RHIC and LHC where the initial energy density significantly exceeds the threshold for the production of decofined and chirally symmetric quark-gluon plasma, and the coherent electromagnetic fields of colliding ions create a pulse of very intense magnetic field. Of particular interest are the following two phenomena caused in the quark-gluon plasma by the axial anomaly: the Chiral Magnetic Effect (CME) and the Chiral Separation Effect (CSE).

The CME is the phenomenon of electric charge separation along the axis of the applied magnetic field in the presence of fluctuating topological charge [1, 2, 3, 4, 5]. The CME in QCD coupled to electromagnetism assumes a chirality asymmetry between left- and right-handed quarks, parametrized by an axial chemical potential μ_A. Such an asymmetry can arise if there is an asymmetry between the topology-changing transitions early in the heavy ion collision. In particular, at finite axial chemical potential μ_A, an external magnetic field induces the vector current $j_i = \bar{\psi}\gamma_i\psi$:

$$\vec{j}_V = \frac{N_c\, e}{2\pi^2}\mu_A\vec{B}; \tag{1.1}$$

in our present convention the current of electric charge is ej_V. Closely related phenomena have been discussed in the physics of primordial electroweak plasma [6] and quantum wires [7]. While the original derivation used the weak coupling methods, the origin of the effect is essentially topological and so the CME is not renormalized even at strong coupling, as was shown by the holographic methods [8, 9, 10, 11, 12, 13]. The evidence for the CME has been found in lattice QCD coupled to electromagnetism, both within the quenched approximation [14, 15, 16] and with light domain wall fermions [17].

Recently, STAR [18, 19] and PHENIX [20, 21] Collaborations at Relativistic Heavy Ion Collider reported experimental observation of charge asymmetry fluctuations. While the interpretation of the observed effect is still under intense discussion, the fluctuations in charge asymmetry have been predicted [1] to occur in heavy ion collisions due to the CME. Additional tests include the correlation between the electric and baryon charge asymmetries [22]. There is an active ongoing discussion of the microscopic mechanisms of CME [23, 24, 25, 26, 27, 28, 29] and of the quantitative estimates of the expected charge asymmetries and of possible backgrounds – see e.g. [30, 31, 32, 33, 34, 35, 36, 37, 38].

The Chiral Separation Effect (CSE) refers to the separation of chiral charge along the axis of external magnetic field at finite density of vector charge (e.g. at finite baryon number density) [39, 40, 41]. The resulting axial current is given by

$$\vec{j}_A = \frac{N_c\, e}{2\pi^2}\mu_V \vec{B}, \tag{1.2}$$

where μ_V is the vector chemical potential. The close connection between CME and CSE can be established for example by the method of dimensional reduction appropriate in the case of a strong magnetic field [42]: the simple relations $J^0_V = J^1_A,\ J^0_A = J^1_V$ between the vector J_V and axial J_A currents in the dimensionally reduced $(1+1)$ theory imply that the density of baryon charge must induce the axial current, and the density of axial charge must induce the current of electric charge (CME); see also Ref.[43]. Since in the strong coupling, short mean free path, regime the plasma represents a fluid (for a recent review, see [44]), a number of recent studies initiated by [41] address the effects of triangle anomalies in hydrodynamics, e.g. [45, 46, 47, 48].

The central observation of the present paper is the following: the connection between the CME and CSE implies the existence of a new type of a collective excitation in the plasma. This excitation stems from the coupling between the density waves of electric and chiral charge. Let us illustrate this statement by a qualitative argument, to be followed by more rigorous derivations in sections 2, 3, and 4. Consider a local fluctuation of electric charge density; according to eq.(1.2) it will induce a local fluctuation of axial current. This fluctuation of axial current would in turn induce a local fluctuation of the axial chemical potential, and thus according to eq.(1.1) a fluctuation of electric current. The resulting fluctuation of electric charge density completes the cycle leading to the excitation that combines the density waves of electric and chiral charges; we will call it the "chiral magnetic wave" (CMW).

Apart from being interesting in its own right, the existence of CMW has important implications for the phenomenology of heavy ion collisions. The CME relies on the fluctuation of the axial charge density and so the net effect is expected to vanish when averaged over many events; one thus relies on measuring the fluctuations of charge asymmetries [1, 49]. On the other hand, since the quark-gluon plasma produced in heavy ion collisions possesses non-zero value of the baryon chemical potential, the CSE can lead to a non-vanishing axial current even after the summation over events is performed. However unfortunately a direct detection of the axial current in heavy ion collisions is very challenging. The CMW should exist even in a neutral plasma and so can induce interesting

observable effects in heavy ion collisions even after the sum over many events is performed; we will return to this topic in the Summary.

The paper is organized as follows. In section 2 we provide a derivation of the CMW based on relativistic magnetohydrodynamics. In section 3 we consider the case of a strong magnetic field and perform a dimensional reduction; in this case the dynamics of CMW is described by the Sine-Gordon equation. In section 4 we describe the CMW at strong coupling using the holographic methods within the Sakai-Sugimoto model. When the electromagnetism is treated dynamically, the CMW mixes with the longitudinal charge wave in the plasma – the plasmon. We consider the mixing of CMW with plasmons in section 6. Finally, in the Summary we outline the main result of the paper and discuss the directions for future studies.

2 Chiral magnetic wave in magnetohydrodynamics

Let us now proceed with the derivation sketched out in the introduction. We will see that there indeed exists a new gapless excitation in a deconfined QCD plasma that propagates along the applied magnetic field; it arises as a dynamical consequence of the underlying triangle anomaly of chiral symmetry. This new excitation is a long wavelength hydrodynamic mode with a dispersion relation that looks like that of sound waves,

$$\omega = \mp v_\chi k - i D_L k^2 + \cdots \quad ; \tag{2.3}$$

however, these propagating modes carry both electric and chiral charges. Since these modes would not exist if it were not for the applied magnetic field or the underlying triangle anomaly, we will call them "the chiral magnetic waves" (CMW). They give rise to several important new transport properties of hot QCD plasma, and affect its thermodynamics; we will further discuss this in section 6. We note that CMW exists even if the background plasma is neutral under either baryonic or axial symmetry, which should make it a generic phenomenon in relativistic plasmas.

For simplicity, let us consider single flavor ($N_F = 1$) massless QCD with chiral symmetry $U(1)_L \times U(1)_R$, or equivalently $U(1)_V \times U(1)_A$ where $V(A)$ denotes vector(axial) respectively. The axial symmetry $U(1)_A$ suffers from both QCD anomaly with gluonic topological density and from the triangle anomaly of global chiral symmetry. The latter is in fact not harmful to the conservation of $U(1)_A$ as long as one does not elevate the global chiral symmetry to a gauged one, while the former indeed breaks the axial $U(1)_A$

symmetry by quantum fluctuations of topological density.

Our starting point is the anomalous generation of vector and axial currents along the applied magnetic field in the presence of axial (vector) chemical potential μ_A (μ_V), as given by eqs. (1.1) and (1.2). We will now re-write these equations in a more suggestive matrix form as

$$\begin{pmatrix} \vec{j}_V \\ \vec{j}_A \end{pmatrix} = \frac{N_c\, e\vec{B}}{2\pi^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_V \\ \mu_A \end{pmatrix} \quad . \tag{2.4}$$

We are interested in small linearized fluctuations of the chiral currents j_A and j_V in the plasma; let us assume that this plasma is neutral, without any background charge density on average. We may then perform a linear expansion of the chemical potentials with respect to small charge densities (j_V^0, j_A^0),

$$\begin{pmatrix} \mu_V \\ \mu_A \end{pmatrix} = \begin{pmatrix} \frac{\partial \mu_V}{\partial j_V^0} & \frac{\partial \mu_V}{\partial j_A^0} \\ \frac{\partial \mu_A}{\partial j_V^0} & \frac{\partial \mu_A}{\partial j_A^0} \end{pmatrix} \begin{pmatrix} j_V^0 \\ j_A^0 \end{pmatrix} + \mathcal{O}\left(\left(j^0\right)^2\right) \equiv \begin{pmatrix} \alpha_{VV} & \alpha_{VA} \\ \alpha_{AV} & \alpha_{AA} \end{pmatrix} \begin{pmatrix} j_V^0 \\ j_A^0 \end{pmatrix} + \mathcal{O}\left(\left(j^0\right)^2\right) \quad . \tag{2.5}$$

Remembering that

$$\mu_i = \frac{\partial \mathcal{F}}{\partial j_i^0} \quad , \quad i = V, A \tag{2.6}$$

where $\mathcal{F}$ is the Helmholtz free energy, the α's appearing above are nothing but the susceptibility matrices of vector/axial charge densities,

$$\alpha_{ij} = \frac{\partial^2 \mathcal{F}}{\partial j_i^0 \partial j_j^0} \quad . \tag{2.7}$$

Considering the parity $\mathcal{P}$ transformation $V \to -V$ and $A \to A$, one concludes that parity invariance of QCD implies that $\alpha_{VA} = \alpha_{AV} = 0$ in the neutral plasma, $\mu_V = \mu_A = 0$. Moreover, a simple large N_c counting shows that

$$\alpha_{VV} \sim \alpha_{AA} \sim \mathcal{O}\left(\frac{1}{N_c}\right) \quad , \tag{2.8}$$

while their difference in a deconfined and chirally symmetric phase is subleading

$$\alpha_{VV} - \alpha_{AA} \sim \mathcal{O}\left(\frac{1}{N_c^2}\right) \quad ; \tag{2.9}$$

we will confirm this within the holographic large N_c Sakai-Sugimoto model in section 4. Independently of this, the vanishing of the difference $\alpha_{VV} - \alpha_{AA}$ can be taken as a signal of chiral symmetry restoration. Therefore, we expect it to be a good approximation to let $\alpha_{VV} = \alpha_{AA} \equiv \alpha$ in the chirally symmetric phase; this leads us to

$$\begin{pmatrix} \vec{j}_V \\ \vec{j}_A \end{pmatrix} = \frac{N_c\, e\vec{B}\alpha}{2\pi^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} j_V^0 \\ j_A^0 \end{pmatrix} \quad . \tag{2.10}$$

