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Physical decomposition of the gauge and gravitational fields

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Abstract

Physical decomposition of the non-Abelian gauge field has recently helped to achieve a meaningful gluon spin. Here we extend this approach to gravity and attempt a meaningful gravitational energy. The metric is unambiguously separated into a pure geometric term which contributes null curvature tensor, and a physical term which represents the true gravitational effect and always vanishes in a flat space-time. By this decomposition the conventional pseudo-tensors of the gravitational stress-energy are easily rescued to produce definite physical result. Our decomposition applies to any symmetric tensor, and has interesting relation to the transverse-traceless decomposition discussed by Arnowitt, Deser and Misner, and by York.

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Gauge invariance is the most elegant and efficient principle for constructing interactions in the present field theories of physics. By requiring field equations to be gauge invariant, the manner of the couplings (and self-couplings) of various fields are strongly constrained. This applies both to the standard model of the strong and electro-weak interactions, and to Einstein's gravitational theory. For the latter case gauge invariance refers to general covariance under arbitrary coordinate transformation. It is rather annoying, however, that a theory built uniquely out of the gauge-invariance requirement does not seem to guarantee gauge invariance for all physical quantities. In hadron physics, e.g., in the two-decade efforts to understand how the nucleon spin originates from the spin and orbital motion of its quark and gluon constituents, one encounters severe difficulty in finding a gauge-invariant description of the gluon spin and orbital angular momentum. Only recently, a solution was obtained in Ref. [1], and further developed in Ref. [2]. A more celebrated and still unsolved gauge-dependence problem is the energy density of the gravitational field. After countless attempts of nearly a century, a convincing solution is still lacking. A reflection of this desperation is the often heard argument that, since the effect of gravity at any point can be eliminated by transiting to a free-fall frame, gravitational energy is intrinsically non-localizable and can at best be quasi-local to a closed two-surface [3, 4].

The key obstacle to constructing all physical quantities gauge-invariantly is the inevitable involvement of the gauge or gravitational field together with their ordinary derivatives, which are all intrinsically gauge dependent. The idea in Refs. [1, 2] is to decompose the gauge field: $A_\mu \equiv \hat{A}_\mu + \bar{A}_\mu$. The aim is that $\hat{A}_\mu$ will be a physical term which is gauge-covariant and always vanishes in the vacuum, and $\bar{A}_\mu$ is a pure-gauge term which solely carries the gauge freedom and has no essential physical effects (particularly, it does not contribute to the electric or magnetic field strength). Equipped with the separate $\hat{A}_\mu$ and $\bar{A}_\mu$, a naively gauge-dependent quantity (such as the gluon spin $\vec{S} = \vec{E} \times \vec{A}$) can easily be rescued to be gauge-invariant, simply by replacing A_μ with $\hat{A}_\mu$, and by replacing the ordinary derivative with the pure-gauge covariant derivative constructed with $\bar{A}_\mu$ instead of A_μ.

Mathematically, a well-defined separation $A_\mu = \hat{A}_\mu + \bar{A}_\mu$ means an unambiguous prescription for constructing $\hat{A}_\mu$ and $\bar{A}_\mu$ out of a given A_μ. The properties (especially, gauge transformations) of $\hat{A}_\mu$ and $\bar{A}_\mu$ are then inherently determined via their mathematical expressions in terms of A_μ. In Refs. [1, 2], it was found that $\hat{A}_\mu$ and $\bar{A}_\mu$ can indeed be solved in terms of A_μ by setting up proper differential equations and boundary conditions, which lead to unique solutions for $\hat{A}_\mu$ and $\bar{A}_\mu$ with desired physical properties. In this paper, we show that this method can be generalized to gravitational theory. The metric tensor $g_{\mu\nu}$ is unambiguously decomposed into the sum of a physical term $\hat{g}_{\mu\nu}$, which represents the true gravitational effect, and a pure geometric term $\bar{g}_{\mu\nu}$, which represents the spurious gravitational effect associated with coordinate choice. Gauge-dependence of the gravitational energy originates exactly from the fact that the metric may contain a spurious gravitational effect. While in a flat space-time the Cartesian coordinate with vanishing affine connection seems a natural choice, in an intrinsically curved space-time no coordinate is obviously more natural than others, hence it is no longer a trivial task to get rid of the spurious gravitational effect. In an accompanying paper [5], we discuss a gauge-fixing approach, by defining a unique physical coordinate which contains no spurious gravitational effect. In this paper, we present the more general field-decomposition approach, by seeking a prescription to identify the geometric $\bar{g}_{\mu\nu}$ for a given metric $g_{\mu\nu}$ in any coordinate.

