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Scalar Contribution to the Graviton Self-Energy during Inflation

Sohyun Park[†] and R. P. Woodard[‡]

Department of Physics
University of Florida
Gainesville, FL 32611

ABSTRACT

We use dimensional regularization to evaluate the one loop contribution to the graviton self-energy from a massless, minimally coupled scalar on a locally de Sitter background. For noncoincident points our result agrees with the stress tensor correlators obtained recently by Perez-Nadal, Roura and Verdaguer. We absorb the ultraviolet divergences using the R^2 and C^2 counterterms first derived by 't Hooft and Veltman, and we take the $D = 4$ limit of the finite remainder. The renormalized result is expressed as the sum of two transverse, 4th order differential operators acting on nonlocal, de Sitter invariant structure functions. In this form it can be used to quantum-correct the linearized Einstein equations so that one can study how the inflationary production of infrared scalars affects the propagation of dynamical gravitons and the force of gravity.

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[†] e-mail: spark@phys.ufl.edu
[‡] e-mail: woodard@phys.ufl.edu

1 Introduction

The linearized equations for all known force fields do two things:

- They give the linearized force fields induced by sources; and
- They describe the propagation of dynamical particles which carry the force but are, in principle, independent of any source.

This is the classic distinction between the constrained and unconstrained parts of a force field. In electromagnetism it amounts to the Coulomb potential versus photons. In gravity there is the Newtonian potential, plus its three relativistic partners, versus gravitons.

Quantum corrections to the linearized field equations derive from how the 0-point fluctuations of various fields in whatever background is assumed, respond to the linearized force fields. These quantum corrections do not change the dichotomy between constrained and unconstrained fields but they can, of course, modify classical results. Around flat space background there is no effect, after renormalization, on the propagation of dynamical photons or gravitons but there are small corrections to the Coulomb and Newtonian potentials. As might be expected, the long distance effects are greatest for the 0-point fluctuations of massless particles and they take the form required by perturbation theory and dimensional analysis [1, 2],

$$\Bigl(\frac{\Delta\Phi}{\Phi}\Bigr)_{\rm Coul.} \sim -\frac{e^2}{\hbar c}\,\ln\Bigl(\frac{r}{r_0}\Bigr) \qquad , \qquad \Bigl(\frac{\Delta\Phi}{\Phi}\Bigr)_{\rm Newt.} \sim -\frac{\hbar G}{c^3 r^2}\;, \tag{1}$$

where r is the distance to the source, r_0 is the point at which the renormalized charge is defined, and the other constants have their usual meanings.

Schrödinger was the first to suggest that the expansion of spacetime can lead to particle production by ripping the virtual particles (which are implicit in 0-point fluctuations) out of the vacuum [3]. Following early work by Imamura [4], the first quantitative results were obtained by Parker [5]. He found that the effect is maximized during accelerated expansion, and for massless particles which are not conformally invariant [6], such as massless, minimally coupled (MMC) scalars and (as noted by Grishchuk [7]) gravitons.

The de Sitter geometry is the most highly accelerated expansion consistent with classical stability. For de Sitter background with Hubble constant H and scale factor $a(t) = e^{Ht}$ it is simple to show that the number of MMC scalars, or either polarization of graviton, created with wave vector $\vec{k}$ is [8],

$$N(t,\vec{k}) = \Bigl(\frac{Ha(t)}{2c\|\vec{k}\|}\Bigr)^2 . \tag{2}$$

It is these particles which comprise the scalar and tensor perturbations produced by inflation [9], the scalar contribution of which has been imaged [10]. Of course the same particles also enter loop diagrams to cause an enormous strengthening of the quantum effects caused by MMC scalars and gravitons. A number of analytic results have been obtained for one loop corrections to the way various particles propagate on de Sitter background and also to how long range forces act:

- In MMC scalar quantum electrodynamics, infrared photons behave as if they had an increasing mass [11], and the charge screening very quickly becomes nonperturbatively strong [12], but there is no big effect on scalars [13];
- For a MMC scalar which is Yukawa-coupled to a massless fermion, infrared fermions behave as if they had an increasing mass [14] but the associated scalars experience no large correction [15];
- For a MMC scalar with a quartic self-interaction, infrared scalars behave as if they had an increasing mass (which persists to two loop order) [16];
- For quantum gravity minimally coupled to a massless fermion, the fermion field strength grows without bound [17]; and
- For quantum gravity plus a MMC scalar, the scalar shows no secular effect but its field strength may acquire a momentum-dependent enhancement [18].

The great omission from this list is how inflationary scalars and gravitons affect gravity, both as regards the propagation of dynamical gravitons and as regards the force of gravity. This paper represents a first step in completing the list.

One includes quantum corrections to the linearized field equation by subtracting the integral of the appropriate one-particle-irreducible (1PI) 2-point function up against the linearized field. For example, a MMC scalar $\varphi(x)$

in a background metric $g_{\mu\nu}(x)$ whose 1PI 2-point function is $-iM^2(x;x')$, would have the linearized effective field equation,

$$\partial_\mu \Big[\sqrt{g} g^{\mu\nu} \partial_\nu \varphi(x) \Big] - \int d^4x' \, M^2(x;x') \varphi(x') = 0 \; . \tag{3}$$

To include gravity on the list we must therefore compute the graviton self-energy, either from MMC scalars or from gravitons, and then use it to correct the linearized Einstein equation. In this paper we shall evaluate the contribution from MMC scalars; a subsequent paper will solve the linearized effective field equations to determine quantum corrections to the propagation of gravitons and the gravitational response to a point mass.

It should be noted that the vastly more complicated contribution from gravitons was derived some time ago [19]. However, that result is not renormalized, and is therefore only valid for noncoincident points. To use the graviton self-energy in an effective field equation such as (3), where the integration carries x'^μ over x^μ, one must extract differential operators until the remaining structure functions are integrable. That is the sort of form we will derive, using dimensional regularization to control the divergences and the standard counterterms to subtract them.

Scalar + Einstein is not perturbatively renormalizable [20], however, ultraviolet divergences can always be absorbed in the sense of Bogoliubov, Parasiuk, Hepp and Zimmerman (BPHZ) [21]. A widespread misconception exists that no valid quantum predictions can be extracted from such an exercise. This is false: while nonrenormalizability does preclude being able to compute *everything*, that is not the same thing as being able to compute *nothing*. The problem with a nonrenormalizable theory is that no physical principle fixes the finite parts of the escalating series of BPHZ counterterms needed to absorb ultraviolet divergences, order-by-order in perturbation theory. Hence any prediction of the theory that can be changed by adjusting the finite parts of these counterterms is essentially arbitrary. However, loops of massless particles make nonlocal contributions to the effective action that can never be affected by local counterterms. These nonlocal contributions typically dominate the infrared. Further, they cannot be affected by whatever modification of ultraviolet physics ultimately results in a completely consistent formalism. As long as the eventual fix introduces no new massless particles, and does not disturb the low energy couplings of the existing ones, the far infrared predictions of a BPHZ-renormalized quantum theory will agree with those of its fully consistent descendant.

It is worthwhile to review the vast body of distinguished work that has exploited this fact. The oldest example is the solution of the infrared problem in quantum electrodynamics by Bloch and Nordsieck [22], long before that theory's renormalizability was suspected. Weinberg [23] was able to achieve a similar resolution for quantum gravity with zero cosmological constant. The same principle was at work in the Fermi theory computation of the long range force due to loops of massless neutrinos by Feinberg and Sucher [24, 25]. Matter which is not supersymmetric generates nonrenormalizable corrections to the graviton propagator at one loop, but this did not prevent the computation of graviton, photon, massless neutrino and scalar loop corrections to the long range gravitational force [1]. In a recent paper [2] we showed explicitly how the arbitrary counterterms — represented by the parameter μ in equations (38-43) of [2] — drop out of the one loop correction to the force law — see equations (44-47) in [2]. The analysis we propose to make exploits the power of low energy effective field theory in the same way, differing from the previous examples only in the detail that our background geometry is locally de Sitter rather than flat. And even the use of effective field theory in cosmology has antecedants [26].

This paper contains five sections. In section 2 we give those of the Feynman rules which are needed for this computation, and we describe the geometry of our D-dimensional, locally de Sitter background. Section 3 derives the relatively simple form for the D-dimensional graviton self-energy with noncoincident points. We show that this version of the result agrees with the flat space limit [27] and with the de Sitter stress tensor correlators recently derived by Perez-Nadal, Roura and Verdaguer [28]. Section 4 undertakes the vastly more difficult reorganization which must be done to isolate the local divergences for renormalization. At the end we subtract off the divergences with the same counterterms originally computed for this model in 1974 by 't Hooft and Veltman [20], and we take the unregulated limit of $D = 4$. Our discussion comprises section 5.

2 Feynman Rules

In this section we derive Feynman rules for the computation. We start by expressing the full metric as

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \kappa h_{\mu\nu} \; , \tag{4}$$

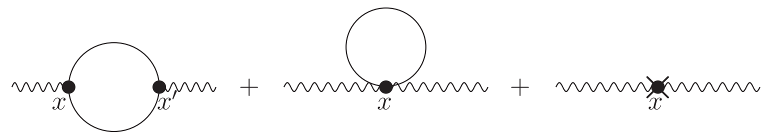

Figure 1: The one loop graviton self-energy from MMC scalars.

where $\overline{g}_{\mu\nu}$ is the background metric, $h_{\mu\nu}$ is the graviton field whose indices are raised and lowered with the background metric, and $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. Expanding the MMC scalar Lagrangian around the background metric we get interaction vertices between the scalar and dynamical gravitons. We take the D-dimensional locally de Sitter space as our background and introduce de Sitter invariant bi-tensors which will be used throughout the calculation. We close this section by providing the MMC scalar propagator on the de Sitter background.

2.1 Interaction Vertices

The Lagrangian which describes pure gravity and the interaction between gravitons and the MMC scalar is,

$$\mathcal{L} = \frac{1}{16\pi G}\Big[R - (D-1)(D-2)H^2\Big] - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} \; . \tag{5}$$

where R is Ricci scalar, G is Newton's constant and H is the Hubble constant.

Computing the one loop scalar contributions to the graviton self-energy consists of summing the 3 Feynman diagrams depicted in Figure 1. The sum of these three diagrams has the following analytic form:

$$\begin{aligned}
&-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x') \\
&\quad = \frac{1}{2}\sum_{I=1}^{2} T_I^{\mu\nu\alpha\beta}(x) \sum_{J=1}^{2} T_J^{\rho\sigma\gamma\delta}(x') \times \partial_\alpha\partial'_\gamma i\Delta(x;x') \times \partial_\beta\partial'_\delta i\Delta(x;x') \\
&\qquad\qquad + \frac{1}{2}\sum_{I=1}^{4} F_I^{\mu\nu\rho\sigma\alpha\beta}(x) \times \partial_\alpha\partial'_\beta i\Delta(x;x') \times \delta^D(x-x') \\
&\qquad\qquad\qquad + 2\sum_{I=1}^{2} C_I^{\mu\nu\rho\sigma}(x) \times \delta^D(x-x') \; .
\end{aligned} \tag{6}$$

The 3-point and 4-point vertex factors $T_I^{\mu\nu\alpha\beta}$ and $F_I^{\mu\nu\rho\sigma\alpha\beta}$ derive from expanding the MMC scalar Lagrangian using (4),

$$-\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} \tag{7}$$

$$= -\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi\overline{g}^{\mu\nu}\sqrt{-\overline{g}} - \frac{\kappa}{2}\partial_\mu\varphi\partial_\nu\varphi\Big(\frac{1}{2}h\overline{g}^{\mu\nu} - h^{\mu\nu}\Big)\sqrt{-\overline{g}}$$
$$-\frac{\kappa^2}{2}\partial_\mu\varphi\partial_\nu\varphi\Big\{\Big[\frac{1}{8}h^2 - \frac{1}{4}h^{\rho\sigma}h_{\rho\sigma}\Big]\overline{g}^{\mu\nu} - \frac{1}{2}hh^{\mu\nu} + h^\mu{}_\rho h^{\rho\nu}\Big\}\sqrt{-\overline{g}} + O(\kappa^3)\ . \tag{8}$$

The resulting 3-point and 4-point vertex factors are given in the Tables 1 and 2, respectively. The procedure to get the counterterm vertex operators $C_I^{\mu\nu\rho\sigma}(x)$ is given in section 4.

I	$T_I^{\mu\nu\alpha\beta}$
1	$-\frac{i\kappa}{2}\sqrt{-\overline{g}}\ \overline{g}^{\mu\nu}\overline{g}^{\alpha\beta}$
2	$+i\kappa\sqrt{-\overline{g}}\ \overline{g}^{\mu(\alpha}\overline{g}^{\beta)\nu}$

Table 1: 3-point vertices $T_I^{\mu\nu\alpha\beta}$ where $\overline{g}_{\mu\nu}$ is the de Sitter background metric, $\kappa^2 \equiv 16\pi G$ and parenthesized indices are symmetrized.

I	$F_I^{\mu\nu\rho\sigma\alpha\beta}$
1	$-\frac{i\kappa^2}{4}\sqrt{-\overline{g}}\ \overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}\overline{g}^{\alpha\beta}$
2	$+\frac{i\kappa^2}{2}\sqrt{-\overline{g}}\ \overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\overline{g}^{\alpha\beta}$
3	$+\frac{i\kappa^2}{2}\sqrt{-\overline{g}}\ \Big[\overline{g}^{\mu(\alpha}\overline{g}^{\beta)\nu}\overline{g}^{\rho\sigma} + \overline{g}^{\mu\nu}\overline{g}^{\rho(\alpha}\overline{g}^{\beta)\sigma}\Big]$
4	$-2i\kappa^2\sqrt{-\overline{g}}\ \overline{g}^{\alpha(\mu}\overline{g}^{\nu)(\rho}\overline{g}^{\sigma)\beta}$

Table 2: 4-point vertices $F_I^{\mu\nu\rho\sigma\alpha\beta}$ where $\overline{g}_{\mu\nu}$ is the de Sitter background metric, $\kappa^2 \equiv 16\pi G$ and parenthesized indices are symmetrized.

These interaction vertices are valid for any background metric $\overline{g}_{\mu\nu}$. In the next two subsections we specialize to a locally de Sitter background and give the scalar propagator $i\triangle(x;x')$ on it.

2.2 Working on de Sitter Space

We specify our background geometry as the open conformal coordinate submanifold of D-dimensional de Sitter space. A spacetime point $x^\mu = (\eta, x^i)$ takes values in the ranges

$$-\infty < \eta < 0 \qquad \text{and} \qquad -\infty < x^i < +\infty \ . \tag{9}$$

In these coordinates the invariant element is,

$$ds^2 \equiv \overline{g}_{\mu\nu} dx^\mu dx^\nu = a^2 \eta_{\mu\nu} dx^\mu dx^\nu \ , \tag{10}$$

where $\eta_{\mu\nu}$ is the Lorentz metric and $a = -1/H\eta$ is the scale factor. The Hubble parameter H is constant for the de Sitter space. So in terms of $\eta_{\mu\nu}$ and a our background metric is

$$\overline{g}_{\mu\nu} \equiv a^2 \eta_{\mu\nu} \ . \tag{11}$$

De Sitter space has the maximum number of space-time symmetries in a given dimension. For our D-dimensional conformal coordinates the $\frac{1}{2}D(D+1)$ de Sitter transformations can be decomposed as follows:

- Spatial transformations - $(D-1)$ transformations.

$$\eta' = \eta \ , \quad x'^i = x^i + \epsilon^i \ . \tag{12}$$

- Rotations - $\frac{1}{2}(D-1)(D-2)$ transformations.

$$\eta' = \eta \ , \quad x'^i = R^{ij} x^j \ . \tag{13}$$

- Dilation - 1 transformation.

$$\eta' = k\eta \ , \quad x'^i = kx^j \ . \tag{14}$$

- Spatial special conformal transformations - $(D-1)$ transformations.

$$\eta' = \frac{\eta}{1 - 2\vec{\theta} \cdot \vec{x} + \parallel \vec{\theta} \parallel^2 x \cdot x} \ , \quad x' = \frac{x^i - \theta^i x \cdot x}{1 - 2\vec{\theta} \cdot \vec{x} + \parallel \vec{\theta} \parallel^2 x \cdot x} \ . \tag{15}$$

It turns out that the MMC scalar contribution to the graviton self-energy is de Sitter invariant. This suggests to express it in terms of the de Sitter length function $y(x;x')$,

$$y(x;x') \equiv aa'H^2\Bigg[\Big\|\vec{x}-\vec{x}'\Big\|^2 - \Big(|\eta-\eta'|-i\epsilon\Big)^2\Bigg] . \tag{16}$$

Except for the factor of $i\epsilon$ (whose purpose is to enforce Feynman boundary conditions) the function $y(x;x')$ is closely related to the invariant length $\ell(x;x')$ from x^μ to x'^μ,

$$y(x;x') = 4\sin^2\Big(\frac{1}{2}H\ell(x;x')\Big) . \tag{17}$$

With this de Sitter invariant quantity $y(x;x')$, we can form a convenient basis of de Sitter invariant bi-tensors. Note that because $y(x;x')$ is de Sitter invariant, so too are covariant derivatives of it. With the metrics $\overline{g}_{\mu\nu}(x)$ and $\overline{g}_{\mu\nu}(x')$, the first three derivatives of $y(x;x')$ furnish a convenient basis of de Sitter invariant bi-tensors [13],

$$\frac{\partial y(x;x')}{\partial x^\mu} = Ha\Big(y\delta^0_\mu + 2a'H\Delta x_\mu\Big) , \tag{18}$$
$$\frac{\partial y(x;x')}{\partial x'^\nu} = Ha'\Big(y\delta^0_\nu - 2aH\Delta x_\nu\Big) , \tag{19}$$
$$\frac{\partial^2 y(x;x')}{\partial x^\mu \partial x'^\nu} = H^2 aa'\Big(y\delta^0_\mu\delta^0_\nu + 2a'H\Delta x_\mu\delta^0_\nu - 2a\delta^0_\mu H\Delta x_\nu - 2\eta_{\mu\nu}\Big) . \tag{20}$$

Here and subsequently $\Delta x_\mu \equiv \eta_{\mu\nu}(x-x')^\nu$.

