New framework for analyzing the effects of small scale inhomogeneities in cosmology

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A new framework for analyzing the effects of small scale inhomogeneities in cosmology

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Abstract

We develop a new, mathematically precise framework for treating the effects of nonlinear phenomena occurring on small scales in general relativity. Our approach is an adaptation of Burnett's formulation of the "shortwave approximation", which we generalize to analyze the effects of matter inhomogeneities as well as gravitational radiation. Our framework requires the metric to be close to a "background metric", but allows arbitrarily large stress-energy fluctuations on small scales. We prove that, within our framework, if the matter stress-energy tensor satisfies the weak energy condition (i.e., positivity of energy density in all frames), then the only effect that small scale inhomogeneities can have on the dynamics of the background metric is to provide an "effective stress-energy tensor" that is traceless and has positive energy density—corresponding to the presence of gravitational radiation. In particular, nonlinear effects produced by small scale inhomogeneities cannot mimic the effects of dark energy. We also develop "perturbation theory" off of the background metric. We derive an equation for the "long-wavelength part" of the leading order deviation of the metric from the background metric, which contains the usual terms occurring in linearized perturbation theory plus additional contributions from the small-scale inhomogeneities. Under various assumptions concerning the absence of gravitational radiation and the non-relativistic behavior of the matter, we argue that the "short wavelength" deviations of the metric from the background metric near a point x should be accurately described by Newtonian gravity, taking into account only the matter lying within a "homogeneity lengthscale" of x. Finally, we argue that our framework should provide an accurate description of the actual universe.

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I. INTRODUCTION

It is generally believed that our universe is very well described on large scales by a Friedmann-Lemaître-Robertson-Walker (FLRW) model. However, on small scales, extremely large departures of the mass density from FLRW models are commonly observed, e.g., on Earth[1], we have $\delta\rho/\rho \sim 10^{30}$. Nevertheless, common sense estimates [1, 2] suggest that (a) the deviation of the metric (as opposed to mass density, which corresponds to second derivatives of the metric) from a FLRW metric are globally very small on all scales except in the immediate vicinity of strong field objects such as black holes and neutron stars, and (b) the terms in Einstein's equation that are nonlinear in the deviation of the metric from a FLRW metric are negligibly small as compared with the linear terms in the deviation from a FLRW metric except in the immediate vicinity of strong field objects. These common sense estimates together with the fact that the motion of matter relative to the rest frame of the cosmic microwave background is non-relativistic strongly suggest that (1) the large scale structure of the universe is well described by a FLRW metric, (2) when averaged on scales sufficiently large that $|\delta\rho/\rho| \ll 1$—i.e., scales of order 100 Mpc in the present universe—the deviations from a FLRW model are well described by ordinary FLRW linear perturbation theory, and (3) on smaller scales, the deviations from a FLRW model (or, for that matter, from Minkowski spacetime) are well described by Newtonian gravity—except, of course, in the immediate vicinity of strong field objects.

The above assumptions underlie the standard cosmological model, which has been remarkably successful in accounting for essentially all cosmological phenomena. Thus, there is good empirical evidence that assumptions (1)–(3) are at least essentially correct. Nevertheless, the situation is quite unsatisfactory from the perspective of having a mathematically consistent theory wherein the assumptions and approximations are justified in a systematic manner. Indeed, it is not even obvious that assumptions (1)–(3) are mathematically consistent [3, 4]. In particular, nonlinear effects play an essential role in Newtonian dynamics, e.g., the fact that the Earth orbits the Sun arises from Einstein's equation as a nonlinear effect in the deviation of the metric from flatness. It is clear that one would get an extremely poor description of small-scale structure in the universe if one neglected the nonlinear terms in Einstein's equation in the deviation of the metric from a FLRW model; for example, galaxies

[1] If we go to scales of atomic nuclei, then $\delta\rho/\rho \sim 10^{43}$.

would not be bound. But if one cannot neglect nonlinear terms in Einstein's equation on small scales, how can one justify neglecting them on large (i.e., ~ 100 Mpc or larger) scales? In addition, since it is not clear exactly what approximations are needed for assumptions (1)–(3) to be valid, it is far from clear as to how one could go about systematically improving these approximations.

Indeed, it is far from obvious, *a priori*, that nonlinearities associated with small-scale inhomogeneities could not produce important effects on the large-scale dynamics of the FLRW model itself, as has been suggested by a number of authors [5–17] as a possible way to account for the effects of "dark energy" without invoking a cosmological constant, a new source of matter, or a modification of Einstein's equation. In fact, the example of gravitational radiation of wavelength much less than the Hubble scale illustrates that it is possible, in principle, for small-scale inhomogeneities in the metric and curvature to affect large-scale dynamics. The dynamics of a FLRW model whose energy content is dominated by gravitational radiation will be very different from one with a similar matter content but no gravitational radiation. It is the nonlinear terms in Einstein's equation associated with the short-wavelength gravitational radiation that are responsible for producing this difference in the large-scale dynamics. Although common-sense estimates indicate that similar effects on large-scale dynamics should not be produced by nonlinear effects of small-scale matter inhomogeneities in our universe, it would be very useful to have a systematic and general approach that can determine exactly what effects small-scale inhomogeneities can and cannot produce on large-scale dynamics.

The main approach that has been taken to investigate the effects of small-scale inhomogeneities on large-scale dynamics has been to consider inhomogeneous models, take spatial averages to define corresponding FLRW quantities, and derive equations of motion for these FLRW quantities [18, 19]. Since, in particular, the spatial average of the square of a quantity does not equal the square of its spatial average, the effective FLRW dynamics of an inhomogeneous universe will differ from that of a homogeneous universe. However, a major difficulty with this approach is that, when the deviations of the metric from that of a FLRW background are not very small, it is not obvious how to interpret the averaged quantities in terms of observable quantities. For example, if the total volume of a spatial region is found to increase with time, this certainly does not imply that observers in this region will find that Hubble's law appears to be satisfied. Further serious difficulties with this approach arise from the fact that the notion of averaging is slicing dependent and the average of tensor quantities over a region in a non-flat spacetime is intrinsically ill defined. In addition, the equations for averaged quantities that have been derived to date are only a partial set of equations—they contain quantities whose evolution is not determined—so it is difficult to analyze what dynamical behavior of the averaged quantities is actually possible. This difficulty is well illustrated by a recent paper of Buchert and Obadia [20], where they suggest that inflationary dynamics may be possible in vacuum spacetimes. However, this conclusion is drawn by simply postulating that a particular functional relation holds between certain averaged quantities under dynamical evolution. In fact, Einstein's equation controls the dynamical evolution of these quantities—so one is not free to postulate additional relations—but the restrictions imposed by Einstein's equation are not considered.

The main purpose of this paper is to develop a framework that allows us to consider spacetimes where there can be significant inhomogeneity and nonlinear dynamics on small scales, yet the framework[2] is capable of describing "average" large-scale behavior in a mathematically precise manner. We seek a framework wherein the approximations are "controlled" in the sense that they can be shown to hold with arbitrarily good accuracy in some appropriate limit. The results obtained within this framework will thereby be theorems, and the only issue that can arise with regard to the applicability of these results to the physical universe is how close the physical universe is to the limiting behavior of the theorems, in which the results hold exactly.

The situation that we wish to describe via our framework is one in which there is a "background spacetime metric", $g^{(0)}_{ab}$, that is supposed to correspond to the metric "averaged" over small scale inhomogeneities. In the case of interest in cosmology, $g^{(0)}_{ab}$ would be taken to be a metric with FLRW symmetry, but our framework does not require this choice, and no restrictions will be placed upon $g^{(0)}_{ab}$ until section IV. The difference, $h_{ab} \equiv g_{ab} - g^{(0)}_{ab}$, between the actual metric g_{ab} and the background metric is assumed to be small everywhere. This precludes the existence of strong field objects such as black holes and neutron stars, but even if such objects are present, by replacing these objects with weak field objects of the same mass, our framework should give a good description of the universe except in the

[2] Our framework will have some significant similarities to the approach of [21] (see also [22]), but our assumptions will be considerably more general and our results will have considerably wider applicability. Our assumptions also will be stated much more precisely. In addition, we will develop perturbation theory within our framework.

immediate vicinity of these objects. However, even though our framework requires h_{ab} to be small, derivatives of h_{ab} (say, with respect to the derivative operator ∇_a of the background metric $g^{(0)}_{ab}$) are *not* assumed to be small. Specifically, quadratic products of $\nabla_c h_{ab}$ are allowed to be of the same order as the curvature of $g^{(0)}_{ab}$. Thus, *a priori*, such terms are allowed to make a significant contribution to the dynamics of $g^{(0)}_{ab}$ itself. Finally, no restrictions are placed upon second derivatives of h_{ab}. In particular, if matter is present, the framework allows $\delta\rho/\rho \gg 1$.

How can one formulate a mathematically precise framework where approximations such as the smallness of $h_{ab} = g_{ab} - g^{(0)}_{ab}$ are "controlled" in the sense that limits can be taken where they hold with arbitrarily good accuracy? The basic idea is to consider a one-parameter family of metrics $g_{ab}(\lambda)$ that has appropriate limiting behavior as $\lambda \to 0$. To illustrate this idea, consider the much simpler case of ordinary perturbation theory, wherein one wishes to describe a situation where not only is $g_{ab} - g^{(0)}_{ab}$ small, but all of its spacetime derivatives are correspondingly small. To describe this in a precise way, we can consider a one-parameter family of metrics $g_{ab}(\lambda, x)$ that is jointly smooth in the parameter λ and the spacetime coordinates x. The limit as $\lambda \to 0$ of this family of metrics clearly exists and defines the background metric $g^{(0)}_{ab}(x) = g_{ab}(0, x)$. If we assume that $g_{ab}(\lambda)$ satisfies Einstein's equation for all $\lambda > 0$, it follows immediately that $g^{(0)}_{ab}$ also satisfies Einstein's equation. The first order perturbation, γ_{ab}, of $g^{(0)}_{ab}$ is defined to be the partial derivative of $g_{ab}(\lambda)$ with respect to λ, evaluated at $\lambda = 0$. It satisfies the linearized Einstein equation, which is derived by taking the partial derivative with respect to λ of Einstein's equation for $g_{ab}(\lambda)$, evaluated at $\lambda = 0$. More generally, the nth order perturbation, $\gamma^{(n)}_{ab}$, of $g^{(0)}_{ab}$ is defined to be the nth partial derivative of $g_{ab}(\lambda)$ with respect to λ evaluated at $\lambda = 0$, and the equation it satisfies is derived by taking the nth partial derivative with respect to λ of Einstein's equation. The perturbative equations for the metric perturbation at each order hold rigorously and exactly. Of course, the issue remains as to how accurately an nth order Taylor series approximation in λ describes a particular metric $g_{ab}(\lambda)$ for some small but finite value of λ. This issue may not be easy to resolve in any specific case. Nevertheless, even if the accuracy of the Taylor approximation cannot be fully resolved, it is far more satisfactory mathematically to derive rigorous results for the perturbative quantities than to make crude arguments about g_{ab} based on the assumption that $h_{ab} = g_{ab} - g^{(0)}_{ab}$ is "small."

To obtain a mathematically precise framework that can be applied to describe situations relevant for cosmology, we also wish to consider a one-parameter family of metrics $g_{ab}(\lambda)$ that approaches a smooth background metric $g^{(0)}_{ab}$ as $\lambda \to 0$. However, we do not want to require that first spacetime derivatives of

$$h_{ab}(\lambda) \equiv g_{ab}(\lambda) - g^{(0)}_{ab} \tag{1}$$

go to zero as $\lambda \to 0$. Indeed, in order to capture the effects we are interested in, it is essential that, *a priori*, the framework allow quadratic products of derivatives of $h_{ab}(\lambda)$ to be of the same order as the curvature of $g^{(0)}_{ab}$ in the limit as $\lambda \to 0$. This suggests that we should consider a one-parameter family wherein, as $\lambda \to 0$, the deviations of $g_{ab}(\lambda)$ from $g^{(0)}_{ab}$ simultaneously become of smaller amplitude and shorter wavelength, in such a way that first spacetime derivatives of $h_{ab}(\lambda)$ remain bounded but do not necessarily go to zero. If $h_{ab}(\lambda) \to 0$ as $\lambda \to 0$ but spacetime derivatives of $h_{ab}(\lambda)$ do not go to zero, then it is easy to see that spacetime derivatives of $h_{ab}(\lambda)$ cannot converge pointwise (i.e., at fixed spacetime points) as $\lambda \to 0$. However, spacetime derivatives of $h_{ab}(\lambda)$ will automatically go to zero when suitably averaged over a spacetime region; more precisely, their "weak limit" exists and vanishes. Similarly, although we cannot require that quadratic products of first spacetime derivatives of $h_{ab}(\lambda)$ approach a limit at fixed spacetime points as $\lambda \to 0$, it is mathematically consistent to require that the weak limit of these quantities exists. As we shall see in the next section, a certain combination of weak limits of quadratic products of first spacetime derivatives of $h_{ab}(\lambda)$ acts as an "effective stress energy tensor", which affects the dynamics of the background metric $g^{(0)}_{ab}$. In this way, the possible effects on FLRW dynamics of small-scale inhomogeneities—which are required to be of small amplitude in the metric but may be of unbounded amplitude in the mass density—can be studied in a mathematically precise manner.

In fact, the issues we confront in attempting to treat the effects of small-scale mass density fluctuations in cosmology are very similar to the issues arising when one attempts to treat the self-gravitating effects of short-wavelength gravitational radiation. In the latter case, one is interested in considering a situation where the amplitude of the gravitational radiation relative to some background metric $g^{(0)}_{ab}$ is small, but the "effective stress-energy tensor" of the gravitational radiation—i.e., products of first spacetime derivatives of $(g_{ab} - g^{(0)}_{ab})$—is comparable to the curvature of $g^{(0)}_{ab}$. A "shortwave approximation" formalism was developed by Isaacson [23, 24] (see also pp. 964–966 of [25]) to treat this situation. The shortwave approximation was put on a rigorous mathematical footing by Burnett [26], who derived the equations satisfied by $g_{ab}^{(0)}$ by considering a one-parameter family of metrics $g_{ab}(\lambda)$ with suitable limiting behavior. In this paper, we shall generalize Burnett's formulation of the shortwave approximation by allowing for the presence of a nonvanishing matter stress-energy tensor T_{ab}. By following Burnett's approach, we shall derive an equation for the "background metric", $g_{ab}^{(0)}$, which takes the form of Einstein's equation with an "averaged" matter stress-energy tensor and an additional "effective stress-energy" contribution arising from the small-scale inhomogeneities.

One of the main results of our paper, proven in the analysis of section II, is that if the true matter stress-energy tensor T_{ab} satisfies the weak energy condition (i.e., if the energy density is positive in all frames), then the effective stress-energy tensor appearing in the equation for $g_{ab}^{(0)}$ must be traceless and must have positive energy density—just as in the vacuum case[3]. In other words, no new effects on large-scale dynamics can arise from small-scale matter inhomogeneities; the only effects that small-scale inhomogeneities of any kind can have on large-scale dynamics corresponds to having gravitational radiation present. Our analysis makes no assumptions of symmetries of the background metric $g_{ab}^{(0)}$ and makes no assumptions about the matter stress-energy tensor T_{ab} other than that it satisfies the weak energy condition. However, if $g_{ab}^{(0)}$ is assumed to have FLRW symmetry, then our results establish that, within our framework, the only effect that small-scale inhomogeneities can have on FLRW dynamics corresponds to the additional presence of an effective $P = \frac{1}{3}\rho$ fluid with $\rho \geq 0$. In particular, within our framework, small-scale inhomogeneities cannot provide an effective source of "dark energy".

In cosmology, in addition to analyzing the dynamics of the background FLRW spacetime, one is interested in analyzing the deviations from the FLRW background. In section III, we undertake a general analysis of perturbation theory within our framework. It is not straightforward to do this because $g_{ab}(\lambda)$ is not differentiable in λ at $\lambda = 0$, so there is no notion of a "linearized metric perturbation" in our framework. However, the weak limit as $\lambda \to 0$ of $[g_{ab}(\lambda) - g_{ab}^{(0)}]/\lambda$ may exist, and, under the assumption that it does, this limit defines a quantity $\gamma_{ab}^{(L)}$, which corresponds closely to what is called the "long wavelength

[3] Our result on the positivity of energy density within this framework is new, i.e., it was not previously shown to hold in the vacuum case by Burnett [26]. Positivity of the effective energy density in the vacuum case was shown by Isaacson [24] only under an additional WKB ansatz.

part" of the metric perturbation in other analyses (see, e.g., [27]). We also write

$$h_{ab}^{(S)}(\lambda) \equiv h_{ab}(\lambda) - \lambda \gamma_{ab}^{(L)} \tag{2}$$

and refer to $h_{ab}^{(S)}$ as the "short wavelength part" of the deviation of the metric from a FLRW model. (Note that in our framework, $h_{ab}^{(S)}$ and $\gamma_{ab}^{(L)}$ have precise mathematical definitions.) Our goal is to derive the equations satisfied by $\gamma_{ab}^{(L)}$ as well as to determine $h_{ab}^{(S)}(\lambda)$ to accuracy $O(\lambda)$. This will yield the spacetime metric to accuracy $O(\lambda)$.

In section III, we will systematically derive the equations satisfied by $\gamma_{ab}^{(L)}$ and $h_{ab}^{(S)}(\lambda)$ in a completely general context. By taking the weak limit of $1/\lambda$ times the difference between the exact Einstein equation for $g_{ab}(\lambda)$ and the effective Einstein equation for $g_{ab}^{(0)}$, we obtain an equation for $\gamma_{ab}^{(L)}$ corresponding to that arising in ordinary linearized perturbation theory. However, in addition to the familiar terms appearing in ordinary linearized perturbation theory, this equation contains additional "source terms" arising from $h_{ab}^{(S)}$ and, if gravitational radiation is present in the background spacetime, this equation also contains additional terms linear in $\gamma_{ab}^{(L)}$ and quadratic in $h_{ab}^{(S)}$. We also obtain additional relations between quantities appearing in this equation by taking weak limits of Einstein's equation multiplied by $h_{ab}^{(S)}/\lambda$ and by $h_{ab}^{(S)}h_{cd}^{(S)}/\lambda$. These relations are used to simplify the perturbation equation for $\gamma_{ab}^{(L)}$. Finally, we write down Einstein's equation for $h_{ab}^{(S)}(\lambda)$. In the vacuum case, we consider the simplifications that can be made to this equation if one is interested only in determining $h_{ab}^{(S)}(\lambda)$ to sufficient accuracy to obtain $g_{ab}^{(0)}$. We compare our approach to that of Isaacson [23, 24] and subsequent works (see, e.g., [25]).

In section IV, we apply our general perturbative analysis to cosmology. We introduce the "generalized wave map gauge" in subsection IVA. In subsection IVB, we make additional assumptions concerning initial conditions and the Newtonian nature of the deviation, $\delta T_{ab} \equiv T_{ab}(\lambda) - T_{ab}^{(0)}$, of the stress-energy tensor from a FLRW model. We argue that, for small λ, $h_{ab}^{(S)}(\lambda)$ should be well approximated (in the wave map gauge) by the Newtonian gravity solution, whereby one needs only take into account the "nearby" matter. In contrast to the rest of the paper—where all of the assumptions are stated in a mathematically precise manner, and all of the results are theorems—in subsection IVB we provide only a sketch of the assumptions needed, and many of our arguments have the character of plausibility arguments rather than proofs. In subsection IVC, we simplify the equation for $\gamma_{ab}^{(L)}$ derived in section III by using our Newtonian assumptions. We show that this equation reduces to the ordinary cosmological perturbation equation with an additional effective source arising from $h^{(S)}_{ab}(\lambda)$, in agreement with a result recently obtained by [27].

In summary, in this paper we introduce a new framework for treating spacetimes whose metric is close to that of a background metric $g^{(0)}_{ab}$ but is such that nonlinear departures from $g^{(0)}_{ab}$ are dynamically important on small scales. We proceed by introducing a suitable one-parameter family of metrics $g_{ab}(\lambda)$ and deriving results in the limit as $\lambda \to 0$. We prove that the small-scale inhomogeneities cannot affect the dynamics of the background metric $g^{(0)}_{ab}$ except by the addition of an effective stress-energy tensor with positive energy density and vanishing trace, which can be interpreted as arising from gravitational radiation. We derive an equation for the "long-wavelength part" of the leading order deviation of the metric from $g^{(0)}_{ab}$, which contains the usual terms occurring in linearized perturbation theory plus additional contributions from the small-scale inhomogeneities. Finally, we argue that the small-scale deviations of the metric from $g^{(0)}_{ab}$ should be accurately described by Newtonian gravity.

Of course, the real universe is not the limit as λ becomes arbitrarily small of the type of a one-parameter family of metrics $g_{ab}(\lambda)$ considered here. Thus, our results do not apply exactly to the real universe—any more than the results of an analysis using ordinary linearized perturbation theory would apply to a real situation. However, in section V we will argue that since the scales in which nonlinear dynamics are important in the present universe (i.e., scales much less than ~ 100 Mpc) are much smaller than the scale of the background curvature (i.e., the Hubble radius ~ 3 Gpc), it seems reasonable to expect that the real universe will be accurately described by a "small λ" approximation to $g_{ab}(\lambda)$ within our formalism. We believe that our analysis thereby goes a long way toward providing a mathematically sound framework that can be used to justify the assumptions and approximations used in cosmology. At the end of section V, we will discuss how these approximations can be improved.

Our notation and sign conventions follow that of [28]. Lower case Latin indices from early in the alphabet $(a, b, c, \dots)$ denote abstract spacetime indices. Greek indices denote components of tensors. Latin indices from mid-alphabet $(i, j, k, \dots)$ denote spatial components of tensors.

II. DYNAMICS OF THE BACKGROUND METRIC

In this section, we will give a precise statement of the assumptions that underlie our framework. We will then analyze the dynamics of the background metric $g^{(0)}_{ab}$ and prove that if the matter stress-energy tensor, T_{ab}, satisfies the weak energy condition, then the "effective stress-energy" contributed by small-scale inhomogeneities must have positive energy density and vanishing trace.