As for gauge theories, we find that the prescription is again a set of defining differential equations, which are displayed most concisely in the form [6]:

$$\bar{R}^\rho{}_{\sigma\mu\nu} \equiv \partial_\mu \bar{\Gamma}^\rho_{\sigma\nu} - \partial_\nu \bar{\Gamma}^\rho_{\sigma\mu} + \bar{\Gamma}^\rho_{\alpha\mu}\bar{\Gamma}^\alpha_{\sigma\nu} - \bar{\Gamma}^\rho_{\alpha\nu}\bar{\Gamma}^\alpha_{\sigma\mu} = 0, \tag{1a}$$

$$g^{ij}\hat{\Gamma}^\rho_{ij} = 0. \tag{1b}$$

The notations require some caution: $\bar{\Gamma}^\rho_{\mu\nu}$ is the purely geometric part of the affine connection. Its relation to $\bar{g}_{\mu\nu}$ is analogous to that of $\Gamma^\rho_{\mu\nu}$ and $g_{\mu\nu}$:

$$\bar{\Gamma}^\rho_{\mu\nu} \equiv \frac{1}{2}\bar{g}^{\rho\sigma}(\partial_\mu \bar{g}_{\sigma\nu} + \partial_\nu \bar{g}_{\sigma\mu} - \partial_\sigma \bar{g}_{\mu\nu}). \tag{2}$$

Here $\bar{g}^{\mu\nu}$ is defined as the inverse of $\bar{g}_{\mu\nu}$, i.e., $\bar{g}^{\mu\rho}\bar{g}_{\rho\nu} = \delta^\mu{}_\nu$. The aim of this choice is that $\bar{R}^\rho{}_{\sigma\mu\nu}$ in (1a) is just the Riemann curvature of $\bar{g}_{\mu\nu}$. It must then be noted that $\hat{g}^{\mu\nu} \equiv g^{\mu\nu} - \bar{g}^{\mu\nu}$ is not the inverse of $\hat{g}_{\mu\nu}$. (In fact, the physical term $\hat{g}_{\mu\nu}$ may not have an inverse at all.) The difference $\Gamma^\rho_{\mu\nu} - \bar{\Gamma}^\rho_{\mu\nu} \equiv \hat{\Gamma}^\rho_{\mu\nu}$ is defined as the physical connection. It is not related to $\hat{g}_{\mu\nu}$ as in Eq. (2).

To comprehend how Eq. (1) is chosen, how it gives solution for $\hat{\Gamma}^\rho_{\sigma\mu}$ and $\bar{\Gamma}^\rho_{\sigma\mu}$ [and further for $\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ by Eq. (2)] with desired properties, and how the solution in turn is employed to solve the gauge-dependence problem of the gravitational energy, it is most helpful to recall the parallel constructions for gauge theories in Refs. [1, 2]. In Abelian case, the gauge field A_μ transforms as $A_\mu \to A'_\mu = A_\mu - \partial_\mu\omega$, which leaves the field strength invariant:

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to F'_{\mu\nu} = F_{\mu\nu}$. The defining equations for the separation $A_\mu = \hat{A}_\mu + \bar{A}_\mu$ are

$$\bar{F}_{\mu\nu} \equiv \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu = 0, \tag{3a}$$

$$\partial_i \hat{A}_i = 0. \tag{3b}$$

Eq. (3a) has very clear physical meaning: the pure-gauge term $\bar{A}_\mu$ gives null field strength. Eq. (3b) can be regarded as the transverse condition for a physical photon with zero mass. But to avoid confusion with the radiation gauge condition $\partial_i A_i = 0$ for the full A_i, it is more helpful to think in a mathematical way that Eq. (3) are the needed differential equations to solve $\hat{A}_\mu$ and $\bar{A}_\mu$. Since $\hat{A}_\mu + \bar{A}_\mu = A_\mu$, it suffices to examine $\hat{A}_\mu$. To this end we rewrite Eq. (3a) as

$$\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu = F_{\mu\nu}. \tag{3a$'$}$$