Acting covariant derivatives generates more basis tensors, for example [13],

$$\frac{D^2 y(x;x')}{Dx^\mu Dx^\nu} = H^2(2-y)\overline{g}_{\mu\nu}(x) , \tag{21}$$
$$\frac{D^2 y(x;x')}{Dx'^\mu Dx'^\nu} = H^2(2-y)\overline{g}_{\mu\nu}(x') . \tag{22}$$

The contraction of any pair of the basis tensors also produces more basis tensors [13],

$$\overline{g}^{\mu\nu}(x)\frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x^\nu} = H^2\Big(4y - y^2\Big) = \overline{g}^{\mu\nu}(x')\frac{\partial y}{\partial x'^\mu}\frac{\partial y}{\partial x'^\nu} , \tag{23}$$

$$\overline{g}^{\mu\nu}(x) \frac{\partial y}{\partial x^{\nu}} \frac{\partial^2 y}{\partial x^{\mu} \partial x'^{\sigma}} = H^2 (2-y) \frac{\partial y}{\partial x'^{\sigma}} , \tag{24}$$

$$\overline{g}^{\rho\sigma}(x') \frac{\partial y}{\partial x'^{\sigma}} \frac{\partial^2 y}{\partial x^{\mu} \partial x'^{\rho}} = H^2 (2-y) \frac{\partial y}{\partial x^{\mu}} , \tag{25}$$

$$\overline{g}^{\mu\nu}(x) \frac{\partial^2 y}{\partial x^{\mu} \partial x'^{\rho}} \frac{\partial^2 y}{\partial x^{\nu} \partial x'^{\sigma}} = 4H^4 \overline{g}_{\rho\sigma}(x') - H^2 \frac{\partial y}{\partial x'^{\rho}} \frac{\partial y}{\partial x'^{\sigma}} , \tag{26}$$

$$\overline{g}^{\rho\sigma}(x') \frac{\partial^2 y}{\partial x^{\mu} \partial x'^{\rho}} \frac{\partial^2 y}{\partial x^{\nu} \partial x'^{\sigma}} = 4H^4 \overline{g}_{\mu\nu}(x) - H^2 \frac{\partial y}{\partial x^{\mu}} \frac{\partial y}{\partial x^{\nu}} . \tag{27}$$

Our basis tensors are naturally covariant, but their indices can of course be raised using the metric at the appropriate point. To save space in writing this out we define the basis tensors with raised indices as differentiation with respect to "covariant" coordinates,

$$\frac{\partial y}{\partial x_{\mu}} \equiv \overline{g}^{\mu\nu}(x) \frac{\partial y}{\partial x^{\nu}} , \tag{28}$$

$$\frac{\partial y}{\partial x'_{\rho}} \equiv \overline{g}^{\rho\sigma}(x') \frac{\partial y}{\partial x'^{\sigma}} , \tag{29}$$

$$\frac{\partial^2 y}{\partial x_{\mu} \partial x'_{\rho}} \equiv \overline{g}^{\mu\nu}(x) \overline{g}^{\rho\sigma}(x') \frac{\partial^2 y}{\partial x^{\nu} \partial x'^{\sigma}} . \tag{30}$$

2.3 Scalar Propagator on de Sitter

From the MMC scalar Lagrangian (5) we see that the propagator obeys

$$\partial_{\mu} \Big[\sqrt{-g} \; \overline{g}^{\mu\nu} \partial_{\nu} \Big] i\triangle(x;x') = \sqrt{-\overline{g}} \square i\triangle(x;x') = i\delta^D(x-x') \tag{31}$$

Although this equation is de Sitter invariant, there is no de Sitter invariant solution for the propagator [29], hence some of the symmetries (12-15) must be broken. We choose to preserve the homogeneity and isotropy of cosmology — relations (12-13) — which corresponds to what is known as the "E3" vacuum [30]. It can be realized in terms of plane wave mode sums by making the spatial manifold T^{D-1}, rather than R^{D-1}, with coordinate radius H^{-1} in each direction, and then using the integral approximation with the lower limit cut off at $k = H$ [31]. The final result consists of a de Sitter invariant function of $y(x;x')$ plus a de Sitter breaking part which depends upon the scale factors at the two points [32],

$$i\triangle(x;x') = A\Big(y(x;x')\Big) + k \ln(aa') . \tag{32}$$

Here the constant k is given as,

$$k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \; , \tag{33}$$

and the function $A(y)$ has the expansion,

$$\begin{aligned} A(y) \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \Biggl\{ & \frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1} \Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2} \Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} - \pi \cot\Bigl(\frac{\pi D}{2}\Bigr) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \\ & + \sum_{n=1}^{\infty} \Biggl[\frac{1}{n} \frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})} \Bigl(\frac{y}{4}\Bigr)^{n} - \frac{1}{n-\frac{D}{2}+2} \frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)} \Bigl(\frac{y}{4}\Bigr)^{n-\frac{D}{2}+2} \Biggr] \Biggr\}. \end{aligned} \tag{34}$$

The infinite series terms of $A(y)$ vanish for $D = 4$, so they only need to be retained when multiplying a potentially divergent quantity, and even then one only needs to include a handful of them. This makes loop computations manageable.

We note that the MMC scalar propagator (32) has a de Sitter breaking term, $k \ln(aa')$. However, the one loop scalar contribution to the graviton self-energy only involves the terms like $\partial_\alpha \partial'_\beta i\triangle(x;x')$, which are de Sitter invariant,

$$\partial_\alpha \partial'_\beta i\Delta(x;x') = \frac{\partial}{\partial x^\alpha} \Bigl\{ A'(y) \frac{\partial y}{\partial x'^\beta} + H a' \delta^0_\beta \Bigr\} = A''(y) \frac{\partial y}{\partial x^\alpha} \frac{\partial y}{\partial x'^\beta} + A'(y) \frac{\partial^2 y}{\partial x^\alpha \partial x'^\beta} \; . \tag{35}$$

Another useful relation follows from the propagator equation,

$$(4y-y^2) A''(y) + D(2-y) A'(y) = (D-1) k \; . \tag{36}$$

3 One Loop Graviton Self-energy

In this section we calculate the first two, primitive, diagrams of Figure 1. It turns out that the contribution from the 4-point vertex (the middle diagram) vanishes in $D = 4$ dimensions. The contribution from two 3-point vertices (the leftmost diagram) is nonzero. For noncoincident points it gives a relatively simple form which agrees with the flat space limit [27] and with the de Sitter stress tensor correlator recently derived by Perez-Nadal, Roura and Verdaguer [28].

3.1 Contribution from 4-Point Vertices

The 4-point contribution from the middle diagram of Figure 1 takes the form,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 4pt}(x;x') \equiv \frac{1}{2}\sum_{I=1}^{4} F_I^{\mu\nu\rho\sigma\alpha\beta}(x) \times \partial_\alpha \partial'_\beta i\triangle(x;x') \times \delta^D(x-x') \; . \qquad (37)$$

Recall that the four 4-point vertices $F_I^{\mu\nu\rho\sigma\alpha\beta}(x)$ are given in Table 2. Owing to the delta function, we need the coincidence limit of the doubly differentiated propagator (35). The coincidence limits of the various tensor factors follow from setting $a' = a$, $\Delta x^\mu = 0$ and $y = 0$ in relations (18-20),

$$\lim_{x'\to x} \frac{\partial y(x;x')}{\partial x^\mu} \;=\; 0 = \lim_{x'\to x} \frac{\partial y(x;x')}{\partial x'^\nu} \; , \qquad (38)$$

$$\lim_{x'\to x} \frac{\partial^2 y(x;x')}{\partial x^\mu \partial x'^\nu} \;=\; = -2H^2 \overline{g}_{\mu\nu} \; . \qquad (39)$$

Hence the coincidence limit of the doubly differentiated propagator can be expressed in terms of $A'(y)$ evaluated at $y = 0$,

$$\lim_{x'\to x} \partial_\alpha \partial'_\beta i\triangle(x;x') = A''(0) \times 0 + A'(0) \times \Big[-2H^2 g_{\mu\nu}\Big] \; . \qquad (40)$$

From the definition (34) of $A(y)$, we see that $A'(y)$ is,

$$A'(y) = \frac{1}{4}\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Bigg\{-\Gamma(\frac{D}{2})\Big(\frac{4}{y}\Big)^{\frac{D}{2}} - \Gamma(\frac{D}{2}+1)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \sum_{n=1}\Bigg[\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^{n-1} - \frac{\Gamma(n+\frac{D}{2}-1)}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+1}\Bigg]\Bigg\} . \qquad (41)$$

Now we recall that, in dimensional regularization, any D-dependent power of zero vanishes. Therefore, only the $n = 1$ term of the infinite series in (41) contributes to the coincidence limit,

$$A'(0) = \frac{1}{4}\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)} \; , \qquad (42)$$

and we have,

$$\lim_{x'\to x} \partial_\alpha \partial'_\beta i\triangle(x;x') = -\frac{1}{2}\frac{H^{D}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)}\, \overline{g}_{\alpha\beta} \; . \qquad (43)$$

Substituting (43), and the 4-point vertices from Table 2, into expression (37) gives,

$$
\begin{aligned}
&-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 4pt}(x;x') \\
&= -\frac{1}{2}\frac{H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}\!+\!1)}\overline{g}_{\alpha\beta} \times i\kappa^2\sqrt{-\overline{g}}\,\Big\{ -\frac{1}{4}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}\overline{g}^{\alpha\beta} + \frac{1}{2}\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\overline{g}^{\alpha\beta} \\
&\quad +\frac{1}{2}\Big[\overline{g}^{\mu(\alpha}\overline{g}^{\beta)\nu}\overline{g}^{\rho\sigma} + \overline{g}^{\mu\nu}\overline{g}^{\rho(\alpha}\overline{g}^{\beta)\sigma}\Big] - 2\overline{g}^{\alpha(\mu}\overline{g}^{\nu)(\rho}\overline{g}^{\sigma)\beta}\Big\}\delta^D(x\!-\!x') \; , \qquad (44) \\
&= \Big(\frac{D\!-\!4}{4}\Big)\frac{i\kappa^2 H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}\!+\!1)}\sqrt{-\overline{g}}\,\Big\{\frac{1}{2}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma} - \overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\Big\}\delta^D(x\!-\!x') \; . \qquad (45)
\end{aligned}
$$

Because the Gamma functions are finite for $D = 4$ dimensions so we can dispense with dimensional regularization and set $D = 4$. At that point the net contribution (45) vanishes.

3.2 Contribution from 3-Point Vertices

The contribution from the leftmost diagram of Figure 1 takes the form,

$$
\begin{aligned}
&-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 3pt}(x;x') \\
&= \frac{1}{2}\sum_{I=1}^{2} T_I^{\mu\nu\alpha\beta}(x)\sum_{J=1}^{2} T_J^{\rho\sigma\gamma\delta}(x') \times \partial_\alpha\partial'_\gamma i\triangle(x;x') \times \partial_\beta\partial'_\delta i\triangle(x;x') \; . \qquad (46)
\end{aligned}
$$

Recall from section 2 that any de Sitter invariant bitensor can be expressed as a linear combination of functions of $y(x;x')$ times the five basis tensors,

$$
\begin{aligned}
&-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 3pt}(x;x') = \sqrt{-\overline{g}}\sqrt{-\overline{g}'}\Big\{\frac{\partial^2 y}{\partial x_\mu \partial x'_{(\rho}}\frac{\partial^2 y}{\partial x'_{\sigma)}\partial x_\nu} \times \alpha(y) \\
&+\frac{\partial y}{\partial x_{(\mu}}\frac{\partial^2 y}{\partial x_{\nu)}\partial x'_{(\rho}}\frac{\partial y}{\partial x'_{\sigma)}} \times \beta(y) + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} \times \gamma(y) \\
&+\overline{g}^{\mu\nu}\overline{g}'^{\rho\sigma} H^4 \times \delta(y) + \Big[\overline{g}^{\mu\nu}\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\overline{g}'^{\rho\sigma}\Big]H^2 \times \epsilon(y)\Big\} \; . \qquad (47)
\end{aligned}
$$

By substituting our result (35) for the mixed second derivative of the scalar propagator, along with the vertices from Table 1, and then making use of the contraction identities (23-27), it is straightforward to obtain expressions for the five coefficient functions,

$$
\begin{aligned}
\alpha(y) &= -\frac{1}{2}\kappa^2 (A')^2 \; , & (48)\\
\beta(y) &= -\kappa^2 A' A'' \; , & (49)\\
\gamma(y) &= -\frac{1}{2}\kappa^2 (A'')^2 \; , & (50)\\
\delta(y) &= -\frac{1}{8}\kappa^2 \Bigl\{ (A'')^2 (4y-y^2)^2 + 2A'A''(2-y)(4y-y^2) \\
&\qquad + (A')^2 \Bigl[4(D-4) - (4y-y^2) \Bigr] \Bigr\} \; , & (51)\\
\epsilon(y) &= \frac{1}{4}\kappa^2 \Bigl[(4y-y^2)(A'')^2 + 2(2-y)A'A'' - (A')^2 \Bigr] \; . & (52)
\end{aligned}
$$

Expressions (48-52) for the coefficient functions have the advantage of being exact for any dimension D, but the disadvantages of being neither very explicit nor very simple functions of $y(x;x')$. We can obtain expressions which are both simple and explicit, and totally adequate for use in the $D=4$ effective field equations, by noting that each pair of terms in the infinite series part of $A(y)$ (34) vanishes for $D=4$ spacetime dimensions. Therefore, it is only neceesary to retain those parts of the infinite series which can potentially multiply potential a divergence. For our computation that turns out to mean only the $n=1$ terms, and we can write the two derivatives as,

$$
\begin{aligned}
A' &= \frac{\Gamma(\frac{D}{2}) H^{D-2}}{4(4\pi)^{\frac{D}{2}}} \Biggl\{ -\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}} - \frac{D}{2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} - \frac{1}{2}\frac{D}{2}\Bigl(\frac{D}{2}+1\Bigr)\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} \\
&\qquad + \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)} + \Bigl(\text{Irrelevant}\Bigr) \Biggr\} \; , & (53)\\
A'' &= \frac{\Gamma(\frac{D}{2}) H^{D-2}}{16(4\pi)^{\frac{D}{2}}} \Biggl\{ \frac{D}{2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}+1} + \Bigl(\frac{D}{2}-1\Bigr)\frac{D}{2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}} \\
&\qquad + \frac{1}{2}\Bigl(\frac{D}{2}-2\Bigr)\frac{D}{2}\Bigl(\frac{D}{2}+1\Bigr)\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \Bigl(\text{Irrelevant}\Bigr) \Biggr\} \; . & (54)
\end{aligned}
$$

Substituting these expansions in (48-52) gives,

$$
\alpha = \frac{K}{2^5} \Biggl\{ -\Bigl(\frac{4}{y}\Bigr)^{D} - D\Bigl(\frac{4}{y}\Bigr)^{D-1} - \frac{D(D+1)}{2}\Bigl(\frac{4}{y}\Bigr)^{D-2}
$$

$$+\frac{2\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}}+\Bigl(\text{Irrelevant}\Bigr)\Biggr\}\;, \qquad (55)$$

$$\beta = \frac{K}{2^7}\Biggl\{D\Bigl(\frac{4}{y}\Bigr)^{D+1}+(D-1)D\Bigl(\frac{4}{y}\Bigr)^{D}+\frac{1}{2}(D-2)D(D+1)\Bigl(\frac{4}{y}\Bigr)^{D-1}$$

$$-\frac{D\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}+1}+\Bigl(\text{Irrelevant}\Bigr)\Biggr\}\;, \qquad (56)$$

$$\gamma = \frac{K}{2^{11}}\Biggl\{-D^2\Bigl(\frac{4}{y}\Bigr)^{D+2}-(D-2)D^2\Bigl(\frac{4}{y}\Bigr)^{D+1}$$

$$-\frac{1}{2}(D^2-3D-2)D^2\Bigl(\frac{4}{y}\Bigr)^{D}+\Bigl(\text{Irrelevant}\Bigr)\Biggr\}\;, \qquad (57)$$

$$\delta = \frac{K}{2^5}\Biggl\{-(D^2-D-4)\Bigl(\frac{4}{y}\Bigr)^{D}-(D^3-5D^2+4D-4)\Bigl(\frac{4}{y}\Bigr)^{D-1}-\frac{1}{2}\Bigl(D^4-8D^3$$

$$+19D^2-28D+8\Bigr)\Bigl(\frac{4}{y}\Bigr)^{D-2}-\frac{8\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}}+(\text{Irrelevant})\Biggr\}\;, \qquad (58)$$

$$\epsilon = \frac{K}{2^8}\Biggl\{(D-2)D\Bigl(\frac{4}{y}\Bigr)^{D+1}+(D^3-5D^2+6D-4)\Bigl(\frac{4}{y}\Bigr)^{D}+\frac{1}{2}D\Bigl(D^3-7D^2$$

$$+12D-12\Bigr)\Bigl(\frac{4}{y}\Bigr)^{D-1}+\frac{D\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}+1}+(\text{Irrelevant})\Biggr\}\;. \qquad (59)$$

where the constant K is,

$$K \equiv \frac{\kappa^2 H^{2D-4}\Gamma^2(\frac{D}{2})}{(4\pi)^D}\;. \qquad (60)$$

3.3 Correspondence with Flat Space

An important and illuminating correspondence limit comes from taking the Hubble constant to zero, with the conformal time going to minus infinity so as to keep the physical time t fixed,

$$\eta = -\frac{1}{H}\,e^{-Ht} = -\frac{1}{H}+t+O(H)\;. \qquad (61)$$

When this is done the background geometry degenerates to flat space and we should recover well-known results [1]. We will also see in the next section that the flat space limit provides crucial guidance in how to reorganize the de Sitter result for renormalization.