As explained in the Introduction, we wish to consider a situation wherein we have a one-parameter family of metrics $g_{ab}(\lambda)$ that approaches a background metric $g^{(0)}_{ab}$, but spacetime derivatives of $g_{ab}(\lambda)$ do not approach the corresponding spacetime derivatives of $g^{(0)}_{ab}$. An example of the type of behavior that we have in mind is for components of $h_{ab}(\lambda) \equiv g_{ab}(\lambda) - g^{(0)}_{ab}$ to behave like $\lambda \sin(x/\lambda)$. In this situation, if we let ∇_a denote the derivative operator associated with $g^{(0)}_{ab}$, we cannot have $\nabla_c h_{ab}(\lambda) \to 0$ pointwise as $\lambda \to 0$. However, suitable spacetime averages of $\nabla_c h_{ab}(\lambda)$ will go to zero. More precisely, if f^{cab} is any smooth tensor field of compact support, we have

$$\begin{aligned} \int f^{cab} \nabla_c h_{ab}(\lambda) &= -\int (\nabla_c f^{cab}) h_{ab}(\lambda) \\ &\to 0 \quad \text{as} \quad \lambda \to 0 \end{aligned} \tag{3}$$

provided only that $h_{ab}(\lambda) \to 0$ locally in L^1, where the volume element in this integral is that associated with $g^{(0)}_{ab}$. If (3) holds for all "test" (i.e., smooth and compact support) tensor fields, f^{cab}, we say that $\nabla_c h_{ab}(\lambda) \to 0$ *weakly*. More generally, if $A_{a_1 \dots a_n}(\lambda)$ is a one-parameter family of tensor fields defined for $\lambda > 0$, we say that $A_{a_1 \dots a_n}(\lambda)$ converges weakly to $B_{a_1 \dots a_n}$ as $\lambda \to 0$ if for all smooth $f^{a_1 \dots a_n}$ of compact support, we have

$$\lim_{\lambda \to 0} \int f^{a_1 \dots a_n} A_{a_1 \dots a_n}(\lambda) = \int f^{a_1 \dots a_n} B_{a_1 \dots a_n} \,. \tag{4}$$

Roughly speaking, the weak limit performs a local spacetime average of $A_{a_1 \dots a_n}(\lambda)$ before letting $\lambda \to 0$.

As noted above, if $g_{ab}(\lambda)$ converges to $g^{(0)}_{ab}$ in a suitably strong sense—locally in L^1 suffices—then spacetime derivatives of $h_{ab}(\lambda)$ automatically converge weakly to zero. However, there is no reason why products of spacetime derivatives of $h_{ab}(\lambda)$ must converge weakly at all, and, if they do converge, one would not expect them to converge to zero. (This latter observation is closely related to the fact that averages of products of quantities are not normally equal to the product of their averages.) As discussed in the Introduction, we wish to consider a situation where first spacetime derivatives of $h_{ab}(\lambda)$ remain bounded but do not necessarily go to zero as $\lambda \to 0$. However, in order to have well defined averaged behavior in the limit as $\lambda \to 0$ we want the weak limit of the nonlinear terms in Einstein's equation to exist. As we shall see below, this will be the case if the weak limit of quadratic products of first spacetime derivatives of $h_{ab}(\lambda)$ exists.

All of the above considerations lead us to consider a one-parameter family of metrics $g_{ab}(\lambda)$, defined for all $\lambda \geq 0$, satisfying the following conditions, which are straightforward generalizations to the non-vacuum case of the conditions considered by Burnett [26] in his formulation of the shortwave approximation. In these conditions, ∇_a denotes an arbitrary fixed (i.e., λ-independent) derivative operator on the spacetime manifold M. For convenience in stating these conditions, we choose an arbitrary Riemannian metric e_{ab} on M and for any tensor field $t_{a_1 \ldots a_n}$ on M we define $|t_{a_1 \ldots a_n}|^2 = e^{a_1 b_1} \ldots e^{a_n b_n} t_{a_1 \ldots a_n} t_{b_1 \ldots b_n}$.

(i) For all $\lambda > 0$, we have

$$G_{ab}(g(\lambda)) + \Lambda g_{ab}(\lambda) = 8\pi T_{ab}(\lambda) \ , \tag{5}$$

where $T_{ab}(\lambda)$ satisfies the weak energy condition, i.e., for all $\lambda > 0$ we have $T_{ab}(\lambda) t^a(\lambda) t^b(\lambda) \geq 0$ for all vectors $t^a(\lambda)$ that are timelike with respect to $g_{ab}(\lambda)$.

(ii) There exists a smooth positive function $C_1(x)$ on M such that

$$|h_{ab}(\lambda, x)| \leq \lambda C_1(x) \ , \tag{6}$$

where $h_{ab}(\lambda, x) \equiv g_{ab}(\lambda, x) - g_{ab}(0, x)$.

(iii) There exists a smooth positive function $C_2(x)$ on M such that

$$|\nabla_c h_{ab}(\lambda, x)| \leq C_2(x) \ . \tag{7}$$

(iv) There exists a smooth tensor field μ_{abcdef} on M such that

$$\operatorname*{w-lim}_{\lambda \to 0} \left[\nabla_a h_{cd}(\lambda) \nabla_b h_{ef}(\lambda)\right] = \mu_{abcdef} \ , \tag{8}$$

where "w-lim" denotes the weak limit.

It follows immediately that $\mu_{ab(cd)(ef)} = \mu_{abcdef}$ and $\mu_{abcdef} = \mu_{baefcd}$, and it is not difficult to show [26] that $\mu_{(ab)cdef} = \mu_{abcdef}$. It also is not difficult to see that if $g_{ab}(\lambda)$ satisfies the above conditions for any choice of ∇_a and e_{ab}, then it satisfies these conditions for all choices of ∇_a and e_{ab}. In our calculations, it will be convenient to choose ∇_a to be the derivative operator associated with the background metric $g^{(0)}_{ab} \equiv g_{ab}(0)$, and in the following, we shall make this choice. We shall also raise and lower indices with $g^{(0)}_{ab}$.

As discussed in the Introduction, the key idea is that our actual spacetime, with all of its inhomogeneities, is described by an element of such a one-parameter family, at some small but finite value of λ. By analyzing the limiting behavior of such one-parameter families at small λ, we hope to attain an excellent approximate description of our universe. However, unlike ordinary perturbative analyses, our one-parameter family $g_{ab}(\lambda)$ is not differentiable in λ at $\lambda = 0$, so we cannot define perturbative quantities or obtain useful equations by differentiation with respect to λ.

Our first task is to derive an equation satisfied by the background metric $g^{(0)}_{ab} = g_{ab}(0)$. This equation will follow directly from Einstein's equation (5) for $g_{ab}(\lambda)$, using the general relationship between the Ricci curvature of $g^{(0)}_{ab}$ and $g_{ab}(\lambda)$, namely

$$R_{ab}(g^{(0)}) = R_{ab}(g(\lambda)) + 2\nabla_{[a}C^{c}{}_{c]b} - 2C^{d}{}_{b[a}C^{c}{}_{c]d}\,, \tag{9}$$

where

$$C^{c}{}_{ab} = \frac{1}{2}g^{cd}(\lambda)\left\{\nabla_a g_{bd}(\lambda) + \nabla_b g_{ad}(\lambda) - \nabla_d g_{ab}(\lambda)\right\} \tag{10}$$

and, again, we remind the reader that ∇_a denotes the derivative operator associated with $g^{(0)}_{ab}$, so that $\nabla_c g^{(0)}_{ab} = 0$. It follows immediately from (9) that

$$\begin{aligned} R_{ab}(g^{(0)}) - \frac{1}{2}g_{ab}(\lambda)g^{cd}(\lambda)R_{cd}(g^{(0)}) + \Lambda g_{ab}(\lambda) \\ = G_{ab}(g(\lambda)) + \Lambda g_{ab}(\lambda) + 2\nabla_{[a}C^{e}{}_{e]b} - 2C^{f}{}_{b[a}C^{e}{}_{e]f} \\ - g_{ab}(\lambda)g^{cd}(\lambda)\nabla_{[c}C^{e}{}_{e]d} + g_{ab}(\lambda)g^{cd}(\lambda)C^{f}{}_{d[c}C^{e}{}_{e]f}\,, \end{aligned} \tag{11}$$

and invoking the Einstein equation for $\lambda > 0$,

$$\begin{aligned} R_{ab}(g^{(0)}) - \frac{1}{2}g_{ab}(\lambda)g^{cd}(\lambda)R_{cd}(g^{(0)}) + \Lambda g_{ab}(\lambda) \\ = 8\pi T_{ab}(\lambda) + 2\nabla_{[a}C^{e}{}_{e]b} - 2C^{f}{}_{b[a}C^{e}{}_{e]f} \\ - g_{ab}(\lambda)g^{cd}(\lambda)\nabla_{[c}C^{e}{}_{e]d} + g_{ab}(\lambda)g^{cd}(\lambda)C^{f}{}_{d[c}C^{e}{}_{e]f}\,. \end{aligned} \tag{12}$$

We now take the weak limit of both sides of (12) as $\lambda \to 0$. It is easy to see that the weak limit of the left side exists and is equal to $G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab}$. Aside from the term $8\pi T_{ab}(\lambda)$, the terms on the right side of (12) all contain a total of precisely two derivatives of $h_{ab}(\lambda)$. These terms can be classified into the following types: (a) Terms linear in $h_{ab}(\lambda)$, corresponding to the linearized Einstein operator acting on $h_{ab}(\lambda)$; (b) terms quadratic in $h_{ab}(\lambda)$, corresponding to the second order Einstein operator acting on $h_{ab}(\lambda)$; (c) terms cubic and higher order in $h_{ab}(\lambda)$. The weak limit of terms of type (a) vanish by the type of argument leading to (3). The terms of type (b) depend upon $h_{ab}(\lambda)$ either in the form $\nabla_a h_{cd}(\lambda)\nabla_b h_{ef}(\lambda)$—which has weak limit μ_{abcdef}—or in the form $h_{cd}(\lambda)\nabla_a\nabla_b h_{ef}(\lambda)$. However, since

$$h_{cd}(\lambda)\nabla_a\nabla_b h_{ef}(\lambda) = \nabla_a\left[h_{cd}(\lambda)\nabla_b h_{ef}(\lambda)\right] - \nabla_a h_{cd}(\lambda)\nabla_b h_{ef}(\lambda) \tag{13}$$

and the weak limit of $\nabla_a\left[h_{cd}(\lambda)\nabla_b h_{ef}(\lambda)\right]$ vanishes, we see that the weak limit of $h_{cd}(\lambda)\nabla_a\nabla_b h_{ef}(\lambda)$ also exists and is equal to $-\mu_{abcdef}$. Finally, it is easily seen that the weak limit of all terms of type (c) vanish.

Since the weak limit of all terms in (12) apart from $T_{ab}(\lambda)$ exist, it follows that the weak limit of $T_{ab}(\lambda)$ itself also must exist, without the necessity to impose any additional assumptions on our one-parameter family. We write

$$T^{(0)}_{ab} \equiv \underset{\lambda\to 0}{\text{w-lim}}\, T_{ab}(\lambda)\,, \tag{14}$$

and we may interpret $T^{(0)}_{ab}$ as representing the matter stress-energy tensor averaged over small scale inhomogeneities. Since $T_{ab}(\lambda)$ satisfies the weak energy condition for all $\lambda > 0$ and since $g_{ab}(\lambda)$ converges uniformly (on compact sets) to $g^{(0)}_{ab}$, it is not difficult to show that $T^{(0)}_{ab}$ also satisfies the weak energy condition, i.e., $T^{(0)}_{ab}t^a t^b \geq 0$ for all timelike vectors t^a with respect to $g^{(0)}_{ab}$. The weak limit of (12) then takes the form

$$G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab} = 8\pi T^{(0)}_{ab} + 8\pi t^{(0)}_{ab}\,, \tag{15}$$

where the "effective gravitational stress-energy tensor" $t^{(0)}_{ab}$ arises from the weak limit of terms of type (b) above and can be expressed entirely in terms of μ_{abcdef}. A lengthy calculation (see [26]) yields

$$\begin{aligned} 8\pi t^{(0)}_{ab} = {} & \frac{1}{8}\left\{-\mu^{c}{}_{c}{}^{de}{}_{de} - \mu^{c}{}_{c}{}^{d}{}_{d}{}^{e}{}_{e} + 2\mu^{cd}{}_{c}{}^{e}{}_{de}\right\} + \frac{1}{2}\mu^{cd}{}_{acbd} - \frac{1}{2}\mu^{c}{}_{ca}{}^{d}{}_{bd} \\ & + \frac{1}{4}\mu_{ab}{}^{cd}{}_{cd} - \frac{1}{2}\mu^{c}{}_{(ab)c}{}^{d}{}_{d} + \frac{3}{4}\mu^{c}{}_{cab}{}^{d}{}_{d} - \frac{1}{2}\mu^{cd}{}_{abcd}\,. \end{aligned} \tag{16}$$

Note that $t^{(0)}_{ab}$ corresponds to the "Isaacson average" of the second order Einstein tensor of $h_{ab}(\lambda)$. It can be shown to be gauge invariant [26].

Following Burnett [26], we decompose μ_{abcdef} into two tensors $\alpha_{abcdef} \equiv \mu_{[c|[ab]|d]ef}$ and $\beta_{abcdef} \equiv \mu_{(abcd)ef}$, so that

$$\mu_{abcdef} = -\frac{4}{3}(\alpha_{c(ab)def} + \alpha_{e(ab)fcd} - \alpha_{e(cd)fab}) \\ +\beta_{abcdef} + \beta_{abefcd} - \beta_{cdefab} \,. \tag{17}$$

Then, we have

$$8\pi t^{(0)}_{ab} = \alpha_a{}^c{}_b{}^d{}_{cd} + \frac{3}{2}\alpha^{cd}{}_{cdab} - 2\alpha_{(a}{}^{cd}{}_{|cd|b)} - \frac{1}{4}g^{(0)}_{ab}\left\{\alpha^{cd}{}_{cd}{}^e{}_e - 2\alpha^{cde}{}_{cde}\right\} \,. \tag{18}$$

Note that the right hand side is independent of β_{abcdef}.

The remainder of this section will be devoted to establishing two key properties of $t^{(0)}_{ab}$, namely, $t^{(0)a}{}_a = 0$ and $t^{(0)}_{ab}t^a t^b \geq 0$ for any timelike vector t^a of the background metric $g^{(0)}_{ab}$. To prove these results, we need to obtain further information about α_{abcdef} from Einstein's equation (12). To do so, it is convenient to re-write this equation in "Ricci form" as

$$R_{ab}(g^{(0)}) - \Lambda g_{ab}(\lambda) = 8\pi T_{ab}(\lambda) - 4\pi g_{ab}(\lambda)g^{cd}(\lambda)T_{cd}(\lambda) + 2\nabla_{[a}C^c{}_{c]b} - 2C^d{}_{b[a}C^c{}_{c]d} \,. \tag{19}$$

We now multiply this equation by $h_{ef}(\lambda)$ and take the weak limit as $\lambda \to 0$. The left side clearly goes to zero. The weak limit of $h_{ef}(\lambda)C^d{}_{b[a}C^c{}_{c]f}$ is also easily seen to vanish. On the other hand, we have

$$\begin{aligned} \operatorname*{w-lim}_{\lambda\to 0} h_{ef}(\lambda)\nabla_{[a}C^c{}_{c]b} &= -\operatorname*{w-lim}_{\lambda\to 0} \nabla_{[a}h_{|ef|}(\lambda)C^c{}_{c]b} \\ &= -\mu_{[b|[ac]|d]ef}g^{(0)cd} \\ &= -\alpha_{acb}{}^c{}_{ef} \,. \end{aligned} \tag{20}$$

Thus, we obtain

$$\alpha_a{}^c{}_{bcef} = 4\pi \operatorname*{w-lim}_{\lambda\to 0} h_{ef}(\lambda)\left(T_{ab}(\lambda) - \frac{1}{2}g_{ab}(\lambda)g^{cd}(\lambda)T_{cd}(\lambda)\right) \,. \tag{21}$$

In particular, the weak limit of the right side of this equation must exist. By similar arguments starting with Einstein's equation in the form (12), it also follows that the weak limit as $\lambda \to 0$ of $h_{ef}(\lambda)T_{ab}(\lambda)$ exists. We write

$$\kappa_{efab} = \operatorname*{w-lim}_{\lambda\to 0} h_{ef}(\lambda)T_{ab}(\lambda) \,. \tag{22}$$

We will now show that the right side of (21) vanishes. We first prove the following lemma.

Lemma. *Let $A(\lambda)$ be a one-parameter family of smooth tensor fields (with indices suppressed) converging uniformly on compact sets to $A(0)$, and let $B(\lambda)$ be a one-parameter family of non-negative smooth functions converging weakly to $B(0)$. Then $A(\lambda)B(\lambda) \to A(0)B(0)$ weakly as $\lambda \to 0$.*

Proof. Let F be a test tensor field—i.e., a smooth tensor field of compact support—with index structure dual to that of A. Let f be a smooth, non-negative function of compact support with $f = 1$ on the support of F. Then $F = fF$. We have

$$\begin{aligned}
&\lim_{\lambda\to 0}\int (A(\lambda)B(\lambda) - A(0)B(0))F \\
&\qquad = \lim_{\lambda\to 0}\int \left[(A(\lambda) - A(0))B(\lambda) + A(0)(B(\lambda) - B(0))\right] F\,. \qquad (23)
\end{aligned}$$

The second term is zero because $B(\lambda) \to B(0)$ weakly and $A(0)F$ is a test function. On the other hand, we have

$$\begin{aligned}
\left|\int (A(\lambda) - A(0))B(\lambda)F\right| &\leq \int |(A(\lambda) - A(0))F||B(\lambda)f| \\
&\leq \sup_{x\in\operatorname{supp} F} |(A(\lambda) - A(0))F| \int B(\lambda)f\,, \qquad (24)
\end{aligned}$$

where we have used the facts that $B(\lambda) \geq 0$ and $f \geq 0$ in the second line. Now let $\epsilon > 0$. Since $A(\lambda) \to A(0)$ uniformly as $\lambda \to 0$ on compact sets, there exists $\lambda_1 > 0$ such that $\sup_{x\in\operatorname{supp} F} |(A(\lambda) - A(0))F| < \epsilon$ for $\lambda < \lambda_1$. Similarly, since $B(\lambda) \to B(0)$ weakly as $\lambda \to 0$, there exists $\lambda_2 > 0$ such that $\int (B(\lambda) - B(0))f < \epsilon$ for all $\lambda < \lambda_2$. Thus, for all $\lambda < \min(\lambda_1, \lambda_2)$, we have

$$\left|\int (A(\lambda) - A(0))B(\lambda)F\right| < \epsilon\left(\int B(0)f + \epsilon\right)\,, \qquad (25)$$

Thus the first term in (23) must be zero as well. □

The vanishing of κ_{efab} (see (22)) is a direct consequence of this lemma. To see this, let t^a be an arbitrary timelike vector field in the metric $g^{(0)}_{ab}$. Then, we have

$$\kappa_{efab}t^a t^b = \operatorname*{w-lim}_{\lambda\to 0} h_{ef}(\lambda)(T_{ab}(\lambda)t^a t^b) \qquad (26)$$

Since by condition (ii) on our one-parameter families, $|g_{ab}(\lambda) - g^{(0)}_{ab}| \leq \lambda C_1(x)$, on any fixed compact region, we can find a λ_0 such that t^a is timelike in the metric $g_{ab}(\lambda)$ for all $\lambda \leq \lambda_0$. Since $T_{ab}(\lambda)$ satisfies the weak energy condition, the function $B(\lambda) \equiv T_{ab}(\lambda)t^a t^b$ is non-negative for all $\lambda \leq \lambda_0$. We previously showed that $T_{ab}(\lambda)$ (and hence B) converges weakly.

Assumption (ii) also directly tells us that $A_{ab}(\lambda) \equiv h_{ab}(\lambda)$ converges uniformly on compact sets. Thus, from the lemma, we immediately conclude that

$$\kappa_{efab} t^a t^b = 0 \tag{27}$$

for all timelike t^a in the metric $g^{(0)}_{ab}$. However, since $\kappa_{efab} = \kappa_{ef(ab)}$, we have $\kappa_{efab} t^a t^b = 0$ for all timelike t^a if and only if $\kappa_{efab} = 0$, as we desired to show.

A similar argument establishes that the second term on the right side of (21) also vanishes. Thus, we obtain

$$\alpha_a{}^c{}_{bcef} = 0\,. \tag{28}$$

Consequently, (18) simplifies to

$$8\pi t^{(0)}_{ab} = \alpha_a{}^c{}_b{}^d{}_{cd}\,. \tag{29}$$

Taking the trace of this equation and again using (28) we obtain our first main result of this section:

Theorem 1. *Given a one-parameter family $g_{ab}(\lambda)$ satisfying assumptions (i)–(iv) above, the effective stress energy tensor $t^{(0)}_{ab}$ appearing in equation* (15) *for the background metric $g^{(0)}_{ab}$ is traceless,*

$$t^{(0)a}{}_a = 0\,. \tag{30}$$

We now show that $t^{(0)}_{ab}$ satisfies the weak energy condition. Let t^a be a timelike unit vector field with respect to $g^{(0)}_{ab}$. We wish to show that $t^{(0)}_{ab} t^a t^b \geq 0$. It is convenient to choose an orthonormal basis of $g^{(0)}_{ab}$ with t^a as the timelike vector. We will use Greek letters $\mu, \nu, \rho, \ldots$ to denote spacetime components in this basis and Latin letters $i, j, k, \ldots$ from mid-alphabet to denote spatial components. Then

$$\begin{aligned} 8\pi t^{(0)}_{ab} t^a t^b &= \alpha_{0\mu 0\nu}{}^{\mu\nu} \\ &= \alpha_{0j0k}{}^{jk} \\ &= \alpha_{ij}{}^i{}_k{}^{jk} \\ &= \frac{1}{4}\left\{\mu_i{}^i{}_{jk}{}^{jk} + \mu_{jki}{}^{ijk} - 2\mu_{jik}{}^{ijk}\right\}\,, \end{aligned} \tag{31}$$

where in the second line we used the antisymmetry of α_{abcdef} in the first two and the second two indices, and in the third line we used (28). Thus, we have expressed $t^{(0)}_{ab} t^a t^b$ entirely in terms of spatial components of μ_{abcdef}, which will be useful for taking advantage of the positive definiteness of the spatial metric.