A clever way to solve is to act on both sides with ∂_i, set $\mu = i$ and sum over i, and use Eq. (3b). This gives

$$\vec{\partial}^2 \hat{A}_\nu = \partial_i F_{i\nu}, \text{ or } \hat{A}_\nu = \frac{1}{\vec{\partial}^2} \partial_i F_{i\nu}, \tag{4}$$

where we have required a natural boundary condition that, for a finite system, the physical term $\hat{A}_\mu$ vanish at infinity, as does the field strength $F_{\mu\nu}$. [7] The explicit solution in Eq. (4) indicates clearly that the physical field $\hat{A}_\mu$ is gauge invariant, and hence the pure-gauge field $\bar{A}_\mu = A_\mu - \hat{A}_\mu$ carries all the gauge freedom and transforms in the same manner as does the full A_μ. Moreover, Eq. (4) tells us that the physical term $\hat{A}_\mu$ vanishes if the field strength $F_{\mu\nu} = 0$.

In non-Abelian case, the gauge transformation is more complicated: $A'_\mu = U A_\mu U^\dagger - \frac{i}{g} U \partial_\mu U^\dagger$. The field strength now contains a self-interaction term, and transforms covariantly instead of invariantly: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \to F'_{\mu\nu} = U F_{\mu\nu} U^\dagger$. It is fairly non-trivial to choose proper defining equations for the non-Abelian $\hat{A}_\mu$ and $\bar{A}_\mu$. They were originally proposed in Ref. [1], and further developed in Ref. [2] to be:

$$\bar{F}_{\mu\nu} \equiv \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu + ig[\bar{A}_\mu, \bar{A}_\nu] = 0, \tag{5a}$$

$$\bar{\mathcal{D}}_i \hat{A}_i \equiv \partial_i \hat{A}_i + ig[\bar{A}_i, \hat{A}_i] = 0. \tag{5b}$$

We will shortly show that Eq. (5) gives solution for $\hat{A}_\mu$ and $\bar{A}_\mu$ with desired gauge-transformation properties:

$$\hat{A}'_\mu = U \hat{A}_\mu U^\dagger, \quad \bar{A}'_\mu = U \bar{A}_\mu U^\dagger - \frac{i}{g} U \partial_\mu U^\dagger. \tag{6}$$

By these properties, $\bar{\mathcal{D}}_\mu = \partial_\mu + ig[\bar{A}_\mu,\ ]$ is a pure-gauge covariant derivative for the adjoint representation, and Eq. (5) is covariant under non-Abelian gauge transformations. Analogous to the Abelian case, Eq. (5) says that $\bar{A}_\mu$ is a pure-gauge field giving null field strength, and the physical field $\hat{A}_\mu$ satisfies a "covariant transverse condition". However, as we remarked in the Abelian case, the real justification for Eq. (5) is that they are the right mathematic equations to solve $\hat{A}_\mu$ and $\bar{A}_\mu$ in terms of A_μ, with desired gauge transformations in (6). Again, we examine $\hat{A}_\mu$ with trivial boundary condition, and rewrite Eq. (5):

$$\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu = F_{\mu\nu} + ig([\hat{A}_\mu - A_\mu, \hat{A}_\nu] - [\hat{A}_\mu, A_\nu]) \tag{7a}$$