Although each independent conformal time diverges under (61), the conformal coordinate separation just goes to the usual temporal separation of flat space,

$$\Delta x^0 \longrightarrow t - t' \; . \tag{62}$$

All scale factors approach unity, and the de Sitter length function goes to H^2 times the invariant interval of flat space,

$$y(x;x') \longrightarrow H^2 \Delta x^2 \; . \tag{63}$$

In the flat space limit the leading behaviors of the various basis tensors are,

$$\frac{\partial y}{\partial x_\mu} \longrightarrow 2 H^2 \Delta x^\mu \quad , \quad \frac{\partial y}{\partial x'_\nu} \longrightarrow -2 H^2 \Delta x^\nu \quad , \quad \frac{\partial y^2}{\partial x_\mu \partial x'_\nu} \longrightarrow -2 H^2 \eta^{\mu\nu} \; . \tag{64}$$

And the leading behaviors for derivatives of the function $A(y)$ are,

$$H^2 A'(y) \;\; \longrightarrow \;\; -\frac{1}{4\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2})}{(\Delta x^2)^{\frac{D}{2}}} \equiv -\frac{1}{4\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2})}{\Delta x^D} \; , \tag{65}$$

$$H^4 A''(y) \;\; \longrightarrow \;\; \frac{1}{4\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}+1)}{(\Delta x^2)^{\frac{D}{2}+1}} \equiv \frac{1}{4\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}+1)}{\Delta x^{D+2}} \; . \tag{66}$$

The 4-point contribution (45) to the graviton self-energy vanishes in the flat space limit, even for $D \neq 4$. We can take the flat space limit of the 3-point contribution (47) in two steps. First, substitute the leading behaviors (63) for $y(x;x')$ and (64) for the basis tensors. Then use expressions (65-66) on the derivatives of $A(y)$. The result is,

$$\begin{aligned} -i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr]_{\rm flat}(x;x') &= \lim_{H \to 0} \kappa^2 \Biggl\{ 4 H^4 \eta^{\mu(\rho} \eta^{\sigma)\nu} \times -\frac{1}{2} (A')^2 \\ & +8 H^6 \Delta x^{(\mu} \eta^{\nu)(\rho} \Delta x^{\sigma)} \times -A' A'' + 16 H^8 \Delta x^\mu \Delta x^\nu \Delta x^\rho \Delta x^\sigma \times -\frac{1}{2} (A'')^2 \\ & +H^4 \eta^{\mu\nu} \eta^{\rho\sigma} \times -\frac{1}{8} \Bigl[16 H^4 \Delta x^4 (A'')^2 + 16 H^2 \Delta x^2 A' A'' + 4 (D-4) (A')^2 \Bigr] \\ & \quad +4 H^6 \Bigl[\eta^{\mu\nu} \Delta x^\rho \Delta x^\sigma + \Delta x^\mu \Delta x^\nu \eta^{\rho\sigma} \Bigr] \times \frac{1}{4} \Bigl[4 H^2 \Delta x^2 (A'')^2 + 4 A' A'' \Bigr] \Biggr\} \; , \qquad (67) \\ &= \frac{\kappa^2 \Gamma^2(\frac{D}{2})}{16 \pi^D} \Biggl\{ \eta^{\mu(\rho} \eta^{\sigma)\nu} \times \Bigl[-\frac{2}{\Delta x^{2D}} \Bigr] + \Delta x^{(\mu} \eta^{\nu)(\rho} \Delta x^{\sigma)} \times \Bigl[\frac{4D}{\Delta x^{2D+2}} \Bigr] \end{aligned}$$

$$+\Delta x^{\mu}\Delta x^{\nu}\Delta x^{\rho}\Delta x^{\sigma}\times\Big[-\frac{2D^2}{\Delta x^{2D+4}}\Big]+\eta^{\mu\nu}\eta^{\rho\sigma}\times\Big[-\frac{1}{2}\frac{(D^2\!-\!D\!-\!4)}{\Delta x^{2D}}\Big]$$
$$+\Big[\eta^{\mu\nu}\Delta x^{\rho}\Delta x^{\sigma}+\Delta x^{\mu}\Delta x^{\nu}\eta^{\rho\sigma}\Big]\times\Big[\frac{D(D\!-\!2)}{\Delta x^{2D+2}}\Big]\Bigg\}\ . \qquad (68)$$

Our result (68) agrees with equation (26) of [27].

3.4 Correspondence with Stress Tensor Correlators

Although the flat space limit (68) will prove a useful guide when we renormalize in the next section, it does not check the purely de Sitter parts of (47). A true de Sitter check is provided by the stress tensor correlators recently derived by Perez-Nadal, Roura and Verdaguer [28]. To exploit their result we first elucidate the relation between the graviton 2-point 1PI function and correlators of the stress tensor. Then we convert their notation to ours.

The Heisenberg equation for the metric field operator coupled to a matter stress tensor $T^{\mu\nu}$ is,

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R+\frac{1}{2}(D\!-\!2)(D\!-\!1)H^2g^{\mu\nu}=\frac{1}{2}\kappa^2T^{\mu\nu}\ . \qquad (69)$$

Perturbation theory is implemented by expressing the full metric $g_{\mu\nu}=\overline{g}_{\mu\nu}+\kappa h_{\mu\nu}$ as the sum of a vacuum solution $\overline{g}_{\mu\nu}$ plus κ times the graviton field $h_{\mu\nu}$. Expanding the left hand side of (69) in powers of the graviton field gives,

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R+\frac{1}{2}(D\!-\!2)(D\!-\!1)H^2g^{\mu\nu}=\kappa\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}-\frac{1}{2}\kappa^2\Delta\mathcal{T}^{\mu\nu}\ , \qquad (70)$$

where the nonlinear terms comprise the graviton pseudo-stress tensor $\Delta\mathcal{T}^{\mu\nu}$. The Lichnerowicz operator of the linear term is,

$$\mathcal{D}^{\mu\nu\rho\sigma}\equiv D^{(\rho}\overline{g}^{\sigma)(\mu}D^{\nu)}-\frac{1}{2}\Big[\overline{g}^{\rho\sigma}D^{\mu\nu}+\overline{g}^{\mu\nu}D^{\rho}D^{\sigma}\Big]$$
$$+\frac{1}{2}\Big[\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}-\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\Big]D^2+(D\!-\!1)\Big[\frac{1}{2}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}-\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\Big]H^2\ , \qquad (71)$$

where D^{μ} is the covariant derivative operator in the background geometry. Substituting these expansions in (69) and rearranging gives,

$$\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}=\frac{1}{2}\kappa\Big(T^{\mu\nu}+\Delta\mathcal{T}^{\mu\nu}\Big)\equiv\frac{1}{2}\kappa\mathcal{T}^{\mu\nu}\ . \qquad (72)$$

We are computing the 1PI graviton 2-point function, which can be obtained from the full graviton 2-point function by eliminating the one particle reducible parts and amputating the external leg propagators. At the one loop order we are working, the one particle reducible part drops out if one computes the correlator of the field minus its expectation value,

$$\delta h_{\mu\nu}(x) \equiv h_{\mu\nu}(x) - \Bigl\langle \Omega \Bigl| h_{\mu\nu}(x) \Bigr| \Omega \Bigr\rangle , \tag{73}$$

$$\delta \mathcal{T}^{\mu\nu}(x) \equiv \mathcal{T}^{\mu\nu}(x) - \Bigl\langle \Omega \Bigl| \mathcal{T}^{\mu\nu}(x) \Bigr| \Omega \Bigr\rangle . \tag{74}$$

To amputate, recall that the graviton propagator obeys,

$$\sqrt{-\overline{g}(x)}\mathcal{D}^{\mu\nu\alpha\beta} i\Bigl[{}_{\alpha\beta}\Delta_{\rho\sigma}\Bigr](x;x') = \delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma)} i\delta^D(x-x') + \Bigl(\text{Gauge Terms}\Bigr) , \tag{75}$$

where "Gauge Terms" refers to the extra pieces needed to complete the projection operator onto whatever gauge condition is employed. (For example, the projection operator for de Donder grauge is given in equation (120) of [33].) This means that external leg propagators are amputated by $-i\sqrt{-\overline{g}}$ times the Lichnerowicz operator. Hence the desired relation between the 2-point graviton 1PI function and a 2-point correlator of the stress tensor is,

$$-i\Bigl[{}^{\mu\nu}\Delta^{\rho\sigma}\Bigr](x;x')$$
$$= \Bigl\langle \Omega \Bigl| \Bigl(-i\sqrt{-\overline{g}}\mathcal{D}^{\mu\nu\alpha\beta}\delta h_{\alpha\beta}(x)\Bigr)\Bigl(-i\sqrt{-\overline{g}}\mathcal{D}^{\rho\sigma\gamma\delta}\delta h_{\gamma\delta}(x')\Bigr)\Bigr| \Omega \Bigr\rangle + O(\kappa^4) , \tag{76}$$
$$= -\frac{1}{4}\kappa^2 \sqrt{-\overline{g}(x)}\,\sqrt{-\overline{g}(x')}\Bigl\langle \Omega \Bigl| \delta\mathcal{T}^{\mu\nu}(x)\delta\mathcal{T}^{\rho\sigma}(x')\Bigr| \Omega \Bigr\rangle + O(\kappa^4) . \tag{77}$$

The expectation value on the right hand side of (77) is the stress tensor correlator $F^{\mu\nu\rho\sigma}$ of Perez-Nadal, Roura and Verdaguer [28].

Perez-Nadal, Roura and Verdaguer actually derived $F^{\mu\nu\rho\sigma}$ for a scalar with arbitrary mass, but we can compare our result (47) for the massless case with their equation (28) [28]

$$\begin{aligned} F_{\mu\nu\rho\sigma} = {} & P(\mu)n_\mu n_\nu n_\rho n_\sigma + Q(\mu)(n_\mu n_\nu \overline{g}_{\rho\sigma} + n_\rho n_\sigma \overline{g}_{\mu\nu}) \\ & + R(\mu)(n_\mu n_\rho \overline{g}_{\nu\sigma} + n_\nu n_\sigma \overline{g}_{\mu\rho} + n_\mu n_\sigma \overline{g}_{\nu\rho} + n_\nu n_\rho \overline{g}_{\mu\sigma}) \\ & + S(\mu)(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} + \overline{g}_{\nu\rho}\overline{g}_{\mu\sigma}) + T(\mu)\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma} . \end{aligned} \tag{78}$$

Note that here they expressed the stress tensor correlator in terms of five basis tensors which are different from ours given in equation (47). Each of these five bitensors are formed as a linear combination of products of the de Sitter invariant bitensors, $n_a, n_{a'}, \overline{g}_{ab}, \overline{g}_{a'b'}$ and $\overline{g}_{ab'}$. The variable μ and bitensors are defined as [28]:

- $\mu(x,x')$: the distance along the shortest geodesic joining x and x', also called the geodesic distance;
- n_a and $n_{a'}$: the unit vectors tangent to the geodesic at the points x and x' respectively, pointing outward from it;
- $\overline{g}_{ab'}$: the parallel propagator which parallel-transports a vector from x to x' along the geodesic;
- $\overline{g}_{ab}$ and $\overline{g}_{a'b'}$: the metric tensors at the points, at the points x and x' respectively.

The distance $\mu(x,x')$ (in our notation $\mu(x,x') = H\ell(x;x')$ which is given in section 2) corresponds to our de Sitter invariant function $y(x,x')$ with the relation,

$$\cos(\mu) \equiv Z = 1 - \frac{y}{2} \; . \tag{79}$$

In comparing their results with ours it is also useful to note the relations between their basis tensors and ours,

$$n_a = \frac{1}{H\sqrt{y(4-y)}} \frac{\partial y}{\partial x^a} \; , \tag{80}$$

$$n_{b'} = \frac{1}{H\sqrt{y(4-y)}} \frac{\partial y}{\partial x'^{b'}} \; , \tag{81}$$

$$\overline{g}_{ab'} = -\frac{1}{2H^2} \Bigl\{ \frac{\partial^2 y}{\partial x^a \partial x'^{b'}} + \frac{1}{4-y} \frac{\partial y}{\partial x^a} \frac{\partial y}{\partial x'^{b'}} \Bigr\} \; . \tag{82}$$

Thus the five basis tensors given in (78) are converted into our basis tensors as,

$$n_a n_b n_{c'} n_{d'} = \frac{1}{H^4 (4y-y^2)^2} \frac{\partial y}{\partial x^a} \frac{\partial y}{\partial x^b} \frac{\partial y}{\partial x'^{c'}} \frac{\partial y}{\partial x'^{d'}} \; ,$$

$$n_a n_b \overline{g}_{c'd'} + n_{c'} n_{d'} \overline{g}_{ab} = \frac{1}{H^2 (4y-y^2)} \Bigl[\overline{g}_{ab} \frac{\partial y}{\partial x'^{c'}} \frac{\partial y}{\partial x'^{d'}} + \frac{\partial y}{\partial x^a} \frac{\partial y}{\partial x^b} \overline{g}_{c'd'} \Bigr] \; ,$$

$$\begin{aligned} 4 n_{(a} \overline{g}_{b)(c'} n_{d')} = & -\frac{2}{H^4 (4y-y^2)} \frac{\partial y}{\partial x^{(a}} \frac{\partial^2 y}{\partial x^{b)} \partial x'^{(c'}} \frac{\partial y}{\partial x'^{d')}} \\ & -\frac{2}{H^4 (4y-y^2)(4-y)} \frac{\partial y}{\partial x^a} \frac{\partial y}{\partial x^b} \frac{\partial y}{\partial x'^{c'}} \frac{\partial y}{\partial x'^{d'}} \; , \end{aligned}$$

$$
\begin{aligned}
2\overline{g}_{a(c'}\overline{g}_{d')b} &= \frac{1}{2H^4}\frac{\partial^2 y}{\partial x^a \partial x'^{(c'}}\frac{\partial^2 y}{\partial x^{d')}\partial x'^b} \\
&\quad + \frac{1}{H^4(4-y)}\frac{\partial y}{\partial x^{(a}}\frac{\partial^2 y}{\partial x^{b)}\partial x'^{(c'}}\frac{\partial y}{\partial x'^{d')}} \\
&\quad + \frac{1}{2H^4}\frac{1}{(4-y)^2}\frac{\partial y}{\partial x^a}\frac{\partial y}{\partial x^b}\frac{\partial y}{\partial x'^{c'}}\frac{\partial y}{\partial x'^{d'}} \; , \\
\overline{g}_{ab}\overline{g}_{c'd'} &= \overline{g}_{ab}\overline{g}_{c'd'} \; . \qquad (83)
\end{aligned}
$$

(Note that we have restored the factor of H which Perez-Nadal, Roura and Veraguer set to unity.)

For a massless, minimally coupled scalar field, the μ-dependent coefficients are [28],

$$
\begin{aligned}
P &= 2G_1^2 \; , \\
Q &= -G_1^2 + 2G_1 G_2 \; , \\
R &= G_1 G_2 \; , \\
S &= G_2^2 \; , \\
T &= \frac{1}{2}G_1^2 - G_1 G_2 + \frac{D-4}{2}G_2^2 \; . \qquad (84)
\end{aligned}
$$

Here the G_1 and G_2 are defined as

$$
\begin{aligned}
G_1(\mu) &= G''(\mu) - G'(\mu)\csc(\mu) \; , \\
G_2(\mu) &= -G'(\mu)\csc(\mu) \; , \qquad (85)
\end{aligned}
$$

where prime stands for derivative with respect to μ.

The comparison can be completed by noting that the Wightman function $G(\mu)$ becomes almost the same as our $A(y)$ for the case of MMC scalar. In the massless limit, their propagator has the formal expansion,

$$
G(\mu) = \frac{H^{D-2}}{(4\pi)^{D/2}} \sum_{n=0}^{\infty} \frac{\Gamma(D-1+n)\Gamma(n)}{\Gamma(\frac{D}{2}+n)} \frac{1}{n!} \Big(\frac{1+Z}{2}\Big)^n \; . \qquad (86)
$$

(Note that we have restored the factor of H^{D-2} which Perez-Nadal, Roura and Veraguer set to unity.) Recalling the hypergeometric function,

$$
{}_2F_1\Big(\alpha, \beta; \gamma; z\Big) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha{+}n)}{\Gamma(\alpha)} \frac{\Gamma(\beta{+}n)}{\Gamma(\beta)} \frac{\Gamma(\gamma)}{\Gamma(\gamma{+}n)} \frac{z^n}{n!} \; , \qquad (87)
$$

we see that $G(Z)$ can be written as,

$$G(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)\Gamma(0)}{\Gamma(\frac{D}{2})} \, {}_2F_1\Bigl(D-1, 0; \frac{D}{2}; 1-\frac{y}{4}\Bigr) \; . \tag{88}$$

Now we use one of the transformation formulae for hypergeometric functions (See for example, 9.131 of [34]) to expand G^+ in powers of $y/4$:

$$\begin{aligned} G(y) = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \Biggl\{ & \frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1} \Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2} \Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} - \Gamma(0) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \\ & + \sum_{n=1}^{\infty} \Biggl[\frac{1}{n} \frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})} \Bigl(\frac{y}{4}\Bigr)^n - \frac{1}{n-\frac{D}{2}+2} \frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)} \Bigl(\frac{y}{4}\Bigr)^{n-\frac{D}{2}+2} \Biggr] \Biggr\} . \end{aligned} \tag{89}$$

So we see that $G(y)$ is the same as the function $A(y)$ except for the replacement,

$$\Gamma(0) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \longrightarrow \pi \cot\Bigl(\frac{\pi D}{2}\Bigr) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \; . \tag{90}$$

This makes no difference because $G(y)$ only enters the stress tensor correlator (78) differentiated (See equations (84-85)). Thus for comparison, we replace the derivatives of G by the ones of A:

$$\begin{aligned} \frac{\partial G}{\partial \mu} &= \sqrt{4y-y^2} G' \equiv \sqrt{4y-y^2} A' \; , \\ \frac{\partial^2 G}{\partial \mu^2} &= (4y-y^2) G'' + (2-y) G' \equiv (4y-y^2) A'' + (2-y) A' \; . \end{aligned} \tag{91}$$

Here the prime stand for derivative with respect to y. Then the coefficients P, Q, R, S and T given in equation (84) are written in terms of y as

$$\begin{aligned} P &= 2(4y-y^2)^2 (A'')^2 - 4y(4y-y^2) A'' A' + 2y^2 (A')^2 \; , \\ Q &= -(4y-y^2)^2 (A'')^2 - 2(2-y)(4y-y^2) A'' A' + (4y-y^2)(A')^2 \; . \\ R &= -2(4y-y^2) A'' A' + 2y (A')^2 \; , \\ S &= 4(A')^2 \; , \\ T &= \frac{1}{2} \Bigl[(4y-y^2)^2 (A'')^2 + 2(2-y)(4y-y^2) A'' A' \\ & \qquad + \{4(D-4) - (4y-y^2)\} (A')^2 \Bigr] \; . \end{aligned} \tag{92}$$

With this equation (92) and the conversion of basis given in equation (83) we can arrange $F_{\mu\nu\rho\sigma}$ for the MMC scalar in terms of our basis tensors,

$$
\begin{aligned}
F_{\mu\nu\rho\sigma} = -\frac{4}{\kappa^2}\Bigg\{ & \frac{\partial^2 y}{\partial x^\mu \partial x'^{(\rho}} \frac{\partial^2 y}{\partial x'^{\sigma)} \partial x^\nu} \times \alpha(y) \\
& + \frac{\partial y}{\partial x^{(\mu}} \frac{\partial^2 y}{\partial x^{\nu)} \partial x'^{(\rho}} \frac{\partial y}{\partial x'^{\sigma)}} \times \beta(y) + \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu} \frac{\partial y}{\partial x'^\rho} \frac{\partial y}{\partial x'^\sigma} \times \gamma(y) \\
& + \overline{g}^{\mu\nu} \overline{g}'^{\rho\sigma} H^4 \times \delta(y) + \Big[\overline{g}^{\mu\nu} \frac{\partial y}{\partial x'^\rho} \frac{\partial y}{\partial x'^\sigma} + \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu} \overline{g}'^{\rho\sigma}\Big] H^2 \times \epsilon(y) \Bigg\} . \qquad (93) \\
& = -\frac{4}{\kappa^2} \times \frac{1}{\sqrt{-\overline{g}(x)}\sqrt{-\overline{g}(x')}} \times -i\Big[{}_{\mu\nu}\Sigma_{\rho\sigma}\Big]_{\rm 3pt}(x;x') . \qquad (94)
\end{aligned}
$$