Aside from the tensor symmetries that arise directly from its definition, the only restrictions on μ_{abcdef} that we have at our disposal come from (28). There is only one equation that can be derived from (28) that involves only spatial components of μ_{abcdef}, namely[4]

$$0 = \alpha_{i\mu}{}^{i\mu}{}_{kl} - \alpha_{0\mu}{}^{0\mu}{}_{kl} \,. \tag{32}$$

This yields

$$\mu_{ij}{}^{ij}{}_{kl} = \mu_{i}{}^{i}{}_{j}{}^{j}{}_{kl} \,. \tag{33}$$

Using this relation, we may re-write (31) as

$$8\pi t^{(0)}_{ab} t^a t^b = \frac{1}{4}\left\{\mu_{i}{}^{i}{}_{jk}{}^{jk} - 2\mu_{jik}{}^{ijk} + 2\mu_{jki}{}^{ijk} - \mu_{i}{}^{i}{}_{j}{}^{j}{}_{k}{}^{k}\right\} \,. \tag{34}$$

For the remainder of our argument, we will work in a small neighborhood of an arbitrary point $P \in M$. We will work in Riemannian normal coordinates x about P adapted to our orthonormal basis. Let f^δ_P be a one-parameter family of smooth, non-negative functions with support contained in a δ-ball centered at P such that

$$\int \left[f^\delta_P(x)\right]^2 \sqrt{-g}\, \mathrm{d}^4 x = 1 \,. \tag{35}$$

An explicit choice of f^δ_P is

$$f^\delta_P(x) = \frac{1}{\delta^2 \sqrt{-g}} F(x/\delta) \,, \tag{36}$$

where F is any smooth, non-negative function of compact support contained in a ball of radius 1 satisfying $\int F^2 d^4x = 1$, but there is no need to make this particular choice. Instead of working with $h_{ab}(\lambda)$, we introduce the quantity

$$\psi_{ab}(\delta, \lambda) \equiv f^\delta_P h_{ab}(\lambda) \,. \tag{37}$$

Note that for any fixed $\delta > 0$ and $\lambda > 0$, ψ_{ab} is smooth and of compact support, so, in particular, ψ_{ab} and all of its spacetime derivatives are in L^2. Furthermore, it follows directly from the properties of $h_{ab}(\lambda)$ and f^δ_P that all components of ψ_{ab} converge uniformly to 0 as

[4] Equation (28) states that the weak limit of h_{ef} times the linearized Ricci tensor vanishes, i.e., it has the character of the linearized vacuum Einstein equation off of flat spacetme. The linearized Hamiltonian constraint—i.e., the vanishing of the time-time component of the linearized Einstein tensor—is the only component of Einstein's equation that can be expressed entirely in terms of spatial derivatives of spatial components of the perturbed metric. Equation (32) corresponds to this equation.

$\lambda \to 0$ at fixed δ. Similarly, components of $\nabla_c \psi_{ab}$ are uniformly bounded in λ and x as $\lambda \to 0$ at fixed δ. Since ψ_{ab} is of fixed compact support, it follows immediately that $\|\psi_{ab}\|_{L^2} \to 0$ and $\|\nabla_c \psi_{ab}\|_{L^2}$ is uniformly bounded as $\lambda \to 0$, where $\|\psi_{ab}\|_{L^2} \equiv \int |\psi_{ab}|^2$.

Since μ_{abcdef} is smooth, it is obvious from (35) and the support properties and positivity of $(f_P^\delta)^2$ that

$$\mu_{\mu\nu\alpha\beta\gamma\rho}(P) = \lim_{\delta \to 0} \int \mu_{\mu\nu\alpha\beta\gamma\rho}(x)(f_P^\delta)^2 \sqrt{-g}\, \mathrm{d}^4 x\,. \tag{38}$$

On the other hand, since $(f_P^\delta)^2$ is a test function, from the definition (8) of μ_{abcdef}, at each fixed δ we have

$$\begin{aligned}\int \mu_{\mu\nu\alpha\beta\gamma\rho}(x)(f_P^\delta)^2 \sqrt{-g}\, \mathrm{d}^4 x &= \lim_{\lambda \to 0} \int \partial_\mu h_{\alpha\beta}(\lambda) \partial_\nu h_{\gamma\rho}(\lambda) (f_P^\delta)^2 \sqrt{-g}\, \mathrm{d}^4 x \\ &= \lim_{\lambda \to 0} \int \partial_\mu \psi_{\alpha\beta} \partial_\nu \psi_{\gamma\rho} \sqrt{-g}\, \mathrm{d}^4 x\,. \end{aligned} \tag{39}$$

Here, in the first line, we replaced the derivative operator ∇_a associated with $g_{ab}^{(0)}$ with the coordinate derivative operator ∂_a associated with Riemannian normal coordinates at P, making use of the fact that the definition of μ_{abcdef} is independent of derivative operator. In the second line, we used $\partial_\mu \psi_{\alpha\beta} = f_P^\delta \partial_\mu h_{\alpha\beta} + h_{\alpha\beta} \partial_\mu f_P^\delta$ and the fact that the resulting terms in (39) with no derivatives on $h_{\alpha\beta}$ vanish in the limit as $\lambda \to 0$. Taking the limit of (39) as $\delta \to 0$, we obtain

$$\mu_{\mu\nu\alpha\beta\gamma\rho}(P) = \lim_{\delta \to 0} \lim_{\lambda \to 0} \int \partial_\mu \psi_{\alpha\beta} \partial_\nu \psi_{\gamma\rho}\, \mathrm{d}^4 x\,, \tag{40}$$

where we have used the fact that $\sqrt{-g} = 1$ at P. Note that it is critical in this equation that the limits be taken in the order specified.

The corresponding formula for $t_{00}^{(0)}$ is

$$t_{00}^{(0)}(P) = \frac{1}{32\pi} \lim_{\delta \to 0} \lim_{\lambda \to 0} \int \mathrm{d}^4 x \left\{ \partial_i \psi_{jk} \partial^i \psi^{jk} - 2 \partial_j \psi_k{}^i \partial_i \psi^{jk} + 2 \partial_j \psi_i{}^i \partial_k \psi^{jk} - \partial_i \psi_j{}^j \partial^i \psi_k{}^k \right\}\,. \tag{41}$$

where, in this equation, indices are raised and lowered with the flat Euclidean spatial metric $\eta_{ij} = \mathrm{diag}(1,1,1)$ corresponding to the spatial components of $g_{ab}^{(0)}$ at P. The major advantage of (41) is that we can apply usual Fourier transform techniques to evaluate the integral appearing on the right side of this equation prior to taking the limit. If we had not "localized" $h_{ab}(\lambda)$ by multiplying it by f_P^δ, the Fourier transform of $h_{ab}(\lambda)$ could have been ill defined and, even if it were well defined, it would contain global information about $h_{ab}(\lambda)$ rather than local information about the behavior of $h_{ab}(\lambda)$ near P.

Our strategy now will be to prove that the quantity

$$\lim_{\lambda\to 0}\int \mathrm{d}^4x\,\{\partial_i\psi_{jk}\partial^i\psi^{jk} - 2\partial_j\psi_k{}^i\partial_i\psi^{jk} + 2\partial_j\psi_i{}^i\partial_k\psi^{jk} - \partial_i\psi_j{}^j\partial^i\psi_k{}^k\} \tag{42}$$

can be expressed as a sum of terms which are either positive, or which converge to zero as $\delta \to 0$. Positivity of $t_{00}^{(0)}$ then follows immediately. We proceed by taking Fourier transforms with respect to the spatial coordinates $\boldsymbol{x}$ only, with the convention

$$\hat{\psi}_{ij}(t,\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3\boldsymbol{x}\,\psi_{ij}(t,\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{43}$$

As previously noted, ψ_{jk} and all of its derivatives are obviously in L^2, since ψ_{jk} is smooth and of compact support. Since the Fourier transform is norm preserving in L^2, we have

$$t_{00}^{(0)}(P) = \frac{1}{32\pi}\lim_{\delta\to 0}\lim_{\lambda\to 0}\int \mathrm{d}t\,\mathrm{d}^3\boldsymbol{k}\,\left\{k_ik^i\hat{\psi}_{jk}\overline{\hat{\psi}^{jk}} - 2k_ik_j\hat{\psi}_k{}^i\overline{\hat{\psi}^{jk}} + 2k_jk_k\hat{\psi}_i{}^i\overline{\hat{\psi}^{jk}} - k_ik^i\hat{\psi}_j{}^j\overline{\hat{\psi}_k{}^k}\right\}\,. \tag{44}$$

We may decompose $\hat{\psi}_{ij}$ into its scalar, vector, and tensor parts as

$$\hat{\psi}_{ij}(t,\boldsymbol{k}) = \hat{\sigma}(t,\boldsymbol{k})k_ik_j - 2\hat{\varphi}q_{ij} + 2k_{(i}\hat{z}_{j)}(t,\boldsymbol{k}) + \hat{s}_{ij}(t,\boldsymbol{k})\,, \tag{45}$$

where $k^i\hat{z}_i = 0 = k^i\hat{s}_{ij}$, and $\hat{s}^i{}_i = 0$. Here q_{ij} is the projection orthogonal to k^i of the Euclidean metric on Fourier transform space. Since the various terms on the right side of (45) are orthogonal at each $\boldsymbol{k}$, it follows immediately that, for example, $|\hat{\varphi}(\boldsymbol{k})|^2 \le \frac{1}{8}\hat{\psi}^{ij}(\boldsymbol{k})\overline{\hat{\psi}}_{ij}(\boldsymbol{k})$. Since $\hat{\psi}_{ij}$ and all powers of k^i times $\hat{\psi}_{ij}$ are in L^2, it follows immediately that $\hat{\varphi}$ and all powers of k^i times $\hat{\varphi}$ are in L^2. Thus, we can freely take Fourier transforms of $\hat{\varphi}$ and all powers of k^i times $\hat{\varphi}$. Furthermore, since the L^2 norm of ψ_{ij}—and, hence, the L^2 norm of $\hat{\psi}_{ij}$—goes to zero as $\lambda \to 0$, it follows immediately that the L^2 norm of $\hat{\varphi}$—and hence the L^2 norm of φ—also goes to zero as $\lambda \to 0$. Similarly, the L^2 norm of $\partial_i\varphi$ must remain uniformly bounded as $\lambda \to 0$. Similar results hold for the other terms appearing on the right side of (45).

Substituting the decomposition (45) in (44) and using the fact that ψ_{ij} is real (which implies that $\overline{\hat{\psi}}(t,\boldsymbol{k}) = \hat{\psi}(t,-\boldsymbol{k})$), we obtain

$$t_{00}^{(0)}(P) = \frac{1}{32\pi}\lim_{\delta\to 0}\lim_{\lambda\to 0}\int \mathrm{d}t\,\mathrm{d}^3\boldsymbol{k}\,\left\{k_ik^i\hat{s}_{jk}\overline{\hat{s}^{jk}} - 8k_ik^i\hat{\varphi}\overline{\hat{\varphi}}\right\}\,. \tag{46}$$

Thus, we see that the "tensor part", $\hat{s}_{ij}$, of $\hat{\psi}_{ij}$ makes a positive contribution to the effective gravitational energy density $t_{00}^{(0)}$. This may be interpreted as saying that, at leading order within this framework, gravitational radiation carries positive energy density. The scalar $\hat{\sigma}$ and the vector part $\hat{z}_i$ do not contribute at all, as might be expected from the fact that these quantities should correspond to "pure gauge". Finally, the scalar $\hat{\varphi}$ makes the negative contribution

$$2E_\varphi = -\frac{1}{4\pi}\int \mathrm{d}t\,\mathrm{d}^3\boldsymbol{k}k_i k^i \hat{\varphi}\overline{\hat{\varphi}} \tag{47}$$

to the effective energy density.

In order to interpret the meaning of $\hat{\varphi}$ and E_φ, we note that by (45), $\hat{\varphi}$ satisfies

$$4k^i k_i \hat{\varphi} = -k^i k_i \hat{\psi}^j{}_j + k^i k^j \hat{\psi}_{ij}\,. \tag{48}$$

In position space, (48) becomes

$$4\partial^i \partial_i \varphi = -\partial^i \partial_i \psi^j{}_j + \partial^i \partial^j \psi_{ij}\,. \tag{49}$$

To put the right side of this equation in a more recognizable form, we return to Einstein's equation (12) and consider its normal-normal component relative to a $t = \text{const}$ surface. This corresponds to the Hamiltonian constraint equation, which has the property that the only terms containing second spacetime derivatives of h_{ab} involve only spatial derivatives of spatial components of h_{ab}. To obtain this equation from (12), we raise both indices with $g^{ab}(\lambda)$ and then take the 00 component. Since we are working in Riemannian normal coordinates about P we also express the background metric as

$$g^{(0)}_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3}R_{\alpha\mu\beta\nu}x^\alpha x^\beta + O(x^3)\,. \tag{50}$$

The terms in the resulting equation that are purely linear in h_{ab}, contain second derivatives of h_{ab}, and do not depend on the background curvature are of the form $\frac{1}{2}\partial^i\partial_i h^j{}_j - \frac{1}{2}\partial^i\partial^j h_{ij}$, i.e. the same combination of derivatives of components[5] as appears in (49). There are also terms which are linear in h_{ab} and contain second derivatives of h_{ab}, which depend on the difference between the exact background metric $g^{(0)}_{ab}$ and η_{ab}, and these can be expressed in the form $U^{ijkl}\partial_i\partial_j h_{kl}$, where $U^{ijkl} = O(x^2)$. The terms in the equation which are nonlinear in h_{ab} that contain second derivatives of h_{ab} can be expressed in the form $\partial_i W^i + Z_1$ where W^i converges to zero uniformly on compact sets as $\lambda \to 0$ and Z_1 is uniformly bounded on compact sets as $\lambda \to 0$. The remaining terms in this equation then take the form $8\pi T^{00} + Z_2$,

[5] These terms correspond to the linearization of the scalar curvature of the spatial metric, as would be expected from the general form of the "Hamiltonian constraint equation".

where Z_2 is uniformly bounded on compact sets as $\lambda \to 0$. Now multiply this equation by f_P^δ. Using (49), we see that the resulting equation takes the form

$$\partial^i \partial_i \varphi(\lambda) = 4\pi f_P^\delta T^{00}(\lambda) + \partial_i \omega^i(\lambda) + \zeta(\lambda) + \frac{1}{2} U^{ijkl} \partial_i \partial_j \psi_{kl}(\lambda) \,, \tag{51}$$

where $\omega^i \equiv \frac{1}{2} f_P^\delta W^i$ is of fixed compact support and converges to zero uniformly as $\lambda \to 0$, and ζ is of fixed compact support and is uniformly bounded as $\lambda \to 0$. Thus, φ satisfies a Poisson-like equation. Furthermore, the position space version of (47) can be written as

$$2E_\varphi = -\frac{1}{4\pi} \int \mathrm{d}t \, \mathrm{d}^3\boldsymbol{x} \partial^i \varphi \partial_i \varphi \ , \tag{52}$$

which is just the usual formula for (twice) the gravitational potential energy in Newtonian gravity! Note that we have not made any Newtonian approximations, nor have we made a special choice of "time vector" t^a.

Thus, we see that the resolution of the issue of whether $t_{00}^{(0)} \geq 0$ depends on a competition between the positive contribution from the tensor modes and the negative contribution, E_φ, arising from a Newtonian-like gravitational potential energy. We will now show that if $T_{00}(\lambda) \geq 0$, then, in fact, $E_\varphi \to 0$ as $\lambda \to 0$ and $\delta \to 0$. Thus, the scalar modes make no contribution in this limit, and the tensor modes always "win".

To prove this, we use (51) to rewrite E_φ as

$$\begin{aligned} E_\varphi &= \frac{1}{8\pi} \int \mathrm{d}t \, \mathrm{d}^3\boldsymbol{x} \varphi \partial^i \partial_i \varphi \\ &= \frac{1}{2} \int \mathrm{d}t \, \mathrm{d}^3\boldsymbol{x} \varphi(\lambda) \left[f_p^\delta T^{00}(\lambda) + \frac{1}{4\pi} \partial_i \omega^i(\lambda) + \frac{1}{4\pi} \zeta(\lambda) + \frac{1}{8\pi} U^{ijkl} \partial_i \partial_j \psi_{kl}(\lambda) \right] \ . \end{aligned} \tag{53}$$

The second term can be written as the time integral of

$$\int \mathrm{d}^3\boldsymbol{x} \varphi \partial_i \omega^i = - \int \mathrm{d}^3\boldsymbol{x} \partial_i \varphi \omega^i \ . \tag{54}$$

By the Schwartz inequality, we have

$$\left| \int \mathrm{d}^3\boldsymbol{x} \partial_i \varphi \omega^i \right| \leq \|\partial_i \varphi\|_{L^2} \|\omega^i\|_{L^2} \ . \tag{55}$$

However, $\|\omega^i\|_{L^2} \to 0$ as $\lambda \to 0$, and we have already noted that $\|\partial_i \varphi\|_{L^2}$ remains uniformly bounded as $\lambda \to 0$. Therefore, we see that the second term vanishes in the limit as λ goes to zero.

To analyze the remaining terms on the right side of (53), suppose we could show that $\varphi(\lambda)$ converges uniformly to 0 on compact sets as $\lambda \to 0$. Then since $\zeta(\lambda)$ is of fixed compact support and is uniformly bounded as $\lambda \to 0$, it follows immediately that $\int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x} \varphi(\lambda)\zeta(\lambda) \to 0$ as $\lambda \to 0$, so the third term in (53) vanishes in the limit as $\lambda \to 0$. On the other hand, if $\varphi(\lambda)$ converges uniformly and $T^{00}(\lambda) \geq 0$, then the first term is exactly of the form to which the above Lemma of this section applies, with $A = \varphi$, $B = T^{00}$, and f_P^δ being the test function with which $A(\lambda)B(\lambda)$ is being smeared. The Lemma then states that the first term in (53) vanishes in the limit as $\lambda \to 0$. Finally, the last term of (53) may be re-written as

$$-\frac{1}{16\pi} \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x} \left[U^{ijkl} \partial_i \varphi(\lambda) \partial_j \psi_{kl}(\lambda) + \partial_i U^{ijkl} \varphi(\lambda) \partial_j \psi_{kl}(\lambda) \right] . \tag{56}$$

If $\varphi(\lambda)$ converges uniformly to zero, then the second of these two terms vanishes in the limit as $\lambda \to 0$ because U^{ijkl} is independent of λ and $\partial_j \psi_{kl}(\lambda)$ is uniformly bounded as $\lambda \to 0$ and is of fixed compact support. In contrast to all the others, the first term above does not converge to zero as $\lambda \to 0$. However, it will still vanish once we subsequently take $\delta \to 0$. To see this, use the fact that on the support of f_P^δ, U^{ijkl} is bounded by a constant times δ^2 (see (50)) to write

$$\left| \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x} U^{ijkl} \partial_i \varphi(\lambda) \partial_j \psi_{kl}(\lambda) \right| \leq C\delta^2 \|\partial_i \varphi\|_{L^2} \|\partial_j \psi_{kl}\|_{L^2} \leq C' \delta^2 \left\|\partial_j \psi_{kl}\right\|^2_{L^2} . \tag{57}$$

where the last inequality follows from the fact that $\|\partial_i \varphi\|^2_{L^2} \leq \frac{1}{8} \|\partial_j \psi_{kl}\|^2_{L^2}$. However, by (40) we have

$$\lim_{\delta \to 0} \lim_{\lambda \to 0} \|\partial_j \psi_{kl}\|^2_{L^2} = \mu^{j}{}_{j}{}^{kl}{}_{kl}(P) . \tag{58}$$

Consequently, the right hand side of (57) vanishes when the limits as $\lambda \to 0$ and $\delta \to 0$ are taken. Thus, we will have proven that $t^{(0)}_{00} \geq 0$ provided only that we show that $\varphi(\lambda)$ converges uniformly to 0 on compact sets as $\lambda \to 0$.

To prove uniform convergence to 0 of $\varphi(\lambda)$ on compact sets, we note that it follows immediately from (49) that

$$4\varphi = -\psi^i{}_i + \chi , \tag{59}$$

where

$$\partial^i \partial_i \chi = \partial^i \partial^j \psi_{ij} . \tag{60}$$

As already noted above, $\psi_{ij}(\lambda)$—and hence $\psi^i{}_i(\lambda)$—converges to 0 uniformly as $\lambda \to 0$. Thus, $\varphi(\lambda)$ will converge to 0 uniformly on compact sets if and only if $\chi(\lambda)$ converges to 0 uniformly on compact sets. We will now prove this by "brute force".

The solution to (60) is

$$\chi(t,\boldsymbol{x}) = -\frac{1}{4\pi}\int \mathrm{d}^3\boldsymbol{x}'\,\frac{\partial_i\partial_j\psi^{ij}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\ . \tag{61}$$

Changing integration variables to $\boldsymbol{y} = \boldsymbol{x}' - \boldsymbol{x}$ and integrating by parts, we obtain

$$\chi(t,\boldsymbol{x}) = -\frac{1}{4\pi}\int \mathrm{d}^3\boldsymbol{y}\,\partial_i\psi^{ij}(t,\boldsymbol{x}+\boldsymbol{y})\frac{y_j}{|\boldsymbol{y}|^3}\,. \tag{62}$$

For any $r_0 > 0$, we can break up the integral appearing on the right side of this equation into an integral over $|\boldsymbol{y}| < r_0$ and an integral over $|\boldsymbol{y}| \geq r_0$. We leave the first integral alone but do another integration by parts on the second integral. We thereby obtain

$$\begin{aligned}\chi(t,\boldsymbol{x}) = &-\frac{1}{4\pi}\int_{|\boldsymbol{y}|<r_0}\mathrm{d}^3\boldsymbol{y}\,\partial_i\psi^{ij}(t,\boldsymbol{x}+\boldsymbol{y})\frac{y_j}{|\boldsymbol{y}|^3} + \frac{1}{4\pi}\int_{|\boldsymbol{y}|=r_0}\mathrm{d}\Omega\, r_0^2\frac{y_i}{r_0}\left(\psi^{ij}(t,\boldsymbol{x}+\boldsymbol{y})\frac{y_j}{r_0^3}\right)\\ &+\frac{1}{4\pi}\int_{|\boldsymbol{y}|>r_0}\mathrm{d}^3\boldsymbol{y}\,\psi^{ij}(t,\boldsymbol{x}+\boldsymbol{y})\frac{\delta_{ij}|\boldsymbol{y}|^2 - y_iy_j}{|\boldsymbol{y}|^5}\,. \end{aligned} \tag{63}$$

Now let $F_1(\lambda) \equiv \sup_{(t,\boldsymbol{x})}|\psi_{ij}|$ and let $F_2(\lambda) \equiv \sup_{(t,\boldsymbol{x})}|\partial_k\psi_{ij}|$. Then $F_1 \to 0$ and F_2 remains bounded as $\lambda \to 0$. It follows straightforwardly from (63) that

$$|\chi(t,\boldsymbol{x})| \leq c_1F_2(\lambda)r_0 + c_2F_1(\lambda) + c_3F_1(\lambda)|\ln(C/r_0)|\ , \tag{64}$$

where c_1, c_2, c_3, and C are constants (i.e., independent of λ, r_0, t, and $\boldsymbol{x}$). This bound holds for all r_0 and all λ. Therefore, as we let $\lambda \to 0$, we are free to choose r_0 to vary with λ in any way that is convenient. Choosing

$$r_0(\lambda) = \exp[-1/\sqrt{F_1(\lambda)}]\,, \tag{65}$$

we obtain the bound

$$|\chi(t,\boldsymbol{x})| \leq C_1\exp[-1/\sqrt{F_1(\lambda)}] + C_2F_1(\lambda) + C_3\sqrt{F_1(\lambda)}\ , \tag{66}$$

from which it follows immediately that $\chi \to 0$ uniformly as $\lambda \to 0$, as we desired to show.