$$\partial_i \hat{A}_i = ig[\hat{A}_i, A_i]. \tag{7b}$$

Due to non-linearity, these are not easy to solve. To proceed, we employ the usual technique of perturbative expansion, which applies when either the coupling constant g or the field amplitude is small. For a small g, e.g., we write $\hat{A}_\mu = \hat{A}_\mu^{(0)} + g\hat{A}_\mu^{(1)} + g^2\hat{A}_\mu^{(2)} + \cdots$. Eq. (7) can then be solved order by order. The zeroth-order term $\hat{A}_\mu^{(0)}$ satisfy the same equations as (3a$'$) and (3b). Its solution is given by Eq. (4), and can in turn be used to solve the equations for the leading non-trivial term $\hat{A}_\mu^{(1)}$:

$$\partial_\mu \hat{A}_\nu^{(1)} - \partial_\nu \hat{A}_\mu^{(1)} = i[\hat{A}_\mu^{(0)} - A_\mu, \hat{A}_\nu^{(0)}] - i[\hat{A}_\mu^{(0)}, A_\nu], \tag{8a}$$

$$\partial_i \hat{A}_i^{(1)} = i[\hat{A}_i^{(0)}, A_i]. \tag{8b}$$

The solution is obtained by the same strategy for Abelian case, and can be further employed to solve the next-order term $\hat{A}_\mu^{(2)}$, and so on. Given validity of this perturbative expansion, the solution to Eq. (7) is unique. This uniqueness has important implications: a) $F_{\mu\nu} = 0$ is necessary and sufficient for $\hat{A}_\mu = 0$; and b) $\hat{A}_\mu$ and $\bar{A}_\mu$ have the gauge transformations as in (6). The proof of b) is as follows: Eq. (6) is solution of Eq. (5) with $\hat{A}_\mu$ and $\bar{A}_\mu$ replaced by $\hat{A}'_\mu$ and $\bar{A}'_\mu$, and since the solution to Eq. (5) is unique, Eq. (6) gives the right gauge transformations.

We now turn to the gravitational equations (1) and (2). Because of non-linearity, we have to rely again on perturbative method, and require that the gravitational field be at most moderately strong. Namely, the magnitude of $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ (with $\eta_{\mu\nu}$ the Minkowski metric) is smaller than 1 and can be treated as an expansion parameter. It then takes a little algebra to show that Eqs. (1) and (2) can be solved similar to the gauge-field equations. We proceed by first looking at the physical connection $\hat{\Gamma}^{\rho}_{\sigma\nu}$, to which we can assign a natural boundary condition that (for a finite system) $\hat{\Gamma}^{\rho}_{\sigma\nu}$ vanish at infinity as does the Riemann curvature $R^{\rho}{}_{\sigma\mu\nu}$. We define an expansion $\hat{\Gamma}^{\rho}_{\sigma\nu} = \hat{\Gamma}^{\rho(1)}_{\sigma\nu} + \hat{\Gamma}^{\rho(2)}_{\sigma\nu} + \cdots$ in orders of $h_{\mu\nu}$. For the first-order term $\hat{\Gamma}^{\rho}_{\sigma\nu}$, we get from Eq. (1)

$$\partial_\mu \hat{\Gamma}^{\rho(1)}_{\sigma\nu} - \partial_\nu \hat{\Gamma}^{\rho(1)}_{\sigma\mu} = R^{\rho}{}_{\sigma\mu\nu}, \tag{9a}$$

$$\hat{\Gamma}^{\rho(1)}_{ii} = 0. \tag{9b}$$

Solution: Set $\mu = \sigma = i$ in Eq. (9a), sum over i, and use Eq. (9b), we get

$$\partial_i \hat{\Gamma}^{\rho(1)}_{i\nu} = R^{\rho}{}_{ii\nu}. \tag{10}$$

Then, act on both sides of Eq. (9a) with ∂_i, set $\mu = i$, sum over i, and use Eq. (10), we obtain the solution

$$\hat{\Gamma}^{\rho(1)}_{\sigma\nu} = \frac{1}{\vec{\partial}^2}(\partial_i R^{\rho}{}_{\sigma i\nu} + \partial_\nu R^{\rho}{}_{ii\sigma}). \tag{11}$$