4 Renormalization

Our result (47) is valid as long as $x'^\mu \neq x^\mu$, either with the exact coefficient functions (48-52) or with the relevant expansions (55-59) for $D = 4$. However, it is not immediately usable in the quantum-corrected, linearized Einstein equations because they involve an integration over x'^μ,

$$
\sqrt{-\overline{g}}\,\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) - \int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm ren}(x;x') h_{\rho\sigma}(x') = \frac{1}{2}\kappa\sqrt{-\overline{g}}\, T^{\mu\nu}_{\rm lin}(x) . \qquad (95)
$$

To obtain a usable form we must express (47) as a product of up to six differential operators acting upon a function of $y(x;x')$ which is integrable in $D = 4$ spacetime dimensions. The derivatives with respect to x^μ can be pulled outside the integral, and those with respect to x'^μ can be partially integrated to act back on the $h_{\rho\sigma}(x')$,[1] leaving an expression for which the $D = 4$ limit could be taken were it not for some factors of $1/(D-4)$. At this stage one adds zero in the form of identities such as,

$$
\Big[\square - \frac{D}{2}\Big(\frac{D}{2}-1\Big)H^2\Big]\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} - \frac{(4\pi)^{\frac{D}{2}}\, i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1) H^{D-2}\sqrt{-\overline{g}}} = 0 . \qquad (96)
$$

We combine (96) with terms which arise from extracting derivatives to segregate the divergences on local, delta function terms, for example,

$$
\frac{1}{D-4}\Big[\square - \frac{D}{2}\Big(\frac{D}{2}-1\Big)H^2\Big]\Big(\frac{4}{y}\Big)^{D-3}
$$

[1] The resulting surface terms can be absorbed by correcting the initial state [35].

$$= \left[\Box - \frac{D}{2}\Big(\frac{D}{2}-1\Big)H^2\right]\left\{\frac{(\frac{4}{y})^{D-3} - (\frac{4}{y})^{\frac{D}{2}-1}}{D-4}\right\} + \frac{(4\pi)^{\frac{D}{2}}\, i\delta^D(x-x')/\sqrt{-g}}{(D-4)\Gamma(\frac{D}{2}-1)H^{D-2}} \; , \qquad (97)$$

$$= -\frac{1}{2}\Big[\Box - 2H^2\Big]\left\{\frac{4}{y}\ln\Big(\frac{y}{4}\Big)\right\} + O(D-4) + \frac{(4\pi)^{\frac{D}{2}}\, i\delta^D(x-x')/\sqrt{-g}}{(D-4)\Gamma(\frac{D}{2}-1)H^{D-2}} \; . \qquad (98)$$

Renormalization consists of subtracting off the divergent delta functions with counterterms. In subsection 4.1 we exhibit the one loop counterterms for quantum gravity. We review how to renormalize the flat space limit (68) in subsection 4.2. That suggests a convenient way of organizing the tensor algebra into two transverse, 4th order differential operators, one with spin zero and the other with spin two. In subsection 4.3 we implement this for de Sitter. The spin zero part is renormalized in subsection 4.4, and the spin two part in subsection 4.5.

4.1 One Loop Counterterms

Gravity + Scalar is not renormalizable in $D = 4$ dimensions [20]. However, the theorem of Bogoliubov, Parasiuk, Hepp and Zimmerman (BPHZ) shows us how to construct local counterterms which absorb the ultraviolet divergences of any quantum field theory to any fixed order in the loop expansion [21]. For quantum gravity at one loop order the necessary counterterms can be taken to be the squares of the Ricci scalar and the Weyl tensor [20]. The problem of quantum gravity is that the Weyl counterterm would destabilize the universe if it were regarded as a fundamental, nonperturbative interaction [36]. We shall therefore consider it only perturbatively, in the sense of effective field theory, as a proxy for the yet unknown ultraviolet completion of quantum gravity. The quantum effects we seek to study derive from infrared virtual scalars with wavelengths on the order of the Hubble radius, and they will manifest as nonlocal and ultraviolet finite contributions to the graviton self-energy which are not affected by how nature resolves the ultraviolet problem of quantum gravity.

Because the background Ricci scalar is nonzero it is useful to reorganize R^2 into a part which is quadratic in the graviton field,

$$R^2 = \Big[R - D(D-1)H^2\Big]^2 + 2D(D-1)H^2 R - D^2(D-1)^2 H^4 \; . \qquad (99)$$

So we will employ four counterterms,

$$\Delta\mathcal{L}_1 \;\; \equiv \;\; c_1\Big[R - D(D-1)H^2\Big]^2\sqrt{-g} \; , \qquad (100)$$

$$\Delta\mathcal{L}_2 \equiv c_2 C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \sqrt{-g} \; , \qquad (101)$$
$$\Delta\mathcal{L}_3 \equiv c_3 H^2 \Bigl[R - (D-1)(D-2)H^2\Bigr] \sqrt{-g} \; , \qquad (102)$$
$$\Delta\mathcal{L}_4 \equiv c_4 H^4 \sqrt{-g} \; . \qquad (103)$$

Of course the divergences can really be eliminated with just $\Delta\mathcal{L}_2$ and the particular linear combination of $\Delta\mathcal{L}_1$, $\Delta\mathcal{L}_3$ and $\Delta\mathcal{L}_4$ which is proportional to just $R^2\sqrt{-g}$. It must therefore be that two linear combinations of the coefficients are finite,

$$\lim_{D\to 4} \Bigl[-2D(D-1)c_1 + c_3\Bigr] = \text{Finite} \; , \qquad (104)$$
$$\lim_{D\to 4} \Bigl[D^2(D-1)^2 c_1 - (D-1)(D-2)c_3 + c_4\Bigr] = \text{Finite} \; . \qquad (105)$$

And the divergent parts of c_1 and c_2 must agree with the values obtained long ago by ‘t Hooft and Veltman [20].

At this point we digress to define two 2nd order differential operators of great importance to our subsequent analysis. They come from expanding the scalar and Weyl curvatures around de Sitter background,

$$R - D(D-1)H^2 \equiv \mathcal{P}^{\mu\nu} \kappa h_{\mu\nu} + O(\kappa^2 h^2) \; , \qquad (106)$$
$$C_{\alpha\beta\gamma\delta} \equiv \mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta} \kappa h_{\mu\nu} + O(\kappa^2 h^2) \; . \qquad (107)$$

From (106) we have,

$$\mathcal{P}^{\mu\nu} = D^\mu D^\nu - \overline{g}^{\mu\nu}\Bigl[D^2 + (D-1)H^2\Bigr] \; , \qquad (108)$$

where D^μ is the covariant derivative operator in de Sitter background. The more difficult expansion of the Weyl tensor gives,

$$\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta} = \mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} + \frac{1}{D-2}\Bigl[\overline{g}_{\alpha\delta}\mathcal{D}^{\mu\nu}_{\beta\gamma} - \overline{g}_{\beta\delta}\mathcal{D}^{\mu\nu}_{\alpha\gamma} - \overline{g}_{\alpha\gamma}\mathcal{D}^{\mu\nu}_{\beta\delta} + \overline{g}_{\beta\gamma}\mathcal{D}^{\mu\nu}_{\alpha\delta}\Bigr]$$
$$+ \frac{1}{(D-1)(D-2)}\Bigl[\overline{g}_{\alpha\gamma}\overline{g}_{\beta\delta} - \overline{g}_{\alpha\delta}\overline{g}_{\beta\gamma}\Bigr]\mathcal{D}^{\mu\nu} \; , \qquad (109)$$

where we define,

$$\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} \equiv \frac{1}{2}\Bigl[\delta^{(\mu}_{\alpha}\delta^{\nu)}_{\delta} D_\gamma D_\beta - \delta^{(\mu}_{\beta}\delta^{\nu)}_{\delta} D_\gamma D_\alpha - \delta^{(\mu}_{\alpha}\delta^{\nu)}_{\gamma} D_\delta D_\beta + \delta^{(\mu}_{\beta}\delta^{\nu)}_{\gamma} D_\delta D_\alpha\Bigr] \; , \qquad (110)$$
$$\mathcal{D}^{\mu\nu}_{\beta\delta} \equiv \overline{g}^{\alpha\gamma}\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} = \frac{1}{2}\Bigl[\delta^{(\mu}_{\delta} D^{\nu)} D_\beta - \delta^{(\mu}_{\beta}\delta^{\nu)}_{\delta} D^2 - \overline{g}^{\mu\nu} D_\delta D_\beta + \delta^{(\mu}_{\beta} D_\delta D^{\nu)}\Bigr] \; , \qquad (111)$$
$$\mathcal{D}^{\mu\nu} \equiv \overline{g}^{\alpha\gamma}\overline{g}^{\beta\delta}\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} = D^{(\mu}D^{\nu)} - \overline{g}^{\mu\nu} D^2 \; . \qquad (112)$$

One obtains the counterterm vertices by functionally differentiating i times each counterterm action twice, and then setting the graviton field to zero. They are,

$$\frac{i\delta\Delta S_1}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h=0} = 2c_1\kappa^2\sqrt{-\overline{g}}\,\mathcal{P}^{\mu\nu}\mathcal{P}^{\rho\sigma}i\delta^D(x-x')\;, \tag{113}$$

$$\frac{i\delta\Delta S_2}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h=0} = 2c_2\kappa^2\sqrt{-\overline{g}}\,\overline{g}^{\alpha\kappa}\overline{g}^{\beta\lambda}\overline{g}^{\gamma\theta}\overline{g}^{\delta\phi}\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}i\delta^D(x-x')\;, \tag{114}$$

$$\frac{i\delta\Delta S_3}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h=0} = -c_3\kappa^2H^2\sqrt{-\overline{g}}\,\mathcal{D}^{\mu\nu\rho\sigma}i\delta^D(x-x')\;, \tag{115}$$

$$\frac{i\delta\Delta S_4}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h=0} = c_4\kappa^2H^4\sqrt{-\overline{g}}\left[\frac{1}{4}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}-\frac{1}{2}\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\right]i\delta^D(x-x')\;. \tag{116}$$

Recall that the Lichnerowicz operator in expression (115) was defined in expression (71). Also note the flat space limits,

$$\frac{i\delta\Delta S_1}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h=0} \longrightarrow 2c_1\kappa^2\Pi^{\mu\nu}\Pi^{\rho\sigma}i\delta^D(x-x')\;, \tag{117}$$

$$\frac{i\delta\Delta S_2}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h=0} \longrightarrow 2c_2\kappa^2\Big(\frac{D-3}{D-2}\Big)\Big[\Pi^{\mu(\rho}\Pi^{\sigma)\nu}-\frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{D-1}\Big]i\delta^D(x-x')\;, \tag{118}$$

$$\frac{i\delta\Delta S_3}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h=0} \longrightarrow 0\;, \tag{119}$$

$$\frac{i\delta\Delta S_4}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big|_{h=0} \longrightarrow 0\;, \tag{120}$$

where we define,

$$\Pi^{\mu\nu} \equiv \partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\;. \tag{121}$$

4.2 Renormalizing the Flat Space Result

Renormalizing the flat space result (68) provides an excellent guide for the vastly more complicated reduction required on de Sitter background. We begin by extracting a 4th order differential operator from each term using the identities,

$$\frac{1}{\Delta x^{2D}} = \frac{\partial^4}{4(D-2)^2(D-1)D}\,\frac{1}{\Delta x^{2D-4}}\;, \tag{122}$$

$$\frac{\Delta x^\mu \Delta x^\nu}{\Delta x^{2D+2}} = \frac{1}{8(D-2)^2(D-1)D}\Bigl\{\partial^\mu\partial^\nu\partial^2 + \frac{\eta^{\mu\nu}\partial^4}{D}\Bigr\}\frac{1}{\Delta x^{2D-4}} \;, \qquad (123)$$

$$\frac{\Delta x^\mu \Delta x^\nu \Delta x^\rho \Delta x^\sigma}{\Delta x^{2D+4}} = \frac{1}{16(D-2)(D-1)D(D+1)}\Bigl\{\partial^\mu\partial^\nu\partial^\rho\partial^\sigma + \frac{6}{D-2}\,\eta^{(\mu\nu}\partial^\rho\partial^{\sigma)}\partial^2 + \frac{3}{(D-2)D}\,\eta^{(\mu\nu}\eta^{\rho\sigma)}\partial^4\Bigr\}\frac{1}{\Delta x^{2D-4}} \;. \qquad (124)$$

Substituting these relations into (68), and then organizing the various derivatives into factors of the transverse operator $\Pi^{\mu\nu}$ of expression (121), gives a manifestly transverse form,

$$-i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr]_{\rm flat}(x;x') = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\Bigl\{-\frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{8(D-1)^2} - \frac{[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{D-1}\Pi^{\mu\nu}\Pi^{\rho\sigma}]}{4(D-2)^2(D-1)(D+1)}\Bigr\}\frac{1}{\Delta x^{2D-4}} \;. \qquad (125)$$

Let us pause at this point to note that we could have guessed most of the form of expression (125). Gauge invariance implies transversality. We also have Poincaré invariance, symmetry under interchange the interchanges $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$, and symmetry under interchange of the primed and unprimed coordinates and indices. All this implies the form,

$$-i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr]_{\rm flat}(x;x') = \Pi^{\mu\nu}\Pi^{\rho\sigma}F_1(\Delta x^2) + \Bigl[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{D-1}\Bigr]F_2(\Delta x^2) \;. \qquad (126)$$

Taking the trace of this and our result (68) against $\eta_{\mu\nu}\eta_{\rho\sigma}$ gives an equation for the spin zero structure function $F_1(\Delta x^2)$,

$$\eta_{\mu\nu}\eta_{\rho\sigma}\times -i\Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr]_{\rm flat} = (D-1)^2\partial^4 F_1(\Delta x^2) = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\times -\frac{(D-2)^2(D-1)D}{2\Delta x^{2D}} \;. \qquad (127)$$

Of course the solution is just what we found in (125) by direct computation,

$$F_1(\Delta x^2) = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\times -\frac{1}{8(D-1)^2}\Bigl(\frac{1}{\Delta x^2}\Bigr)^{D-2} \;. \qquad (128)$$

Determining the spin two structure function $F_2(\Delta x^2)$ is done by first acting the derivatives on the spin zero structure function,

$$\Pi^{\mu\nu}\Pi^{\rho\sigma}F_1 = \eta^{\mu(\rho}\eta^{\sigma)\nu}\times 8F_1'' + \Delta x^{(\mu}\eta^{\nu)(\rho}\Delta x^{\sigma)}\times 32F_1''' + \Delta x^\mu\Delta x^\nu\Delta x^\rho\Delta x^\sigma \times 16F_1'''' + \eta^{\mu\nu}\eta^{\rho\sigma}\times\Bigl[4(D^2-3)F_1'' + 16(D+1)\Delta x^2 F_1''' + 16\Delta x^4 F_1''''\Bigr] + \Bigl[\eta^{\mu\nu}\Delta x^\rho\Delta x^\sigma + \Delta x^\mu\Delta x^\nu\eta^{\rho\sigma}\Bigr]\times\Bigl[-8(D+3)F_1''' - 16\Delta x^2 F_1''''\Bigr] \;. \qquad (129)$$

We subtract these from each tensor factor in (68) and then act the spintwo operator $[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{D-1}\Pi^{\mu\nu}\Pi^{\rho\sigma}]$ on $F_2(\Delta x^2)$ to read off an equation for each of the five tensor factors,

$$\eta^{\mu(\rho}\eta^{\sigma)\nu} \Rightarrow \frac{4(D-2)D(D+1)}{D-1}\, F_2'' + 16(D+1)\Delta x^2 F_2''' + 16\Delta x^4 F_2'''' = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\Bigg\{-\frac{D}{D-1}\frac{1}{\Delta x^{2D}}\Bigg\} , \tag{130}$$

$$\Delta x^{(\mu}\eta^{\nu)(\rho}\Delta x^{\sigma)} \Rightarrow -\frac{16D(D+1)}{D-1}\, F_2''' - 32\Delta x^2 F_2'''' = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\Bigg\{\frac{4D}{D-1}\frac{1}{\Delta x^{2D}}\Bigg\} , \tag{131}$$

$$\Delta x^\mu\Delta x^\nu\Delta x^\rho\Delta x^\sigma \Rightarrow 16\Big(\frac{D-2}{D-1}\Big)F_2'''' = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\Bigg\{-\frac{4D}{D-1}\frac{1}{\Delta x^{2D}}\Bigg\} , \tag{132}$$

$$\eta^{\mu\nu}\eta^{\rho\sigma} \Rightarrow -\frac{4}{D-1}\Big[(D-2)(D+1)F_2'' + 4(D+1)\Delta x^2 F_2''' + 4\Delta x^4 F_2''''\Big] = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\Bigg\{\frac{1}{D-1}\frac{1}{\Delta x^{2D}}\Bigg\} , \tag{133}$$

$$\Big[\eta^{\mu\nu}\Delta x^\rho\Delta x^\sigma + \Delta x^\mu\Delta x^\nu\eta^{\rho\sigma}\Big] \Rightarrow \frac{16}{D-1}\Big[(D+1)F_2''' + \Delta x^2 F_2''''\Big] = 0 . \tag{134}$$

Each of these equations has the same solution, which of course agrees with (125),

$$F_2(\Delta x^2) = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D} \times -\frac{1}{4(D-2)^2(D-1)(D+1)}\Big(\frac{1}{\Delta x^2}\Big)^{D-2} . \tag{135}$$

We note for future reference that a particular linear combination of the five relations (130-134) gives a second order equation for $F_2(\Delta x^2)$,

$$(133) + \Delta x^2(134) = -\frac{4}{D-1}\,(D-2)(D+1)F_2'' = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\Bigg\{\frac{1}{D-1}\frac{1}{\Delta x^{2D}}\Bigg\} . \tag{136}$$