We have thus proven

Theorem 2. *Given a one-parameter family $g_{ab}(\lambda)$ satisfying assumptions (i)–(iv) above, the effective stress energy tensor $t^{(0)}_{ab}$ appearing in equation* (15) *for the background metric $g^{(0)}_{ab}$ satisfies the weak energy condition, i.e.,*

$$t^{(0)}_{ab}t^at^b \geq 0 \tag{67}$$

for all t^a that are timelike with respect to $g^{(0)}_{ab}$.

It should be emphasized that all of the results of this section apply to an arbitrary one-parameter family $g_{ab}(\lambda)$ satisfying assumptions (i)–(iv). In particular, no symmetry or other assumptions concerning the background metric, $g^{(0)}_{ab}$, were made. However, if FLRW symmetry is assumed for $g^{(0)}_{ab}$ as well as for the (weak limit of) the matter stress-energy tensor, $T^{(0)}_{ab}$, then $t^{(0)}_{ab}$ must also have this symmetry. It then follows immediately from Theorems 1 and 2 that $t^{(0)}_{ab}$ must have the form of a perfect fluid with $P = \frac{1}{3}\rho$ and $\rho \geq 0$. In particular, the effective stress-energy tensor arising from nonlinear terms in Einstein's equation associated with short-wavelength inhomogeneities cannot produce effects similar to those of dark energy.

III. PERTURBATION THEORY

In the previous section, we obtained the equation satisfied by the background metric, $g^{(0)}_{ab}$, derived key properties of the effective stress-energy tensor $t^{(0)}_{ab}$, and thereby proved that small scale inhomogeneities cannot mimic the effects of dark energy on large scale dynamics. However, in cosmology and other contexts, we wish to know not only the dynamical behavior of $g^{(0)}_{ab}$ but also the dynamical behavior of the deviations from $g^{(0)}_{ab}$, as this is needed to describe the formation and growth of structures in the universe. In particular, we would like to obtain the equations satisfied by $h_{ab}(\lambda)$ to sufficient accuracy that $h_{ab}(\lambda)$ can be determined to first order in λ, i.e., any deviations from an exact solution (over a compact spacetime region) go to zero faster than λ as $\lambda \to 0$. As already mentioned in the introduction, if we were in the context of ordinary perturbation theory where $g_{ab}(\lambda, x)$ is jointly differentiable in λ and x, we would define

$$\gamma^{(1)}_{ab} \equiv \left.\frac{\partial g_{ab}(\lambda)}{\partial \lambda}\right|_{\lambda=0} = \lim_{\lambda\to 0} \frac{g_{ab}(\lambda) - g^{(0)}_{ab}}{\lambda} \,. \tag{68}$$

To derive the equation satisfied by $\gamma^{(1)}_{ab}$, we differentiate the Einstein equation with respect to λ, at $\lambda = 0$. The result is an equation that sets the linearized Einstein operator acting on $\gamma^{(1)}_{ab}$ equal to the derivative of the stress energy tensor with respect to λ, evaluated at $\lambda = 0$. We would then take $h_{ab}(\lambda) = \lambda\gamma^{(1)}_{ab}$. However, in the context of our framework, $g_{ab}(\lambda, x)$ is not differentiable in λ at $\lambda = 0$, so we cannot even define a notion of a "metric perturbation" by differentiating $g_{ab}(\lambda, x)$.

Of course, the (exact) equation satisfied by $h_{ab}(\lambda)$ is simply the equation obtained by substituting $g_{ab}(\lambda) = g^{(0)}_{ab} + h_{ab}(\lambda)$ into Einstein's equation (12). However, this is not any more useful in practice than simply asserting that $g_{ab}(\lambda)$ must be a solution of Einstein's equation for all λ; if we could solve Einstein's equation exactly, there would be no need to develop a perturbative formalism. The key idea needed to obtain a more useful version of Einstein's equation is that, although nonlinearities may be important on small scales, there should be a simpler, linear description on large scales. The key idea needed to implement this description is the observation that although the ordinary (pointwise or uniform) limit of $[g_{ab}(\lambda) - g^{(0)}_{ab}]/\lambda$ does not exist in the context of our framework—$g_{ab}(\lambda)$ is not differentiable—there is no reason why the *weak limit* of this quantity cannot exist.

Thus, a natural generalization of the conventional linearized metric perturbation is

$$\gamma^{(L)}_{ab} \equiv \underset{\lambda\to 0}{\text{w-lim}} \frac{g_{ab}(\lambda) - g^{(0)}_{ab}}{\lambda} . \tag{69}$$

Here we have replaced the ordinary limit of (68) with a weak limit, which "averages away" the small scale inhomogeneities. This quantity thereby corresponds closely to the notion of the "long wavelength part" of the metric perturbation that appears in other analyses (see, e.g., [27]). We will discuss this further in section V below. The remainder of the perturbation will be denoted

$$h^{(S)}_{ab}(\lambda) \equiv h_{ab}(\lambda) - \lambda\gamma^{(L)}_{ab} , \tag{70}$$

and will be referred to as the "short wavelength part" of the deviation[6] of the metric from $g^{(0)}_{ab}$. In subsection IVB, we will argue that, under suitable Newtonian assumptions in cosmology, to leading order in λ, $h^{(S)}_{ab}$ depends only *locally* on the matter distribution and is well approximated by a Newtonian gravity solution. We emphasize that, within our framework, short and long wavelength perturbations have a very different character. The long wavelength part of a perturbation has a well-defined description in the $\lambda \to 0$ limit, namely $\gamma^{(L)}_{ab}$. On the other hand, the short wavelength part, $h^{(S)}_{ab}(\lambda)$, is defined only for $\lambda > 0$ and has no description in terms of a limit as $\lambda \to 0$.

We can obtain an equation for $\gamma^{(L)}_{ab}$ by taking the difference of the exact Einstein equation (12) for $g_{ab}(\lambda)$ and the background Einstein equation (15) for $g^{(0)}_{ab}$, dividing by λ, and taking the weak limit as $\lambda \to 0$. However, in order to ensure that $\gamma^{(L)}_{ab}$ is well defined and satisfies

[6] If we were to consider higher order perturbation theory, then we would also subtract from h_{ab} higher order in λ "long wavelength" contributions to define $h^{(S)}_{ab}(\lambda)$. However, we shall only be concerned with first order perturbation theory in this paper.

a well-defined equation, we must append to assumptions (i)–(iv) of section II the following additional assumptions on our one-parameter family $g_{ab}(\lambda)$:

(v) There exist smooth tensor fields $\gamma^{(L)}_{ab}$, $\mu^{(1)}_{abcdef}$, $\nu^{(1)}_{abcde}$ and $\omega^{(1)}_{abcdefgh}$ on M such that

(a)
$$\underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}h_{ab}(\lambda) = \gamma^{(L)}_{ab}\,, \tag{71}$$

(b)
$$\underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}\left[\nabla_{(a}h^{(S)}_{|cd|}(\lambda)\nabla_{b)}h^{(S)}_{ef}(\lambda) - \mu_{abcdef}\right] = \mu^{(1)}_{abcdef}\,, \tag{72}$$

(c)
$$\underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}\left[h^{(S)}_{bc}(\lambda)\nabla_a h^{(S)}_{de}(\lambda)\right] = \nu^{(1)}_{abcde}\,, \tag{73}$$

(d)
$$\underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}\left[h^{(S)}_{cd}(\lambda)\nabla_a h^{(S)}_{ef}(\lambda)\nabla_b h^{(S)}_{gh}(\lambda)\right] = \omega^{(1)}_{abcdefgh}\,. \tag{74}$$

In the following, we shall assume that our one-parameter family $g_{ab}(\lambda)$ satisfies assumptions (i)–(iv) of section II together with assumption (v) above. In this section we will make no additional assumptions about $g_{ab}(\lambda)$, so all of the results obtained in this section should hold, e.g., for self-gravitating gravitational radiation in a background without any symmetries. In section IV, we shall specialize to the case of Newtonian-like cosmological perturbations off of a background metric with FLRW symmetry, and will make numerous additional assumptions and simplifications.

The newly defined first order backreaction tensors, $\mu^{(1)}_{abcdef}$, $\nu^{(1)}_{abcde}$ and $\omega^{(1)}_{abcdefgh}$, possess certain tensor symmetries as a direct consequence of their definitions. Clearly, as with the zeroth order quantity μ_{abcdef}, we have $\mu^{(1)}_{(ab)(cd)(ef)} = \mu^{(1)}_{abcdef}$ and $\mu^{(1)}_{abcdef} = \mu^{(1)}_{baefcd}$. However, in contrast with μ_{abcdef}, the symmetry under interchange of the first two indices had to be built directly into the definition of $\mu^{(1)}_{abcdef}$, rather than derived. Indeed,

$$\begin{aligned}
&\underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}\left[\nabla_a h_{cd}(\lambda)\nabla_b h_{ef}(\lambda) - \mu_{abcdef}\right] \\
&= \mu^{(1)}_{abcdef} + \underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}\nabla_{[a}h_{|cd|}(\lambda)\nabla_{b]}h_{ef}(\lambda) \\
&= \mu^{(1)}_{abcdef} + \underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}\left[\nabla_{[a}\left(h_{|cd|}(\lambda)\nabla_{b]}h_{ef}(\lambda)\right) - h_{cd}(\lambda)\nabla_{[a}\nabla_{b]}h_{ef}(\lambda)\right] \\
&= \mu^{(1)}_{abcdef} + \nabla_{[a}\nu^{(1)}_{b]cdef} - 2\,\underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}h_{cd}(\lambda)R_{ab(e}{}^{g}h_{f)g}(\lambda) \\
&= \mu^{(1)}_{abcdef} + \nabla_{[a}\nu^{(1)}_{b]cdef}\,.
\end{aligned} \tag{75}$$

What this calculation also illustrates is that, because of the factor of $1/\lambda$, we may no longer freely drop total derivative terms when taking weak limits, and in general we will pick up terms of the form $\nabla_a \nu^{(1)}_{bcdef}$. It follows also that $\nu^{(1)}_{a(bc)(de)} = \nu^{(1)}_{abcde}$ and $\nu^{(1)}_{abcde} = -\nu^{(1)}_{adebc}$. Finally, $\omega^{(1)}_{ab(cd)(ef)(gh)} = \omega^{(1)}_{abcdefgh}$, $\omega^{(1)}_{abcdefgh} = \omega^{(1)}_{bacdghef}$, and

$$\omega^{(1)}_{abcdefgh} - \omega^{(1)}_{bacdefgh} = \omega^{(1)}_{baefcdgh} - \omega^{(1)}_{abefcdgh} \,. \tag{76}$$

We also note that definitions (b) and (c) would be unchanged if $h^{(S)}_{ab}$ were replaced by h_{ab}, and (d) would be unchanged if any $h^{(S)}_{ab}$ which is being differentiated were replaced by h_{ab}.

We now subtract the background Einstein equation (15) from the exact Einstein equation (12), divide by λ, and then take the weak limit as $\lambda \to 0$. A very lengthy calculation, performed with the help of the *xAct* tensor manipulation package [29] for *Mathematica*, yields

$$\begin{aligned}
&\nabla^c\nabla_{(a}\gamma^{(L)}_{b)c} - \frac{1}{2}\nabla^c\nabla_c\gamma^{(L)}_{ab} - \frac{1}{2}\nabla_a\nabla_b\gamma^{(L)c}{}_{c} - \frac{1}{2}g^{(0)}_{ab}\left(\nabla^c\nabla^d\gamma^{(L)}_{cd} - \nabla^c\nabla_c\gamma^{(L)d}{}_{d}\right)\\
&+\frac{1}{2}g^{(0)}_{ab}R^{cd}(g^{(0)})\gamma^{(L)}_{cd} - \frac{1}{2}R(g^{(0)})\gamma^{(L)}_{ab} + \Lambda\gamma^{(L)}_{ab} + \frac{1}{8}\gamma^{(L)}_{ab}\left(\mu^{c}{}_{c}{}^{d}{}_{d}{}^{e}{}_{e} + \mu^{c}{}_{c}{}^{de}{}_{de} - 2\mu^{cd}{}_{c}{}^{e}{}_{de}\right)\\
&+\gamma^{(L)cd}\Bigg(\frac{1}{2}\mu_{abc}{}^{e}{}_{de} - \frac{1}{2}\mu_{c(ab)d}{}^{e}{}_{e} - \frac{1}{2}\mu^{e}{}_{(ab)ecd} + \frac{3}{4}\mu_{cdab}{}^{e}{}_{e} - \frac{1}{2}\mu_{cda}{}^{e}{}_{be} - \mu_{c}{}^{e}{}_{abde} + \mu_{c}{}^{e}{}_{e(ab)d}\\
&+\frac{3}{4}\mu^{e}{}_{eabcd} - \frac{1}{2}\mu^{e}{}_{eacbd} + \frac{1}{8}g^{(0)}_{ab}\left\{-\mu_{cd}{}^{e}{}_{e}{}^{f}{}_{f} - \mu_{cd}{}^{ef}{}_{ef} + 4\mu_{c}{}^{e}{}_{d}{}^{f}{}_{ef} - 2\mu^{e}{}_{ecd}{}^{f}{}_{f} + 2\mu^{ef}{}_{cedf}\right\}\Bigg)\\
=\;& 8\pi T^{(1)}_{ab} + \frac{1}{8}g^{(0)}_{ab}\left\{-\mu^{(1)c}{}_{c}{}^{de}{}_{de} - \mu^{(1)c}{}_{c}{}^{d}{}_{d}{}^{e}{}_{e} + 2\mu^{(1)cd}{}_{c}{}^{e}{}_{de}\right\} + \frac{1}{2}\mu^{(1)cd}{}_{acbd}\\
&-\frac{1}{2}\mu^{(1)c}{}_{ca}{}^{d}{}_{bd} + \frac{1}{4}\mu^{(1)}{}_{ab}{}^{cd}{}_{cd} - \frac{1}{2}\mu^{(1)c}{}_{(ab)c}{}^{d}{}_{d} + \frac{3}{4}\mu^{(1)c}{}_{cab}{}^{d}{}_{d} - \frac{1}{2}\mu^{(1)cd}{}_{abcd}\\
&+\frac{1}{8}g^{(0)}_{ab}\left\{2\omega^{(1)c}{}_{c}{}^{de}{}_{de}{}^{f}{}_{f} + 2\omega^{(1)c}{}_{c}{}^{de}{}_{d}{}^{f}{}_{ef} + \omega^{(1)cd}{}_{cd}{}^{e}{}_{e}{}^{f}{}_{f} + \omega^{(1)cd}{}_{cd}{}^{ef}{}_{ef} - 4\omega^{(1)cd}{}_{c}{}^{e}{}_{d}{}^{f}{}_{ef} - 2\omega^{(1)cdef}{}_{decf}\right\}\\
&-\frac{1}{2}\omega^{(1)}{}_{ab}{}^{cd}{}_{c}{}^{e}{}_{de} + \frac{1}{2}\omega^{(1)}{}_{(a}{}^{c}{}_{|c|}{}^{d}{}_{b)d}{}^{e}{}_{e} + \frac{1}{2}\omega^{(1)}{}_{(a}{}^{cde}{}_{b)cde} - \frac{1}{8}\omega^{(1)c}{}_{cab}{}^{d}{}_{d}{}^{e}{}_{e} - \frac{1}{8}\omega^{(1)c}{}_{cab}{}^{de}{}_{de} - \frac{3}{4}\omega^{(1)c}{}_{c}{}^{de}{}_{abde}\\
&+\frac{1}{2}\omega^{(1)c}{}_{c}{}^{de}{}_{adbe} + \frac{1}{4}\omega^{(1)cd}{}_{abd}{}^{e}{}_{ce} - \frac{3}{4}\omega^{(1)cd}{}_{cdab}{}^{e}{}_{e} + \frac{1}{2}\omega^{(1)cd}{}_{cda}{}^{e}{}_{be} + \frac{1}{2}\omega^{(1)cd}{}_{c}{}^{e}{}_{abde} - \frac{1}{2}\omega^{(1)cd}{}_{c}{}^{e}{}_{adbe}\\
&+\frac{1}{2}\omega^{(1)cd}{}_{d}{}^{e}{}_{abce} - \frac{1}{2}\omega^{(1)cd}{}_{d}{}^{e}{}_{aebc} + \frac{1}{4}g^{(0)}_{ab}\left\{2\nabla_d\nu^{(1)c}{}_{c}{}^{de}{}_{e} + \nabla_e\nu^{(1)c}{}_{c}{}^{d}{}_{d}{}^{e}\right\} - \frac{1}{4}\nabla_{(a}\nu^{(1)c}{}_{b)c}{}^{d}{}_{d}\\
&+\frac{1}{4}\nabla_c\nu^{(1)}{}_{(ab)}{}^{cd}{}_{d} - \frac{1}{2}\nabla_c\nu^{(1)c}{}_{ab}{}^{d}{}_{d} - \nabla_d\nu^{(1)}{}_{(ab)}{}^{c}{}_{c}{}^{d} + \nabla_d\nu^{(1)c}{}_{abc}{}^{d} + \frac{1}{2}\nabla_d\nu^{(1)c}{}_{c(ab)}{}^{d} \,.
\end{aligned} \tag{77}$$

Here, we have written

$$T^{(1)}_{ab} \equiv \underset{\lambda\to 0}{\text{w-lim}}\, \frac{T_{ab}(\lambda) - T^{(0)}_{ab}}{\lambda} \,. \tag{78}$$

This weak limit exists by virtue of assumption (v) and the fact that $g_{ab}(\lambda)$ satisfies Einstein's equation.

Our equation (77) for $\gamma^{(L)}_{ab}$ takes the form of a modified linearized Einstein equation. The terms on the left side are linear in $\gamma^{(L)}_{ab}$ and, in addition to the usual terms appearing in the linearized Einstein tensor, contain terms proportional to μ_{abcdef}. The right side contains, in addition to the matter source $T^{(1)}_{ab}$, numerous "effective source terms" arising from $\mu^{(1)}_{abcdef}$, $\nu^{(1)}_{abcde}$ and $\omega^{(1)}_{abcdefgh}$.

Some relations between $\mu^{(1)}_{abcdef}$, $\nu^{(1)}_{abcde}$ and $\omega^{(1)}_{abcdefgh}$ can be derived from Einstein's equation. We previously derived the relation $\alpha_{aeb}{}^{e}{}_{cd} = 0$ (see (28)) by starting with Einstein's equation in "Ricci form" (19), multiplying it by $h_{ef}(\lambda)$ and taking the weak limit as $\lambda \to 0$. In a similar manner, if we multiply (19) by $h^{(S)}_{cd}(\lambda)h^{(S)}_{ef}(\lambda)/\lambda$, we obtain the following equation satisfied by $\omega^{(1)}_{abcdefgh}$:

$$\begin{aligned}
&\omega^{(1)g}{}_{acdefbg} + \omega^{(1)g}{}_{bcdefag} - \omega^{(1)g}{}_{gcdefab} - \omega^{(1)}{}_{abcdef}{}^{g}{}_{g} \\
&+\omega^{(1)g}{}_{aefcdbg} + \omega^{(1)g}{}_{befcdag} - \omega^{(1)g}{}_{gefcdab} - \omega^{(1)}{}_{abefcd}{}^{g}{}_{g} = 0\,.
\end{aligned} \tag{79}$$

In deriving this, we used the fact that

$$\operatorname*{w-lim}_{\lambda\to 0} \frac{1}{\lambda} h^{(S)}_{ab}(\lambda) h^{(S)}_{cd}(\lambda) T_{ef}(\lambda) = 0\,, \tag{80}$$

which follows from our lemma from section 2 and the assumption that $T_{ab}(\lambda)$ satisfies the weak energy condition, in the same way that we showed $\kappa_{abcd} = 0$ (see (27)).

Similarly, if we subtract the background Einstein equation in "Ricci form" from (19), multiply by $h^{(S)}_{cd}(\lambda)/\lambda$, and take the $\lambda \to 0$ weak limit, we can derive an analog of (28) satisfied by $\mu^{(1)}_{abcdef}$,

$$\begin{aligned}
&\frac{1}{2}\mu^{(1)}{}_{abcd}{}^{e}{}_{e} + \frac{1}{2}\mu^{(1)e}{}_{cdab} - \mu^{(1)e}{}_{(a|cd|b)e} \\
&= 8\pi\kappa^{(1)}{}_{cdab} - 4\pi g^{(0)}_{ab}\kappa^{(1)}{}_{cd}{}^{e}{}_{e} + \frac{1}{2}\gamma^{(L)ef}\left(\mu_{abefcd} + \mu_{efabcd} - 2\mu_{e(ab)fcd}\right) + \frac{1}{4}\omega^{(1)}{}_{abcd}{}^{ef}{}_{ef} \\
&\quad+\frac{1}{2}\omega^{(1)}{}_{ab}{}^{ef}{}_{cdef} - \frac{1}{2}\omega^{(1)}{}_{(a}{}^{e}{}_{|cd|b)e}{}^{f}{}_{f} - \omega^{(1)}{}_{(a}{}^{e}{}_{|e|}{}^{f}{}_{b)fcd} + \frac{1}{4}\omega^{(1)e}{}_{ecdab}{}^{f}{}_{f} - \frac{1}{2}\omega^{(1)e}{}_{ecda}{}^{f}{}_{bf} + \frac{1}{2}\omega^{(1)ef}{}_{cdafbe} \\
&\quad+\frac{1}{2}\omega^{(1)ef}{}_{efabcd} + \frac{1}{2}\nabla_{(a}\nu^{(1)}{}_{b)cd}{}^{e}{}_{e} + \frac{1}{2}\nabla_{(a}\nu^{(1)e}{}_{b)ecd} + \frac{1}{2}\nabla_{e}\nu^{(1)}{}_{(ab)}{}^{e}{}_{cd} - \frac{1}{2}\nabla_{e}\nu^{(1)e}{}_{abcd}\,.
\end{aligned} \tag{81}$$

In this equation we have defined the quantity

$$\kappa^{(1)}_{abcd} = \operatorname*{w-lim}_{\lambda\to 0} \frac{1}{\lambda} h^{(S)}_{ab}(\lambda) T_{cd}(\lambda)\,. \tag{82}$$

The existence of this limit is guaranteed by Einstein's equation together with our other assumptions. Note that since $h^{(S)}_{ab}(\lambda)/\lambda$ does not converge uniformly to zero as $\lambda \to 0$

(although it is uniformly bounded), the weak energy condition does not imply the vanishing of $\kappa^{(1)}_{abcd}$ as it did for κ_{abcd}.