It is natural to diagonalize the equation above by going to the chiral basis

$$j_L^\mu \equiv \frac{1}{2}\left(j_V^\mu - j_A^\mu\right) \quad , \quad j_R^\mu \equiv \frac{1}{2}\left(j_V^\mu + j_A^\mu\right) \quad . \tag{2.11}$$

In terms of chiral currents, our previous assumptions and the definition of α's are easily translated to

$$\alpha = \frac{1}{2}\left(\frac{\partial \mu_L}{\partial j_L^0}\right) = \frac{1}{2}\left(\frac{\partial^2 \mathcal{F}}{\partial j_L^0 \partial j_L^0}\right) = \frac{1}{2}\left(\frac{\partial \mu_R}{\partial j_R^0}\right) = \frac{1}{2}\left(\frac{\partial^2 \mathcal{F}}{\partial j_R^0 \partial j_R^0}\right) \quad . \tag{2.12}$$

The (2.10) then leads to two decoupled relations

$$\vec{j}_{L,R} = \mp\left(\frac{N_c e \vec{B}\alpha}{2\pi^2}\right) j_{L,R}^0 \quad , \tag{2.13}$$

where one should keep in mind the definite sign in front of the right-hand side depending on the chirality of the currents.

One can view the above expression as the leading constitutive equation for the currents in the long wavelength derivative expansion of hydrodynamics. Indeed, our starting point (1.1,1.2) is strictly valid only when the variation of chemical potentials is sufficiently slow; for a finite frequency/momentum these expression gets modified resulting in frequency/momentum dependent chiral magnetic conductivity [50, 8, 24, 32]. The equation (2.13) is the first leading term in the derivative expansion, while the next leading-order correction to the chiral magnetic conductivity will be ∂^2 or $\omega^2 \sim k^2$ in frequency/momentum space. However, there is an important first-order derivative term in any constitutive equation of conserved current: a diffusion term $-D\vec{\nabla} j^0$, with a diffusion constant D. In our case, we will be interested only in the waves propagating along the magnetic field direction which we call *longitudinal*; thus on general grounds, the constitutive relation including the next leading-order diffusion term reads as

$$\vec{j}_{L,R} = \mp\left(\frac{N_c\ e\vec{B}\alpha}{2\pi^2}\right) j_{L,R}^0 - D_L \frac{\vec{B}(\vec{B}\cdot\vec{\nabla})}{B^2} j_{L,R}^0 + \cdots \quad , \tag{2.14}$$

with a longitudinal diffusion constant D_L. Although we discuss only longitudinal dynamics in this paper, it would also be interesting to study the transverse dynamics with the transverse diffusion constant D_T.

A similar constitutive equation was written previously by Son and Surowka [41]. There is however one point that will appear important for us: while Ref.[41] considers a weak magnetic field B and treats it in the linear approximation, we are claiming that (2.13)

and (2.14) are valid for *arbitrary* strength of eB non-perturbatively. This is equivalent to the validity of our starting point (1.1),(1.2) for arbitrarily large eB, which is not at all trivial and is a consequence of the absence of corrections to the axial anomaly. It is also important to note that although (1.1),(1.2) look linear in eB, this linearity is only apparent. Given a fixed density $j^0_{V,A}$, the chemical potentials $\mu_{V,A}$ in general may well depend on the dynamics of underlying microscopic theory, such as coupling constants, temperature, as well as magnetic field eB *non-linearly*, so that the currents in (1.1),(1.2) can in fact be very non-linear in these parameters, see e.g. [51, 52, 53]. The statement of (1.1),(1.2) is that these dependencies can be absorbed into the chemical potentials $\mu_{V,A}$. Therefore, one expects that α and D_L are in general non-linear functions of eB, temperature T, etc. In particular, they would also depend on the coupling constant, so that it is meaningful to study them in the strong coupling regime via holographic QCD as we do in our section 4. Let us mention Ref.[54] that presents a diagrammatic proof of (1.1),(1.2) perturbatively in the coupling constant, and Ref.[10] that proved this relation in the Sakai-Sugimoto model using the two-derivative approximation. We will present a strong coupling proof with full DBI action of holographic QCD in section 4 with arbitrary strength of eB, which presumably includes non-linear effects of derivatives as well. Therefore we expect that the equations (1.1),(1.2) and hence (2.14) hold universally.

Our next step is to combine (2.14) with the conservation law $\partial_\mu j^\mu_{L,R} = 0$. We take $\vec{B} = B\hat{x}^1$ and consider only longitudinal gradient ∂_1, which results in

$$\left(\partial_0 \mp \frac{N_c eB\alpha}{2\pi^2}\partial_1 - D_L\partial_1^2\right) j^0_{L,R} = 0 \quad . \tag{2.15}$$

This describes a directional wave, or *chiral wave*, of charge densities whose direction of motion is correlated with its chirality. The velocity is given by

$$v_\chi = \frac{N_c eB\alpha}{2\pi^2} = \frac{N_c eB}{4\pi^2}\left(\frac{\partial\mu_L}{\partial j^0_L}\right) = \frac{N_c eB}{4\pi^2}\left(\frac{\partial\mu_R}{\partial j^0_R}\right) \quad . \tag{2.16}$$

As we discussed in the previous paragraph, one expects that v_χ and D_L are non-trivial functions of eB, T, and the coupling constant, so they are interesting dynamical quantities to compute in any model. In frequency/momentum space, the above equation takes the form

$$\omega = \mp v_\chi k - iD_L k^2 + \cdots \quad , \tag{2.17}$$

as a hydrodynamic dispersion relation. Our main observation is the new first term in the dispersion relation which makes the mode propagating instead of simply diffusing. It

exists only if 1) triangle anomaly exists and 2) there is a background magnetic field. We stress that this new chiral mode of electric and chiral charge transport is present even if the plasma is neutral on average.

3 Chiral magnetic wave in strong magnetic field: dimensional reduction to $(1+1)$ Sine-Gordon problem

The expression (2.16) for the velocity v_χ of the Chiral Magnetic Wave (CMW) shows that $v_\chi \sim eB$. What happens when eB becomes large? We will now show that in the limit $eB \to \infty$ the velocity v_χ stays finite and reaches the velocity of light. To understand this, let us first examine the spectrum of charged fermions in magnetic field; for massless fermions, the energies of Landau levels are given by

$$E_n = \sqrt{2eB(n + \frac{1}{2} - s_z) + p_z^2}; \tag{3.18}$$

we assume that magnetic field $\vec{B}$ is directed along the z axis. Note that the lowest Landau level (LLL) with zero energy (at vanishing momentum p_z) is not spin-degenerate, and all excited Landau levels are – therefore the net chirality $s_z \cdot p_z$ is carried only by the LLLs.

When magnetic field is large compared to the temperature T, $eB \gg T^2$, the fermions stay "frozen" in the LLL and the transverse and longitudinal (along $\vec{z}$) dynamics are independent. In this case one can perform a dimensional reduction to the $(1+1)$ dimensional theory where the only allowed direction of motion is along the magnetic field. The transverse density of states is given by $eB/(2\pi)$, the density of LLs in the transverse plane. The longitudinal phase space density for a Fermi-momentum is simply $p_F/(2\pi)$. Therefore for the density of N_c left-handed fermions j_L^0 with $p_F = \mu_L$ (for massless particles the Fermi-momentum and the chemical potential μ are equal) is

$$j_L^0 = N_c \, \frac{eB}{2\pi} \, \frac{\mu_L}{2\pi}, \tag{3.19}$$

and the derivative is given by

$$\frac{\partial \mu_L}{\partial j_L^0} = \frac{1}{N_c} \, \frac{4\pi^2}{eB}. \tag{3.20}$$

Substituting this expression into (2.16), we obtain

$$v_\chi = 1; \tag{3.21}$$

therefore in the limit of strong magnetic field $eB \gg T^2$ the CMW indeed propagates with the velocity of light.

We can deduce a more detailed information about the dynamics of the CMW in strong magnetic field by making use of bosonization procedure. As is well known, bosonization approach is very powerful in the studies of $(1+1)$ dimensional systems [55, 56]. This is easy to understand as in one spatial dimension the produced fermion and an anti-fermion never separate and propagate together as a composite bosonic excitation – even if they do not interact at all!

The conservation of vector current $\partial_\mu j^\mu_V = 0$ can be ensured by introducing a boson field φ and choosing

$$j^\mu_V = \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \varphi; \tag{3.22}$$

this way the vector current is always conserved independently of the equations of motion. The corresponding choice for the axial current is

$$j^\mu_A = \epsilon^{\mu\nu} j^V_\nu = \frac{1}{\sqrt{\pi}} \partial^\mu \varphi. \tag{3.23}$$

For zero quark mass and in the absence of background electric field, the axial current should be conserved:

$$\partial_\mu j^\mu_A = \Box \varphi = 0. \tag{3.24}$$

Therefore the conservation of axial current leads to the wave equation for the bosonic excitation φ. The variation of φ in space and time causes variations of both charge and chiral densities; therefore it is natural to identify the wave defined by (3.24) with the CMW. Since (for massless quarks) there is no mass term in (3.24), this CMW propagates with the velocity of light.