Even after extracting the 4th order differential operators from the integration of (95), the factor of $1/\Delta x^{2D-4}$ is logarithmically divergent. We must therefore extract one more d'Alembertian,

$$\Big(\frac{1}{\Delta x^2}\Big)^{D-2} = \frac{\partial^2}{2(D-3)(D-4)}\Big(\frac{1}{\Delta x^2}\Big)^{D-3} . \tag{137}$$

After this final derivative is extracted the integrand converges, however, we still cannot take the $D=4$ limit owing to the factor of $1/(D-4)$. The solution is to add zero in the form of the identity,

$$\partial^2\Bigl(\frac{1}{\Delta x^2}\Bigr)^{\frac{D}{2}-1} - \frac{4\pi^{\frac{D}{2}}\, i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1)} = 0 \; . \tag{138}$$

To make this dimensionally consistent with (137) we must multiply by the dimensional regualrization mass scale μ raised to the $(D-4)$ power,

$$\begin{aligned}
&\Bigl(\frac{1}{\Delta x^2}\Bigr)^{D-2} \\
&\quad = \frac{\partial^2}{2(D-3)(D-4)}\Biggl\{\frac{1}{\Delta x^{2D-6}} - \frac{\mu^{D-4}}{\Delta x^{D-2}}\Biggr\} + \frac{4\pi^{\frac{D}{2}}\mu^{D-4} i\delta^D(x-x')}{2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} \; , &(139)\\
&\quad = -\frac{1}{4}\partial^2\Biggl\{\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2} + O(D-4)\Biggr\} + \frac{4\pi^{\frac{D}{2}}\mu^{D-4} i\delta^D(x-x')}{2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} \; . &(140)
\end{aligned}$$

The divergences have now been segregated on delta function terms which can be removed with local counterterms. From expressions (117-120) we see that the counterterms make the following contribution to the graviton self-energy,

$$\begin{aligned}
-i\Bigl[{}^{\mu\nu}\Delta\Sigma^{\rho\sigma}\Bigr]_{\rm flat}(x;x') &= \Pi^{\mu\nu}\Pi^{\rho\sigma}\Biggl\{2c_1\kappa^2 i\delta^D(x-x')\Biggr\} \\
&\quad + \Bigl[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{D-1}\Bigr]\Biggl\{2\Bigl(\frac{D-3}{D-2}\Bigr)c_2\kappa^2 i\delta^D(x-x')\Biggr\} \; . \qquad (141)
\end{aligned}$$

The delta function terms will be entirely absorbed by choosing the constants c_1 and c_2 as,

$$\begin{aligned}
c_1 &= \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{2^8\pi^{\frac{D}{2}}}\,\frac{(D-2)}{(D-1)^2(D-3)(D-4)} \; , &(142)\\
c_2 &= \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{2^8\pi^{\frac{D}{2}}}\,\frac{2}{(D+1)(D-1)(D-3)^2(D-4)} \; . &(143)
\end{aligned}$$

Of course the divergent parts agree with the results obtained long ago by ‘t Hooft and Veltman [20], with the arbitrary finite parts represented by μ.

The fully renormalized graviton self-energy (for flat space background) is,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\substack{\rm ren \\ \rm flat}} = \lim_{D\to 4}\Big\{-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm flat}(x;x') - i\Big[{}^{\mu\nu}\Delta\Sigma^{\rho\sigma}\Big]_{\rm flat}(x;x')\Big\} \; , \tag{144}$$

$$= \Pi^{\mu\nu}\Pi^{\rho\sigma}\partial^2\Big\{\frac{\kappa^2}{2^9 3^2\pi^4}\,\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big\} + \Big[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{3}\Pi^{\mu\nu}\Pi^{\rho\sigma}\Big]\partial^2\Big\{\frac{\kappa^2}{2^{10}3^1 5^1\pi^4}\,\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big\} \; . \tag{145}$$

4.3 The de Sitter Structure Functions

We must now extend the flat space ansatz (126) to de Sitter and determine the resulting structure functions by comparison with the explicit result (47) of section 3. As before, gauge invariance implies transversality, which suggests that we make use of the differential operators $\mathcal{P}^{\mu\nu}$ and $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}$ which were defined in expressions (108) and (109), respectively. In place of Poincaré invariance we now have de Sitter invariance. We also have symmetry under the interchanges $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$, and under interchange of the primed and unprimed coordinates and indices. A simple generalization is,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = \sqrt{-\overline{g}(x)}\,\mathcal{P}^{\mu\nu}(x)\sqrt{-\overline{g}(x')}\,\mathcal{P}^{\rho\sigma}(x')\Big\{\mathcal{F}_1(y)\Big\} + \sqrt{-\overline{g}(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\sqrt{-\overline{g}(x')}\,\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')\Big\{\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big(\frac{D-2}{D-3}\Big)\mathcal{F}_2(y)\Big\} , \tag{146}$$

where the bitensor $\mathcal{T}^{\alpha\kappa}$ is,[2]

$$\mathcal{T}^{\alpha\kappa}(x;x') \equiv -\frac{1}{2H^2}\,\frac{\partial^2 y(x;x')}{\partial x_\alpha \partial x'_\kappa} \; . \tag{147}$$

As in flat space, the second term is traceless.

Note the flat space limits of the bitensor and the two structure functions,

$$\lim_{H\to 0}\mathcal{T}^{\alpha\kappa} = \eta^{\kappa\lambda} \quad , \quad \lim_{H\to 0}\mathcal{F}_1(y) = F_1(\Delta x^2) \quad , \quad \lim_{H\to 0}\mathcal{F}_2(y) = F_2(\Delta x^2) \; . \tag{148}$$

[2]One could actually employ any bitensor — for example, the parallel transport matrix (82) — which reduces to $\eta^{\alpha\kappa}$ in the flat space limit. Different choices for $\mathcal{T}^{\alpha\kappa}(x;x')$ make corresponding changes in the subdominant parts of the spin two structure function $\mathcal{F}_2(y)$. We have not troubled to determine the "simplest" choice.

These limits mean one can immediately read off the most singular parts of the expansions for each structure function from the corresponding flat space result,

$$\mathcal{F}_1(y) \;\; = \;\; \frac{\kappa^2 H^{2D-4} \Gamma^2(\frac{D}{2})}{(4\pi)^D} \Biggl\{ \frac{-1}{8(D\!-\!1)^2} \Bigl(\frac{4}{y}\Bigr)^{D-2} + \dots \Biggr\} , \qquad (149)$$

$$\mathcal{F}_2(y) \;\; = \;\; \frac{\kappa^2 H^{2D-4} \Gamma^2(\frac{D}{2})}{(4\pi)^D} \Biggl\{ \frac{-1}{4(D\!-\!3)(D\!-\!2)(D\!-\!1)(D\!+\!1)} \Bigl(\frac{4}{y}\Bigr)^{D-2} + \dots \Biggr\} . \qquad (150)$$

The interesting de Sitter physics we seek to elucidate derives from the subdominant terms.

Just as for the flat space limit, we can obtain an equation for the spin zero structure function by tracing (146) and then comparing with the trace of the explicit computation (47). Tracing the ansatz gives,

$$\frac{\overline{g}_{\mu\nu}(x)}{\sqrt{-\overline{g}(x)}} \times \frac{\overline{g}_{\rho\sigma}(x')}{\sqrt{-\overline{g}(x')}} \times -i \Bigl[\mbox{}^{\mu\nu} \Sigma^{\rho\sigma}\Bigr](x;x') = (D\!-\!1)^2 \Bigl[\square \!+\! D H^2\Bigr] \Bigl[\square' \!+\! D H^2\Bigr] \mathcal{F}_1(y) \; . \qquad (151)$$

Tracing the explicit result (47), substituting (48-52), and then making use of (36) gives,

$$\frac{\overline{g}_{\mu\nu}(x)}{\sqrt{-\overline{g}(x)}} \times \frac{\overline{g}_{\rho\sigma}(x')}{\sqrt{-\overline{g}(x')}} \times -i \Bigl[\mbox{}^{\mu\nu} \Sigma^{\rho\sigma}\Bigr]_{\rm 3pt}(x;x') = H^4 \Biggl\{ \Bigl[4D \!-\! (4y\!-\!y^2)\Bigr] \alpha$$

$$+ (2\!-\!y)(4y\!-\!y^2) \beta + (4y\!-\!y^2)^2 \gamma + D^2 \delta + 2D (4y\!-\!y^2) \epsilon \Biggr\} , \qquad (152)$$

$$= \frac18 (D\!-\!2)^2 \kappa^2 H^4 \Biggl\{ \Bigl[(4y\!-\!y^2) - 4D\Bigr] (A')^2$$

$$- 2 (2\!-\!y) (4y\!-\!y^2) A' A'' - (4y\!-\!y^2)^2 (A'')^2 \Biggr\} , \qquad (153)$$

$$= -\frac18 (D\!-\!1)^2 (D\!-\!2)^2 \kappa^2 H^4 \Biggl\{ \frac{4}{D\!-\!1} \, (A')^2 + \Bigl[(2\!-\!y) A' - k\Bigr]^2 \Biggr\} . \qquad (154)$$

Now note that the primed and unprimed scalar d'Alembertian's agree when acting on any function of only $y(x;x')$. Equating (151) and (154) and expanding implies,

$$\Bigl[\frac{\square}{H^2} \!+\! D\Bigr]^2 \mathcal{F}_1(y) = -\frac18 (D\!-\!2)^2 \kappa^2 \Biggl\{ \frac{4}{D\!-\!1} \, (A')^2 + \Bigl[(2\!-\!y) A' - k\Bigr]^2 \Biggr\} . \qquad (155)$$

$$= -\frac{K}{32}\frac{(D-2)^2}{(D-1)}\Biggl\{D\Bigl(\frac{4}{y}\Bigr)^{D} + (D-2)^2\Bigl(\frac{4}{y}\Bigr)^{D-1} + \frac{1}{2}(D^3-7D^2+16D-8)\Bigl(\frac{4}{y}\Bigr)^{D-2} + \Bigl(\text{Irrelevant}\Bigr)\Biggr\} , \qquad (156)$$

where the constant K was defined in (60) and "Irrelevant" means terms which are both integrable at coincidence, and which vanish in $D = 4$ dimensions.

Let us first note that we can find a Green's function for the differential operator $[\square/H^2 + D]$. To see this, act the operator on some function $f(y)$ which is free of the unique power $y^{\frac{D}{2}-1}$ which produces a delta function,

$$\Bigl[\frac{\square}{H^2}+D\Bigr]f(y) = (4y-y^2)f'' + D(2-y)f' + Df \; . \qquad (157)$$

Now note that $f_1(y) = (2 - y)$ is a homogeneous solution, which means we can factor to obtain a first order equation (and hence solvable) for the second solution,

$$f_1(y) = (2-y) \Longrightarrow f_2(y) \equiv f_1(y)g(y) \quad \text{with} \quad g'(y) = \frac{1}{(4y-y^2)^{\frac{D}{2}} f_1^2(y)} \; . \qquad (158)$$

With the two, linearly independent solutions one can construct a Green's function,

$$G_1(y;y') = \theta((y-y')\Bigl[f_2(y)f_1(y')-f_1(y)f_2(y')\Bigr](4y'-y'^2)^{\frac{D}{2}-1} \; . \qquad (159)$$

Hence we can solve (156) to obtain on integral epxression for the spin zero structure function,

$$\mathcal{F}_1(y) = \left[\frac{1}{\frac{\square}{H^2}+D}\right]^2\Biggl\{\text{Right hand side of (156)}\Biggr\} \; . \qquad (160)$$

Although we will eventually make use of the Green's function (159), it is best to delay this until the point at which one can set $D = 4$. For the more singular terms the best strategy is to exploit the fact that the "source" terms on the right hand side of (156) upon which we wish to act the inverse of $[\square/H^2 + D]^2$ are just powers of y. Consider acting the operator upon a power $p - 2 \neq \frac{D}{2} - 1$ or $\frac{D}{2} - 2$ (those powers produce delta functions),

$$\Bigl[\frac{\square}{H^2}+D\Bigr]^2\Bigl(\frac{4}{y}\Bigr)^{p-2} = (p-2)(p-1)(p-1-\frac{D}{2})(p-\frac{D}{2})\Bigl(\frac{4}{y}\Bigr)^{p} + (p-2)(p-1-\frac{D}{2}) \times\Bigl[D(2p-1)-2(p-1)^2\Bigr]\Bigl(\frac{4}{y}\Bigr)^{p-1} + (p-1)^2(D-p+2)^2\Bigl(\frac{4}{y}\Bigr)^{p-2} . \qquad (161)$$

We can therefore develop a recursive procedure for reducing the power of the source,

$$\Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \Big(\frac{4}{y}\Big)^p = \frac{1}{(p-2)(p-1)(p-1-\frac{D}{2})(p-\frac{D}{2})}\Big(\frac{4}{y}\Big)^{p-2} - \Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \times\Big\{\frac{[D(2p-1)-2(p-1)^2]}{(p-1)p-\frac{D}{2})}\Big(\frac{4}{y}\Big)^{p-1} + \frac{(p-1)(D+2-p)^2}{(p-2)(p-1-\frac{D}{2})(p-\frac{D}{2})}\Big(\frac{4}{y}\Big)^{p-2}\Big\} \tag{162}$$

The strategy is to apply this until the source is integrable, at which point the dimension can be set to $D = 4$ (unless there are factors of $1/(D-4)$) and the $D = 4$ Green's function can be used to obtain the full solution for $\mathcal{F}_1(y)$.

It is useful to examine the sorts of terms generated when this recursive procedure is applied to the source terms on the right hand side of (156). The most singular term introduces no factors of $1/(D-4)$, nor does it produce remainder terms different from those in the original source term (156),

$$\Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \Big(\frac{4}{y}\Big)^D = \frac{4}{(D-2)D(D-2)(D-1)}\Big(\frac{4}{y}\Big)^{D-2} - \Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \Big\{\frac{2(3D-2)}{D(D-1)}\Big(\frac{4}{y}\Big)^{D-1} + \frac{16(D-1)}{(D-2)D(D-2)}\Big(\frac{4}{y}\Big)^{D-2}\Big\} . \tag{163}$$

Neither statement is true for the remaining two source terms,

$$\Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \Big(\frac{4}{y}\Big)^{D-1} = \frac{4}{(D-4)(D-2)(D-3)(D-2)}\Big(\frac{4}{y}\Big)^{D-3} - \Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \Big\{\frac{2(5D-8)}{(D-2)(D-2)}\Big(\frac{4}{y}\Big)^{D-2} + \frac{36(D-2)}{(D-4)(D-2)(D-3)}\Big(\frac{4}{y}\Big)^{D-3}\Big\} , \tag{164}$$

$$\Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \Big(\frac{4}{y}\Big)^{D-2} = \frac{4}{(D-6)(D-4)(D-4)(D-3)}\Big(\frac{4}{y}\Big)^{D-4} - \Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \Big\{\frac{2(7D-18)}{(D-4)(D-3)}\Big(\frac{4}{y}\Big)^{D-3} + \frac{64(D-3)}{(D-6)(D-4)(D-4)}\Big(\frac{4}{y}\Big)^{D-4}\Big\} . \tag{165}$$

These relations allow the the spin zero structure function to be expressed as a "quotient" and a "remainder" of the form,

$$\mathcal{F}_1(y) = \mathcal{Q}_1(y) + \Big[\frac{1}{\frac{\Box}{H^2}+D}\Big]^2 \mathcal{R}_1(y) , \tag{166}$$

$$\mathcal{Q}_1(y) = -K\Biggl\{f_{1a}\Bigl(\frac{4}{y}\Bigr)^{D-2} + \frac{f_{1b}}{D-4}\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{f_{1c}}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{D-4}\Biggr\} , \qquad (167)$$

$$\mathcal{R}_1(y) = -K\Biggl\{\frac{f_{1d}}{D-4}\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{f_{1e}}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{D-4} + \Bigl(\text{Irrelevant}\Bigr)\Biggr\} , \qquad (168)$$

where the coefficients are,

$$f_{1a} = \frac{1}{8(D-1)^2} , \qquad (169)$$

$$f_{1b} = \frac{D(D^2-5D+2)}{8(D-3)(D-1)^2} , \qquad (170)$$

$$f_{1c} = \frac{D^2(D^4-12D^3+39D^2-16D-36)}{16(D-6)(D-3)(D-1)^2} , \qquad (171)$$

$$f_{1d} = -\frac{8}{3} + \frac{79}{9}(D-4) + O\Bigl((D-4)^2\Bigr) , \qquad (172)$$

$$f_{1e} = \frac{32}{3} - \frac{64}{9}(D-4) - \frac{274}{9}(D-4)^2 + O\Bigl((D-4)^3\Bigr) . \qquad (173)$$

Although the powers y^{D-3} and y^{D-4} in the remainder term of (166) are integrable, the factors of $1/(D-4)$ they carry preclude us setting $D=4$ and then obtaining an explicit form using the $D=4$ Green's function. In the next subsection we will see how to add zero so as to localize the divergences, and then absorb them into counterterms. For now, let us assume $\mathcal{F}_1(y)$ has been derived and explain the procedure for computing the spin two structure function $\mathcal{F}_2(y)$.