We can simplify the above equations as follows. As with the background case, define $\alpha^{(1)}_{abcdef} = \mu^{(1)}_{[c|[ab]|d]ef}$ and $\beta^{(1)}_{abcdef} = \mu^{(1)}_{(abcd)ef}$, and express the equations in terms of these quantities. A similar breakup is also possible for $\omega^{(1)}_{abcdefgh}$. First, split it into the parts with are symmetric and antisymmetric in the first two indices, $\omega^{(1,S)}_{abcdefgh} = \omega^{(1)}_{(ab)cdefgh}$ and $\omega^{(1,A)}_{abcdefgh} = \omega^{(1)}_{[ab]cdefgh}$. By (76), we have

$$\omega^{(1,A)}_{abcdefgh} = -\omega^{(1,A)}_{abefcdgh}\,. \tag{83}$$

The symmetric part is then decomposed into $\omega^{(1,\alpha)}_{abcdefgh} = \omega^{(1,S)}_{[e|[a|cd|b]|f]gh}$ and $\omega^{(1,\beta)}_{abcdefgh} = \omega^{(1,S)}_{(ab|cd|ef)gh}$, with inverse transformation

$$\begin{aligned}\omega^{(1,S)}_{abcdefgh} = -\frac{4}{3}\Big(&\omega^{(1,\alpha)}_{e(a|cd|b)fgh} + \omega^{(1,\alpha)}_{g(a|cd|b)hef} - \omega^{(1,\alpha)}_{g(e|cd|f)hab}\Big)\\ &+\omega^{(1,\beta)}_{abcdefgh} + \omega^{(1,\beta)}_{abcdghef} - \omega^{(1,\beta)}_{efcdghab}\,.\end{aligned} \tag{84}$$

Substituting for our new quantities, (79) can be rewritten as

$$\omega^{(1,\alpha)}{}_{a}{}^{g}{}_{cdbgef} + \omega^{(1,\alpha)}{}_{a}{}^{g}{}_{efbgcd} = 0\,, \tag{85}$$

whereas (81) becomes

$$\begin{aligned}\alpha^{(1)}{}_{a}{}^{e}{}_{becd} = {}& 4\pi\kappa^{(1)}{}_{cdab} - 2\pi g^{(0)}_{ab}\kappa^{(1)}{}_{cd}{}^{e}{}_{e} + \alpha_{aebfcd}\gamma^{(L)ef} + \frac{1}{4}\omega^{(1,A)}{}_{(a}{}^{e}{}_{b)ecd}{}^{f}{}_{f} - \frac{1}{2}\omega^{(1,A)}{}_{(a}{}^{e}{}_{b)}{}^{f}{}_{cdef}\\ &-\frac{1}{4}\omega^{(1,A)ef}{}_{aebfcd} + \frac{1}{2}\omega^{(1,\alpha)}{}_{a}{}^{e}{}_{cdb}{}^{f}{}_{ef} + \omega^{(1,\alpha)}{}_{(a}{}^{e}{}_{|cde|}{}^{f}{}_{b)f} + \omega^{(1,\alpha)}{}_{a}{}^{e}{}_{e}{}^{f}{}_{bfdc} - \frac{1}{4}\omega^{(1,\alpha)ef}{}_{baefdc}\\ &+\frac{1}{4}\nabla_{(a}\nu^{(1)}{}_{b)cd}{}^{e}{}_{e} + \frac{1}{4}\nabla_{(a}\nu^{(1)e}{}_{b)ecd} + \frac{1}{4}\nabla_{e}\nu^{(1)}{}_{(ab)}{}^{e}{}_{cd} - \frac{1}{4}\nabla_{e}\nu^{(1)e}{}_{abcd}\,.\end{aligned} \tag{86}$$

Finally we can use (85) and (86) to simplify our version of the linearized Einstein equation, (77), resulting in

$$
\begin{aligned}
&\nabla^c\nabla_{(a}\gamma^{(L)}_{b)c} - \frac{1}{2}\nabla^c\nabla_c\gamma^{(L)}_{ab} - \frac{1}{2}\nabla_a\nabla_b\gamma^{(L)c}{}_c - \frac{1}{2}g^{(0)}_{ab}\left(\nabla^c\nabla^d\gamma^{(L)}_{cd} - \nabla^c\nabla_c\gamma^{(L)d}{}_d\right) \\
&+\frac{1}{2}g^{(0)}_{ab}R^{cd}(g^{(0)})\gamma^{(L)}_{cd} - \frac{1}{2}R(g^{(0)})\gamma^{(L)}_{ab} + \Lambda\gamma^{(L)}_{ab} + 2\gamma^{(L)cd}\alpha_{(a}{}^e{}_{b)cde} \\
= {}& 8\pi T^{(1)}_{ab} + \alpha^{(1)}{}_a{}^c{}_b{}^d{}_{cd} - 2\pi\kappa^{(1)}{}_{ab}{}^c{}_c - 8\pi\kappa^{(1)}{}_{(a}{}^c{}_{b)c} + 2\pi g^{(0)}_{ab}\left\{\kappa^{(1)c}{}_c{}^d{}_d - \kappa^{(1)cd}{}_{cd}\right\} \\
&-\frac{1}{4}\omega^{(1,A)}{}_{(a}{}^c{}_{b)}{}^d{}_{cd}{}^e{}_e + \frac{1}{2}\omega^{(1,A)}{}_{(a}{}^c{}_{b)}{}^d{}_c{}^e{}_{de} + \frac{1}{8}\omega^{(1,A)cd}{}_{abc}{}^e{}_{de} + \frac{1}{4}\omega^{(1,A)cd}{}_{acbd}{}^e{}_e \\
&+\frac{1}{16}g^{(0)}_{ab}\left\{\omega^{(1,A)cd}{}_c{}^e{}_{de}{}^f{}_f + 2\omega^{(1,A)cd}{}_c{}^e{}_d{}^f{}_{ef}\right\} - 2\omega^{(1,\alpha)}{}_{(a}{}^{ced}{}_{b)dce} - \omega^{(1,\alpha)}{}_{(a}{}^c{}_{b)d}{}^{de}{}_{ce} - \omega^{(1,\alpha)}{}_{(a}{}^{ce}{}_{b)}{}^d{}_{cde} \\
&+\frac{1}{8}\omega^{(1,\alpha)cd}{}_{abcd}{}^e{}_e + \frac{1}{4}\omega^{(1,\alpha)cd}{}_{bac}{}^e{}_{de} + \frac{1}{8}g^{(0)}_{ab}\left\{\omega^{(1,\alpha)c}{}_{dec}{}^{edf}{}_f - 2\omega^{(1,\alpha)c}{}_{dec}{}^{efd}{}_f\right\} \\
&-\frac{1}{2}\nabla_{(a}\nu^{(1)c}{}_{b)c}{}^d{}_d + \frac{1}{4}\nabla_{(a}\nu^{(1)c}{}_{b)}{}^d{}_{cd} + \frac{1}{4}\nabla_c\nu^{(1)c}{}_{ab}{}^d{}_d - \frac{3}{4}\nabla_d\nu^{(1)}{}_{(ab)}{}^c{}_c{}^d + \frac{1}{4}\nabla_d\nu^{(1)c}{}_{abc}{}^d \\
&-\frac{1}{2}\nabla_d\nu^{(1)c}{}_{(a}{}^d{}_{b)c} + \frac{1}{4}g^{(0)}_{ab}\left\{\nabla_d\nu^{(1)c}{}_c{}^{de}{}_e + \nabla_e\nu^{(1)c}{}_c{}^d{}_d{}^e\right\}\,.
\end{aligned}
\tag{87}
$$

Equations (85), (86) and (87) describe the long wavelength perturbations. It should be noted that, just as β_{abcdef} was absent from our background equations, β_{abcdef}, $\beta^{(1)}_{abcdef}$, and $\omega^{(1,\beta)}_{abcdefgh}$ are all absent from our perturbation equations.

Equations (85), (86) and (87) have been written down in an arbitrary gauge. We will make a specific choice of gauge in subsection IVA below, but for now we note that we can apply any one-parameter family of diffeomorphisms, ϕ_λ, to $g_{ab}(\lambda)$ that preserves conditions (i)–(v). Burnett [26] has analyzed the properties of gauge transformations associated with one-parameter families of diffeomorphisms that are not smooth in λ. Here, we simply note that any smooth, one-parameter group of diffeomorphisms ϕ_λ generates gauge transformations that are easily seen to preserve conditions (i)–(v). Under such gauge transformations, it is not difficult to see that $\gamma^{(L)}_{ab} \to \gamma^{(L)}_{ab} + \mathcal{L}_\xi g^{(0)}_{ab}$, where ξ^a is the vector field that generates ϕ_λ and $\mathcal{L}$ denotes the Lie derivative. Thus, $\gamma^{(L)}_{ab}$ has the same gauge freedom arising from smooth ϕ_λ as in ordinary linearized perturbation theory. This freedom can be used to impose the same types of gauge conditions on $\gamma^{(L)}_{ab}$ as in ordinary linearized perturbation theory. It is also not difficult to see that $h^{(S)}_{ab}(\lambda) \to \phi^*_\lambda h^{(S)}_{ab}(\lambda) + j_{ab}(\lambda)$, where $j_{ab}(\lambda) = O(\lambda^2)$ and is jointly smooth in λ and the spacetime point. By using this gauge transformation property of $h^{(S)}_{ab}(\lambda)$, it is possible to show that μ_{abcdef}, $\nu^{(1)}_{abcde}$, $\omega^{(1)}_{abcdefgh}$, and $\kappa^{(1)}_{abcd}$ are gauge invariant under gauge transformations arising from smooth ϕ_λ, whereas $\mu^{(1)}_{abcdef} \to \mu^{(1)}_{abcdef} + \mathcal{L}_\xi\mu_{abcdef}$ and $T^{(1)}_{ab} \to T^{(1)}_{ab} + \mathcal{L}_\xi T^{(0)}_{ab}$.

We turn our attention now to the short wavelength perturbations. Without making any approximations it is straightforward to write down an equation satisfied by $h_{ab}^{(S)}(\lambda)$: Simply substitute $g_{ab}(\lambda) = g_{ab}^{(0)} + \lambda\gamma_{ab}^{(L)} + h_{ab}^{(S)}(\lambda)$ into the exact Einstein equation. We may write this equation in the form

$$G_{ab}^{(1)}(g^{(0)}, h^{(S)}(\lambda)) + \Lambda h_{ab}^{(S)}(\lambda) = 8\pi T_{ab}(\lambda) - G_{ab}(g^{(0)}) - \Lambda g_{ab}^{(0)} - \lambda G_{ab}^{(1)}(g^{(0)}, \gamma^{(L)}) \\ -\lambda\Lambda\gamma_{ab}^{(L)} - \sum_{n=2}^{\infty} G_{ab}^{(n)}(g^{(0)}, \lambda\gamma^{(L)} + h^{(S)}(\lambda)) \,, \tag{88}$$

where we have grouped linear terms in $h_{ab}^{(S)}(\lambda)$ on the left hand side. Here, $G_{ab}^{(n)}(g^{(0)}, \lambda\gamma^{(L)} + h^{(S)}(\lambda))$ denotes the nth order Einstein tensor expanded about $g_{ab}^{(0)}$ of the perturbation $\lambda\gamma_{ab}^{(L)} + h_{ab}^{(S)}(\lambda)$.

Unfortunately, it does not appear possible to simplify (88) to obtain suitable approximate solutions without introducing additional assumptions. Of course, if we do not simplify (88), then we have not made any progress beyond asserting that we must solve Einstein's equation. In the next section, we will introduce additional assumptions relevant to the case of cosmological perturbations and argue that to obtain an accurate description of the metric to order λ, we may replace (88) by the equations of Newtonian gravity with local matter sources.

For the remainder of this section, we shall compare our general analysis to that given by Isaacson [23, 24] and others (see, e.g., [25]), who were interested in describing the self-gravitating effects of gravitational radiation. We therefore restrict attention to the vacuum case ($T_{ab}(\lambda) = 0$). These authors work with the quantity $h_{ab}(\lambda)$ rather than introducing μ_{abcdef}. Suppose one is merely interested in determining the background metric $g_{ab}^{(0)}$, i.e., one is not interested in obtaining an accurate (to order λ) description of the deviation of the metric from $g_{ab}^{(0)}$. Then one would need only to calculate $h_{ab}(\lambda)$ to sufficient accuracy that one could determine μ_{abcdef} and, thereby, $t_{ab}^{(0)}$ (see (16)). In particular, one would not be interested in computing $\gamma_{ab}^{(L)}$, so one could ignore the equations we have derived above for $\gamma_{ab}^{(L)}$. Furthermore, in order to obtain $h_{ab}^{(S)}(\lambda)$ to sufficient accuracy, it appears plausible that one could make $O(1)$ modifications to (88) as $\lambda \to 0$ and still determine $h_{ab}^{(S)}(\lambda)$ to sufficient accuracy, provided that these $O(1)$ modifications have vanishing weak limit. Here, by the phrase "it appears plausible" we mean that we believe it is likely that one could introduce additional reasonable assumptions on the one-parameter family $g_{ab}(\lambda)$ so that these modifications to (88) could be made without affecting $g_{ab}^{(0)}$. The reason for this belief is that $O(1)$ error terms in (88) should—under suitable further assumptions similar to ones indicated at the end of subsection IVB below—give rise to $O(\lambda^2)$ errors in $h^{(S)}_{ab}(\lambda)$, which should not affect μ_{abcdef}.

A candidate modification of (88) in the vacuum case would be to drop the entire right side of this equation (together with the term $\Lambda h^{(S)}_{ab}(\lambda)$ on the left side), to obtain simply the linearized Einstein equation for $h^{(S)}_{ab}$ off of $g^{(0)}_{ab}$,

$$G^{(1)}_{ab}(g^{(0)}, h^{(S)}(\lambda)) = 0\,. \tag{89}$$

However, if $G_{ab}(g^{(0)}) \neq 0$, the linearized Einstein equation off of $g^{(0)}_{ab}$ does not appear to have an initial value formulation, so this modification of (88) is probably not suitable. (Note that the linearized Einstein equation off of a non-solution is not gauge invariant, so one cannot employ a choice of gauge to simplify the equation and/or put it in hyperbolic form.) A better candidate modification would be an equation of the same form that the linearized Einstein equation would take in the Lorenz gauge if perturbed off of a vacuum spacetime. Since the linearized Einstein equation off of a non-vacuum spacetime is inconsistent with the Lorenz gauge condition, there is no unique choice of such an equation—in particular, one can add new terms involving the background Ricci tensor—and, indeed, Isaacson [23, 24] and Misner, Thorne, and Wheeler [25] give slightly different forms of the proposed equation (compare Eq. (5.12) of [23] with Eq. (35.68) of [25]). In fact, since terms involving the product of the background curvature with $h^{(S)}_{ab}$ are $O(\lambda)$, it would appear simplest to drop all of these terms and work with the wave equation

$$\nabla^c \nabla_c \bar{h}^{(S)}_{ab} = 0\,, \tag{90}$$

where $\bar{h}^{(S)}_{ab} = h^{(S)}_{ab} - \frac{1}{2} g^{(0)}_{ab} h^{(S)c}{}_c$. As with the equations used in [23] and [25], this equation is inconsistent with the Lorenz gauge condition $\nabla^a \bar{h}^{(S)}_{ab} = 0$. However, if one constrains the initial data for $\bar{h}^{(S)}_{ab}$ for solutions to (90) so that $\nabla^a \bar{h}^{(S)}_{ab}$ and its first time derivative vanish initially, then the Lorenz gauge condition should hold to $O(\lambda)$ at later times (in compact regions of spacetime). Thus, the solutions to (90) with these initial data restrictions should satisfy Einstein's equation (88) to the desired $O(1)$ accuracy. It should also be possible to impose the gauge condition $h^{(S)a}{}_a = 0$ to $O(\lambda)$ accuracy.

If the above arguments are correct, then in order to obtain the possible background metrics $g^{(0)}_{ab}$, it should suffice to simultaneously solve (15) and (90) (or equivalently, Eq. (5.12) of [23] or Eq. (35.68) of [25]) with appropriate restrictions on initial data, to obtain $g^{(0)}_{ab}$ and a one-parameter family $h^{(S)}_{ab}(\lambda)$ satisfying our conditions (ii)–(iv), where, in (15), $t^{(0)}_{ab}$ is given by (16) and μ_{abcdef} is given by (8) with $h_{ab}(\lambda)$ replaced by our one-parameter family of solutions to (90). One could then attempt to establish properties of $t^{(0)}_{ab}$—in particular, the vanishing of its trace and the positivity of its energy density—by working with the expressions for it in terms of $h^{(S)}_{ab}(\lambda)$ in a particular gauge. Needless to say, numerous extremely murky mathematical issues arise if one proceeds in this manner. Our analysis of section II, as well as the work of Burnett [26] in the vacuum case, proved rigorous results about $t^{(0)}_{ab}$ without introducing any approximate equations satisfied by $h^{(S)}_{ab}(\lambda)$, and thus completely bypassed these murky issues.

Finally, we note that if one wished to know the deviation of the metric from $g^{(0)}_{ab}$ to $O(\lambda)$—as would not necessarily be of interest in studying gravitational radiation but is of considerable interest in cosmology—then, of course, it would be necessary to know $\gamma^{(L)}_{ab}$. Although we argued above that one has considerable freedom in modifying the equation satisfied by $h^{(S)}_{ab}$ without affecting $g^{(0)}_{ab}$, it is clear that one cannot make any modification to the equation (87) satisfied by $\gamma^{(L)}_{ab}$ without introducing $O(1)$ errors in $\gamma^{(L)}_{ab}$ and thus $O(\lambda)$ errors in $g_{ab}(\lambda)$. Furthermore, in order to calculate the effective source term $\alpha^{(1)}{}_a{}^c{}_b{}^d{}_{cd}$ in (87), it appears that one would need to know $h^{(S)}_{ab}$ to $O(\lambda^2)$, in which case one could not drop the quadratic terms in $h^{(S)}_{ab}$ in (88). Thus, in the case of self-gravitating gravitational radiation, if one wished to know the deviation of the metric from $g^{(0)}_{ab}$ to $O(\lambda)$, one would have to solve (15), (87), and some suitable simplification of (88). This would comprise an extremely complicated system. Fortunately, in the case of cosmology, we will now argue that, under additional assumptions, significant simplifications occur.

IV. COSMOLOGICAL PERTURBATION THEORY

Up to this point we have not assumed any symmetries or other special properties of the background metric $g^{(0)}_{ab}$. We also have not made any restrictions on the matter content, $T_{ab}(\lambda)$, other than that it satisfy the weak energy condition, nor have we imposed any restrictions on the perturbations. In this section, we will be concerned with the case of main interest in cosmology, where $g^{(0)}_{ab}$ has FLRW symmetry, there is negligible gravitational radiation content (in particular, $t^{(0)}_{ab} = 0$), and the matter content satisfies suitable Newtonian assumptions. We will argue that, under these assumptions, to leading order in

λ, $h^{(S)}_{ab}(\lambda)$ is given by local Newtonian gravity, i.e., in a neighborhood of any point x, $h^{(S)}_{ab}$ can be calculated to sufficient accuracy using Newtonian gravity, taking into account only the matter distribution within a suitable neighborhood of x. (The effects of more distant matter are taken into account by $\gamma^{(L)}_{ab}$.) With our "no gravitational radiation" assumption for the background and our Newtonian approximation for $h^{(S)}_{ab}$, our equation (87) for $\gamma^{(L)}_{ab}$ simplifies considerably, yielding the linearized Einstein equation with an additional effective source that agrees with recent results of [27] (see also [21]).

In our analysis, it will be important to make a convenient choice of gauge. Since we are taking nonlinear effects at small scales into account, we cannot simply impose the usual cosmological gauge choices for perturbations, i.e., we must make our gauge choice at the nonlinear level. When studying non-linear perturbations off of a flat background it is often convenient to work with "wave map coordinates" (usually called "harmonic coordinates" in the literature), particularly in the context of the post-Newtonian expansion [30]. Since our background metric $g^{(0)}_{ab}$ is not flat, the usual definition of wave-map coordinates is not convenient, but we may instead impose a generalized wave-map gauge condition with respect to the background metric $g^{(0)}_{ab}$ [31].

Our gauge choice will be introduced in subsection A in the context of a general background metric $g^{(0)}_{ab}$ (i.e., without assuming FLRW symmetry). In subsection B, we restrict to a FLRW background, we make our Newtonian assumptions, and argue that $h^{(S)}_{ab}$ is given by local Newtonian gravity. It should be emphasized that the arguments of section B have the character of plausibility arguments rather than proofs. Finally, the simplifications to the equation for $\gamma^{(L)}_{ab}$ will be obtained in subsection C.

A. Generalized wave map (harmonic) gauge

Given a one-parameter family of metrics $g_{ab}(\lambda)$ on our spacetime manifold M that satisfies our assumptions (i)–(v), we may apply any one-parameter family of diffeomorphisms, $\phi(\lambda)$: $M \to M$, which preserve these conditions, where without loss of generality, we may assume that $\phi(0)$ is the identity map. As already noted in the paragraph below (87), it is clear that any $\phi(\lambda)$ that is jointly smooth in λ and the spacetime point x will preserve conditions (i)–(v), but there also should exist a wide class of $\phi(\lambda)$ that are not smooth in λ that preserve these conditions. The properties of such $\phi(\lambda)$ and the transformations that they induce on μ_{abcdef} were analyzed by [26] (under his assumptions, which differ slightly from our assumptions (i)–(iv)). Unfortunately, although Burnett's analysis can be used to prove important properties of gauge transformations, such as the invariance of $t^{(0)}_{ab}$ under all allowed gauge transformations, it is very difficult to prove any existence results that establish that specific gauge conditions can be imposed on an arbitrary one parameter family of metrics $g_{ab}(\lambda)$ satisfying our conditions.