It is convenient to split the field φ into left- and right-moving components:

$$\varphi(x,t) = \varphi_L(x - vt) + \varphi_R(x + vt); \tag{3.25}$$

in terms of these fields the original fermion fields are

$$\Psi_R(x - vt) = \frac{1}{\sqrt{2\pi}}\, e^{-i\sqrt{4\pi}\varphi_R(x-vt)}; \quad \Psi_L(x + vt) = \frac{1}{\sqrt{2\pi}}\, e^{i\sqrt{4\pi}\varphi_L(x+vt)}. \tag{3.26}$$

Upon quantization of the boson fields, the fermions (3.26) represent coherent states of φ_R, φ_L. Viceversa, these boson fields describe the collective electric and chiral charge density fluctuations of the underlying fermion quark fields.

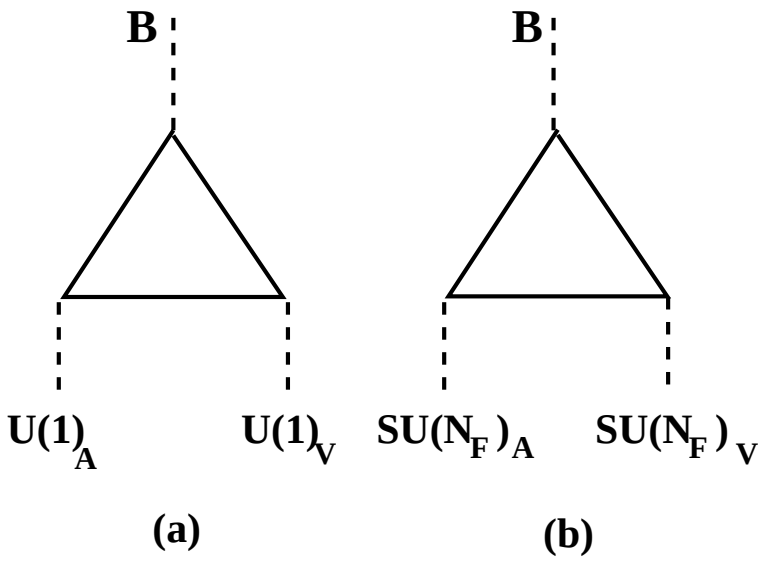

Figure 1: Anomalous triangle diagrams of chiral magnetic waves for (a) abelian and (b) non-abelian flavor symmetries in the presence of external magnetic field B. The quark lines are dressed propagators including the background magnetic field.

The quark mass term in the original Lagrangian can be re-written with the help of (3.26) in the following form:

$$M\bar{\Psi}\Psi = M(\bar{\Psi}_R\Psi_L + \bar{\Psi}_L\Psi_R) = \frac{M}{\pi}\cos\left(\sqrt{4\pi}\varphi\right), \tag{3.27}$$

and leads to the interaction term for bosons; the bosonic Lagrangian thus describes the Sine-Gordon theory. This correspondence between the fermionic and bosonic description has been discovered by Coleman [55] and Mandelstam [56]. The kinks of the Sine-Gordon theory describe the fermions, and the fluctuations of the boson field - collective fermion–anti-fermion excitations. Coleman's theorem [57] forbids spontaneous breaking of a continuous symmetry in $(1+1)$ dimensions. The underlying reason for this no-go theorem is the presence of strong infra-red fluctuations that dominate two-point functions and destroy the long-range order in two dimensions. This means that the massless (in the limit of massless quarks) boson φ is not a Goldstone boson*. This is consistent with our interpretation of the field φ as of a collective density wave of electric and chiral charges. At finite density of baryon or chiral charge, this density wave propagates on top of a "chiral spiral" [59] – the winding configuration of the background field $\bar{\varphi}$ causing the chiral magnetic effect in $(1+1)$ dimensional description [42, 43].

*There exists however a way around Coleman's theorem uncovered by Witten [58]: for N-component field at $N \to \infty$, when the number of degrees of freedom diverges at each point, the two-point functions can exhibit long-range order and the Goldstone phenomenon can still be realized.

One can also generalize the above to non-abelian version of chiral magnetic waves. As Figure.1 illustrates, essentially the same type of triangle diagrams for mixed $U(1)_V \times SU(N_F)_V \times SU(N_F)_A$ would result in

$$\begin{pmatrix} \vec{j}_V^a \\ \vec{j}_A^a \end{pmatrix} = \frac{N_c e \vec{B} \alpha}{2\pi^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} j_V^{0a} \\ j_A^{0a} \end{pmatrix} \quad , \tag{3.28}$$

for non-abelian $SU(N_F)$ components of currents $j_{V,A}^{\mu} = j_{V,A}^{\mu a} t^a$ with $\mathrm{tr}_F(t^a t^b) = \frac{1}{2}\delta^{ab}$. One then gets to the same conclusion on the emergence of the chiral/directional CMWs for each non-abelian component of $SU(N_F)$. Upon the 1+1 dimensional reduction with strong magnetic field, these non-abelian chiral magnetic waves should be described by non-abelian bosonization [60] of $SU(N_F)$ symmetry. More precisely, the theory of N_F Dirac fermions in fundamental representation of $SU(N_c)$ in bosonized description can be represented as

$$\mathcal{L} = N_c \mathcal{L}\left(SU(N_F)\right) + \mathcal{L}\left(U(1)\right) + N_F \mathcal{L}\left(SU(N_c)\right) \quad , \tag{3.29}$$

where $\mathcal{L}(G)$ represents the Wess-Zumino-Witten model of group G with level 1 [60]. After integrating over QCD $SU(N_c)$ dynamics, one is left with the first two pieces as a low energy effective theory [61, 62, 63, 64]. The $U(1)$ part is what we have discussed above, while the non-abelian $SU(N_F)$ part describes the non-abelian chiral magnetic wave as a propagating group field $g(x) \in SU(N_F)$.

4 Chiral magnetic wave in holographic QCD

In this section, we intend to study chiral magnetic wave at strong coupling in the framework of holographic QCD, particularly using the model by Sakai and Sugimoto [65]. The Sakai-Sugimoto model is a right place to look for the phenomenon because it includes the relevant triangle anomalies of QCD chiral symmetry in terms of 5D Chern-Simons terms and the existence of chiral magnetic wave is robust as long as the right anomalies exist. What is non-trivial will be the details of dispersion relations such as wave velocity and diffusion constant, which do depend on the strong dynamics and the stength of the applied magnetic field.

The Sakai-Sugimoto model in deconfined phase consists of two separated $D8$ and $\overline{D8}$ branes that touch the black hole horizon of the background geometry as shown in Figure 2. This geometric separation of two probe branes correctly indicates that interactions between left-handed quarks and right-handed quarks are sub-leading in large N_c expansion,

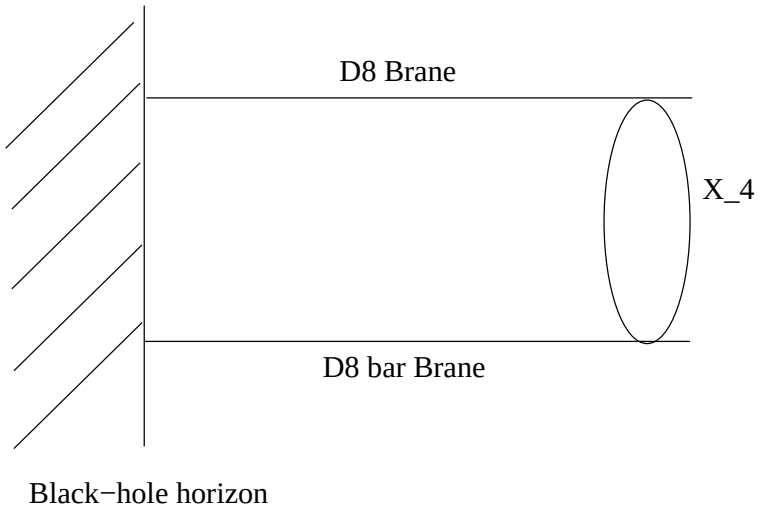

Figure 2: A schematic picture of $D8-\overline{D8}$ branes in deconfined phase of the Sakai-Sugimoto model.

when there is no chiral condensate in the deconfined phase. Because of this decoupling, one can study dynamics of each $D8$ probe brane independently at least in leading order approximation. The background geometry is a warped product of a 5D black hole and $S^1 \times S^4$ where $D8/\overline{D8}$ branes are separated along S^1. They wrap all other dimensions except S^1 to make up 9 dimensional world-volume on which the action is given by a Dirac-Born-Infeld action plus a Chern-Simons term;

$$S_{D8/\overline{D8}} = -\mu_8 \int d^9\xi \, e^{-\phi} \sqrt{\det\left(g^* + 2\pi l_s^2 F\right)} \mp \frac{\mu_8 \left(2\pi l_s^2\right)^3}{3!} \int F_4^{RR} \wedge A \wedge F \wedge F \quad , \quad (4.30)$$

with $\mu_p = (2\pi)^{-p} l_s^{-(p+1)}$. After integrating over S^4, one arrives at an effective 5 dimensional world-volume action of $D8/\overline{D8}$ branes embedded in the 5 dimensional black hole space-time of the metric

$$ds_{5D}^2 = \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(-f(U)dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2\right) + 2dUdt \quad , \quad f(U) = 1 - \left(\frac{U_T}{U}\right)^3 \quad , \quad (4.31)$$

where U is the holographic direction whose UV(IR) boundary is sitting at $U \to \infty (U = U_T)$, and the metric is written in Eddington-Finkelstein coordinate. Note that Eddington-Finkelstein coordinate is suitable to describe future event horizon in a smooth manner, and the in-going boundary conditions of perturbed modes at the future horizon reduce to simple regularity in this coordinate. The temperature T of the dual 4 dimensional QCD

plasma is related to the parameters in the above by

$$T = \frac{3}{4\pi}\left(\frac{U_T}{R^3}\right)^{\frac{1}{2}} \quad , \quad R^3 = \pi g_s N_c l_s^3 \quad . \tag{4.32}$$