This can then be employed to solve the second-order term $\hat{\Gamma}^{\rho(2)}_{\sigma\nu}$. From Eq. (1), we have

$$\partial_\mu \hat{\Gamma}^{\rho(2)}_{\sigma\nu} - \partial_\nu \hat{\Gamma}^{\rho(2)}_{\sigma\mu} = (\hat{\Gamma}^{\rho(1)}_{\alpha\mu} - \Gamma^{\rho}_{\alpha\mu})\hat{\Gamma}^{\alpha(1)}_{\sigma\nu} - \hat{\Gamma}^{\rho(1)}_{\alpha\mu}\Gamma^{\alpha}_{\sigma\nu} - (\hat{\Gamma}^{\rho(1)}_{\alpha\nu} - \Gamma^{\rho}_{\alpha\nu})\hat{\Gamma}^{\alpha(1)}_{\sigma\mu} + \hat{\Gamma}^{\rho(1)}_{\alpha\nu}\Gamma^{\alpha}_{\sigma\mu}, \tag{12a}$$

$$\hat{\Gamma}^{\rho(2)}_{ii} = h^{ij}\hat{\Gamma}^{\rho(1)}_{ij}. \tag{12b}$$

Here $h^{\mu\nu} \equiv \eta^{\mu\nu} - g^{\mu\nu}$. Though looking tedious, Eq. (12) can be solved similar to Eq. (9). The solution can be further employed to continue the perturbative procedure up to any desired order, in principle.

Having separated the affine connection, we can use Eq. (2) to solve the metric separation, $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \hat{g}_{\mu\nu}$. It is useful to define $h_{\mu\nu} \equiv \bar{h}_{\mu\nu} + \hat{h}_{\mu\nu}$, thus $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$ and $\hat{g}_{\mu\nu} = \hat{h}_{\mu\nu}$. We again look at the physical term $\hat{h}_{\mu\nu}$ which can be assigned a trivial boundary condition. As for $\hat{\Gamma}^{\rho}_{\mu\nu}$, we define an expansion $\hat{h}_{\mu\nu} \equiv \hat{h}^{(1)}_{\mu\nu} + \hat{h}^{(2)}_{\mu\nu} + \cdots$ in orders of $h_{\mu\nu}$. From Eq. (2), we derive the first-order equation

$$\partial_\mu \hat{h}^{(1)}_{\sigma\nu} + \partial_\nu \hat{h}^{(1)}_{\sigma\mu} - \partial_\sigma \hat{h}^{(1)}_{\mu\nu} = 2\eta_{\rho\sigma}\hat{\Gamma}^{\rho(1)}_{\mu\nu} \tag{13}$$

Interchange σ, ν in Eq. (13) and add the result back to Eq. (13), we get

$$\partial_\mu \hat{h}^{(1)}_{\sigma\nu} = \eta_{\rho\sigma}\hat{\Gamma}^{\rho(1)}_{\mu\nu} + \eta_{\rho\nu}\hat{\Gamma}^{\rho(1)}_{\mu\sigma}. \tag{14}$$

Act on both sides with ∂_i, set $\mu = i$ and sum over i, we obtain

$$\vec{\partial}^2 \hat{h}^{(1)}_{\sigma\nu} = \partial_i(\eta_{\rho\sigma}\hat{\Gamma}^{\rho(1)}_{i\nu} + \eta_{\rho\nu}\hat{\Gamma}^{\rho(1)}_{i\sigma}) = \eta_{\rho\sigma}R^{\rho}{}_{ii\nu} + \eta_{\rho\nu}R^{\rho}{}_{ii\sigma}. \tag{15}$$

where in the second step we have used Eq. (10). Since this is the first-order equation, indices can be lowered by the Minkowski metric. Then by noticing the symmetry property of $R_{\sigma ii\nu}$, we finally obtain the solution

$$\begin{aligned}\hat{h}^{(1)}_{\sigma\nu} &= 2\frac{1}{\vec{\partial}^2}R^{(1)}_{\sigma ii\nu} \\ &= h_{\sigma\nu} - \frac{1}{\vec{\partial}^2}(h_{\nu i,i\sigma} + h_{\sigma i,i\nu} - h_{ii,\sigma\nu}).\end{aligned} \tag{16}$$

Here and below a comma is used to denote derivative when too many occur. The superscript on $R^{(1)}_{\sigma ii\nu}$ is to remind that it is computed to first-order in $h_{\mu\nu}$. Rigorously speaking, the second expression requires that $h_{\mu\nu}$ (not just $R^{\rho}{}_{\sigma\mu\nu}$) vanish at infinity.