In this section, we will assume that we can impose the "generalized wave-map gauge condition" on the metric $g_{ab}(\lambda)$, namely

$$g^{ab}(\lambda)C^c{}_{ab}(\lambda) = 0\,, \tag{91}$$

where $C^c{}_{ab}(\lambda)$ is given by (10). Note that this condition depends upon the background metric, $g^{(0)}_{ab}$, since the derivative operator, ∇_a, of $g^{(0)}_{ab}$ appears in the definition of $C^c{}_{ab}(\lambda)$. We can impose the gauge condition (91) on $g_{ab}(\lambda)$ by applying the diffeomorphism $x^\mu \to \phi^\mu(\lambda, x)$ to $g_{ab}(\lambda)$, where x^μ are arbitrarily chosen coordinates on M and ϕ^μ satisfies[7]

$$\frac{1}{\sqrt{-g(\lambda)}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g(\lambda)}g^{\mu\nu}(\lambda)\frac{\partial \phi^{\bar{\alpha}}}{\partial x^\nu}\right) + \Gamma^{(0)\bar{\alpha}}{}_{\bar{\mu}\bar{\nu}}g^{\mu\nu}(\lambda)\frac{\partial \phi^{\bar{\mu}}}{\partial x^\mu}\frac{\partial \phi^{\bar{\nu}}}{\partial x^\nu} = 0\,. \tag{92}$$

This equation is a (nonlinear) wave equation, so we can always find (local) solutions. To see that solutions to this equation give rise to the condition (91), we note that if we use $\phi^\mu(\lambda, x)$ as coordinates for the λth spacetime, then we may replace $\partial\phi^\alpha/\partial x^\beta$ by $\delta^\alpha{}_\beta$, and (92) reduces to

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g(\lambda)}}\partial_\mu\left(\sqrt{-g(\lambda)}g^{\mu\alpha}(\lambda)\right) + \Gamma^{(0)\alpha}{}_{\mu\nu}g^{\mu\nu}(\lambda) \\ &= g^{\mu\nu}(\lambda)\Gamma^\alpha_{\mu\nu}(\lambda) - g^{\mu\nu}(\lambda)\Gamma^{(0)\alpha}{}_{\mu\nu} \\ &= -g^{\mu\nu}(\lambda)C^\alpha{}_{\mu\nu}\,. \end{aligned} \tag{93}$$

We will refer to the coordinates $\phi^\mu(\lambda, x)$ as *generalized wave map coordinates* for $g_{ab}(\lambda)$ relative to the coordinates x^μ for $g^{(0)}_{ab}$. Note that in the case where $g^{(0)}_{ab} = \eta_{ab}$ and x^μ are Minkowski coordinates, the generalized wave map gauge condition reduces to the condition $g^{\mu\nu}(\lambda)\Gamma^\alpha_{\mu\nu}(\lambda) = 0$, and the coordinates $\phi^\mu(\lambda, x)$ satisfy a linear wave equation. Such coordinates are usually referred to as "harmonic coordinates" in the literature.

[7] The diffeomorphisms defined by (92) do not depend on the choice of coordinates x^μ, since this equation can be derived from the coordinate invariant action

$$E[\phi] = \int_M g^{\mu\nu}(\lambda)g^{(0)}_{\bar{\mu}\bar{\nu}}\frac{\partial\phi^{\bar{\mu}}}{\partial x^\mu}\frac{\partial\phi^{\bar{\nu}}}{\partial x^\nu}\sqrt{-g(\lambda)}\,\mathrm{d}^4x\,.$$

Although we can always (locally) solve (92) and thus (locally) put each $g_{ab}(\lambda)$ in our one-parameter family in wave map gauge, we have no guarantee that the resulting new one-parameter family of metrics will satisfy our conditions (i)–(v). In the following, we shall simply assume that this is the case, i.e., that we have a one-parameter family of metrics $g_{ab}(\lambda)$ that satisfies conditions (i)–(v) as well as our gauge condition (91). This corresponds to a strengthening of our assumptions.

When the generalized wave map gauge condition is satisfied, it is very convenient to work with the variable

$$\mathfrak{h}^{ab}(\lambda) \equiv g^{(0)ab} - \sqrt{\frac{g(\lambda)}{g^{(0)}}} g^{ab}(\lambda) \,. \tag{94}$$

instead of $h_{ab}(\lambda) = g_{ab}(\lambda) - g^{(0)}_{ab}$. (Note that $\sqrt{g(\lambda)/g^{(0)}}$ is the proportionality factor between the volume elements of $g_{ab}(\lambda)$ and $g^{(0)}_{ab}$, so this quantity does not depend on any choice of coordinates x^μ on M.) We have

$$\begin{aligned}
\nabla_b \mathfrak{h}^{ab}(\lambda) &= -\sqrt{\frac{g(\lambda)}{g^{(0)}}} \nabla_b g^{ab}(\lambda) - \sqrt{\frac{g(\lambda)}{g^{(0)}}} g^{ab}(\lambda) C^c{}_{cb} \\
&= -\sqrt{\frac{g(\lambda)}{g^{(0)}}} \left(\nabla_b g^{ab}(\lambda) + \frac{1}{2} g^{ab}(\lambda) g^{cd}(\lambda) \nabla_b g_{cd}(\lambda) \right) \\
&= \sqrt{\frac{g(\lambda)}{g^{(0)}}} g^{bc}(\lambda) C^a{}_{bc} \\
&= 0 \,,
\end{aligned} \tag{95}$$

where on the first line we used the fact that

$$C^b{}_{ba} = \frac{1}{2} \nabla_a \log \left(\frac{g(\lambda)}{g^{(0)}} \right) \,. \tag{96}$$

Note that in linearized gravity, $\mathfrak{h}^{ab}$ reduces to the the trace-reversed metric perturbation, i.e., $\mathfrak{h}^{ab}(\lambda) \to \bar{h}^{ab}(\lambda) = h^{ab}(\lambda) - \frac{1}{2} g^{(0)ab} h^c{}_c(\lambda)$, and (95) reduces to the Lorenz gauge condition $\nabla_a \bar{h}^{ab} = 0$. One should keep in mind, though, that beyond lowest order in λ, $\mathfrak{h}^{ab}(\lambda)$ is not the trace-reversed metric perturbation.

We now express the exact Einstein equation in wave-map gauge in terms of $\mathfrak{h}^{ab}$ and the background derivative operator. Starting with (12), we use the background equation (15), the gauge condition $\nabla_b \mathfrak{h}^{ab} = 0$, as well as the fact that

$$\nabla_a g^{bc}(\lambda) = -\sqrt{\frac{g^{(0)}}{g(\lambda)}} \left(\nabla_a \mathfrak{h}^{bc} - \frac{1}{2} g^{bc}(\lambda) g_{de}(\lambda) \nabla_a \mathfrak{h}^{de} \right) \,, \tag{97}$$

to show (after a long computation) that

$$
\begin{aligned}
&\nabla^c\nabla_c\mathfrak{h}^{ab} - 2R^a{}_{cd}{}^b(g^{(0)})\mathfrak{h}^{cd} + 2R_c{}^{(a}(g^{(0)})\mathfrak{h}^{b)c} - R(g^{(0)})\mathfrak{h}^{ab} - g^{(0)ab}R_{cd}(g^{(0)})\mathfrak{h}^{cd} \\
&-2G^{ab}(g^{(0)})g^{(0)}_{cd}\mathfrak{h}^{cd} + 2\Lambda\left(\mathfrak{h}^{ab} - \frac{1}{2}g^{(0)ab}g^{(0)}_{cd}\mathfrak{h}^{cd}\right) \\
= &-16\pi\frac{g(\lambda)}{g^{(0)}}\left(T^{ab}(\lambda) - T^{(0)ab} + \mathfrak{t}^{ab}(\lambda) - t^{(0)ab}\right) .
\end{aligned}
\tag{98}
$$

Here, the terms that are non-linear in $\mathfrak{h}^{ab}$ have been absorbed into the quantity,

$$
\begin{aligned}
16\pi\frac{g(\lambda)}{g^{(0)}}\mathfrak{t}^{ab}(\lambda) \equiv &-2G^{ab}\left(1 - \frac{g(\lambda)}{g^{(0)}} - g^{(0)}_{cd}\mathfrak{h}^{cd}\right) + R_{cd}(g^{(0)})\mathfrak{h}^{cd}\mathfrak{h}^{ab} - 2R_{ced}{}^{(a}(g^{(0)})\mathfrak{h}^{b)e}\mathfrak{h}^{cd} \\
&-\mathfrak{h}^{cd}\nabla_c\nabla_d\mathfrak{h}^{ab} + \nabla_d\mathfrak{h}^{ca}\nabla_c\mathfrak{h}^{db} + g^{ef}(\lambda)g_{cd}(\lambda)\nabla_e\mathfrak{h}^{ca}\nabla_f\mathfrak{h}^{db} \\
&+\frac{1}{2}g^{ab}(\lambda)g_{cd}(\lambda)\nabla_e\mathfrak{h}^{fc}\nabla_f\mathfrak{h}^{ed} - 2g_{cd}(\lambda)g^{f(a}(\lambda)\nabla_e\mathfrak{h}^{b)c}\nabla_f\mathfrak{h}^{ed} \\
&+\frac{1}{8}\left(2g^{ag}(\lambda)g^{bh}(\lambda) - g^{ab}(\lambda)g^{gh}(\lambda)\right)\left(2g_{cd}(\lambda)g_{ef}(\lambda) - g_{ed}(\lambda)g_{cf}(\lambda)\right)\nabla_h\mathfrak{h}^{ed}\nabla_g\mathfrak{h}^{cf} \\
&-2\Lambda\left(\frac{g(\lambda)}{g^{(0)}}\left(g^{ab}(\lambda) - g^{(0)ab}\right) + \mathfrak{h}^{ab} - \frac{1}{2}g^{(0)ab}g^{(0)}_{cd}\mathfrak{h}^{cd}\right) .
\end{aligned}
\tag{99}
$$

Equation (98) is of the form

$$
\mathscr{L}^{ab}(\mathfrak{h}) = -16\pi S^{ab} . \tag{100}
$$

where $\mathscr{L}^{ab}$ takes the form of a linear wave operator acting on $\mathfrak{h}^{ab}$. Consequently, re-introducing our (arbitrarily chosen) coordinates x^μ of the background spacetime, we may rewrite (98) in the following equivalent integral form:

$$
\mathfrak{h}^{\alpha\beta}(\lambda, x) = 4\int_M G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x, x')S^{\mu'\nu'}(\lambda, x')\sqrt{-g^{(0)}(x')}\,\mathrm{d}^4x' + \mathfrak{h}^{\alpha\beta}_{\text{hom}}(\lambda, x) , \tag{101}
$$

where $G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x, x')$ is the retarded Green's function for $\mathscr{L}^{ab}$ and $\mathfrak{h}^{\alpha\beta}_{\text{hom}}$ is a solution to $\mathscr{L}^{ab}(\mathfrak{h}_{\text{hom}}) = 0$. We emphasize that (101) is not a *solution* to (98) since the source $S^{\alpha\beta}$ depends on $\mathfrak{h}^{\alpha\beta}$. Rather, it is simply a re-writing of (98) in an integral form.

B. Local Newtonian gravity

For the remainder of this section, we restrict attention to the case where the background spacetime $g^{(0)}_{ab}$ has FLRW symmetry. It will be useful to work in coordinates where the metric components are nonsingular. Thus, instead of the more common choice of polar-type coordinates, we will write the background metric in the form

$$
ds^{(0)2} = -d\tau^2 + a^2(\tau)\left(1 + k(x^2 + y^2 + z^2)/4\right)^{-2}[dx^2 + dy^2 + dz^2] , \tag{102}
$$

where $k = 0, \pm 1$, depending on the spatial curvature.

The "long wavelength part" of the leading order in λ part of $\mathfrak{h}^{ab}(\lambda)$ is given by

$$\bar{\gamma}^{(L)ab} \equiv \underset{\lambda \to 0}{\text{w-lim}} \frac{\mathfrak{h}^{ab}}{\lambda} = \gamma^{(L)ab} - \frac{1}{2} g^{(0)ab} \gamma^{(L)c}{}_{c} \,. \tag{103}$$

It follows directly from (95) that $\bar{\gamma}^{(L)ab}$ satisfies the Lorenz gauge condition

$$\nabla_a \bar{\gamma}^{(L)ab} = 0 \,. \tag{104}$$

The "short wavelength part" of $\mathfrak{h}^{ab}(\lambda)$ is given by

$$\mathfrak{h}^{(S)ab}(\lambda) = \mathfrak{h}^{ab}(\lambda) - \lambda \bar{\gamma}^{(L)ab} \,. \tag{105}$$

By (98), it satisfies (in the notation of (100))

$$\mathscr{L}^{ab}(\mathfrak{h}^{(S)}(\lambda)) = -16\pi \left(S^{ab}(\lambda) - \lambda \underset{\lambda' \to 0}{\text{w-lim}} \frac{S^{ab}(\lambda')}{\lambda'} \right) , \tag{106}$$

where the weak limit appearing in this equation exists by virtue of assumption (v) of section III. Note that $S^{ab}(\lambda)$ still depends on the full perturbation, i.e., $\mathfrak{h}^{ab}$ cannot be replaced by $\mathfrak{h}^{(S)ab}$ in the expression for S^{ab}. Equation (106) can be rewritten in integral form as,

$$\begin{aligned} \mathfrak{h}^{(S)\alpha\beta}(\lambda, x) = 4 \int_M & G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x, x') \left(S^{\mu'\nu'}(\lambda, x') - \lambda \underset{\lambda' \to 0}{\text{w-lim}} \frac{S^{\mu'\nu'}(\lambda', x')}{\lambda'} \right) \sqrt{-g^{(0)}(x')} \, \mathrm{d}^4 x' \\ & + \mathfrak{h}^{(S)}_{\text{hom}}{}^{\alpha\beta}(\lambda, x) \,. \end{aligned} \tag{107}$$

Our aim for the remainder of this subsection is to argue that—in the absence of gravitational radiation and under suitable assumptions concerning the behavior of the matter distribution $T_{ab}(\lambda)$—to leading order in λ, $\mathfrak{h}^{(S)ab}(\lambda)$ near point x is described by Newtonian gravity, taking into account only the matter distribution in a suitable local neighborhood of x. However, in order to derive this conclusion, we must make significant additional assumptions about our one-parameter family $g_{ab}(\lambda)$, and severe difficulties arise if one attempts to formulate these assumptions in a mathematically precise manner. The reason is that, although there are simple, precise limits that one can take of general relativistic spacetimes to obtain Newtonian gravity [32], these limits would not be compatible with retaining the cosmological background spacetime $g^{(0)}_{ab}$, and thus would not be suitable for our use. We believe that it should be possible to concoct a mathematically consistent set of assumptions that would enable us to rigorously justify our conclusions below, but we do not see a simple and/or elegant way of formulating such assumptions, and we do not feel that it would be illuminating to attempt to derive our results from a complicated list of technical assumptions whose intrinsic plausibility is not much greater than that of the conclusions we wish to draw. Thus, in the discussion below in this subsection, although we will clearly indicate the nature of the assumptions that are needed, we will not attempt to formulate all of our assumptions in a mathematically precise manner, and we will thereby resort to plausibility arguments to obtain our conclusions.

We are interested in obtaining $\mathfrak{h}^{(S)ab}(\lambda)$ near a point x at a time roughly corresponding to the present time in the actual universe. We first argue that although the retarded Green's function integral extends all the way back to the "big bang", it should suffice to integrate only over the "recent universe" (corresponding, say, to $z \leq 1000$ in the present, actual universe). There are two reasons why this should be so: (1) The universe is expected to be very nearly homogeneous and isotropic in the distant past, so the source term $\left(S^{ab} - \lambda \, \text{w-lim}_{\lambda' \to 0} \left[S^{ab}/\lambda'\right]\right)$ should be negligibly small. (2) The nature of the retarded Green's function in an expanding universe is such as to make the influence of distant sources small (on account of redshift and intensity diminution). Similarly, we assume that the gravitational radiation content of the present universe arising from the "big bang" is negligible. Consequently, we discard the last term, $\mathfrak{h}^{(S)}_{\text{hom}}{}^{\alpha\beta}$, in (107). Thus, our integral relation for $\mathfrak{h}^{(S)ab}$ becomes

$$\mathfrak{h}^{(S)\alpha\beta}(\lambda, x) = 4 \int_{\mathcal{W}} G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x, x') \left(S^{\mu'\nu'}(\lambda, x') - \lambda \underset{\lambda' \to 0}{\text{w-lim}} \frac{S^{\mu'\nu'}(\lambda', x')}{\lambda'} \right) \sqrt{-g^{(0)}(x')} \, \mathrm{d}^4 x' \,, \tag{108}$$

where $\mathcal{W}$ is the (compact) region corresponding to the "recent universe" (i.e., $z \leq 1000$ in the present, actual universe).

Next, we argue that, in order to calculate $\mathfrak{h}^{(S)\alpha\beta}$ to $O(\lambda)$ at x, it suffices to perform the integral in (108) only over a small neighborhood $\mathcal{V}$ of x. In other words, we argue that—for any fixed neighborhood, $\mathcal{V}$, of x—as $\lambda \to 0$, the contribution to the integral in (108) from the region $x' \in \mathcal{W} \setminus \mathcal{V}$ should vanish faster than λ as $\lambda \to 0$, i.e.,

$$\int_{\mathcal{W} \setminus \mathcal{V}} G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x, x') \left(\frac{S^{\mu'\nu'}(\lambda, x')}{\lambda} - \underset{\lambda' \to 0}{\text{w-lim}} \frac{S^{\mu'\nu'}(\lambda', x')}{\lambda'} \right) \sqrt{-g^{(0)}(x')} \, \mathrm{d}^4 x' \to 0 \,. \tag{109}$$

To see this, we note that if $G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x, x')$ were smooth (and if the sharp boundaries of the region of integration were replaced by smooth cut-off functions), then we would be integrating the quantity $(S^{\mu\nu}/\lambda - \text{w-lim}_{\lambda'\to 0}[S^{\mu\nu}/\lambda'])$ with a test function. Since, clearly, the weak limit as $\lambda \to 0$ of this quantity is 0, it follows immediately that (109) would hold. Of course, $G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x,x')$ is not smooth, so (109) cannot be expected to hold without further restrictions on S^{ab}. However, in a normal neighborhood of x, the retarded Green's function $G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x,x')$ for the linear operator $\mathscr{L}^{ab}$ takes the form

$$G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x,x') = U^{\alpha\beta}{}_{\mu'\nu'}(x,x')\delta_+(\sigma) + V^{\alpha\beta}{}_{\mu'\nu'}(x,x')\theta_+(-\sigma)\,, \tag{110}$$

where $U^{\alpha\beta}{}_{\mu'\nu'}(x,x')$ and $V^{\alpha\beta}{}_{\mu'\nu'}(x,x')$ are smooth bitensors and σ is the squared geodesic distance between x and x' (see [33] for details). Thus, apart from the singularity at $x' = x$ (which is excluded from the region of integration in (109)), the singularities of $G_{\text{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x,x')$ are of the form of a restriction to the past lightcone (i.e., $\delta_+(\sigma)$) and a cutoff at the past lightcone (i.e., $\theta_+(-\sigma)$). If S^{ab} is not rapidly varying with time (as should be the case under our Newtonian assumptions below), these singularities should be quite benign, so it seems not unreasonable that (109) will hold under suitable assumptions.

We have argued that (109) should hold for an arbitrary neighborhood $\mathcal{V}$ of x. However, the vanishing of the contribution to $\mathfrak{h}^{(S)\alpha\beta}(\lambda,x)$ from outside of $\mathcal{V}$ to $O(\lambda)$ holds only in the limit as $\lambda \to 0$, and the smaller we take $\mathcal{V}$, the smaller we must take λ in order to get a good approximation to $\mathfrak{h}^{(S)\alpha\beta}(\lambda,x)$ by integrating only over $\mathcal{V}$. At any finite λ, we cannot take $\mathcal{V}$ to be arbitrarily small and still get a good approximation to $\mathfrak{h}^{(S)\alpha\beta}(\lambda,x)$. How large must we take $\mathcal{V}$ at finite λ?

To propose an answer to this question, we restrict consideration to the case where the matter content satisfies suitable Newtonian behavior (at least in the "recent universe"). We shall assume that as $\lambda \to 0$ we have $T_{00}(\lambda) = O(1/\lambda)$, $T_{0i}(\lambda) = O(1/\lambda^{1/2})$ and $T_{ij}(\lambda) = O(1)$, so that, for small λ, the energy density is much greater than the momentum density, and the momentum density is much greater than the stress. In addition we shall assume that spatial differentiation of components of the stress-tensor results in blow-up as $\lambda \to 0$ that is a factor of λ^{-1} faster than the undifferentiated components (so, e.g., $\partial_i T_{00}(\lambda) = O(1/\lambda^2)$), but that time differentiation results in a blow-up of only a factor of $\lambda^{-1/2}$ faster (so, e.g., $\partial_0 T_{00}(\lambda) = O(1/\lambda^{3/2})$).

We now introduce the notion of the *scale of homogeneity*, $l(\lambda,\tau_0)$ at cosmic time τ_0 as follows. First, we define a fiducial window function which is to be used for averaging. Fix a time interval $\Delta\tau \ll \tau_0$ and let V_{R,x_0} to be the "cylinder", centered at x_0, of "height" $\Delta\tau$

and proper spatial radius R. Let $\chi_{R,x_0}(x)$ be a smooth non-negative function which is equal to one on V_{R,x_0}, and falls rapidly to zero outside of this region. Let $\mathcal{H}$ denote the "Hubble volume" relative to x at time τ_0, i.e., the ball of proper radius equal to the Hubble radius, R_H, centered at point x. We define

$$l(\lambda,\tau_0) = \inf\left\{R : \left|\int \left(T_{00}(\lambda) - T^{(0)}_{00}\right)\chi_{R,\tau_0,\boldsymbol{x}_0}\right| < \left|\int T^{(0)}_{00}\chi_{R,\tau_0,\boldsymbol{x}_0}\right|, \forall \boldsymbol{x}_0 \in \mathcal{H}\right\}. \tag{111}$$

In other words l is the smallest radius such that averaging over a ball of radius l centered at any point x_0 lying within the Hubble volume relative to x always yields $|\delta\rho|/\rho^{(0)} < 1$. Note that our definition of $l(\lambda,\tau_0)$ depends on our choice of "window function" $\chi_{R,x_0}(x)$.

We now claim that $l(\lambda,\tau_0) \to 0$ as $\lambda \to 0$ (and thus, in particular, l is always finite at sufficiently small λ). To prove this, we note that if this result did not hold, we could find an $l_0 > 0$ and a sequence $\{\lambda_n : n \in \mathbb{N}\}$ converging to zero, such that $l(\lambda_n,\tau_0) > l_0$ for all $n \in \mathbb{N}$. Consequently, there would exist a sequence of points $\{\boldsymbol{x}_n\} \subset \mathcal{H}$ such that

$$\left|\int \left(T_{00}(\lambda_n) - T^{(0)}_{00}\right)\chi_{l_0,\tau_0,\boldsymbol{x}_n}\right| \geq \left|\int T^{(0)}_{00}\chi_{l_0,\tau_0,\boldsymbol{x}_n}\right|. \tag{112}$$

Since $\mathcal{H}$ is compact, there exists a subsequence—which we also denote as $\{\boldsymbol{x}_n\}$—converging to some $\boldsymbol{z} \in \mathcal{H}$. By the triangle inequality, we have

$$\left|\int \left(T_{00}(\lambda_n) - T^{(0)}_{00}\right)\chi_{l_0,\tau_0,\boldsymbol{z}}\right| + \left|\int \left(T_{00}(\lambda_n) - T^{(0)}_{00}\right)\left[\chi_{l_0,\tau_0,\boldsymbol{x}_n} - \chi_{l_0,\tau_0,\boldsymbol{z}}\right]\right| \geq \left|\int T^{(0)}_{00}\chi_{l_0,\tau_0,\boldsymbol{x}_n}\right|. \tag{113}$$

Taking the limit as $n \to \infty$, we see that the first term on the left side vanishes because $T_{00}(\lambda)$ converges weakly to $T^{(0)}_{00}$ and the second term vanishes by the lemma of section II. On the other hand, the right side is bounded away from 0, thus yielding a contradiction, thereby proving the desired result that $l(\lambda,\tau_0) \to 0$ as $\lambda \to 0$.