The resulting 5 dimensional world-volume action of the probe brane is

$$\begin{aligned} S_{D8/\overline{D8}} &= -CR^{\frac{9}{4}} \int d^4x dU\, U^{\frac{1}{4}} \sqrt{\det\left(g^*_{5D} + 2\pi l_s^2 F\right)} \\ &\mp \frac{N_c}{96\pi^2} \int d^4x dU\, \epsilon^{MNPQR} A_M F_{NP} F_{QR} \quad , \quad C = \frac{N_c^{\frac{1}{2}}}{3\cdot 2^5 \pi^{\frac{11}{2}} g_s^{\frac{1}{2}} l_s^{\frac{15}{2}}} \quad , \end{aligned} \tag{4.33}$$

where the sign of the last Chern-Simons term depends on the chirality of the probe $D8/\overline{D8}$ branes. The ϵ-symbol above is numerical one. It is straightforward to write down the equation of motion of the world-volume gauge field A_M from the above action, which reads as

$$CR^{\frac{9}{4}}\pi l_s^2\, \partial_N \left[U^{\frac{1}{4}} \sqrt{\det\left(g^* + 2\pi l_s^2 F\right)} \left(\left(g^* + 2\pi l_s^2 F\right)^{-1}\right)^{[MN]}\right] \mp \frac{N_c}{32\pi^2} \epsilon^{MNPQR} F_{NP} F_{QR} = 0, \tag{4.34}$$

where $[MN] = MN - NM$ is anti-symmetrization. In deriving the above as well as for later convenience, it is useful to be reminded of the following expansion

$$\sqrt{\det\left(1+\delta A\right)} = 1 + \frac{1}{2}\mathrm{tr}\left(\delta A\right) + \frac{1}{8}\left[\mathrm{tr}\left(\delta A\right)\right]^2 - \frac{1}{4}\mathrm{tr}\left(\left(\delta A\right)^2\right) + \mathcal{O}\left(\left(\delta A\right)^3\right) \quad . \tag{4.35}$$

For our purpose of studying chiral magnetic wave, we first need to construct a background solution of having a constant magnetic field of *arbitrary strength* pointing, say, x^3 direction. It is in fact easy to show that the trivially constant $F_{12} \equiv B$ solves both Bianchi identity and the equation of motion (4.34), so the *holographic* background configuration of a constant B will be our starting point of looking for a signal of chiral magnetic wave. In making contact between our $D8/\overline{D8}$ branes and the QCD chiral symmetry, we work in the case of $N_F = 1$ for simplicity, neglecting $U(1)_A$-anomaly induced by gluons as a leading large N_c approximation. This is appropriate as long as we focus on triagle anomalies which are encoded in the 5D Chern-Simons terms in holographic QCD. Therefore, chiral symmetry is $U(1)_L \times U(1)_R$, and $D8(\overline{D8})$-brane dynamics captures $U(1)_L(U(1)_R)$ chiral dynamics of QCD holographically. The electromagnetism lies in the diagonal combination of $U(1)_L$ and $U(1)_R$, while axial symmetry is the other orthogonal combination, so one has a dictionary

$$eA_{EM} = \frac{1}{2}\left(A_L + A_R\right) \quad , \quad A_a = \frac{1}{2}\left(-A_L + A_R\right) \quad , \tag{4.36}$$

where $A_L(A_R)$ is the external potential that couples to the chiral current $J_L(J_R)$, and e is the electromagnetic coupling constant. Therefore, having a constant electromagnetic B means having a constant F_{12} on each $D8$ and $\overline{D8}$ brane with

$$F_{12}^{D8} = F_{12}^{\overline{D8}} = eB \quad , \tag{4.37}$$

and we will assume this in the following.

Our next task is to expand linearly around this background to study hydrodynamics of *longitudinal* charge/current fluctuations. As we discuss in previous sections, these longitudinal charge/current fluctuations would have had leading diffusive dispersion relation $\omega \sim -iDk^2$ in the absence of triangle anomalies. Chiral magnetic wave is an anomaly-induced modification of this into a leading propagating dispersion relation

$$\omega \sim \mp v_\chi k - iD_L k^2 + \cdots \quad , \tag{4.38}$$

where the velocity v_χ as well as the *longitudinal* diffusion constant D_L are expected to depend on strong coupling dynamics and the magnitude of the magnetic field B. The above dispersion relation due to triangle anomalies is our main objective we will be heading to in this section. Especially, v_χ as a function of B will be a new physical quantity that we compute in the framework of holographic QCD. Because v_χ should vanish in the absence of anomalies and the magnetic field, it will be an odd function of the product of Chern-Simons coefficient and the magnetic field B. We will confirm this expectation later. We also emphasize that the sign of the leading term in (4.38) is fixed by the sign of Chern-Simons term, that is, by the chirality one is looking at, so left-handed charge/current fluctuations would have a definite propagating direction with respect to the magnetic field direction, while the right-handed ones would propagate in the opposite way. Therefore, one can achieve *chirality separation* in this way as a dynamical consequence of triangle anomalies of QCD. Recall that no charge in the background, either axial or baryonic, needs to be present for the effect to take place, and only magnetic field applied to a deconfined plasma with an underlying triangle anomaly is sufficient to have the phenomenon.

Studying linearized fluctuations from the constant $F_{12} = eB$ background requires expanding the 5D action (4.33) quadratically in terms of relavant fluctuation fields, and it is tedious but straightforward to do it using the formula (4.35). As we are interested in longitudinal charge/current fluctuations, it is sufficient to consider $(\delta F_{tU}, \delta F_{3U}, \delta F_{t3})$ fluctuations of "helicity" 0 only. Because of residual $SO(2)$ rotation symmetry of the constant F_{12} background, other non-zero helicity modes simply decouple from the above

modes at linearized level of equations of motion. After a sizable amount of computation, one arrives at

$$\begin{aligned} S^{(2)}_{D8/\overline{D8}} &= \int d^4xdU\,\frac{1}{2}\left[A(U)\left(\delta F_{tU}\right)^2 - B(U)\left(\delta F_{3U}\right)^2 + 2C(U)\left(\delta F_{t3}\right)\left(\delta F_{3U}\right)\right] \\ &\mp \frac{N_c eB}{8\pi^2}\int d^4xdU\left[\delta A_U\delta F_{t3} - \delta A_3\delta F_{tU} + \delta A_t\delta F_{3U}\right] \quad , \end{aligned} \tag{4.39}$$

where the three functions that appear in the coefficients are given by

$$\begin{aligned} A(U) &= C\left(2\pi l_s^2\right)^2 U\left[U^3 + R^3\left(2\pi l_s^2 eB\right)^2\right]^{\frac{1}{2}} \quad , \\ B(U) &= C\left(2\pi l_s^2\right)^2 U f(U)\left[U^3 + R^3\left(2\pi l_s^2 eB\right)^2\right]^{\frac{1}{2}} \quad , \\ C(U) &= C\left(2\pi l_s^2\right)^2 U\left(\frac{R}{U}\right)^{\frac{3}{2}}\left[U^3 + R^3\left(2\pi l_s^2 eB\right)^2\right]^{\frac{1}{2}} \quad . \end{aligned} \tag{4.40}$$

The second line in (4.39) is from the Chern-Simons term that represents triangle anomaly of QCD holographically. It is easy to keep track of that by a combination $N_c eB$ which is the product of anomaly coefficient and the magnetic field. From the above quadratic expansion of the action, one easily writes down the linearized equations of motion as

$$\begin{aligned} \partial_U\left(A(U)\delta F_{tU}\right) + C(U)\left(\partial_3\delta F_{3U}\right) \mp \frac{N_c eB}{4\pi^2}\delta F_{3U} &= 0 \quad , \\ \partial_U\left(B(U)\delta F_{3U}\right) + C(U)\left(\partial_t\delta F_{3U}\right) - \partial_U\left(C(U)\delta F_{t3}\right) \mp \frac{N_c eB}{4\pi^2}\delta F_{tU} &= 0 \quad , \\ A(U)\left(\partial_t\delta F_{tU}\right) - B(U)\left(\partial_3\delta F_{3U}\right) + C(U)\left(\partial_3\delta F_{t3}\right) \pm \frac{N_c eB}{4\pi^2}\delta F_{t3} &= 0 \quad . \end{aligned} \tag{4.41}$$

To proceed to hydrodynamic analysis from the above, one invokes low frequency/momentum expansion in solving (4.41) order by order with right boundary conditions. As mentioned before, simple regualarity at the future horizon $U = U_T$ in Eddington-Finkelstein coordinate is equivalent to implementing in-coming boundary conditions, and one should also impose normalizability on the modes at the UV boundary $U \to \infty$. Near $U \to \infty$, (4.41) gives two asymptotic behaviors, either $\mathcal{O}(1)$ or $\mathcal{O}(U^{-\frac{3}{2}})$, and the normalizability picks up the latter only. In holography, $U \to \infty$ boundary value of A_μ corresponds to the value of an external gauge potential that couples to the field theory currents. Because we do not have such external gauge potential in the system, we have to put the boundary condition corresponding to A_μ vanishing at $U \to \infty$. Assuming the frequency/momentum factor $e^{-i\omega t + ikx^3}$ or equivalently replacing $\partial_t = -i\omega$ and $\partial_3 = ik$, and working in a gauge