The spin zero part of the graviton self-energy can be expressed as a sum of the five de Sitter invariant bitensors times functions of y,

$$\mathcal{P}^{\mu\nu}(x)\times\mathcal{P}^{\rho\sigma}(x')\times\mathcal{F}_1(y) = \frac{\partial^2 y}{\partial x_\mu \partial x'_{(\rho}}\frac{\partial^2 y}{\partial x'_{\sigma)}\partial x_\nu}\times\alpha_1(y) + \frac{\partial y}{\partial x_{(\mu}}\frac{\partial^2 y}{\partial x_{\nu)}\partial x'_{(\rho}}\frac{\partial y}{\partial x'_{\sigma)}}$$
$$\times\beta_1(y) + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma}\times\gamma_1(y) + H^4\overline{g}^{\mu\nu}(x)\overline{g}^{\rho\sigma}(x')\times\delta_1(y)$$
$$+H^2\Bigl[\overline{g}^{\mu\nu}(x)\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\overline{g}^{\rho\sigma}(x')\Bigr]\times\epsilon_1(y) , \qquad (174)$$

Here the spin zero coefficient functions are,

$$\alpha_1 = 2\mathcal{F}_1'' , \qquad (175)$$

$$
\begin{aligned}
\beta_1 &= 4\mathcal{F}_1''' \;, & (176)\\
\gamma_1 &= \mathcal{F}_1'''' \;, & (177)\\
\delta_1 &= (4y-y^2)^2\mathcal{F}_1'''' + 2(D+1)(2-y)(4y-y^2)\mathcal{F}_1''' - 4(4y-y^2)\mathcal{F}_1'' \\
&\qquad +(D^2-3)(2-y)^2\mathcal{F}_1'' + (D-1)^2(2-y)\mathcal{F}_1' + (D-1)^2\mathcal{F}_1 \;, & (178)\\
\epsilon_1 &= -(4y-y^2)\mathcal{F}_1'''' - (D+3)(2-y)\mathcal{F}_1''' + (D+1)\mathcal{F}_1'' \;. & (179)
\end{aligned}
$$

Of course the spin two contribution can be reduced to the same form,

$$
\begin{aligned}
&\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x) \times \mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x') \times \Big\{ \mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big(\frac{D-2}{D-3}\Big)\mathcal{F}_2(y)\Big\} \\
&\qquad = \frac{\partial^2 y}{\partial x_\mu \partial x'_{(\rho}} \frac{\partial^2 y}{\partial x'_{\sigma)}\partial x_\nu} \times \alpha_2(y) + \frac{\partial y}{\partial x_{(\mu}} \frac{\partial^2 y}{\partial x_{\nu)}\partial x'_{(\rho}} \frac{\partial y}{\partial x'_{\sigma)}} \times \beta_2(y) \\
&\qquad\qquad + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} \times \gamma_2(y) + H^4 \overline{g}^{\mu\nu}(x)\overline{g}^{\rho\sigma}(x') \times \delta_2(y) \\
&\qquad\qquad + H^2\Big[\overline{g}^{\mu\nu}(x)\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\overline{g}^{\rho\sigma}(x')\Big] \times \epsilon_2(y) \;, \qquad (180)
\end{aligned}
$$

Determining the coefficient functions is an extremely tedious exercise that was done by computer. The results for each coefficient function are expressed as an expansion in powers of derivatives of the spin two structure function, for example,

$$
\alpha_2 = \sum_{k=0}^{4} \alpha_{2k} \frac{d^k \mathcal{F}_2}{dy^k} \;. \qquad (181)
$$

The various coefficients, which are functions of D and y, are reported in Tables 3-7.

Now recall the second order equation (136) we were able to find for the flat space structure function $F_2(\Delta x^2)$ by adding δ and $\Delta x^2 \epsilon$. After long contemplation of the bewildering data in Tables 3-7 it becomes apparent that a similar second order equation for $\mathcal{F}_2(y)$ derives from the combination,

$$
\begin{aligned}
&\delta_2(y) + (4y-y^2)\epsilon_2(y) = \Big[\delta(y) - \delta_1(y)\Big] + (4y-y^2)\Big[\epsilon(y) - \epsilon_1(y)\Big] \;, & (182)\\
&\quad = -\Big(\frac{D+1}{D-1}\Big)\Big\{(D-2)\mathcal{F}_2'' - (D-3)\Big[(4y-y^2)\mathcal{F}_2'' \\
&\qquad\qquad + 2(D+1)(2-y)\mathcal{F}_2' - D(D+1)\mathcal{F}_2\Big]\Big\} . & (183)
\end{aligned}
$$

	Coefficient of F_2
α_{20}	$-(D-3)D^2(D+1)^2\Big[-4(D-2)+(D-1)(4y-y^2)\Big]$
β_{20}	$2(D-3)(D-1)D^2(D+1)^2(2-y)$
γ_{20}	$(D-3)(D-1)D^2(D+1)^2$
δ_{20}	$4(D-3)D(D+1)^2\Big[-4(D-2)+D(4y-y^2)\Big]$
ϵ_{20}	$-4(D-3)D^2(D+1)^2$

Table 3: Coefficient of F_2: each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

	Coefficient of F_2'
α_{21}	$4(D-3)(D+1)^2(2-y)\Big[-2(D-2)D+(D-1)(D+1)(4y-y^2)\Big]$
β_{21}	$8(D-3)(D+1)^2\Big[-3D^2+(D-1)(D+1)(4y-y^2)\Big]$
γ_{21}	$-4(D-3)(D-1)(D+1)^3(2-y)$
δ_{21}	$-16(D-3)(D+1)^2(2-y)\Big[-2(D-2)+(D+1)(4y-y^2)\Big]$
ϵ_{21}	$16(D-3)(D+1)^3(2-y)$

Table 4: Coefficient of F_2': each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

Hence we can express the equation for $\mathcal{F}_2(y)$ as,

$$\mathcal{D}\mathcal{F}_2 = -\Big(\frac{D-1}{D+1}\Big)\Bigg\{\Big[\delta(y)-\delta_1(y)\Big]+(4y-y^2)\Big[\epsilon(y)-\epsilon_1(y)\Big]\Bigg\}, \tag{184}$$

where the second order operator $\mathcal{D}$ is,

$$\mathcal{D} \equiv 4(D-2)\Big(\frac{d}{dy}\Big)^2 -(D-3)\Bigg[(4y-y^2)\Big(\frac{d}{dy}\Big)^2+2(D+1)(2-y)\frac{d}{dy}-D(D+1)\Bigg], \tag{185}$$

$$= 4\Big(\frac{d}{dy}\Big)^2+(D-3)\Bigg[(2-y)^2\Big(\frac{d}{dy}\Big)^2-2(D+1)(2-y)\frac{d}{dy}+D(D+1)\Bigg]. \tag{186}$$

	Coefficient of F_2''
α_{22}	$2\Big[8(D-2)^2 D(D+1) - 4(D+1)(3D^3-8D^2-6D+12)(4y-y^2) +(D-3)(D-1)(3D^2+9D+7)(4y-y^2)^2\Big]$
β_{22}	$-4(2-y)\Big[-2D(D+1)(3D^2-5D-10) +(D-3)(D-1)(3D^2+9D+7)(4y-y^2)\Big]$
γ_{22}	$-2\Big[-12(D^4-D^3-7D^2+D+10) +(D-3)(D-1)(3D^2+9D+72)(4y-y^2)\Big]$
δ_{22}	$-8\Big[8(D-2)^2(D+1) - 2(D+1)(6D^2-11D-18)(4y-y^2) +(D-3)(3D^2+9D+7)(4y-y^2)^2\Big]$
ϵ_{22}	$8\Big[-2(D+1)(5D^2-6D-24) + (D-3)(3D^2+9D+7)(4y-y^2)\Big]$

Table 5: Coefficient of F_2'': each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

The source term on the right hand side of (184) has the form,

$$
\begin{aligned}
-\Big(\frac{D-1}{D+1}\Big)\Big\{\Big[\delta(y)-\delta_1(y)\Big] + (4y-y^2)\Big[\epsilon(y)-\epsilon_1(y)\Big]\Big\} \\
= K\Big\{s_a\Big(\frac{4}{y}\Big)^D + \frac{s_b}{D-4}\Big(\frac{4}{y}\Big)^{D-1} + \frac{s_c}{D-4}\Big(\frac{4}{y}\Big)^{D-2} + s_{c'}\Big(\frac{4}{y}\Big)^{\frac{D}{2}} \\
+\frac{s_d}{D-4}\Big(\frac{4}{y}\Big)^{D-3} + \frac{s_e}{(D-4)^2}\Big(\frac{4}{y}\Big)^{D-4} + \Big(\text{Irrelevant}\Big)\Big\} + \mathcal{R} \; , \qquad (187)
\end{aligned}
$$

where the remainder term $\mathcal{R}$ derives from the remainder $\mathcal{R}_1$ of $\mathcal{F}_1$,

$$
\begin{aligned}
\mathcal{R} = \Big(\frac{D-1}{D+1}\Big)\Big\{(D-1)(2-y)(4y-y^2)\Big(\frac{\partial}{\partial y}\Big)^3 - D(D-1)(4y-y^2)\Big(\frac{\partial}{\partial y}\Big)^2 \\
+4(D^2-3)\Big(\frac{\partial}{\partial y}\Big)^2 + (D-1)^2(2-y)\Big(\frac{\partial}{\partial y}\Big) + (D-1)^2\Big\}\Big[\frac{1}{\frac{\square}{H^2}+D}\Big]^2 \mathcal{R}_1 \; . \qquad (188)
\end{aligned}
$$

The coefficients in (187) are,

$$
s_a = -\frac{1}{16(D+1)} \; , \qquad (189)
$$

	Coefficient of F_2'''
α_{23}	$-4(D-1)(2-y)(4y-y^2)\Big[-2(D-2)(D+1) +(D-3)(D+2)(4y-y^2)\Big]$
β_{23}	$-8\Big[4(D-2)D(D+1)-(5D^3-8D^2-23D+22)(4y-y^2) +(D-3)(D-1)(D+2)(4y-y^2)^2\Big]$
γ_{23}	$4(2-y)\Big[-4(D-2)(D^2-5)+(D-3)(D-1)(D+2)(4y-y^2)\Big]$
δ_{23}	$16(2-y)(4y-y^2)\Big[-2(D-2)(D+1)+(D-3)(D+2)(4y-y^2)\Big]$
ϵ_{23}	$-16(2-y)\Big[-2(D-2)(D+1)+(D-3)(D+2)(4y-y^2)\Big]$

Table 6: Coefficient of F_2''': each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

$$s_b = -\frac{(D-2)D}{16(D-1)} , \tag{190}$$

$$s_c = -\frac{(D-4)(D-2)D(D+3)}{32(D-6)(D-1)} , \tag{191}$$

$$s_{c'} = -\frac{(D-4)(D-1)\Gamma(D)}{16(D+1)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)} , \tag{192}$$

$$s_d = -\frac{7}{5}+\frac{263}{100}(D-4)+O\Big((D-4)^2\Big) , \tag{193}$$

$$s_e = \frac{18}{5}-\frac{18}{25}(D-4)-\frac{11331}{1000}(D-4)^2+O\Big((D-4)^3\Big) . \tag{194}$$

Just as for the differential operator $(\frac{\square}{H^2}+D)$, it is straightforward to construct a Green's function to invert $\mathcal{D}$. The first step is to change variables in the second form (186),

$$w \equiv \sqrt{\frac{D-3}{4}}\,(2-y) \Longrightarrow \mathcal{D} = (D-3)\Big[(1+w^2)\Big(\frac{d}{dw}\Big)^2+2(D+1)w\frac{d}{dw}+D(D+1)\Big] . \tag{195}$$

The homogeneous equation $\mathcal{D}f(w)=0$ gives rise to a simple, 2-term recursion relation which generates even and odd solutions. These series solutions can be expressed as hypergeometric functions that reduce to elementary func-

	Coefficient of F_2''''
α_{24}	$-(D-1)(4y-y^2)^2\Big[-4(D-2)+(D-3)(4y-y^2)\Big]$
β_{24}	$2(D-1)(2-y)(4y-y^2)\Big[-4(D-2)+(D-3)(4y-y^2)\Big]$
γ_{24}	$\Big[4(D-2)-(D-3)(4y-y^2)\Big]\Big[4(D-2)-(D-1)(4y-y^2)\Big]$
δ_{24}	$4(4y-y^2)^2\Big[-4(D-2)+(D-3)(4y-y^2)\Big]$
ϵ_{24}	$-4(4y-y^2)\Big[-4(D-2)+(D-3)(4y-y^2)\Big]$

Table 7: Coefficient of F_2'''': each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

tions for $D=4$,

$$f_e(w) \;=\; {}_2F_1\Big(\frac{D}{2},\frac{D+1}{2};\frac{1}{2};w^2\Big) \longrightarrow \frac{(1-6w^2+w^4)}{(1+w^2)^4} \;, \qquad (196)$$

$$f_o(w) \;=\; w\times {}_2F_1\Big(\frac{D+1}{2},\frac{D+2}{2};\frac{3}{2};w^2\Big) \longrightarrow \frac{(w-w^3)}{(1+w^2)^4} \;. \qquad (197)$$

Because we again have both homogeneous solutions it is simple to write down a Green's function,

$$G_2(w;w') = \frac{\theta(w-w')}{D-3}\Big[f_o(w)f_e(w')-f_e(w)f_o(w')\Big](1+w'^2)^D \;. \qquad (198)$$

As was the case for it spin zero cousin (159), the spin two Green's function (198) is not simple to use for arbitrary D. We therefore adopt the same strategy we used for $\mathcal{F}_1$, of recursively extracting powers until the remainder is integrable and the $D=4$ forms can be employed. Acting $\mathcal{D}$ on a power gives,

$$\mathcal{D}\Big(\frac{4}{y}\Big)^{p-2} = \frac{1}{4}(D-2)(p-2)(p-1)\Big(\frac{4}{y}\Big)^p$$
$$+(D-3)(p-2)(D+2-p)\Big(\frac{4}{y}\Big)^{p-1}+(D-3)(D+2-p)(D+3-p)\Big(\frac{4}{y}\Big)^{p-2}. \qquad (199)$$

Hence we conclude,

$$\frac{1}{\mathcal{D}}\Big(\frac{4}{y}\Big)^p = \frac{4}{(D-2)(p-2)(p-1)}\Big(\frac{4}{y}\Big)^{p-2}$$

$$-\frac{4}{\mathcal{D}}\Biggl\{\frac{(D\!-\!3)(D\!+\!2\!-\!p)}{(D\!-\!2)(p\!-\!1)}\Bigl(\frac{4}{y}\Bigr)^{p-1}\!+\frac{(D\!-\!3)(D\!+\!2\!-\!p)(D\!+\!3\!-\!p)}{(D\!-\!2)(p\!-\!2)(p\!-\!1)}\Bigl(\frac{4}{y}\Bigr)^{p-2}\Biggr\}. \tag{200}$$

For the four powers of relevance expression (200) gives,

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{D} = \frac{4}{(D\!-\!2)^2(D\!-\!1)}\Bigl(\frac{4}{y}\Bigr)^{D-2} - \frac{1}{\mathcal{D}}\Biggl\{\frac{8(D\!-\!3)}{(D\!-\!2)(D\!-\!1)}\Bigl(\frac{4}{y}\Bigr)^{D-1} + \frac{24(D\!-\!3)}{(D\!-\!2)^2(D\!-\!1)}\Bigl(\frac{4}{y}\Bigr)^{D-2}\Biggr\}, \tag{201}$$

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{D-1} = \frac{4}{(D\!-\!3)(D\!-\!2)^2}\Bigl(\frac{4}{y}\Bigr)^{D-3} - \frac{1}{\mathcal{D}}\Biggl\{\frac{12(D\!-\!3)}{(D\!-\!2)^2}\Bigl(\frac{4}{y}\Bigr)^{D-2} + \frac{48}{(D\!-\!2)^2}\Bigl(\frac{4}{y}\Bigr)^{D-3}\Biggr\}, \tag{202}$$

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{D-2} = \frac{4}{(D\!-\!4)(D\!-\!3)(D\!-\!2)}\Bigl(\frac{4}{y}\Bigr)^{D-4} - \frac{1}{\mathcal{D}}\Biggl\{\frac{16}{(D\!-\!2)}\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{80}{(D\!-\!4)(D\!-\!2)}\Bigl(\frac{4}{y}\Bigr)^{D-4}\Biggr\}, \tag{203}$$

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}} = \frac{16}{(D\!-\!4)(D\!-\!2)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} - \frac{4}{\mathcal{D}}\Biggl\{\frac{(D\!-\!3)(D\!+\!4)}{(D\!-\!2)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{(D\!-\!3)(D\!+\!4)(D\!+\!6)}{(D\!-\!4)(D\!-\!2)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2}\Biggr\}. \tag{204}$$

These relations allow the spin two structure function to be expressed as a "quotient" and "remainder" of the form,

$$\mathcal{F}_2 = \mathcal{Q}_2(y) + \frac{1}{\mathcal{D}}\mathcal{R}_2(y) \; , \tag{205}$$

$$\mathcal{Q}_2 = -K\Biggl\{f_{2a}\Bigl(\frac{4}{y}\Bigr)^{D-2}\!+\frac{f_{2b}}{D\!-\!4}\Bigl(\frac{4}{y}\Bigr)^{D-3}\!+\frac{f_{2c}}{(D\!-\!4)^2}\Bigl(\frac{4}{y}\Bigr)^{D-4}\!+\frac{f_{2c'}}{D\!-\!4}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2}\Biggr\}, \tag{206}$$

$$\mathcal{R}_2 = -K\Biggl\{\frac{f_{2d}}{D\!-\!4}\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{f_{2e}}{(D\!-\!4)^2}\Bigl(\frac{4}{y}\Bigr)^{D-4} + \Bigl(\text{Irrelevant}\Bigr)\Biggr\} + \mathcal{R} \; , \tag{207}$$

where the coefficients are,

$$f_{2a} = \frac{1}{4(D-2)^2(D-1)(D+1)} \; , \tag{208}$$

$$f_{2b} = \frac{D^4\!-\!3D^3\!-\!8D^2\!+\!60D\!-\!96}{4(D\!-\!3)(D\!-\!2)^3(D\!-\!1)(D\!+\!1)} \; , \tag{209}$$

$$f_{2c} =$$
$$\frac{D^8-8D^7-13D^6+348D^5-1136D^4-2^{10}D^3+15056D^2-38208D+34560}{8(D-6)(D-3)(D-2)^4(D-1)(D+1)} , \tag{210}$$
$$f_{2c'} = \frac{(D-4)(D-1)\Gamma(D)}{(D-2)^2(D+1)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)} , \tag{211}$$
$$f_{2d} = \frac{17}{5} + \frac{161}{300}(D-4) + O\Bigl((D-4)^2\Bigr) ,$$
$$f_{2e} = \frac{82}{5} + \frac{243}{25}(D-4) + \frac{13343}{3000}(D-4)^2 + O\Bigl((D-4)^3\Bigr) . \tag{212}$$

4.4 Renormalizing the Spin Zero Structure Function

Recall the form (166) we obtained for the spin zero struncture function from taking the trace of the graviton self-energy,

$$\mathcal{F}_1(y) = \mathcal{Q}_1(y) + \Bigl[\frac{1}{\frac{\Box}{H^2}+D}\Bigr]^2 \mathcal{R}_1(y) . \tag{213}$$

Recall also that the quotient $\mathcal{Q}_1(y)$ and the remainder $\mathcal{R}_1(y)$ are given in relations (167-173). From these expressions we perceive three sorts of ultraviolet divergences:

- The factor of $(\frac{4}{y})^{D-2}$ in $\mathcal{Q}_1$, which has a finite coefficient but is still not integrable in $D = 4$ dimensions;
- The factors of $\frac{1}{D-4}(\frac{4}{y})^{D-3}$ in $\mathcal{Q}_1$ and $\mathcal{R}_1$ which are integrable in $D = 4$ dimensions but have divergent coefficients that preclude taking the unregulated limits; and
- The factors of $(\frac{1}{D-4})^2(\frac{4}{y})^{D-4}$ in $\mathcal{Q}_1$ and $\mathcal{R}_1$ which are integrable in $D = 4$ dimensions but have even more divergent coefficients.