Returning to the question posed four paragraphs above, since $T_{00}(\lambda)$ provides the dominant contribution to the source term S^{ab}, it seems clear that if, at finite λ, $\mathcal{V}(\lambda)$ is chosen to be so small that it does not include all source contributions lying within a homogeneity scale $l(\lambda,\tau_0)$ about point x, then we cannot expect the source contributions from outside of $\mathcal{V}(\lambda)$ to consistently average to zero to a good approximation. On the other hand, if $\mathcal{V}(\lambda)$ is of order of the homogeneity scale or larger, then it seems plausible that (with suitable additional assumptions) a good approximation to $\mathfrak{h}^{(S)\alpha\beta}(\lambda,x)$ will be obtained. Thus, we have argued that $\mathfrak{h}^{(S)\alpha\beta}(\lambda, x)$ should be well approximated by

$$\mathfrak{h}^{(S)\alpha\beta}(\lambda, x) = 4 \int_{\mathcal{V}(\lambda)} G_{\mathrm{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(x, x') \left(S^{\mu'\nu'}(\lambda, x') - \lambda \operatorname*{w-lim}_{\lambda' \to 0} \frac{S^{\mu'\nu'}(\lambda', x')}{\lambda'} \right) \sqrt{-g^{(0)}(x')}\, \mathrm{d}^4 x' \,, \tag{114}$$

where $\mathcal{V}(\lambda)$ may be taken to be a "cylinder" centered at x of proper spatial radius L and proper time "height" $2L$, where $L \gtrsim l(\lambda, \tau_0)$. Note that in order to obtain the conclusion that we need only integrate over the local region $\mathcal{V}(\lambda)$, it is essential that we have removed the "long wavelength part" of $\mathfrak{h}^{\alpha\beta}$.

We now assume that $l(\lambda, \tau_0) \ll R_C$ so that we can choose L such that $L \ll R_C$, where R_C denotes the length scale of curvature of the background metric $g^{(0)}_{ab}$ (i.e., the Hubble radius, assuming spatial curvature is negligible). In the actual universe at the present time, we have $R_C \sim 3000$ Mpc and $l(\lambda, \tau_0) \lesssim 100$ Mpc, so we should easily satisfy all required criteria with $L \sim 100$ Mpc. In that case, it should suffice to take only the leading order terms in the Hadamard expansion for $U^{\alpha\beta}{}_{\mu'\nu'}$ and $V^{\alpha\beta}{}_{\mu'\nu'}$ as well as the leading order approximation to σ in the Green's function expression (110). This yields

$$\begin{aligned} G_{\mathrm{ret}}{}^{\alpha\beta}{}_{\mu'\nu'}(\tau, \mathbf{0}, \tau', \boldsymbol{x}') = {}& \delta^{(\alpha}_{\mu'} \delta^{\beta)}_{\nu'} \frac{\delta\left(\tau' - \tau + a(\tau) r'\right)}{a(\tau) r'} \\ & + V^{\alpha\beta}{}_{\mu'\nu'}(\tau, \mathbf{0}, \tau, \mathbf{0}) \theta\left(-\tau' + \tau - a(\tau) r'\right) \,, \end{aligned} \tag{115}$$

where we have now put the field evaluation point x at the spatial origin of our coordinate system (102). Since $V^{\alpha\beta}{}_{\mu'\nu'}(\tau, \mathbf{0}, \tau, \mathbf{0})$ is proportional to the curvature of $g^{(0)}_{ab}$, it is clear that the contribution of the second term will be down by a factor of $(L/R_C)^2$ from the first term, so we neglect this contribution. We also neglect "retardation effects", i.e., the difference between evaluating the source at time $\tau - a(\tau) r'$ and time τ. Our formula (114) for $\mathfrak{h}^{(S)\alpha\beta}$ then reduces to

$$\mathfrak{h}^{(S)\alpha\beta}(\lambda, \tau, \mathbf{0}) \approx 4 \int \mathrm{d}\Omega' \int_0^{\frac{L}{a(\tau)}} \frac{1}{r'} \left(S^{\alpha\beta}(\lambda, \tau, \boldsymbol{x}') - \lambda \operatorname*{w-lim}_{\lambda' \to 0} \frac{S^{\alpha\beta}(\lambda', \tau, \boldsymbol{x}')}{\lambda'} \right) a^2(\tau) r'^2 \, \mathrm{d}r' \,. \tag{116}$$

Next, we assume that $l(\lambda, \tau_0)$ not only goes to zero as $\lambda \to 0$ (as we have proven above) but is $O(\lambda)$ as $\lambda \to 0$, i.e., $l(\lambda, \tau_0)/\lambda$ remains bounded[8] as $\lambda \to 0$. Thus, if we choose L such that L/λ remains bounded as $\lambda \to 0$, by inspection of (116), it is then clear that terms in S^{ab} that are $o(1/\lambda)$ as $\lambda \to 0$ will make only $o(\lambda)$ contributions to $\mathfrak{h}^{(S)\alpha\beta}$. However, we assumed

[8] This precludes behavior wherein, e.g., $T_{00}(\lambda)$ is $O(1)$ as $\lambda \to 0$ but its scale of spatial variation goes as $\lambda^{1/2}$ rather than λ. Such behavior would not be excluded by our previous assumptions.

above that as $\lambda \to 0$, we have $T_{00}(\lambda) = O(1/\lambda)$, $T_{0i}(\lambda) = O(1/\lambda^{1/2})$ and $T_{ij}(\lambda) = O(1)$. We now make a final, additional assumption that terms of the form $\nabla_c \nabla_d \mathfrak{h}^{ab}$ are $O(1/\lambda)$. In that case, it follows immediately that $\mathfrak{t}^{ab}$ is $O(1)$ as $\lambda \to 0$ (see (99)). It then follows that to obtain $\mathfrak{h}^{(S)\alpha\beta}$ to $O(\lambda)$ accuracy, we need only take into account the contribution to S^{ab} from T_{00}. Thus, to $O(\lambda)$ accuracy, we obtain

$$\mathfrak{h}^{(S)00}(\lambda, \tau, \mathbf{0}) = 4 \int \mathrm{d}\Omega' \int_0^{\frac{L}{a(\tau)}} \frac{1}{r'} \left(T^{00}(\lambda, \tau, \boldsymbol{x}') - T^{(0)00}(\tau, \boldsymbol{x}') - \lambda T^{(1)00}(\tau, \boldsymbol{x}') \right) a^2(\tau) r'^2 \, \mathrm{d}r' \tag{117}$$

and $\mathfrak{h}^{(S)\mu\nu} = 0$ for all other components of $\mathfrak{h}^{(S)ab}$. We write

$$\phi \equiv -\frac{1}{4} \mathfrak{h}^{(S)00} \,. \tag{118}$$

Then $\phi(x)$ differs from the familiar formula for the gravitational potential arising in ordinary Newtonian gravity due to the matter lying within a ball of proper radius L about x only in the following ways: (a) There is a factor of $a^2(\tau)$ in (117), which arises from the trivial scaling difference between the spatial coordinates of (102) and ordinary Cartesian coordinates. (b) $T^{(0)00}$ is subtracted in the integrand of (117) because the FLRW time slicing differs from a locally Minkowskian time slicing (see section IA of [1]); equivalently, the effects of $T^{(0)00}$ have already been taken into account via the dynamics of the FLRW background. (c) $\lambda T^{(1)00}$ is subtracted in the integrand of (117) because its effects were already taken into account by $\gamma^{(L)}_{ab}$. As discussed above, this subtraction of $\lambda T^{(1)00}$ gives the integral much better convergence properties. Thus, we conclude that the leading order short wavelength deviation from the FLRW background $g^{(0)}_{ab}$ is described by Newtonian gravity, taking into account only the matter distribution lying within a region about x whose size is of order the homogeneity lengthscale.

The motion of matter is given by

$$0 = \nabla_a(\lambda) T^{ab}(\lambda) = \nabla_a T^{ab}(\lambda) + C^a{}_{ac}(\lambda) T^{cb}(\lambda) + C^b{}_{ac}(\lambda) T^{ac}(\lambda) \,. \tag{119}$$

We may write

$$C^a{}_{bc}(\lambda) = C^{(S)a}{}_{bc}(\lambda) + \lambda C^{(1)a}{}_{bc} \,, \tag{120}$$

where, to leading order in wave map gauge, we have

$$C^{(S)0}{}_{0i} = \nabla_i \phi \,, \tag{121}$$

$$C^{(S)i}{}_{00} = \nabla^i \phi \,, \tag{122}$$

$$C^{(S)i}{}_{jk} = g^{(0)}_{jk} \nabla^i \phi - 2\delta^i{}_{(j} \nabla_{k)} \phi \,, \tag{123}$$

(with other components zero), and

$$C^{(1)a}{}_{bc} = \frac{1}{2} g^{(0)cd} \left\{ \nabla_a \gamma^{(L)}_{bd} + \nabla_b \gamma^{(L)}_{ad} - \nabla_d \gamma^{(L)}_{ab} \right\} . \tag{124}$$

The dominant terms in (119) arise from $C^{(S)i}{}_{00}(\lambda) T^{00}(\lambda)$ and correspond to the ordinary Newtonian gravitational effects on the motion of matter. Although, for small λ, the contributions from $C^{(1)a}{}_{bc}$ will be much smaller than those arising from ϕ, it is important not to discard the terms in $C^{(1)a}{}_{bc}$ since they can produce large scale, coherent motions.

C. Behavior of $\gamma^{(L)}_{ab}$ with dust source

In this subsection, we will simplify the rather complicated equations for $\gamma^{(L)}_{ab}$ derived in section III under the assumption that $\mathfrak{h}^{(S)ab}(\lambda)$ is of "Newtonian form". More precisely, we assume that the following quantities are uniformly bounded as $\lambda \to 0$:

$$\begin{aligned} &\tfrac{1}{\lambda}\mathfrak{h}^{(S)00}\,, \ \tfrac{1}{\lambda^{1/2}}\nabla_0 \mathfrak{h}^{(S)00}\,, \quad \nabla_i \mathfrak{h}^{(S)00}\,, \\ &\qquad\qquad \tfrac{1}{\lambda}\nabla_0 \mathfrak{h}^{(S)0j}\,, \quad \tfrac{1}{\lambda^{1/2}}\nabla_i \mathfrak{h}^{(S)0j}\,, \\ &\qquad\qquad\qquad\qquad \tfrac{1}{\lambda}\nabla_i \mathfrak{h}^{(S)jk}\,. \end{aligned} \tag{125}$$

The remaining components ($\mathfrak{h}^{(S)0i}$, $\mathfrak{h}^{(S)ij}$, and $\nabla_0 \mathfrak{h}^{(S)ij}$) are assumed to be $o(\lambda)$. These assumptions can be justified by generalizations of the arguments made in the previous subsection, but here we will simply assume that they are valid.

We will also assume that the matter stress-energy takes the form of "dust"

$$T_{ab}(\lambda) = \rho(\lambda) u_a(\lambda) u_b(\lambda) \,, \tag{126}$$

where $u^a(\lambda)$ has norm -1 with respect to the metric $g_{ab}(\lambda)$, and $\rho(\lambda) \geq 0$. (Recall that we have incorporated a cosmological constant into Einstein's equation, so the possible presence of "dark energy" of that form has already been taken into account. This assumption of the dust form of the stress-energy tensor as opposed to a more general form of non-relativistic matter is made here mainly for the purpose of obtaining definite equations involving familiar quantities.) We further assume that as $\lambda \to 0$, $u_a(\lambda)$ converges uniformly to $u^{(0)a}$, where in the coordinates of (102), we have $u^{(0)\mu} = (1,0,0,0)$. Note that since $u_a(\lambda) \to u^{(0)}_a$ uniformly as $\lambda \to 0$, it follows by our lemma of section II and the positivity of $\rho(\lambda)$ that

$$T^{(0)}_{ab} = \rho^{(0)} u^{(0)}_a u^{(0)}_b \,, \tag{127}$$

where $\rho^{(0)} = \text{w-lim}_{\lambda\to 0}\,\rho(\lambda)$.

Let $v^a(\lambda)$ denote the peculiar velocity of the dust relative to the "Hubble flow" $u^{(0)a}$, i.e., $v^a(\lambda)$ is the projection of $u^a(\lambda)$ orthogonal to $u^{(0)a}$ in the metric $g^{(0)}_{ab}$. In accord with usual Newtonian limits, we assume that $v^a(\lambda)/\lambda^{1/2}$ is uniformly bounded as $\lambda \to 0$. It follows that

$$u^a(\lambda) = \left(1 + \frac{1}{2}h_{cd}(\lambda)u^{(0)c}u^{(0)d} + \frac{1}{2}v_c(\lambda)v^c(\lambda)\right)u^{(0)a} + v^a(\lambda) + o(\lambda)\,, \tag{128}$$

or, equivalently,

$$u_a(\lambda) = g_{ab}(\lambda)u^b(\lambda) = \left(1 + \frac{1}{2}h_{cd}(\lambda)u^{(0)c}u^{(0)d} + \frac{1}{2}v_c(\lambda)v^c(\lambda)\right)u^{(0)}_a + v_a(\lambda) + h_{ab}(\lambda)u^{(0)b} + o(\lambda)\,. \tag{129}$$

The results we shall now derive in this subsection will be rigorous consequences of the assumptions that have been stated above. We shall first show that, under the above assumptions[9], the quantities μ_{abcdef}, $\omega^{(1,\alpha)}_{abcdefgh}$, and $\nu^{(1)}_{abcde}$ all vanish, so all of the "backreaction tensors" that appear in the equations for $\gamma^{(L)}_{ab}$ of section III vanish, except for $\mu^{(1)}_{abcdef}$ and $\kappa^{(1)}_{abcd}$.

Taking the weak limit of $\mathfrak{h}^{cd}$ times (98) (or, equivalently, using (28) directly) we find that our Newtonian assumptions imply

$$\underset{\lambda\to 0}{\text{w-lim}}\, \partial_i\phi\partial^i\phi = 0\,, \tag{130}$$

where ϕ was defined by (118). Since the spatial metric $g^{(0)}_{ij}$ is positive definite, this implies that for any test function f, we have $\|\partial_i(f\phi)\|_{L^2} \to 0$ as $\lambda \to 0$. Now, $h^{(S)}_{ab} = \mathfrak{h}^{(S)}_{ab} - \frac{1}{2}g^{(0)}_{ab}\mathfrak{h}^{(S)c}{}_c + O(\mathfrak{h}\mathfrak{h})$, so the only possible contributions to $\omega^{(1)}_{abcdefgh}$ come from terms proportional to

$$\underset{\lambda\to 0}{\text{w-lim}}\, \frac{1}{\lambda}\phi\partial_i\phi\partial_j\phi\,. \tag{131}$$

Let f be any test function, and let g be a non-negative test function which is equal to 1 on

[9] In fact, we can show, without any Newtonian assumptions or gauge choice, that if $t^{(0)}_{ab} = 0$, then $\alpha_{abcdef} = 0$ and $\omega^{(1,\alpha)}_{abcdefgh} = 0$. However, this is not enough to tell us anything about $\nu^{(1)}_{abcde}$ or the other components of μ_{abcdef} or $\omega^{(1)}_{abcdefgh}$.

the support of f. Then we have

$$\begin{aligned}\left|\int \left(\underset{\lambda\to 0}{\text{w-lim}}\frac{1}{\lambda}\phi\partial_i\phi\partial_j\phi\right) f\,\mathrm{d}^4x\right| &= \left|\lim_{\lambda\to 0}\int \frac{1}{\lambda}\phi\partial_i\phi\partial_j\phi f\,\mathrm{d}^4x\right| \\ &= \left|\lim_{\lambda\to 0}\int \frac{1}{\lambda}\phi\partial_i\phi\partial_j\phi f g^2\,\mathrm{d}^4x\right| \\ &\le C\lim_{\lambda\to 0}\int |\partial_i(g\phi)\partial_j(g\phi)|\,\mathrm{d}^4x \\ &\le C\lim_{\lambda\to 0}\|\partial_i(g\phi)\|_{L^2}\|\partial_j(g\phi)\|_{L^2} \\ &= 0\,. \end{aligned} \tag{132}$$

Thus, we have $\omega^{(1)}_{abcdefgh} = 0$. Similar arguments show that $\mu_{abcdef} = 0$. Finally, $\nu^{(1)}_{abcde}$ can only depend on terms proportional to

$$\underset{\lambda\to 0}{\text{w-lim}}\frac{1}{\lambda}\phi\partial_i\phi = \underset{\lambda\to 0}{\text{w-lim}}\,\partial_i\left(\frac{\phi^2}{2\lambda}\right) = 0\,, \tag{133}$$

so it vanishes as well, as we desired to show. With this simplification, (87) reduces to

$$\begin{aligned}&-\frac{1}{2}\nabla^c\nabla_c\bar{\gamma}^{(L)}_{ab} + R_a{}^{cd}{}_b(g^{(0)})\bar{\gamma}^{(L)}_{cd} + R^d{}_{(a}(g^{(0)})\bar{\gamma}^{(L)}_{b)d} + \frac{1}{2}g^{(0)}_{ab}R^{cd}(g^{(0)})\bar{\gamma}^{(L)}_{cd} - \frac{1}{2}R(g^{(0)})\bar{\gamma}^{(L)}_{ab} \\ &+\Lambda\left(\bar{\gamma}^{(L)}_{ab} - \frac{1}{2}g^{(0)}_{ab}\bar{\gamma}^{(L)c}{}_c\right) \\ &= 8\pi T^{(1)}_{ab} + \alpha^{(1)}{}_a{}^c{}_b{}^d{}_{cd} - 2\pi\kappa^{(1)}{}_{ab}{}^c{}_c - 8\pi\kappa^{(1)}{}_{(a}{}^c{}_{b)c} + 2\pi g^{(0)}_{ab}\left\{\kappa^{(1)c}{}_c{}^d{}_d - \kappa^{(1)cd}{}_{cd}\right\}\,. \end{aligned} \tag{134}$$

where $\bar{\gamma}^{(L)}_{ab} = \gamma^{(L)}_{ab} - \frac{1}{2}g^{(0)}_{ab}\gamma^{(L)c}{}_c$, and where we have used the wave map gauge condition, $\nabla^a\bar{\gamma}^{(L)}_{ab} = 0$ (see (104)).

Next, we evaluate $\kappa^{(1)}_{abcd}$. With our above assumptions of slowly moving dust matter, we have

$$\kappa^{(1)}_{abcd} = \underset{\lambda\to 0}{\text{w-lim}}\frac{1}{\lambda}h^{(S)}_{ab}(\lambda)\rho(\lambda)u_a(\lambda)u_b(\lambda) = u^{(0)}_c u^{(0)}_d\,\underset{\lambda\to 0}{\text{w-lim}}\frac{1}{\lambda}h^{(S)}_{ab}(\lambda)\rho(\lambda)\,. \tag{135}$$

The second equality follows from our lemma of section II because $\left[u_c(\lambda) - u^{(0)}_c\right]h^{(S)}_{ab}(\lambda)/\lambda \to 0$ uniformly as $\lambda\to 0$ and $\rho(\lambda)\ge 0$. Using the Newtonian assumption that all components, except for the time-time component, of $\mathfrak{h}^{(S)}_{ab}(\lambda)/\lambda$ converge uniformly to zero as $\lambda\to 0$, the lemma of section II tells us furthermore that the only non-vanishing components of $\kappa^{(1)}_{abcd}$ are

$$\kappa^{(1)}_{0000} = -2\,\underset{\lambda\to 0}{\text{w-lim}}\frac{1}{\lambda}\rho(\lambda)\phi(\lambda)\,, \tag{136}$$

$$\kappa^{(1)}_{ij00} = -2g^{(0)}_{ij}\,\underset{\lambda\to 0}{\text{w-lim}}\frac{1}{\lambda}\rho(\lambda)\phi(\lambda)\,. \tag{137}$$

To evaluate $\mu^{(1)}_{abcdef}$, it is more convenient to work with the quantity

$$\begin{aligned}\bar{\mu}^{(1)}{}_{ab}{}^{cdef} &\equiv \underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}\left[\nabla_{(a}\mathfrak{h}^{(S)cd}\nabla_{b)}\mathfrak{h}^{(S)ef} - \underset{\lambda'\to 0}{\text{w-lim}}\left[\nabla_{(a}\mathfrak{h}^{(S)cd}\nabla_{b)}\mathfrak{h}^{(S)ef}\right]\right] \\ &= \underset{\lambda\to 0}{\text{w-lim}}\,\frac{1}{\lambda}\left[\nabla_{a}\mathfrak{h}^{(S)cd}\nabla_{b}\mathfrak{h}^{(S)ef}\right] ,\end{aligned} \tag{138}$$

where the second line follows by the same type of argument as used above to show the vanishing of $\omega^{(1)}_{abcdefgh}$. Since $h^{(S)}_{ab} = \mathfrak{h}^{(S)}_{ab} - \frac{1}{2}g^{(0)}_{ab}\mathfrak{h}^{(S)c}{}_c + O(\mathfrak{h}\mathfrak{h})$, it is not difficult to see that $\bar{\mu}^{(1)}_{abcdef}$ can be expressed straightforwardly in terms of $\mu^{(1)}_{abcdef}$ and vice-versa. Our Newtonian assumptions (125) directly imply that the only potentially non-zero components of $\bar{\mu}^{(1)}_{abcdef}$ are

$$\begin{array}{l}\bar{\mu}^{(1)}_{000000}\,,\ \bar{\mu}^{(1)}_{0i0000}\,,\ \bar{\mu}^{(1)}_{0i000j}\,,\ \bar{\mu}^{(1)}_{ij0000}\,,\\ \bar{\mu}^{(1)}_{ij000k}\,,\ \bar{\mu}^{(1)}_{ij00kl}\,,\ \bar{\mu}^{(1)}_{ij0k0l}\,,\end{array} \tag{139}$$

together with the components related to these by symmetries. However, it also follows that $\bar{\mu}^{(1)}_{ij00kl} = 0$, since

$$\begin{aligned}\left|\int f\bar{\mu}^{(1)}_{ij00kl}\,\mathrm{d}^4x\right| &= \left|\lim_{\lambda\to 0}\int f\frac{1}{\lambda}\nabla_i\mathfrak{h}^{(S)}_{00}\nabla_j\mathfrak{h}^{(S)}_{kl}\,\mathrm{d}^4x\right| \\ &\le C\lim_{\lambda\to 0}\int\left|f\nabla_i\mathfrak{h}^{(S)}_{00}\right|\,\mathrm{d}^4x \\ &\le C'\lim_{\lambda\to 0}\|\nabla_i(f\phi)\|_{L^2} \\ &= 0\,.\end{aligned} \tag{140}$$

where in the second line we used that fact that $\nabla_j\mathfrak{h}^{(S)}_{kl}/\lambda$ is uniformly bounded as $\lambda\to 0$.