$A_U = 0$, the equations of motion become

$$\begin{aligned}
\partial_U \left[A(U)\left(\partial_U \delta A_t\right)\right] + ikC(U)\left(\partial_U \delta A_3\right) \mp \frac{N_c eB}{4\pi^2}\left(\partial_U \delta A_3\right) = 0, \\
\partial_U \left[B(U)\left(\partial_U \delta A_3\right)\right] - i\omega C(U)\left(\partial_U \delta A_3\right) - \partial_U \left[C(U)\left(i\omega \delta A_3 + ik\delta A_t\right)\right] \mp \frac{N_c eB}{4\pi^2}\left(\partial_U \delta A_t\right) = 0, \\
-i\omega A(U)\left(\partial_U \delta A_t\right) - ikB(U)\left(\partial_U \delta A_3\right) + ikC(U)\left(i\omega \delta A_3 + ik\delta A_t\right) \pm \frac{N_c eB}{4\pi^2}\left(i\omega \delta A_3 + ik\delta A_t\right) = 0,
\end{aligned} \tag{4.42}$$

which should be solved in perturbative expansion of (ω, k). What one expects is that imposing boundary conditions restricts the solution space such that ω in hydrodynamic expansion is determined once k is given, and the relation $\omega = \omega(k)$ is called the *dispersion relation*.

There are several methods in literature to solve similar kinds of equations in hydrodynamic expansion [66], but we will follow our own method which seems most convenient to us. We first assume that $\omega(k)$ is an analytic power series in k, which is expected based on hydrodynamics,

$$\omega(k) = \sum_{n\geq 1} a_n k^n = v_\chi k - iD_L k^2 + \cdots \quad . \tag{4.43}$$

Inserting this to (4.42), then one can take k as the only expansion parameter in solving (4.42) systematically, along which a_n should also be determined order by order. Because (4.42) is linear in $(\delta A_t, \delta A_3)$ one can always rescale them so that they start their k-expansion as

$$(\delta A_t, \delta A_3) = \sum_{n\geq 0} (A_t^{(n)}, A_3^{(n)}) k^n = (A_t^{(0)}, A_3^{(0)}) + (A_t^{(1)}, A_3^{(1)}) k + \cdots \quad , \tag{4.44}$$

where $(A_t^{(0)}, A_3^{(0)})$ cannot vanish simultaneously by definition. It is straightforward to insert (4.43) and (4.44) into (4.42), and solve order by order in k.

At $\mathcal{O}(k^0)$, one gets the equations

$$\partial_U \left[A(U)\left(\partial_U A_t^{(0)}\right)\right] \mp \frac{N_c eB}{4\pi^2}\left(\partial_U A_3^{(0)}\right) = 0 \quad , \tag{4.45}$$

$$\partial_U \left[B(U)\left(\partial_U A_3^{(0)}\right)\right] \mp \frac{N_c eB}{4\pi^2}\left(\partial_U A_t^{(0)}\right) = 0 \quad , \tag{4.46}$$

$$-iv_\chi A(U)\left(\partial_U A_t^{(0)}\right) - iB(U)\left(\partial_U A_3^{(0)}\right) \pm \frac{N_c eB}{4\pi^2}\left(iv_\chi A_3^{(0)} + iA_t^{(0)}\right) = 0 \quad . \tag{4.47}$$

Integrating the first two equations (4.45) and (4.46) gives us

$$A(U)\left(\partial_U A_t^{(0)}\right) \mp \frac{N_c eB}{4\pi^2} A_3^{(0)} = C_1 \quad , \tag{4.48}$$

$$B(U)\left(\partial_U A_3^{(0)}\right) \mp \frac{N_c eB}{4\pi^2} A_t^{(0)} = C_2 \quad , \tag{4.49}$$

with two integration constants, while the last equation (4.47) simply becomes

$$-iv_\chi C_1 - iC_2 = 0 \quad \Longrightarrow v_\chi = -\frac{C_2}{C_1} \quad , \tag{4.50}$$

so that v_χ will be determined once $C_{1,2}$ (or more precisely their ratio) are fixed by imposing relevant boundary conditions. Considering (4.48) at $U \to \infty$, one first fixes

$$C_1 = \lim_{U\to\infty} A(U)\left(\partial_U A_t^{(0)}\right) \quad , \tag{4.51}$$

and solving $A_3^{(0)}$ gives

$$A_3^{(0)} = \pm\left(\frac{4\pi^2}{N_c eB}\right)\left(A(U)\left(\partial_U A_t^{(0)}\right) - C_1\right) \quad . \tag{4.52}$$

Inserting this into (4.49), one gets a second order differential equation for $A_t^{(0)}$,

$$B(U)\partial_U\left(A(U)\left(\partial_U A_t^{(0)}\right)\right) - \left(\frac{N_c eB}{4\pi^2}\right)^2 A_t^{(0)} = \pm C_2\left(\frac{N_c eB}{4\pi^2}\right) \quad . \tag{4.53}$$

Because $B(U_T) = 0$ at the horizon, the regularity boundary condition imples that

$$C_2 = \mp\left(\frac{N_c eB}{4\pi^2}\right) A_t^{(0)}(U_T) \quad , \tag{4.54}$$

and one can write the solution for $A_t^{(0)}$ as

$$A_t^{(0)} = \tilde{A} + A_t^{(0)}(U_T) \quad , \tag{4.55}$$

where $\tilde{A}$ satisfies

$$B(U)\partial_U\left(A(U)\left(\partial_U \tilde{A}\right)\right) - \left(\frac{N_c eB}{4\pi^2}\right)^2 \tilde{A} = 0 \quad , \tag{4.56}$$

with the boundary condition $\tilde{A}(U_T) = 0$ at the horizon. This uniquely determines $\tilde{A}$ up to rescaling. Note that $A_t^{(0)}(U_T)$ is free up to this point, and the final boundary condition we need to impose is to demand vanishing $A_t^{(0)}$ at $U \to \infty$, and from (4.55) this fixes $A_t^{(0)}(U_T)$ as

$$A_t^{(0)}(U_T) = -\lim_{U\to\infty} \tilde{A}(U) \quad , \tag{4.57}$$

so that C_2 is finally

$$C_2 = \pm \left(\frac{N_c eB}{4\pi^2} \right) \lim_{U\to\infty} \tilde{A}(U) \quad . \tag{4.58}$$

Observe that C_1 is also given by $\tilde{A}$ as

$$C_1 = \lim_{U\to\infty} A(U) \left(\partial_U \tilde{A} \right) \quad . \tag{4.59}$$

Therefore, the complete solution at $\mathcal{O}(k^0)$ with the right boundary conditions can be written solely in terms of $\tilde{A}$ as above, and it is unique up to overall rescaling. Especially v_χ is well-posed and given by

$$v_\chi = -\frac{C_2}{C_1} = \mp \left(\frac{N_c eB}{4\pi^2} \right) \lim_{U\to\infty} \left(\frac{\tilde{A}(U)}{A(U) \left(\partial_U \tilde{A} \right)} \right) \quad . \tag{4.60}$$

As expected, v_χ is proportional to the anomaly coefficient, and its sign depends on the chirality and the chiral magnetic wave is uni-directional.

Discussions in the previous sections independently argue that v_χ should be given by

$$v_\chi = \mp \left(\frac{N_c eB}{4\pi^2} \right) \left(\frac{\partial \mu}{\partial j^0} \right)_{j^0=0} \quad , \tag{4.61}$$

where (μ, j^0) are chemical potential and charge density for either $U(1)_L$ or $U(1)_R$. Indeed we can confirm this expectation in our final formula (4.60) which is fully non-linear in eB, so that one can consider this as a strong coupling proof of the relation. First note that to compute $\left(\frac{\partial \mu}{\partial j^0} \right)_{j^0=0}$ one only needs a linear perturbation of μ or j^0 to the system of our background B field, and the relevant equations of motion for them are precisely given by our previous one (4.41) with additional assumption of space-time homogeneity $\partial_t = \partial_3 = 0$. Then the equations simplify exactly to the previous (4.48) and (4.49) with suitable boundary conditions. In the second equation considering the horizon point $U = U_T$, we have $B(U_T) = 0$ and we demand that A_t vanishes at the horizon, so that $C_2 = 0$. Combining the two equations after removing A_3, one easily arrives at that A_t satisfies the same equation (4.56) that $\tilde{A}$ satisfies, and moreover they share the same boundary condition at the horizon $U = U_T$, so that they are in fact the same object $A_t = \tilde{A}$. On the other hand, gauge/gravity dictionary tells us that up to linear order,

$$\mu = \lim_{U\to\infty} A_t(U) \quad , \quad j^0 = \lim_{U\to\infty} A(U) \left(\partial_U A_t \right) \quad , \tag{4.62}$$

so that one has

$$\left(\frac{\partial \mu}{\partial j^0} \right)_{j^0=0} = \lim_{U\to\infty} \left(\frac{\tilde{A}(U)}{A(U) \left(\partial_U \tilde{A} \right)} \right) \quad , \tag{4.63}$$

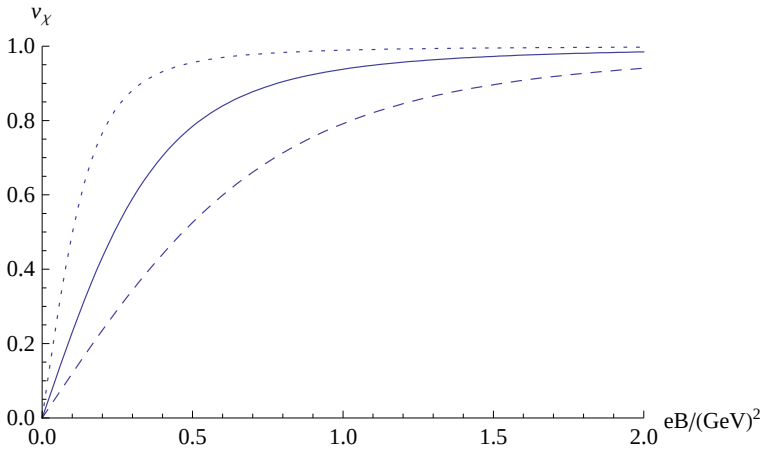

Figure 3: Numerical result for v_χ in the Sakai-Sugimoto model with $T = 150$ MeV(dotted), $T = 200$ MeV(plain), and $T = 250$ MeV(dashed).

which proves the relation (4.61) at strong coupling fully non-linearly in eB. As the validity of (4.61) is equivalent to (1.1),(1.2), this constitutes a holographic proof of (1.1),(1.2). Note that there is a subtlety regarding the possible contribution of the Bardeen counterterm [10]; this issue was settled in Refs. [9, 11] that show that the Bardeen counterterm does not affect the CME.