In this subsection we will explain how to localize all three divergences onto delta function terms which can be absorbed by the counterterms (113), (115) and (116). We will also take the unregulated limits of the remaining, finite parts, and use the $D = 4$ Green's function (159) to obtain an explicit result for the renormalized structure function.

In dealing with the factor of $(\frac{4}{y})^{D-2}$ in $\mathcal{Q}_1$, the first step is to extract a d'Alembertian,

$$\Bigl(\frac{4}{y}\Bigr)^{D-2} = \frac{2}{(D-4)(D-3)} \Biggl[\frac{\Box}{H^2}\Bigl(\frac{4}{y}\Bigr)^{D-3} - 2(D-3)\Bigl(\frac{4}{y}\Bigr)^{D-3}\Biggr] . \tag{214}$$

The resulting factors of $(\frac{4}{y})^{D-3}$ are integrable in $D=4$ dimensions, at which point we could take the unregulated limit except for the factor of $1/(D-4)$ in (214). We can localize the divergence on a delta function by adding zero in the form of the identity (96),

$$\Bigl(\frac{4}{y}\Bigr)^{D-2} = \frac{2}{(D-4)(D-3)}\Biggl\{\frac{\square}{H^2}\Biggl[\Bigl(\frac{4}{y}\Bigr)^{D-3} - \Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1}\Biggr] -2(D-3)\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{D}{2}\Bigl(\frac{D}{2}-1\Bigr)\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2}-1)}\frac{i\delta^D(x-x')}{H^D\sqrt{-g}}\Biggr\} , \qquad (215)$$

$$= -\Bigl[\frac{\square}{H^2}-2\Bigr]\Bigl\{\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)\Bigr\} - \frac{4}{y} + O(D-4) + \frac{2(4\pi)^{\frac{D}{2}} i\delta^D(x-x')/\sqrt{-g}}{(D-4)(D-3)\Gamma(\frac{D}{2}-1)H^D} . \qquad (216)$$

We turn now to the factors of $\frac{1}{D-4}(\frac{4}{y})^{D-3}$ and $(\frac{1}{D-4})^2(\frac{4}{y})^{D-4}$ in $\mathcal{Q}_1$ and $\mathcal{R}_1$. The key relations for resolving these terms follow from (96),

$$\Bigl[\frac{\square}{H^2}+D\Bigr]^2\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} = \frac{1}{16}D^2(D+2)^2\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2}-1)H^D\sqrt{-g}}\Bigl[\frac{\square}{H^2}+D+\frac{1}{4}D(D+2)\Bigr]i\delta^D(x-x') , \qquad (217)$$

$$\Bigl[\frac{\square}{H^2}+D\Bigr]^2\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} = -\frac{1}{4}(D-4)(D^2+2D-4)\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{1}{16}(D-2)^2(D+4)^2\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} - \frac{(D-4)(4\pi)^{\frac{D}{2}}}{2\Gamma(\frac{D}{2}-1)H^D\sqrt{-g}}i\delta^D(x-x') , \qquad (218)$$

$$\Bigl[\frac{\square}{H^2}+D\Bigr]^2 1 = D^2 . \qquad (219)$$

One add zero using these relations so as to resolve the problematic terms in $\mathcal{Q}_1$, and the remainder automatically resolves the problematic terms in $\mathcal{R}_1$,

$$\frac{f_{1b}}{D-4}\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{f_{1c}}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{D-4} + \Biggl[\frac{1}{\frac{\square}{H^2}+D}\Biggr]^2\Biggl\{\frac{f_{1d}}{D-4}\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{f_{1e}}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{D-4}\Biggr\}$$

$$= \frac{f_{1b}}{D-4}\Biggl\{\Bigl(\frac{4}{y}\Bigr)^{D-3} - \Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1}\Biggr\} + \frac{f_{1c}}{(D-4)^2}\Biggl\{\Bigl(\frac{4}{y}\Bigr)^{D-4} - 2\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} + 1\Biggr\} + \Biggl[\frac{1}{\frac{\square}{H^2}+D}\Biggr]^2\Biggl\{\frac{f_{1d}}{D-4}\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{[D^2(D+2)^2 f_{1b} - 8(D^2+2D-4)f_{1c}]}{16(D-4)}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1}$$

$$+\frac{f_{1e}}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{D-4} + \frac{(D-2)^2(D+4)^2 f_{1c}}{8(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} - \frac{D^2 f_{1c}}{(D-4)^2}$$

$$+\frac{(4\pi)^{\frac{D}{2}}/\sqrt{-\overline{g}}}{\Gamma(\frac{D}{2}-1)H^D}\Biggl[\frac{f_{1b}}{D-4}\Bigl[\frac{\square}{H^2}+D\Bigr] + \frac{D(D+2)f_{1b}-4f_{1c}}{4(D-4)}\Biggr] i\delta^D(x-x')\Biggr\}, \quad (220)$$

$$= \frac{1}{18}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr) - \frac{1}{6}\times\ln^2\Bigl(\frac{y}{4}\Bigr) + O(D-4) + \Biggl[\frac{1}{\frac{\square}{H^2}+4}\Biggr]^2\Biggl\{\frac{4}{3}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)$$

$$+\frac{8}{3}\times\frac{4}{y} + \frac{8}{3}\ln^2\Bigl(\frac{y}{4}\Bigr) - 8\ln\Bigl(\frac{y}{4}\Bigr) + \frac{1}{3}\Biggr\} + \Biggl[\frac{1}{\frac{\square}{H^2}+D}\Biggr]^2\Biggl\{\frac{(4\pi)^{\frac{D}{2}}/\sqrt{-\overline{g}}}{\Gamma(\frac{D}{2}-1)H^D}$$

$$\times\Biggl[\frac{f_{1b}}{D-4}\Bigl[\frac{\square}{H^2}+D\Bigr] + \frac{D(D+2)f_{1b}-4f_{1c}}{4(D-4)}\Biggr] i\delta^D(x-x')\Biggr\}. (221)$$

Employing expressions (216) and (221) in (166) allows us to separate the spin zero structure function into a finite part and a divergent part,

$$\mathcal{F}_1 = \mathcal{F}_{1R} + O(D-4) + \Delta\mathcal{F}_1 \; . \quad (222)$$

The finite part consists of the renormalized spin zero structure function,

$$\mathcal{F}_{1R} = \frac{\kappa^2 H^4}{(4\pi)^4}\Biggl\{\frac{\square}{H^2}\Biggl[\frac{1}{72}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)\Biggr] - \frac{1}{12}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr) + \frac{1}{72}\times\frac{4}{y} + \frac{1}{6}\ln^2\Bigl(\frac{y}{4}\Bigr)\Biggr\}$$

$$+\frac{\kappa^2 H^4}{(4\pi)^4}\Biggl[\frac{1}{\frac{\square}{H^2}+4}\Biggr]^2\Biggl\{-\frac{4}{3}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr) - \frac{8}{3}\times\frac{4}{y} - \frac{8}{3}\ln^2\Bigl(\frac{y}{4}\Bigr) + 8\ln\Bigl(\frac{y}{4}\Bigr) - \frac{1}{3}\Biggr\}. (223)$$

The divergent part consists of $[\frac{\square}{H^2}+D]^{-2}$ acting on a sum of three local terms,

$$\Delta\mathcal{F}_1 = \frac{\kappa^2 H^{D-4}(\frac{D}{2}-1)\Gamma(\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\Biggl[\frac{1}{\frac{\square}{H^2}+D}\Biggr]^2\Biggl\{\frac{-2f_{1a}}{(D-4)(D-3)}\Biggl[\frac{\square}{H^2}+D\Biggr]^2\frac{i\delta^D(x-x')}{\sqrt{-\overline{g}}}$$

$$-\frac{f_{1b}}{D-4}\Biggl[\frac{\square}{H^2}+D\Biggr]\frac{i\delta^D(x-x')}{\sqrt{-\overline{g}}} - \Biggl[\frac{D(D+2)f_{1b}-4f_{1c}}{4(D-4)}\Biggr]\frac{i\delta^D(x-x')}{\sqrt{-\overline{g}}}\Biggr\}. (224)$$

Of course one cancels $\Delta\mathcal{F}_1$ with counterterms. From expressions (113-116) we wee that the four counterterms contribute to the graviton self-energy as,

$$-i\Bigl[{}^{\mu\nu}\Delta\Sigma^{\rho\sigma}\Bigr](x;x') = \sqrt{-\overline{g}}\Biggl[2c_1\kappa^2\mathcal{P}^{\mu\nu}\mathcal{P}^{\rho\sigma} + 2c_2\kappa^2\overline{g}^{\alpha\kappa}\overline{g}^{\beta\lambda}\overline{g}^{\gamma\theta}\overline{g}^{\delta\phi}\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}$$

$$-c_3\kappa^2 H^2\mathcal{D}^{\mu\nu\rho\sigma} + c_4\kappa^2 H^4\sqrt{-\overline{g}}\Bigl[\frac{1}{4}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma} - \frac{1}{2}\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\Bigr]\Biggr] i\delta^D(x-x') \; . (225)$$

Tracing as we did in (151) gives,

$$\frac{\overline{g}_{\mu\nu}(x)}{\sqrt{-\overline{g}(x)}} \times \frac{\overline{g}_{\rho\sigma}(x')}{\sqrt{-\overline{g}(x')}} \times -i\Big[{}^{\mu\nu}\Delta\Sigma^{\rho\sigma}\Big](x;x') = (D-1)^2 H^4 \Bigg[2c_1\kappa^2\Big[\frac{\square}{H^2} + D\Big]^2$$
$$+0 - \frac{1}{2}\Big(\frac{D-2}{D-1}\Big)c_3\kappa^2\Big[\frac{\square}{H^2} + D\Big] + \frac{D(D-2)}{4(D-1)^2}c_4\kappa^2\Bigg]\frac{i\delta^D(x-x')}{\sqrt{-\overline{g}}} . \quad (226)$$

We can entirely absorb $\Delta\mathcal{F}_1$ by making the choices,

$$c_1 = \frac{H^{D-4}(\frac{D}{2}-1)\Gamma(\frac{D}{2})}{(\pi)^{\frac{D}{2}}} \times \frac{f_{1a}}{(D-4)(D-3)}$$
$$= \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times \frac{(D-2)}{(D-4)(D-3)(D-1)^2} , \quad (227)$$
$$c_3 = \frac{H^{D-4}(\frac{D}{2}-1)\Gamma(\frac{D}{2})}{(\pi)^{\frac{D}{2}}} \times -2\Big(\frac{D-1}{D-2}\Big) \times \frac{f_{1b}}{D-4}$$
$$= \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times -\frac{2D(D^2-5D+2)}{(D-4)(D-3)(D-1)} , \quad (228)$$
$$c_4 = \frac{H^{D-4}(\frac{D}{2}-1)\Gamma(\frac{D}{2})}{(\pi)^{\frac{D}{2}}} \times \frac{4(D-1)^2}{D(D-2)} \times \Bigg[\frac{D(D+2)f_{1b} - 4f_{1c}}{4(D-4)}\Bigg]$$
$$= \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times -\frac{D(D^3-11D^2+24D+12)}{(D-6)(D-3)(D-2)} . \quad (229)$$

The linear combinations (104) and (105) are finite,

$$-2(D-1)Dc_1 + c_3 = \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times \frac{-2D^2}{(D-3)(D-1)} , \quad (230)$$
$$(D-1)^2D^2c_1 - (D-2)(D-1)c_3 + c_4$$
$$= \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times \frac{D(D^3-6D^2+8D-24)}{(D-6)(D-3)} . \quad (231)$$

Therefore neither the Newton constant nor the cosmological constant requires a divergent renormalization, although we are free to continue making the finite renormalizations of these constants which are implied by equations (227-229).

It remains to act the $D=4$ Green's function (159) twice on the renormalized remainder term in expression (223). The result is,

$$\left[\frac{1}{\frac{\square}{H^2}+4}\right]^2\left\{-\frac{4}{3}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)-\frac{8}{3}\times\frac{4}{y}-\frac{8}{3}\ln^2\Bigl(\frac{y}{4}\Bigr)+8\ln\Bigl(\frac{y}{4}\Bigr)-\frac{1}{3}\right\}$$
$$=-\frac{1}{3}\times\frac{y}{4}\ln^2\Bigl(\frac{y}{4}\Bigr)+\frac{1}{3}\times\frac{y}{4}\ln\Bigl(\frac{y}{4}\Bigr)$$
$$-\frac{7}{540}(12\pi^2+265)\times\frac{y}{4}+\frac{84\pi^2-131}{1080}$$
$$+\frac{1}{9}\times\frac{y}{4}\ln(\frac{y}{4})-\frac{1}{45}\ln(\frac{y}{4})+\frac{1}{45}\times\frac{4}{4-y}\ln(\frac{y}{4})$$
$$-\frac{1}{30}(2-y)\Bigl[7\mathrm{Li}_2(1-\frac{y}{4})-2\mathrm{Li}_2(\frac{y}{4})+5\ln(1-\frac{y}{4})\ln(\frac{y}{4})\Bigr]$$
$$+\frac{43}{216}\times\frac{4}{4-y}-\frac{5}{6}\times\frac{y}{4}\ln(1-\frac{y}{4})-\frac{1}{20}\ln(1-\frac{y}{4})+\frac{7}{90}\times\frac{4}{y}\ln(1-\frac{y}{4})\ . \tag{232}$$

Here $\mathrm{Li}_2(z)$ is the dilogarithm function,

$$\mathrm{Li}_2(z)\equiv-\int_0^z dt\,\frac{\ln(1-t)}{t}=\sum_{k=1}^{\infty}\frac{z^k}{k^2}\ . \tag{233}$$

Hence our final result for the renormalized spin zero structure function is,

$$\mathcal{F}_{1R}=\frac{\kappa^2H^4}{(4\pi)^4}\left\{\frac{\square}{H^2}\left[\frac{1}{72}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)\right]-\frac{1}{12}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)+\frac{1}{72}\times\frac{4}{y}+\frac{1}{6}\ln^2\Bigl(\frac{y}{4}\Bigr)\right.$$
$$+\frac{1}{45}\times\frac{4}{4-y}\ln(\frac{y}{4})-\frac{1}{45}\ln(\frac{y}{4})+\frac{43}{216}\times\frac{4}{4-y}-\frac{5}{6}\times\frac{y}{4}\ln(1-\frac{y}{4})$$
$$+\frac{7}{90}\times\frac{4}{y}\ln(1-\frac{y}{4})-\frac{1}{20}\ln(1-\frac{y}{4})-\frac{7(12\pi^2+265)}{540}\times\frac{y}{4}$$
$$+\frac{84\pi^2-131}{1080}-\frac{1}{3}\times\frac{y}{4}\ln^2\Bigl(\frac{y}{4}\Bigr)+\frac{4}{9}\times\frac{y}{4}\ln\Bigl(\frac{y}{4}\Bigr)$$
$$\left.-\frac{1}{30}(2-y)\Bigl[7\mathrm{Li}_2(1-\frac{y}{4})-2\mathrm{Li}_2(\frac{y}{4})+5\ln(1-\frac{y}{4})\ln(\frac{y}{4})\Bigr]\right\}\ . \tag{234}$$

4.5 Renormalizing the Spin Two Structure Function

Recall the form (205) we obtained for the spin two structure function,

$$\mathcal{F}_2(y)=\mathcal{Q}_2(y)+\frac{1}{\mathcal{D}}\mathcal{R}_2(y)\ , \tag{235}$$

where the second order differential operator $\mathcal{D}$ was defined in (186). Recall also that the quotient $\mathcal{Q}_2(y)$ and the remainder $\mathcal{R}_2(y)$ are given in relations (206-212). These expression imply that $\mathcal{F}_2$ harbors the same sort of ultraviolet divergences as $\mathcal{F}_1$:

- The factor of $(\frac{4}{y})^{D-2}$ in $\mathcal{Q}_2$, which has a finite coefficient but is still not integrable in $D=4$ dimensions;
- The factors of $\frac{1}{D-4}(\frac{4}{y})^{D-3}$ in $\mathcal{Q}_2$ and $\mathcal{R}_2$ which are integrable in $D=4$ dimensions but have divergent coefficients that preclude taking the unregulated limits; and
- The factors of $(\frac{1}{D-4})^2(\frac{4}{y})^{D-4}$ in $\mathcal{Q}_2$ and $\mathcal{R}_2$ which are integrable in $D=4$ dimensions but have even more divergent coefficients.

Only the leading divergence requires a new counterterm. It is handled by first extracting another derivative and then adding zero in the form (96), just as we did in equations (214) and (216). The final result is,

$$-K f_{2a}\Bigl(\frac{4}{y}\Bigr)^{D-2} = \frac{\kappa^2 H^4}{(4\pi)^4}\Biggl\{\frac{\square}{H^2}\Biggl[\frac{1}{240}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)\Biggr]-\frac{1}{120}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)+\frac{1}{240}\times\frac{4}{y}\Biggr\}$$
$$+O(D-4)-\frac{\kappa^2 H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}}\times\frac{4i\delta^D(x-x')/\sqrt{-g}}{(D-4)(D-3)(D-2)(D-1)(D+1)} \; . \tag{236}$$

Comparing expressions (114) and (146) implies that the divergent part can be entirely absorbed by choosing the coefficient c_2 of the Weyl counterterm (101) to be,

$$c_2 = \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}}\times\frac{2}{(D-4)(D-3)^2(D-1)(D+1)} \; . \tag{237}$$

Of course the divergent part agrees with [20].