To further simplify $\bar{\mu}^{(1)}_{abcdef}$, we now appeal to (86). With the simplifications arising from the vanishing of μ_{abcdef}, $\omega^{(1,\alpha)}_{abcdefgh}$, and $\nu^{(1)}_{abcde}$ together with the wave map gauge condition $\nabla_a\mathfrak{h}^{(S)ab} = 0$ (see (95)), we obtain

$$\frac{1}{4}\bar{\mu}^{(1)e}{}_{eabcd} = 4\pi\kappa^{(1)}_{cdab} - 2\pi g^{(0)}_{cd}\kappa^{(1)e}{}_{eab}\,. \tag{141}$$

From the form we have derived above for $\kappa^{(1)}_{abcd}$, it follows immediately that

$$\bar{\mu}^{(1)i}{}_{i0j0k} = \bar{\mu}^{(1)a}{}_{a0j0k} = 0\,. \tag{142}$$

Again by the positive definiteness of the spatial metric, it follows that $\|\nabla_i(f\mathfrak{h}^{(S)}_{0j})/\lambda^{1/2}\|_{L^2}\to 0$ as $\lambda\to 0$, which implies $\bar{\mu}^{(1)}_{ij0k0l} = 0$ as well. By using similar Schwartz inequality-type arguments, it also follows that $\bar{\mu}^{(1)}_{ij000k} = 0$. The wave map gauge condition $\nabla_a\mathfrak{h}^{(S)ab} = 0$

then yields

$$\begin{aligned}
\bar{\mu}^{(1)}_{0i000j} &= \bar{\mu}^{(1)k}{}_{i0k0j} = 0\,, \\
\bar{\mu}^{(1)}_{0i0000} &= \bar{\mu}^{(1)j}{}_{i0j00} = 0\,, \\
\bar{\mu}^{(1)}_{000000} &= \bar{\mu}^{(1)i}{}_{00i00} = 0\,.
\end{aligned} \tag{143}$$

The only nonvanishing components which remain are

$$\bar{\mu}^{(1)}_{ij0000} = 16\Xi_{ij}\,, \tag{144}$$

where

$$\Xi_{ij} \equiv \operatorname*{w-lim}_{\lambda\to 0} \frac{1}{\lambda}\nabla_i\phi\nabla_j\phi\,. \tag{145}$$

Furthermore, by (141) and our previous expression for $\kappa^{(1)}_{abcd}$, we obtain

$$\Xi^i{}_i = -4\pi \operatorname*{w-lim}_{\lambda\to 0} \frac{1}{\lambda}\rho(\lambda)\phi(\lambda)\,. \tag{146}$$

Finally, expressing $\mu^{(1)}_{abcdef}$ in terms of $\bar{\mu}^{(1)}_{abcdef}$, we find that the nonvanishing components of $\mu^{(1)}_{abcdef}$ are

$$\mu^{(1)}_{ij0000} = 4\Xi_{ij}\,, \tag{147}$$

$$\mu^{(1)}_{ijkl00} = 4g^{(0)}_{kl}\Xi_{ij}\,, \tag{148}$$

$$\mu^{(1)}_{ijklmn} = 4g^{(0)}_{kl}g^{(0)}_{mn}\Xi_{ij}\,. \tag{149}$$

Next, we consider $T^{(1)}_{ab}$. We have,

$$\begin{aligned}
T^{(1)}_{ab} &= \operatorname*{w-lim}_{\lambda\to 0} \frac{1}{\lambda}\left(T_{ab}(\lambda) - T^{(0)}_{ab}\right) \\
&= \operatorname*{w-lim}_{\lambda\to 0} \frac{1}{\lambda}\left(\rho(\lambda)u_a(\lambda)u_b(\lambda) - \rho^{(0)}u^{(0)}_a u^{(0)}_b\right) \\
&= \operatorname*{w-lim}_{\lambda\to 0} \frac{1}{\lambda}\Big(\left[\rho(\lambda) - \rho^{(0)}\right] u^{(0)}_a u^{(0)}_b + \rho(\lambda)v_a(\lambda)v_b(\lambda) + 2\rho(\lambda)u^{(0)}_{(a}v_{b)}(\lambda) \\
&\quad +\rho(\lambda)u^{(0)}_a u^{(0)}_b\left[h_{cd}(\lambda)u^{(0)c}u^{(0)d} + v_c(\lambda)v^c(\lambda)\right] + 2\rho(\lambda)u^{(0)c}u^{(0)}_{(a}h_{b)c}(\lambda)\Big) \\
&= \rho^{(1)}u^{(0)}_a u^{(0)}_b + p^{(1)}_{ab} + 2u^{(0)}_{(a}P^{(1)}_{b)} + u^{(0)}_a u^{(0)}_b \kappa^{(1)cd}{}_{cd} + \rho^{(0)}u^{(0)}_a u^{(0)}_b u^{(0)c}u^{(0)d}\gamma^{(L)}_{cd} \\
&\quad +u^{(0)}_a u^{(0)}_b p^{(1)c}{}_c + 2\kappa^{(1)c}{}_{(ab)c} + 2\rho^{(0)}u^{(0)c}u^{(0)}_{(a}\gamma^{(L)}_{b)c} \\
&= \rho^{(1)}u^{(0)}_a u^{(0)}_b + 2u^{(0)}_{(a}P^{(1)}_{b)} + p^{(1)}_{ab} + u^{(0)}_a u^{(0)}_b p^{(1)c}{}_c + u^{(0)}_a u^{(0)}_b \kappa^{(1)cd}{}_{cd} + 2\kappa^{(1)c}{}_{(ab)c} \\
&\quad +u^{(0)}_a u^{(0)}_b T^{(0)cd}\bar{\gamma}^{(L)}_{cd} + 2T^{(0)c}{}_{(a}\bar{\gamma}^{(L)}_{b)c} - \frac{1}{2}T^{(0)}_{ab}\bar{\gamma}^{(L)c}{}_c\,.
\end{aligned} \tag{150}$$

Here, we have introduced the quantities

$$\rho^{(1)} \equiv \underset{\lambda\to 0}{\text{w-lim}} \left[\rho(\lambda) - \rho(0)\right]/\lambda\,, \tag{151}$$

$$P_a^{(1)} \equiv \underset{\lambda\to 0}{\text{w-lim}}\, \rho(\lambda) v_a(\lambda)/\lambda\,, \tag{152}$$

and

$$p_{ab}^{(1)} \equiv \underset{\lambda\to 0}{\text{w-lim}}\, \rho(\lambda) v_a(\lambda) v_b(\lambda)/\lambda\,. \tag{153}$$

Now, for a pressureless fluid, ρu^a is a conserved current, whose integrated flux through a spacelike hypersurface can be interpreted as (proportional to) the "number of particles" (often referred to as the "number of baryons") in the fluid. The perturbation, $\rho_M^{(1)}$, of the density (relative to the background metric $g_{ab}^{(0)}$) of particles on a $\tau =$ const. hypersurface is given by

$$\rho_M^{(1)} \boldsymbol{\epsilon}_{abc}^{(0)} \equiv \underset{\lambda\to 0}{\text{w-lim}}\, \frac{1}{\lambda}\left[T_{de}(\lambda) u^d(\lambda) n^e(\lambda) \boldsymbol{\epsilon}_{abc}(\lambda) - T_{de}^{(0)} u^{(0)d} n^{(0)e} \boldsymbol{\epsilon}_{abc}^{(0)}\right]\,, \tag{154}$$

where $n^a(\lambda)$ and $\boldsymbol{\epsilon}_{abc}(\lambda)$ are the unit normal and 3-volume of the hypersurface in the metric $g_{ab}(\lambda)$ and $n^{(0)a}$ and $\boldsymbol{\epsilon}_{abc}^{(0)}$ are the corresponding quantities for $g_{ab}^{(0)}$. We obtain

$$\rho_M^{(1)} = \rho^{(1)} + \frac{1}{2} p^{(1)i}{}_i + \frac{3}{4\pi}\Xi^i{}_i + \frac{1}{2}\rho^{(0)}\left(\bar{\gamma}_{00}^{(L)} - \frac{1}{2}\bar{\gamma}^{(L)c}{}_c\right)\,. \tag{155}$$

We now return to our perturbation equation (134) for $\bar{\gamma}_{ab}^{(L)}$. We use the above explicit expressions for $\kappa_{abcd}^{(1)}$ and $\mu_{abcdef}^{(1)}$, we use (150) to substitute for $T_{ab}^{(1)}$, and we use (155) to eliminate $\rho^{(1)}$ in favor of $\rho_M^{(1)}$. Bringing all terms explicitly involving $\bar{\gamma}_{ab}^{(L)}$ to the left side, we obtain

$$\begin{aligned}
&-\frac{1}{2}\nabla^c\nabla_c\bar{\gamma}_{ab}^{(L)} + R_a{}^{cd}{}_b(g^{(0)})\bar{\gamma}_{cd}^{(L)} + R^d{}_{(a}(g^{(0)})\bar{\gamma}_{b)d}^{(L)} + \frac{1}{2}g_{ab}^{(0)} R^{cd}(g^{(0)})\bar{\gamma}_{cd}^{(L)} - \frac{1}{2}R(g^{(0)})\bar{\gamma}_{ab}^{(L)} \\
&+\Lambda\left(\bar{\gamma}_{ab}^{(L)} - \frac{1}{2}g_{ab}^{(0)}\bar{\gamma}^{(L)c}{}_c\right) - 4\pi T_{ab}^{(0)}\bar{\gamma}_{cd}^{(L)} u^{(0)c} u^{(0)d} - 16\pi T^{(0)c}{}_{(a}\bar{\gamma}_{b)c}^{(L)} + 2\pi T_{ab}^{(0)}\bar{\gamma}^{(L)c}{}_c \\
&\equiv 8\pi\Theta_{ab}\,,
\end{aligned} \tag{156}$$

where the components of Θ_{ab} are explicitly given by

$$\Theta_{00} = \rho_M^{(1)} + \frac{1}{2}p^{(1)i}{}_i - \frac{1}{8\pi}\Xi^i{}_i\,, \tag{157}$$

$$\Theta_{0i} = P_i^{(1)}\,, \tag{158}$$

$$\Theta_{ij} = p_{ij}^{(1)} + \frac{1}{4\pi}\Xi_{ij} - \frac{1}{8\pi}g_{ij}^{(0)}\Xi^k{}_k\,. \tag{159}$$

This expression for the "effective perturbed stress-energy" of matter agrees with the expression obtained by [27] (see also [21]).

All of the terms on the left side of (156) together with the terms $\rho_M^{(1)}$ and $P_i^{(1)}$ appearing in Θ_{ab} would be present in ordinary linearized perturbation theory. We previously showed in section II that, in the absence of gravitational radiation, small scale inhomogenities cannot affect the dynamics of the background metric. We now see that, under the assumptions stated at the beginning of this subsection, the only effect that small scale inhomogeneities have on long wavelength perturbations is to add the terms involving $p_{ij}^{(1)}$ and Ξ_{ij} to the effective perturbed stress-energy Θ_{ab}. These additional terms provide precisely the contributions to the energy density and stresses that one would expect from kinetic motions and Newtonian gravitational potential energy and stresses. In particular, for the energy density, these terms have the effect of shifting the proper mass density appearing in the FLRW background to an "ADM mass density" (to first order in λ). Although the presence of these terms is very important as a matter of principle, they should be extremely small compared with $\rho_M^{(1)}$ (and even with $P_i^{(1)}$).

Finally, we note that although we have written (156) in wave map gauge (104), we may use the gauge freedom with respect to smooth, one-parameter groups of diffeomorphisms discussed in the paragraph below (87) to impose any other desired gauge condition on $\gamma_{ab}^{(L)}$. Since the left side of (156) together with the terms $\rho_M^{(1)}$ and $P_i^{(1)}$ in Θ_{ab} have the same gauge transformation properties as in ordinary linearized perturbation theory, these terms will take the same form as in ordinary linearized perturbation theory when transformed to the new gauge. On the other hand, the terms in Θ_{ab} involving Ξ_{ij} and $p_{ij}^{(1)}$ are gauge invariant under the transformations induced by a smooth, one-parameter group of diffeomorphisms. This enables one to write down the form of (156) in other gauges.

V. APPLICABILITY OF OUR FORMALISM TO THE REAL UNIVERSE

In this paper, we have developed a formalism that enables one to take full account of small scale non-linearities in Einstein's equation. This formalism is, in essence, an adaptation of Burnett's formulation of the "shortwave approximation" for analyzing the self-gravitating effects of short-wavelength gravitational radiation. The key idea of our formalism is to consider an idealized, one-parameter family of metrics, $g_{ab}(\lambda)$, with the property that, as $\lambda \to 0$, the deviation of $g_{ab}(\lambda)$ from a "background metric" $g_{ab}^{(0)}$ goes to zero in proportion to λ, but the scales over which the metric and stress-energy vary also go to zero in proportion to λ. In principle, this allows the small-scale nonlinear terms in Einstein's equation to have a significant effect on the large scale dynamics of $g^{(0)}_{ab}$. However, we proved in section II that the only such effects that can actually arise correspond to the presence of an effective stress-energy associated with gravitational radiation. No new effects can arise from the presence of matter (provided that the matter satisfies the weak energy condition), and no effects can arise that in any way mimic "dark energy".

In sections III and IV, we applied our formalism to analyze the leading order deviations of $g_{ab}(\lambda)$ from $g^{(0)}_{ab}$. Within our formalism, these deviations can naturally be split into "long wavelength" and "short wavelength" parts. The long wavelength part satisfies a modified version of the usual linearized perturbation equation. We argued that, in the case of cosmological perturbations with nearly Newtonian sources, the short wavelength part of the leading order deviation of $g_{ab}(\lambda)$ from $g^{(0)}_{ab}$ is described by Newtonian gravity, taking into account only the local matter distribution.

However, our actual universe is not an idealized limit of spacetimes with inhomogeneities on arbitrarily small scales, but is a particular spacetime with finite amplitude, finite wavelength deviations from a FLRW model. Quantities in our formalism like the "long wavelength perturbation," $\gamma^{(L)}_{ab}$, are defined by taking weak limits as $\lambda \to 0$. How do quantities like $\gamma^{(L)}_{ab}$ that arise in our formalism correspond to quantities observed in the actual universe? Furthermore, to what extent are we justified in applying our results to the actual universe, i.e., how do we know whether the actual universe is sufficiently "close" to the idealized limit considered in our formalism that results derived using our formalism should hold to a good approximation?

The above questions can also be asked in the context of ordinary linearized perturbation theory. In this context, one also deals with a limit of a one-parameter family of metrics $g_{ab}(\lambda, x)$ (now assumed to be jointly smooth in (λ, x)) and the formalism obtains results for idealized quantities like $\partial g_{ab}(\lambda, x)/\partial\lambda|_{\lambda=0}$. Nevertheless, in the case of ordinary perturbation theory, the above questions can be answered in a relatively straightforward manner. In an actual ("finite λ") spacetime with metric g_{ab}, one may introduce a background metric $g^{(0)}_{ab}$ and identify the difference, $h_{ab} = g_{ab} - g^{(0)}_{ab}$, with $\lambda\partial g_{ab}(\lambda, x)/\partial\lambda|_{\lambda=0}$. A fundamental criterion for the validity of ordinary perturbation theory is that $|h_{\mu\nu}| \ll 1$ in some orthonormal basis of $g^{(0)}_{ab}$. However, we also need to satisfy conditions that state that first and second spacetime derivatives of $h_{\mu\nu}$ are sufficiently "small," since these quantities appear in the nonlinear terms that are being neglected in Einstein's equation. In particular, the criteria for the applicability of ordinary linear perturbation theory cannot be satisfied in a cosmological spacetime with $|\delta\rho| > \rho_0$, as occurs in the real universe. Indeed, it is for this reason that we have developed the formalism of this paper. It should be noted that even in the case of spacetimes that do satisfy the basic criteria for the validity of ordinary linear perturbation theory, it would be very difficult to obtain precise error estimates for this approximation.

If one wishes to apply the formalism of this paper to a particular spacetime, such as our universe, it also is necessary to introduce a background metric $g^{(0)}_{ab}$. As in ordinary perturbation theory, it is essential that the difference, $h_{ab} = g_{ab} - g^{(0)}_{ab}$, satisfy $|h_{\mu\nu}| \ll 1$ in some orthonormal basis of $g^{(0)}_{ab}$. The main advantage of our formalism over ordinary perturbation theory is that it imposes much weaker restrictions on spacetime derivatives of $h_{\mu\nu}$. In particular, for cosmological spacetimes, it allows $|\delta\rho| \gg \rho_0$ on small scales. However, it is clear that some further restrictions in addition to $|h_{\mu\nu}| \ll 1$ must hold for the results derived using our formalism to be a good approximation. Although it would be extremely difficult to formulate mathematically precise criteria for the validity of applying our formalism to a given spacetime—much more difficult than for ordinary perturbation theory—we now shall propose a rough criterion for its validity.

In essence, taking the weak limit as $\lambda \to 0$ of quantities that arise in our formalism corresponds to taking spacetime averages of these quantities over arbitrarily small regions of spacetime before letting $\lambda \to 0$. For a given, fixed spacetime, such as our universe, the weak limits in our formalism should thus be identified with spacetime averages over small regions. Thus, for example, the quantity $\lambda\gamma^{(L)}_{ab}(x)$ [where $\gamma^{(L)}_{ab} \equiv \text{w-lim}_{\lambda\to 0}\, h_{ab}(\lambda)/\lambda$] should be identified with the spacetime average of h_{ab} over a suitable region, $\mathcal{R}_x$, centered on x. In order that the spacetime under consideration be be "sufficiently close" to the idealized limit $\lambda \to 0$ of our formalism that our results should apply, it is necessary that $\mathcal{R}_x$ be small compared with the curvature scale, R_C, of $g^{(0)}_{ab}$; indeed, the notion of "averaging" would be highly ambiguous if this were not the case. However, it also is necessary that $\mathcal{R}_x$ be large enough that our perturbative approximations should apply. In particular, $\lambda\gamma^{(L)}_{ab}$ must satisfy the same "smallness" conditions as required in ordinary linearized perturbation theory, i.e., its spacetime derivatives must be appropriately small. Similarly, the perturbative quantity $\lambda T^{(1)}_{00}$ (see (78)) should correspond to the spacetime average of $T_{00} - T^{(0)}_{00}$. In order that our approximations should apply, we must choose $\mathcal{R}_x$ large enough that $|\lambda T^{(1)}_{00}| \ll T^{(0)}_{00}$.

The condition that the spacetime average of $T_{00} - T_{00}^{(0)}$ be less than $T_{00}^{(0)}$ was considered in section IVB above and used to define the homogeneity scale, l (see (111)). Similar "homogeneity scales" can be defined for all other "long wavelength" quantities arising in the analysis of section III. Since our averaging region must be large compared with these homogeneity scales but small compared with the scale of the curvature of the background metric, it is clear that a necessary condition for the applicabiliity of our formalism is that these homogeneity scales be small compared with the curvature of the background metric. *We believe that this condition, together with* $|h_{\mu\nu}| \ll 1$*, should also be sufficient for the applicability of our formalism*, since it should guarantee the appropriate "smallness" of all long wavelength perturbative quantities.

In the case of our universe, it seems reasonable to assume that the only relevant homogeneity scale is the density homogeneity scale, l, defined by (111). In the present universe, we have $l \lesssim 100$ Mpc whereas the scale of the curvature of the background metric (i.e., the present Hubble radius) is of order $R_C \sim 3000$ Mpc. Thus $l \ll R_C$. In addition, common sense estimates indicate that, except in the immediate vicinity of black holes or neutron stars, we have $|h_{\mu\nu}| \leq 10^{-5}$. Thus, if we define "long wavelength perturbations" by averaging on scales of order $L \sim 100$ Mpc, the criteria we have proposed for the validity of our approximations should be satisfied. *Consequently, we believe that the results of this paper should be applicable to our universe to an excellent approximation.*

If this is the case, then the following conclusions can be drawn: (1) As shown in section II, the only effect that small scale inhomogeneities can have on the leading (i.e., zeroth) order large scale dynamics of our universe is that of a $P = \frac{1}{3}\rho$ fluid, corresponding to the presence of gravitational radiation. In particular, small scale inhomogeneities cannot mimic the effects of "dark energy." (2) The deviation, h_{ab}, of the metric from a FLRW model can be broken up into "long wavelength" and "short wavelength" parts. The long wavelength part corresponds in our universe to averaging h_{ab} over a spatial scale of order $L \sim 100$ Mpc. As analyzed fully in section III and subsection IVC, it satisfies a linear equation with an additional source term due to the short wavelength part. As argued in subsection IVB, the short wavelength part should be described by Newtonian gravity, taking into account only the matter within a proper distance of order $L \sim 100$ Mpc of the point under consideration.

Thus, the analysis of this paper goes a long way towards justifying many of the key assumptions made in cosmology. In particular, if the matter in the universe is non-relativistic, it suggests that to calculate structure formation in the universe on all scales—from the Hubble radius or larger down to arbitrarily small scales—it should be a good approximation to evolve the long wavelength part of the deviation of the metric from a FLRW model by (156), to calculate the short wavelength part by (117), and to calculate the motion of the matter by (119) together with (120)–(124).

In addition, our analysis suggests how further improvements can be made to get more accurate approximations. To improve upon the description of the "long wavelength" part of the deviation of the metric from a FLRW model one could go to higher order perturbation theory. In particular, the second order correction to $\gamma_{ab}^{(L)}$ would be defined in our framework by

$$\gamma_{ab}^{(2)} = \operatorname*{w-lim}_{\lambda \to 0} \frac{g_{ab}(\lambda) - g_{ab}^{(0)} - \lambda \gamma_{ab}^{(L)}}{\lambda^2}\,, \tag{160}$$

assuming, of course, that this weak limit exists. The equations satisfied by $\gamma_{ab}^{(2)}$—taking full account of the small scale inhomogeneities—could be derived by the methods used in section III. The description of the short wavelength part of the deviation of the metric from a FLRW model could be improved by a more accurate treatment of the integral relation (108). However, the investigation of such improvements is beyond the scope of this paper.

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