It is interesting to see how v_χ depends on the magnitude of the magnetic field eB in a non-linear way because $(A(U), B(U))$ contain eB as in (4.40), although the necessary analysis inevitably involves numerical study. To perform numeric analysis, we have to specify parameters of the model. First of all, one can always put $2\pi l_s^2 \equiv 1$ for simplicity because this factor will eventually cancel out in any well-defined field theory observables. One can easily check that this is the case for v_χ as well. By fitting to the observed ρ-meson mass and the pion decay constant, Sakai-Sugimoto fixed the parameters as

$$N_c = 3 \quad , \quad g_{YM}^2 N_c \sim 17 \quad , \quad M_{KK} \sim 0.94\,\mathrm{GeV} \quad , \tag{4.64}$$

where g_{YM}^2 and M_{KK} are related to g_s by

$$g_{YM}^2 = 2\pi l_s M_{KK} g_s \quad . \tag{4.65}$$

Recall that what matters for us is simply the parameters R^3 and C, and in terms of the above parameters, one has

$$C \sim 0.0211 \quad , \quad R^3 \sim 1.44 \quad . \tag{4.66}$$

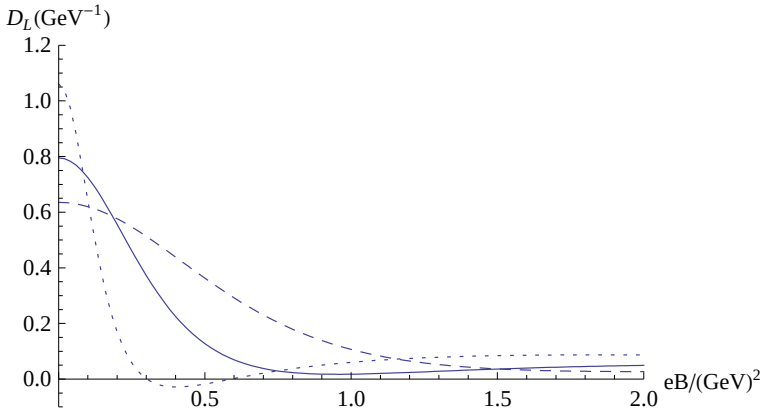

Figure 4: Numerical result for D_L in the Sakai-Sugimoto model with $T = 150$ MeV(dotted), $T = 200$ MeV(plain), and $T = 250$ MeV(dashed).

For the temperature, we take $T = (150, 200, 250)$ MeV as an illustrative purpose. Note that this model has deconfinement phase transition at $T_c = \frac{M_{KK}}{2\pi} \sim 150$ MeV [67]. We plot our numeric result of v_χ as a function of eB in Figure 3.

Limited analytic results for v_χ are available for two extreme regions, either $eB \to 0$ or $eB \to \infty$. For this purpose as well as an easier numerical analysis, it is convenient to consider the combination

$$V(U) \equiv \frac{\tilde{A}(U)}{A(U)\left(\partial_U \tilde{A}\right)} \quad , \tag{4.67}$$

in terms of which the equation (4.56) becomes a simple first order differential equation

$$\partial_U V(U) = \frac{1}{A(U)} - \left(\frac{N_c eB}{4\pi^2}\right)^2 \frac{1}{B(U)} V^2(U) \quad , \tag{4.68}$$

with a boundary condition $V(U_T) = 0$. From (4.60), v_χ is then simply given by

$$v_\chi = \mp \left(\frac{N_c eB}{4\pi^2}\right) V(\infty) \quad , \tag{4.69}$$

which seems technically much easier. With this formulation, it is not difficult to derive the following results;

Weak field limit : $eB \to 0$

$$v_\chi \sim \mp \left(\frac{N_c eB}{4\pi^2}\right) \int_{U_T}^{\infty} \frac{dU'}{A(U')} + \mathcal{O}\left(eB\right)^3 = \mp \frac{27}{8\pi^2} \frac{M_{KK}(eB)}{(g_{YM}^2 N_c) T^3} + \mathcal{O}\left(eB\right)^3 \quad . \tag{4.70}$$

Strong field limit : $eB \to \infty$

$$v_\chi \to \mp 1 \quad \text{(speed of light)} \quad . \tag{4.71}$$

Note that the strong field limit gives us the same result that one expects in the weak coupling Landau level picture that is discussed in section 3. It might come as a surprise because the dynamical degrees of freedom in our holographic model are still mesonic degrees of freedom represented by world-volume gauge fields on $D8$-branes. The Sakai-Sugimoto model operates with mesonic degrees of freedom, so it is difficult to trace directly the meaning of our results in terms of the Dirac eigenmodes of quarks [40]. However, we find that our result for the velocity of the chiral magnetic wave in the strong field limit, $v_\chi \to 1$, seems natural in terms of the Landau level picture. Indeed, in the strong field limit the quarks occupy only the lowest Landau levels that are chiral, so we indeed are dealing with left- and right-handed Dirac modes. In this limit, and for vanishing quark mass, the propagation of the wave must occur with the velocity of light, and this is what we obtain from holography. This may represent a facet of quark-hadron duality, and it would be interesting to explore this further.

One can go on to the next order expansion in k to compute the longitudinal diffusion constant D_L. As the computation is conceptually straightforward, we only present our numerical results in our Figure 4. For $eB = 0$, it reduces to the known diffusion constant in the model

$$D_L \to C(U_T) \int_{U_T}^{\infty} \frac{dU'}{A(U')} = \frac{1}{2\pi T} \quad , \quad eB \to 0 \quad , \tag{4.72}$$

while in general it is a complicated function of eB and the temperature T. As one can observe in Figure 4, D_L generally gets decreased as eB increases, which seems physically sensible because a larger magnetic field would align quasi-particles more efficiently, so that their microscopic longitudinal velocity diffusion would be smaller. What is interesting is that for some range of eB in a low temperature, say $T = 150$ MeV, one seems to have a *negative* value of D_L. Looking at the dispersion relation

$$\omega = \mp v_\chi k - i D_L k^2 + \cdots \quad , \tag{4.73}$$

this signals an instability for sufficiently small k or long wave-length fluctuations, which is precisely similar to Gregory-Laflamme instability in a gravity system [68]. In the case of gravity system, Gubser-Mitra conjecture [69] links this dynamic instability with a thermodynamic instability, and it would be interesting to study whether our instability is also related to some kind of thermodynamic instability. We leave this for future studies.

5 Dynamical Electromagnetism : Mixing Chiral Magnetic Wave with Plasmons

The previous sections treat electromagnetism as providing only a non-dynamical external magnetic field, and neglecting dynamical nature of electromagnetic field. This is valid in the limit $e \to 0$ while keeping eB finite. However, to describe the real world more precisely, it would be desirable to go beyond this approximation including dynamical electromagnetism. In general, dynamical electromagnetic field in the plasma couples to longitudinal fluctuations of vector charge density inducing the plasma waves. The chiral magnetic wave (more precisely its projection onto the vector $U(1)_V$-part) also involves longitudinal charge density fluctuations; it is thus natural to expect them to mix with each other, resulting in interesting modifications of their dispersion relations. This will be the main topic of this section. We also point out that for non-Abelian $SU(N_F)$ chiral magnetic waves there is no mixing with plasmons as far as one does not introduce gauging of $SU(N_F)$, so that the previous discussion stays intact for non-Abelian CMWs.[†]

Having dynamical electromagnetism, one first needs to include the Maxwell equation as a new dynamical equation of motion in addition to the conservation laws of currents,

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = e j_V^\nu \quad . \tag{5.74}$$

What is important for us is how to determine the current j_V^μ. Typical discussions of plasma waves proceed by assuming a linear retarded response of the plasma to the electromagnetic field A^μ,

$$j_V^\mu = e\Pi^{\mu\nu}(\omega, k) A_\nu \quad , \quad k_\mu \Pi^{\mu\nu} = 0 \quad , \tag{5.75}$$

where we work in the frequency-momentum space. Combined with (5.74) this would result in a self-contained equation for A_μ from which one can extract the plasmons. Let us review this procedure briefly as we are going to extend it by including the chiral magnetic waves and diffusion terms. Assuming a definite frequency-momentum $e^{-i\omega t + ikx^1}$, we focus on longitudinal polarizations (A_0, A_1), the only ones where we expect the emergence of plasmons. Because one has a gauge freedom of shifting A_μ by k_μ, one can use this to remove A_1 and work with A_0 only, which simplifies the analysis significantly. From (5.74) and (5.75), one then obtains

$$k^2 A^0 = e^2 \Pi^{00}(\omega, k) A^0 \quad , \quad k\omega A^0 = e^2 \Pi^{10}(\omega, k) A^0 \quad . \tag{5.76}$$

[†]To be more precise, EM charge is a sum of $U(1)$ and I_3 of $SU(2)_F$ for $N_F = 2$, so the situation can be more complicated in general. We leave a full discussion as a simple extension of ours to the readers.