It turns out that the lower divergences of $\mathcal{F}_2$ are canceled by the three factors we added to $\mathcal{Q}_1$ to cancel its lower divergences,

$$\delta\mathcal{Q}_1 = K\Biggl\{\frac{f_{1b}}{D-4}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1}+\frac{2f_{1c}}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2}-\frac{f_{1c}}{(D-4)^2}\Biggr\}. \tag{238}$$

These changes in $\mathcal{Q}_1$ induce changes in the source term upon which we act $\mathcal{D}^{-1}$ to get $\mathcal{F}_2$,

$$\delta S \equiv \Bigl(\frac{D-1}{D+1}\Bigr)\Bigl\{(D-1)(2-y)(4y-y^2)\delta\mathcal{Q}_1''' - D(D-1)(4y-y^2)\delta\mathcal{Q}_1'' + 4(D^2-3)\delta\mathcal{Q}_1'' + (D-1)^2(2-y)\delta\mathcal{Q}_1' + (D-1)^2\delta\mathcal{Q}_1\Bigr\} , \tag{239}$$

$$= K\Bigl\{\frac{\delta s_b}{D-4}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}+1} + \frac{\delta s_c}{D-4}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}} + \frac{\delta s_d}{D-4}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{\delta s_e}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} + \frac{\delta s_{e'}}{(D-4)^2}\Bigr\}. \tag{240}$$

Here the coefficients are,

$$\delta s_b = -\frac{1}{16}(D-2)(D-1)Df_{1b} , \tag{241}$$

$$\delta s_c = \frac{(D-2)(D-1)}{16(D+1)}\Bigl[-(D-1)(D^2-2D-4)f_{1b} + 2(D-3)f_{1c}\Bigr] , \tag{242}$$

$$\delta s_d = \frac{(D-1)^2}{8(D+1)}\Bigl[D^3 f_{1b} - (D^2+2D-4)f_{1c}\Bigr] , \tag{243}$$

$$\delta s_e = \frac{(D-2)^2(D-1)^2(D+2)}{4(D+1)} f_{1c} , \tag{244}$$

$$\delta s_{e'} = -\frac{(D-1)^3}{(D+1)} f_{1c} . \tag{245}$$

To infer the corresponding changes in the spin two quotient and remainder we need to invert $\mathcal{D}$ on $(\frac{4}{y})^{\frac{D}{2}+1}$, $(\frac{4}{y})^{\frac{D}{2}}$ and 1. The second one was given in (204). From expression (200) we find,

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}+1} = \frac{16}{(D-2)^2 D}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} - \frac{4}{\mathcal{D}}\Bigl\{\frac{(D-3)(D+2)}{(D-2)D}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}} + \frac{(D-3)(D+2)(D+4)}{(D-2)^2 D}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1}\Bigr\}, \tag{246}$$

$$\frac{1}{\mathcal{D}}\Bigl(1\Bigr) = \frac{1}{(D-3)D(D+1)} . \tag{247}$$

Although we want to move all the $(\frac{4}{y})^{\frac{D}{2}+1}$ and $(\frac{4}{y})^{\frac{D}{2}}$ terms from the remainder to the quotient, we must allow for an arbitrary amount $\delta f_{2c'}$ of the 1 term.

Hence the changes in the quotient and the remainder take the form,

$$\delta\mathcal{Q}_2 = K\Big\{\frac{\delta f_{2b}}{D-4}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \frac{\delta f_{2c}}{(D-4)^2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} + \frac{\delta f_{2c'}}{(D-4)^2}\Big\}, \tag{248}$$

$$\delta\mathcal{R}_2 = K\Big\{\frac{\delta f_{2d}}{D-4}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \frac{\delta f_{2e}}{(D-4)^2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} + \frac{\delta f_{e'}}{(D-4)^2}\Big\}. \tag{249}$$

The various coefficients are,

$$\delta f_{2b} = \frac{16}{(D-2)^2 D} \times \delta s_b \,, \tag{250}$$

$$\delta f_{2c} = -\frac{64(D-3)(D+2)}{(D-2)^3 D} \times \delta s_b + \frac{16}{(D-2)^2} \times \delta s_c \,, \tag{251}$$

$$\delta f_{2d} = \frac{4(D-3)(D+2)(D+4)(3D-10)}{(D-2)^3 D} \times \delta s_b - \frac{4(D-3)(D+4)}{(D-2)^2} \times \delta s_c + \delta s_d \,, \tag{252}$$

$$\delta f_{2e} = \frac{16(D-3)^2(D+2)(D+4)(D+6)}{(D-2)^3 D} \times \delta s_b - \frac{4(D-3)(D+4)(D+6)}{(D-2)^2} \times \delta s_c + \delta s_e \,, \tag{253}$$

$$\delta f_{2e'} = -(D-3)D(D+1)\delta f_{2c'} + \delta s_{e'} \,. \tag{254}$$

It is possible to make the combination $\mathcal{Q}_2 + \delta\mathcal{Q}_2$ possess a finite unregulated limit by choosing,

$$\delta f_{2c'} = 1 - \frac{271}{60}(D-4) + \frac{11057}{3600}(D-4)^2 \,. \tag{255}$$

With this choice the renormalized spin two quotient is,

$$\mathcal{Q}_{2R} = \frac{\kappa^2 H^4}{(4\pi)^4}\Big\{\frac{\square}{H^2}\Big[\frac{1}{240}\times\frac{4}{y}\ln\Big(\Big(\frac{y}{4}\Big)\Big] - \frac{1}{120}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big) + \frac{1}{240}\times\frac{4}{y} + \frac{1}{12}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big) - \frac{7}{30}\times\frac{4}{y} + \frac{1}{4}\ln^2\Big(\frac{y}{4}\Big) - \frac{119}{60}\ln\Big(\frac{y}{4}\Big)\Big\}. \tag{256}$$

Choosing (255) also produces a finite result for the spin two remainder term,

$$\mathcal{R}_{2R} = \frac{\kappa^2 H^4}{(4\pi)^4}\Big\{\frac{17}{10}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big) - \frac{149}{30}\times\frac{4}{y} - \frac{41}{10}\ln^2\Big(\frac{y}{4}\Big) + \frac{193}{6}\ln\Big(\frac{y}{4}\Big) + \frac{359}{20}$$

$$+\frac{32}{15(4-y)^3}\Big[90\Big(\frac{y}{4}\Big)^4-291\Big(\frac{y}{4}\Big)^3+333\Big(\frac{y}{4}\Big)^2-152\Big(\frac{y}{4}\Big)$$
$$+21\Big]\Big(\frac{4}{y}\Big)\ln(\frac{y}{4})+\frac{4}{45(4-y)^3}\Big[432\Big(\frac{y}{4}\Big)^3-792\Big(\frac{y}{4}\Big)^2-288\Big(\frac{y}{4}\Big)$$
$$+991-474\Big(\frac{4}{y}\Big)-84\Big(\frac{4}{y}\Big)^2\Big]-\frac{7}{60}(\frac{4}{y})^3\ln(1-\frac{y}{4})-\frac{9}{10}\ln^2(\frac{y}{4})\Big\}\;. \tag{257}$$

Acting the $D=4$ Green's function (198) on the remainder and adding the result to the quotient gives our final result for the renormalized spin two structure function (recall the definition (233) of the dilogarithm function),

$$\mathcal{F}_{2R}=\frac{\kappa^2H^4}{(4\pi)^4}\Big\{\frac{\square}{H^2}\Big[\frac{1}{240}\times\frac{4}{y}\ln(\Big(\frac{y}{4}\Big)\Big]+\frac{3}{40}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big)-\frac{11}{48}\times\frac{4}{y}+\frac{1}{4}\ln^2\Big(\frac{y}{4}\Big)$$
$$-\frac{119}{60}\ln\Big(\frac{y}{4}\Big)+\frac{4096}{(4y-y^2-8)^4}\Big[\Big[-\frac{47}{15}\Big(\frac{y}{4}\Big)^8+\frac{141}{10}\Big(\frac{y}{4}\Big)^7$$
$$-\frac{2471}{90}\Big(\frac{y}{4}\Big)^6+\frac{34523}{720}\Big(\frac{y}{4}\Big)^5-\frac{132749}{1440}\Big(\frac{y}{4}\Big)^4+\frac{38927}{320}\Big(\frac{y}{4}\Big)^3$$
$$-\frac{10607}{120}\Big(\frac{y}{4}\Big)^2+\frac{22399}{720}\Big(\frac{y}{4}\Big)-\frac{3779}{960}\Big]\frac{4}{4-y}+\Big[\frac{193}{30}\Big(\frac{y}{4}\Big)^4-\frac{131}{10}\Big(\frac{y}{4}\Big)^3$$
$$+\frac{7}{20}\Big(\frac{y}{4}\Big)^2+\frac{379}{60}\Big(\frac{y}{4}\Big)-\frac{193}{120}\Big]\ln(2-\frac{y}{2})+\Big[-\frac{14}{15}\Big(\frac{y}{4}\Big)^5-\frac{1}{5}\Big(\frac{y}{4}\Big)^4$$
$$+\frac{19}{2}\Big(\frac{y}{4}\Big)^3-\frac{889}{60}\Big(\frac{y}{4}\Big)^2+\frac{143}{20}\Big(\frac{y}{4}\Big)-\frac{13}{20}-\frac{7}{60}\Big(\frac{4}{y}\Big)\Big]\ln(1-\frac{y}{4})$$
$$+\Big[-\frac{476}{15}\Big(\frac{y}{4}\Big)^9+160\Big(\frac{y}{4}\Big)^8-\frac{5812}{15}\Big(\frac{y}{4}\Big)^7+\frac{8794}{15}\Big(\frac{y}{4}\Big)^6$$
$$-\frac{18271}{30}\Big(\frac{y}{4}\Big)^5+\frac{54499}{120}\Big(\frac{y}{4}\Big)^4-\frac{59219}{240}\Big(\frac{y}{4}\Big)^3+\frac{1917}{20}\Big(\frac{y}{4}\Big)^2$$
$$-\frac{1951}{80}\Big(\frac{y}{4}\Big)+\frac{367}{120}\Big]\frac{4}{4-y}\ln(\frac{y}{4})+\Big[4\Big(\frac{y}{4}\Big)^7-12\Big(\frac{y}{4}\Big)^6+20\Big(\frac{y}{4}\Big)^5$$
$$-20\Big(\frac{y}{4}\Big)^4+15\Big(\frac{y}{4}\Big)^3-7\Big(\frac{y}{4}\Big)^2+\Big(\frac{y}{4}\Big)\Big]\frac{4-y}{4}\ln^2(\frac{y}{4})$$
$$+\Big[\frac{367}{30}\Big(\frac{y}{4}\Big)^4-\frac{4121}{120}\Big(\frac{y}{4}\Big)^3+\frac{237}{16}\Big(\frac{y}{4}\Big)^2+\frac{1751}{240}\Big(\frac{y}{4}\Big)-\frac{367}{120}\Big]\ln(\frac{y}{2})$$
$$+\frac{1}{64}(y^2-8)\Big[4(2-y)-(4y-y^2)\Big]\Big[\frac{1}{5}\mathrm{Li}_2(1-\frac{y}{4})+\frac{7}{10}\mathrm{Li}_2(\frac{y}{4})\Big]\Big]\Big\}\;. \tag{258}$$

5 Discussion

We have derived two forms for the one loop contribution to the graviton self-energy from a massless, minimally coupled scalar on de Sitter background. The first form (47) is fully dimensionally regulated, with the ultraviolate divergences neither localized nor subtracted off with counterterms. This version of the result agrees with the stress tensor correlator recently computed by Perez-Nadal, Roura and Verdaguer [28]. Our second form is fully renormalized, with the unregulated limit taken,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x') = \sqrt{-\overline{g}(x)}\,\mathcal{P}^{\mu\nu}(x)\sqrt{-\overline{g}(x')}\,\mathcal{P}^{\rho\sigma}(x')\Big[\mathcal{F}_{1R}(y)\Big] \\ +\sqrt{-\overline{g}(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\sqrt{-\overline{g}(x')}\mathcal{P}^{\rho\sigma}_{\alpha\beta\gamma\delta}(x')\Big[\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\mathcal{F}_{2R}(y)\Big]\ . \quad (259)$$

In this expression the spin zero operator $\mathcal{P}^{\mu\nu}$ was defined in (108), the spin two operator $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}$ was defined in (109), and the bitensor $\mathcal{T}^{\alpha\kappa}$ was given in (147). Our results for the renormalized spin zero and spin two structure functions are expressions (234) and (258), respectively.

Our final form (259) is manifestly transverse, as required by gauge invariance. It is also de Sitter invariant, despite the fact that the massless, minimally coupled propagator breaks de Sitter invariance [29], because the de Sitter breaking term drops out of mixed second derivatives (35). Our result agrees with the flat space limit [27]. And the divergent parts of the counterterms we used to subtract off the divergences agree with those found long ago by 't Hooft and Veltman [20]. We actually included finite renormalizations of Newton's constant and of the cosmological constant. Such renormalizations are presumably necessary when considering the effective field equations of quantum gravity if the parameters Λ and G are to have their correct physical meanings.

The point of this exercise has been to quantum correct the linearized Einstein equation,

$$\sqrt{-\overline{g}}\,\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}(x) - \int d^4x'\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x')h_{\rho\sigma}(x') = \frac{1}{2}\kappa\sqrt{-\overline{g}}\,T^{\mu\nu}_{\rm lin}(x)\ , \quad (260)$$

where $\mathcal{D}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator (71) specialized to de Sitter background. In a future paper we will employ this effective field equation to work out the one loop quantum corrections to mode functions for dynamical gravitons and to the gravitational response to a stationary point mass. It is worthwhile closing this paper with an adumbration of the procedure and some of the physical considerations.

Our first comment is that one must use the Schwinger-Keldysh formalism [37, 38] to correctly describe the quantum response from a prepared initial state. This amounts to replacing the in-out self-energy in (260) by the sum of two of the four Schwinger-Keldysh self-energies,

$$\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x') \longrightarrow \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big]_{++}(x;x') + \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big]_{+-}(x;x') \; . \tag{261}$$

At the one loop order we are working $[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}]_{++}(x;x')$ agrees exactly with the in-out result (259) we have derived. To get $[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}]_{+-}(x;x')$, at this order, one simply adds a minus sign and replaces the de Sitter length function $y(x;x')$ everywhere with,

$$y(x;x') \longrightarrow y_{+-}(x;x') \equiv H^2 a(\eta) a(\eta') \Big[\Vert \vec{x} - \vec{x}' \Vert^2 - (\eta - \eta' + i\epsilon)^2 \Big] \; . \tag{262}$$

It will be seen that the $++$ and $+-$ self-energies cancel unless the point x'^μ is on or inside the past light-cone of x^μ. That makes the effective field equation (260) causal. When x'^μ is on or inside the past light-cone of x^μ the $+-$ self-energy is the complex conjugate of the $++$ one, which makes the effective field equation (260) real.

Our second comment concerns the various derivative operators in expression (259). Because the second order operators $\mathcal{P}^{\mu\nu}(x)$ and $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)$ act on x^μ, they can be pulled outside of the integration over x'^μ. The same is true for the covariant scalar d'Alembertian acting on the most singular terms of the two structure functions. The second order operators $\mathcal{P}^{\rho\sigma}(x')$ and $\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')$ can be partially integrated to act on the graviton field $h_{\rho\sigma}(x')$. This will give no spatial surface terms because the two self-energies cancel for x'^μ outside the past light-cone of x^μ. Nor will there be any temporal surface terms at the upper limit, because the integrand vanishes like $(\eta - \eta')^3$. There *will* be temporal surface terms at the lower limit. We conjecture that these are all absorbed by perturbative corrections to the initial state [35].

Our third comment is that, because we only know the self-energy at order κ^2, all we can do is to solve (260) perturbatively by expanding the graviton field and the self-energy in powers of κ^2,

$$h_{\mu\nu}(x) = h^{(0)}_{\mu\nu}(x) + \kappa^2 h^{(1)}_{\mu\nu}(x) + O(\kappa^4) \; , \tag{263}$$

$$\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x') = \kappa^2 \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{1}\Big](x;x') + O(\kappa^4) \; . \tag{264}$$

Of course $h^{(0)}_{\mu\nu}(x)$ obeys the classical, linearized Einstein equation. Given this solution, the corresponding one loop correction is defined by the equation,

$$\sqrt{-\overline{g}(x)}\,\mathcal{D}^{\mu\nu\rho\sigma} h^{(1)}_{\rho\sigma}(x) = \int d^4x' \left[{}^{\mu\nu}\Sigma^{\rho\sigma}_1\right](x;x') h^{(0)}_{\rho\sigma}(x') \; . \tag{265}$$

We are interested in the one loop corrections to two sorts of classical solutions. The first is a dynamical graviton of wave vector $\vec{k}$. The classical solution for this takes the form [39],

$$h^{(0)}_{\rho\sigma}(x) = \epsilon_{\rho\sigma}(\vec{k}) u(\eta,k) e^{i\vec{k}\cdot\vec{x}} \; , \tag{266}$$

where the tree order mode function is,

$$u(\eta,k) = \frac{H}{\sqrt{2k^3}} \Bigl[1 - \frac{ik}{Ha}\Bigr] \exp\Bigl[\frac{ik}{Ha}\Bigr] \; , \tag{267}$$

and the polarization tensor obeys all the same relations as in flat space,

$$0 = \epsilon_{0\mu} = k_i \epsilon_{ij} = \epsilon_{jj} \quad \text{and} \quad \epsilon_{ij}\epsilon^*_{ij} = 1 \; . \tag{268}$$

The second classical solution we wish to correct is the linearized response to a stationary point mass M [39],

$$h^{(0)}_{00}(x) = a^2 \times \frac{2GM}{a\|\vec{x}\|} \; , \; h^{(0)}_{0i}(x) = 0 \; , \; h^{(0)}_{ij}(x) = a^2 \times \frac{2GM}{a\|\vec{x}\|} \times \delta_{ij} \; . \tag{269}$$

The one loop corrections we seek to compute represent the response (of either dynamical gravitons or the force of gravity) to the vast ensemble of infrared scalars which are produced by inflation. It is simple to show that the occupation number for *each mode* with wave number $\vec{k}$ grows like [8],

$$N(k,\eta) = \Bigl(\frac{Ha(\eta)}{2k}\Bigr)^2 \tag{270}$$

This growth is balanced by expansion of the 3-volume so that the number density of infrared particles with $0 < k < Ha$ remains fixed,

$$n(\eta) = \int \frac{d^3k}{(2\pi a)^3}\, \theta(Ha - k) N(k,\eta) = \frac{H^3}{8\pi^2} \; . \tag{271}$$

The constant density of virtual scalars in flat space background has no effect at all on dynamical gravitons (after field strength renormalization) [20], so we expect that dynamical gravitons on de Sitter will likewise suffer no important quantum corrections. The virtual scalars of flat space do induce a correction to the classical potential [1, 2] and we expect one as well on de Sitter background. On dimensional grounds the flat space result must (and does) take the form,

$$\Phi_{\rm flat} = -\frac{GM}{r} \Biggl\{ 1 + {\rm constant} \times \frac{G}{r^2} + O(G^2) \Biggr\} \; . \tag{272}$$

On de Sitter background there is a dimensionally consistent alternative provided by the Hubble parameter H and the secular growth driven by continuous particle production,

$$\Phi_{\rm dS} = -\frac{GM}{r} \Biggl\{ 1 + {\rm constant} \times G H^2 \ln(a) + O(G^2) \Biggr\} \; . \tag{273}$$

If such a correction were to occur its natural interpretation would be as a time dependent renormalization of the Newton constant. The physical origin of the effect (if it is present) would be that virtual infrared quanta which emerge near the source tend to collapse to it, leading to a progressive increase in the source.

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