These two equations are in fact equivalent as can be seen using the Ward identity $k_\mu \Pi^{\mu 0} = \omega \Pi^{00} - k\Pi^{10} = 0$, and one has a non-trivial solution of A^0 only if (ω, k) satisfies the *plasmon dispersion relation*,

$$k^2 = e^2 \Pi^{00}(\omega, k) \quad . \tag{5.77}$$

One expects plasma waves in long wave-length regime $k \to 0$ with finite frequency, so one expands Π^{00} in powers of $\frac{k^2}{\omega^2}$ as

$$\Pi^{00} \sim \omega_0^2 \frac{k^2}{\omega^2} \left(1 + c_{eff}^2 \frac{k^2}{\omega^2} \right) + \cdots \quad , \tag{5.78}$$

upon which the dispersion relation becomes

$$\omega^2 = e^2 \omega_0^2 + c_{eff}^2 k^2 + \mathcal{O}(k^4) \quad ; \tag{5.79}$$

this looks like a massive excitation of mass squared $\omega_p^2 \equiv e^2 \omega_0^2$ with an effective speed of light c_{eff}. The parameter ω_0 is typically of the order of temperature T, and the plasma frequency is about $\omega_p \sim eT$ which can be small compared to T in weak-coupling limit; the plasmon thus can be an important ingredient in describing the hydrodynamic properties of the system.

We will have to modify this picture in two aspects for our case of a constant background magnetic field by including chiral magnetic waves and diffusion terms. Because the magnetic field B is constant while the plasma waves involve only longitudinal electric field fluctuations, we can decouple them, and treat A_μ from now on as an *additional* fluctuation of the gauge field on top of the background magnetic field. Looking at the chiral magnetic effect (2.10), one first notices that the *anomalously* induced spatial currents are not directly related to retarded response to the gauge field fluctuations A^μ; they are induced from the *total* charge densities $j^0_{V,A}$ without any regard to how these charge densities appear. We will discuss these charge densities below. Also, we have to modify (2.10) for finite frequency-momentum by introducing chiral magnetic conductivity $\sigma(\omega, k)$ [50, 8] because it generally depends on (ω, k),

$$\begin{pmatrix} \vec{j}_V \\ \vec{j}_A \end{pmatrix} = \vec{B}\alpha \begin{pmatrix} 0 & \sigma(\omega,k) \\ \tilde{\sigma}(\omega,k) & 0 \end{pmatrix} \begin{pmatrix} j^0_V \\ j^0_A \end{pmatrix} \equiv \frac{N_c e \vec{B} \alpha}{2\pi^2} \begin{pmatrix} 0 & \left(\frac{\sigma(\omega,k)}{\sigma_0}\right) \\ \left(\frac{\tilde{\sigma}(\omega,k)}{\sigma_0}\right) & 0 \end{pmatrix} \begin{pmatrix} j^0_V \\ j^0_A \end{pmatrix} \quad . \tag{5.80}$$

where $\sigma_0 = \frac{N_c e}{2\pi^2}$ is the zero frequency-momentum limit we have used before. In general, at finite (ω, k), we expect $\sigma \neq \tilde{\sigma}$. Therefore the total spatial currents $\vec{j}_{V,A}$ will represent

the sum of the retarded response to A^μ given by (5.75) and the anomalously induced contribution (5.80),

$$\begin{aligned} j_V^1 &= e\Pi^{10}A^0 + \frac{N_c eB\alpha}{2\pi^2}\left(\frac{\sigma(\omega,k)}{\sigma_0}\right) j_A^0 - ikD_L j_V^0 \quad , \\ j_A^1 &= e\Pi_{AV}^{10}A^0 + \frac{N_c eB\alpha}{2\pi^2}\left(\frac{\tilde{\sigma}(\omega,k)}{\sigma_0}\right) j_V^0 - ikD_L j_A^0 \quad , \end{aligned} \tag{5.81}$$

where we keep our focus on longitudinal components only, and we also included the diffusion terms proportional to D_L. Note that we also include the induced axial current from response to A^0 through Π_{AV}^{10} because this term indeed exists in the presence of background magnetic field B (think of triagle diagram of external B and A^0 which couples to axial current). It is important to keep in mind that the charge densities $j_{V,A}^0$ appearing on the right hand side are *total* charge densities that may come from *both* response to A^μ as well as additional fluctuations due to chiral magnetic effects. Once we write down (5.81), we don't and can't specify charge densities because they are free up to dynamical equations of Maxwell equation and current conservation laws. To be more precise, we have three dynamic equations; $\nu = 0$-component of Maxwell equation and two current conservation laws for $j_{V,A}^\mu$ (the $\nu = 1$ Maxwell equation becomes equivalent to $\nu = 0$ once j_V^μ-conservation is imposed due to gauge invariance). They are homogeneous linear equations in terms of three variables (A_0, j_V^0, j_A^0), so that non-zero solutions exist if and only if the 3×3 coefficient matrix has zero determinant. This constraint on (ω, k) will give us the dispersion relation.

Although it is not necessary, it is convenient to decompose j_V^0 as

$$j_V^0 = e\Pi^{00}A^0 + \delta j_V^0 \quad , \tag{5.82}$$

to visualize additional fluctuation δj_V^0 to the retarded response explicitly. Let us then write down the three independent dynamical equations mentioned above. The $\nu = 0$-component Maxwell equation is

$$k^2 A^0 = e^2\Pi^{00}A^0 + e\delta j_V^0 \quad , \tag{5.83}$$

while the vector current conservation, $\partial_\mu j_V^\mu = 0$, looks as

$$-i\omega\delta j_V^0 + ik\frac{N_c eB\alpha}{2\pi^2}\left(\frac{\sigma}{\sigma_0}\right) j_A^0 + k^2 D_L\left(e\Pi^{00}A^0 + \delta j_V^0\right) = 0 \quad . \tag{5.84}$$

Finally, for axial current conservation, there is an important modification to its conservation due to triangle anomaly we are considering (note that we are still neglecting QCD

anomaly from gluons). Recall that axial current becomes anomalous in the presence of non-zero electromagnetic $\vec{E}\cdot\vec{B}\neq 0$ due to triangle anomaly,

$$\partial_\mu j^\mu_A = \frac{e^2 N_c}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \frac{e^2 N_c}{2\pi^2}\vec{E}\cdot\vec{B} \quad . \tag{5.85}$$

Remember that our N_c quarks have charge e in this paper. We already have a background magnetic field $\vec{B}=B\hat{x}^1$, while dynamical longitudinal plasma fluctuations we are considering have a longitudinal electric field fluctuation $E_1=\partial_1 A^0 = ikA^0$ in Fourier space, so that one has locally non-vanishing $\vec{E}\cdot\vec{B}$ that affects axial current conservation law as in (5.85). The resulting (modified) conservation law of j^μ_A gives us

$$-i\omega j^0_A + ike\Pi^{01}_{AV}A^0 + ik\frac{N_c eB\alpha}{2\pi^2}\left(\frac{\tilde{\sigma}}{\sigma_0}\right)\left(e\Pi^{00}A^0+\delta j^0_V\right)+k^2 D_L j^0_A = ik\frac{N_c e^2 B}{2\pi^2}A^0 \quad , \tag{5.86}$$

where the right hand side is the anomalous contribution that we discussed. The above equations (5.83),(5.84), and (5.86) are the main equations for $(A^0,\delta j^0_V, j^0_A)$ from which one can obtain dispersion relations.

As an easy application as well as an illustration, let us turn off the magnetic field for a moment and consider the diffusion effects only, which may be called *diffusive plasmons*. In this case, one has $\Pi^{10}_{AV}=0$ due to $B=0$, and axial current decouples with the usual diffusion $\omega=-iD_Lk^2$, while $(A^0,\delta j^0_V)$ system becomes

$$\begin{aligned}\left(k^2-e^2\Pi^{00}\right)A^0 &= e\delta j^0_V \quad , \\ \left(i\omega-k^2D_L\right)\delta j^0_V &= k^2 eD_L\Pi^{00}A^0 \quad ,\end{aligned} \tag{5.87}$$

which has non-zero solutions if and only if

$$\left(k^2-e^2\Pi^{00}(\omega,k)\right)\left(i\omega-k^2D_L\right) = k^2e^2D_L\Pi^{00}(\omega,k) \quad , \tag{5.88}$$

which gives the dispersion relation. Upon expanding Π^{00} as in (5.78), one can solve the above for small k as

$$\omega^2 = \omega_p^2 + \left(c^2_{eff}-i\omega_p D_L\right)k^2 + \mathcal{O}(k^4) \quad , \tag{5.89}$$

where $\omega_p = e\omega_0$ is the plasma frequency.

Going back to our interesting case of non-zero magnetic field $B\neq 0$, it is straightforward to study (5.83),(5.84), and (5.86) in complete generality, but we will restrict ourselves to the case with $D_L=0$ for simplicity in this paper, leaving their full analysis

including D_L to the future. The system then becomes

$$\begin{aligned} k^2 A^0 &= e^2 \Pi^{00} A^0 + e\delta j_V^0 \quad , \\ -i\omega \delta j_V^0 + ik \frac{N_c e B \alpha}{2\pi^2} \left(\frac{\sigma}{\sigma_0}\right) j_A^0 &= 0 \quad , \\ -i\omega j_A^0 + ike\Pi_{AV}^{10} A^0 + ik \frac{N_c e B \alpha}{2\pi^2} \left(\frac{\tilde{\sigma}}{\sigma_0}\right) \left(e\Pi^{00} A^0 + \delta j_V^0\right) &= ik \frac{N_c e^2 B}{2\pi^2} A^0 \quad , \end{aligned} \tag{5.90}$$

which mixes all three fluctuations together. From the first equation, one can replace A^0 with δj_V^0, and inserting it into the other two equations, one gets

$$\begin{aligned} -i\omega \delta j_V^0 + ik \frac{N_c e B \alpha}{2\pi^2} \left(\frac{\sigma}{\sigma_0}\right) j_A^0 &= 0 \quad , \\ -i\omega j_A^0 + ik \frac{N_c e B \alpha}{2\pi^2} \left(\frac{\tilde{\sigma}}{\sigma_0}\right) \frac{\left(k^2 - \frac{e^2}{\alpha}\frac{\sigma_0}{\tilde{\sigma}} + \frac{2\pi^2 e}{N_c B \alpha}\frac{\sigma_0}{\tilde{\sigma}}\Pi_{AV}^{10}\right)}{k^2 - e^2\Pi^{00}} \delta j_V^0 &= 0 \quad , \end{aligned} \tag{5.91}$$

from which one gets the dispersion equation

$$\omega^2 = v_\chi^2 k^2 \left(\frac{\sigma(\omega,k)}{\sigma_0}\right) \left(\frac{\tilde{\sigma}(\omega,k)}{\sigma_0}\right) \frac{\left(k^2 - \frac{e^2}{\alpha}\frac{\sigma_0}{\tilde{\sigma}(\omega,k)} + \frac{2\pi^2 e}{N_c B \alpha}\frac{\sigma_0}{\tilde{\sigma}(\omega,k)}\Pi_{AV}^{10}(\omega,k)\right)}{k^2 - e^2\Pi^{00}(\omega,k)} \quad , \tag{5.92}$$

where $v_\chi = \frac{N_c e B \alpha}{2\pi^2}$ as before. This equation is our master equation that governs mixing between chiral magnetic waves and plasma waves.

For small magnetic field B and $\omega \sim \omega_p \sim eT \ll T$, one expects that chiral magnetic conductivities are approximately the zero-frequency value $\sigma \approx \tilde{\sigma} \approx \sigma_0$, and moreover anomaly triangle diagram gives us

$$\Pi_{AV}^{10}(\omega,k) \to \frac{N_c e B}{2\pi^2} \quad \text{as} \quad \omega, k \to 0 \quad , \tag{5.93}$$

so that the numerator in the right hand side of (5.92) becomes simplified. One then uses the previous expansion (5.78) of Π^{00} to solve the above equation to get

$$\omega^2 = \omega_p^2 + \left(v_\chi^2 + c_{eff}^2\right) k^2 + \mathcal{O}(k^4) \quad , \tag{5.94}$$

which describes effects from chiral magnetic wave to the plasma waves. Note that the effect exists only with finite k, and this makes sense because chiral magnetic waves disappear in $k \to 0$ limit.

Another interesting limit is an infinitely large $B \to \infty$ limit, where one expects effective reduction to 1+1 dimensional theory. In fact, making electromagnetism dynamical

corresponds to 1+1 dimensional QED with N_c massless Dirac fermions, or the Schwinger model [70]. It has been known for long time that the photon in the model becomes massive due to 1+1 dimensional axial anomaly,

$$m_\gamma^2 = \frac{N_c e_{eff}^2}{\pi} \quad , \tag{5.95}$$

where e_{eff}^2 is an effective 1+1 dimensional QED coupling constant. As the 4-dimensional triangle anomaly (5.85) correctly reduces to 1+1 dimensional axial anomaly in the presence of background magnetic field B, one should be able to reproduce this Schwinger phenomenon from our master equation (5.92) in the limit $B \to \infty$.

One can be more quantitative to test this connection. To find e_{eff}^2, it is useful to consider a transverse area of

$$\int d^2x_T = \frac{2\pi}{eB} \quad , \tag{5.96}$$

to have a single lowest Landau level system per each 4D fermion because the transverse density of LLL is $eB/2\pi$. Thinking of fermion kinetic term, the proper normalization between 4D fermion and 2D fermion is

$$\psi_{4D} = \sqrt{\frac{eB}{2\pi}}\psi_{2D} \quad , \tag{5.97}$$

which will be useful shortly when we discuss about Π^{00}. Because one is looking at only longitudinal dynamics of 4D $U(1)$ gauge field, the only relevant dynamical field is F_{01}, and the gauge field action indeed reduces to 1+1 dimensional QED action as

$$\frac{1}{2e^2}\int d^4x \ (F_{01})^2 = \frac{1}{2e^2}\frac{2\pi}{eB}\int d^2x \ (F_{01})^2 \equiv \frac{1}{2e_{eff}^2}\int d^2x \ (F_{01})^2 \quad , \tag{5.98}$$

so that $e_{eff}^2 = \frac{e^3 B}{2\pi}$, and the Schwinger photon mass should be

$$m_\gamma^2 = \frac{N_c e^3 B}{2\pi^2} \quad . \tag{5.99}$$

To reproduce this from our equation (5.92), note that as $B \to \infty$, the expected $\omega^2 = m_\gamma^2$ is also infinite and one naturally expects that chiral magnetic conductivities go to zero in this limit as the system cannot respond to arbitrary fast perturbations; $(\sigma, \tilde{\sigma}) \to 0$ as $\omega \to \infty$. Therefore, the solution of (5.92) in this limit is found simply by demanding that the denominator vanishes or

$$k^2 = e^2\Pi^{00}(\omega, k) \quad , \tag{5.100}$$

where Π^{00} should be given by the effective 1+1 dimensional theory. To find it, recall from (5.97) that $j^\mu_{4D} = \frac{eB}{2\pi} j^\mu_{2D}$, so that

$$\Pi^{00} \sim \int d^4x e^{-i\omega t+ikx} \langle j^0_{4D}(x) j^0_{4D}(0)\rangle = \frac{2\pi}{eB}\left(\frac{eB}{2\pi}\right)^2 \int d^2x e^{-i\omega t+ikx} \langle j^0_{2D}(x) j^0_{2D}(0)\rangle = \left(\frac{eB}{2\pi}\right)\Pi^{00}_{2D}. \tag{5.101}$$

As we have N_c 1+1 dimensional Dirac fermions, Π^{00}_{2D} is N_c times that of a single Dirac fermion, which can be found most easily by bosonization to a single real scalar field ϕ such that

$$j^\mu_V = \sqrt{\frac{1}{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi \quad , \quad j^\mu_A = \sqrt{\frac{1}{\pi}}\partial^\mu\phi \quad , \tag{5.102}$$

where ϕ is normalized to have a standard kinetic term $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. Then, Π^{00}_{2D} is easily computed as

$$\Pi^{00}_{2D} = \frac{N_c}{\pi}\langle(\partial_1\phi)(\partial_1\phi)\rangle = \frac{N_c}{\pi}\frac{(ik)(-ik)}{\omega^2-k^2} = \frac{N_c}{\pi}\frac{k^2}{\omega^2-k^2} \quad , \tag{5.103}$$

so that the equation (5.100) becomes

$$k^2 = e^2\Pi^{00}_{4D} = \frac{N_c e^3 B}{2\pi^2}\frac{k^2}{\omega^2-k^2} \quad , \tag{5.104}$$

which indeed gives us $\omega^2 = m_\gamma^2 + k^2$ reproducing the Schwinger model result.

Therefore the plasmon in the dimensionally reduced theory can be seen as a result of the interaction of the dynamical photon with the chiral magnetic waves.

6 Summary

We have demonstrated that the Chiral Magnetic and Chiral Separation Effects (CME and CSE) in relativistic plasmas subjected to magnetic field imply the existence of a new type of a collective excitation in the plasma - the Chiral Magnetic Wave (CMW). This excitation represents the density waves of electric and chiral charge coupled by the triangle anomaly. In strong magnetic field the CMW propagates with the velocity of light, $v_\chi \to 1$. In weak magnetic field, the velocity v_χ decreases; the result of the holographic computation is shown in Figure 3. At weak coupling, this decrease of the velocity of the CMW can be understood as originating from the admixture of the excited Landau levels.

The existence of CMW in the quark-gluon plasma has important implications for the phenomenology of heavy ion collisions. The CME relies on the fluctuation of the axial

charge density and so the net effect is expected to vanish when averaged over many events; one thus relies on measuring the fluctuations of charge asymmetries [1, 49]. On the other hand, the CMW should exist even in a neutral plasma, and so does not require the presence of the axial or baryon chemical potentials. Since it represents the coupled density waves of electric and chiral charges propagating along the direction of the applied magnetic field (that in heavy ion collisions is perpendicular to the reaction plane), the CMW can induce dynamical, reaction plane dependent, fluctuations of electric charge. The azimuthal angle dependence of these fluctuations will be determined by the wavelength of the CMW excitation. We will return to the consideration of phenomenology related to CMW in heavy ion collisions in a forthcoming publication [71].

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Note added: After this paper was submitted, we learned of an interesting paper by G. Newman [72]. In the section VI of that paper, the author arrives at the results similar to ours in section II. We thank D. Son for bringing this work to our attention.

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