For the second-order term $\hat{h}^{(2)}_{\sigma\nu}$, we derive from Eq. (2)

$$\begin{aligned}\hat{g}^{(2)}_{\sigma\nu,\mu} + \hat{g}^{(2)}_{\sigma\mu,\nu} - \hat{g}^{(2)}_{\mu\nu,\sigma} = 2\eta_{\rho\sigma}\hat{\Gamma}^{\rho(2)}_{\mu\nu} + \eta^{\alpha\rho}\hat{h}^{(1)}_{\rho\sigma}(h_{\alpha\nu,\mu} + h_{\alpha\mu,\nu} - h_{\mu\nu,\alpha}) \\ +\eta^{\alpha\rho}(h_{\rho\sigma} - \hat{h}^{(1)}_{\rho\sigma})(\hat{h}^{(1)}_{\alpha\nu,\mu} + \hat{h}^{(1)}_{\alpha\mu,\nu} - \hat{h}^{(1)}_{\mu\nu,\alpha}).\end{aligned} \tag{17}$$

Solution of $\hat{h}^{(2)}_{\sigma\mu}$ is similar to $\hat{h}^{(1)}_{\sigma\mu}$, though more tedious. The perturbative solution for $\hat{h}_{\mu\nu}$ can be continued to the same order as $\hat{\Gamma}^{\rho}_{\mu\nu}$.

After obtaining $\hat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu} = g_{\mu\nu} - \hat{g}_{\mu\nu}$, we must remark on how $\bar{g}^{\mu\nu}$ and $\hat{g}^{\mu\nu}$ are computed. By definition, $\bar{g}^{\mu\nu}$ is the inverse of $\bar{g}_{\mu\nu}$. Then, $\hat{g}^{\mu\nu}$ is computed as $g^{\mu\nu} - \bar{g}^{\mu\nu}$. At lowest order, $h^{\mu\nu}$, $\bar{g}^{\mu\nu}$, and $\hat{g}^{\mu\nu}$ are just related to $h_{\mu\nu}$, $\bar{g}_{\mu\nu}$, and $\hat{g}_{\mu\nu}$ by the Minkowski metric. But this property is lost at higher orders.

The solutions we obtain show the desired property that the physical terms $\hat{\Gamma}^{\rho}_{\sigma\mu}$ and $\hat{h}_{\mu\nu}$ vanish if and only if $R^{\rho}{}_{\sigma\mu\nu} = 0$, i.e., the space-time is intrinsically flat. It is also illuminating to look at the property of the pure geometric terms $\bar{\Gamma}^{\rho}_{\sigma\mu}$ and $\bar{h}_{\mu\nu}$. To this end we rewrite Eq. (1b) as $g^{ij}\bar{\Gamma}^{\rho}_{ij} = g^{ij}\Gamma^{\rho}_{ij}$. This indicates that in order to have $\bar{\Gamma}^{\rho}_{\sigma\mu} = 0$ (so that the spurious gravitational effect is absent), it is necessary that $g^{ij}\Gamma^{\rho}_{ij} = 0$. On the other hand, given validity of our perturbative expansion, $g^{ij}\Gamma^{\rho}_{ij} = 0$ will lead uniquely to $\bar{\Gamma}^{\rho}_{\sigma\mu} = 0$. We therefore name a coordinate in which $g^{ij}\Gamma^{\rho}_{ij} = 0$ the "pertinent coordinate". (Similarly, in gauge theories, the radiation gauge $\partial_i A_i = 0$ leads to the solution for the pure-gauge field $\bar{A}_\mu = 0$, and can be termed the "pertinent gauge" [8].) The pertinency condition $g^{ij}\Gamma^{\rho}_{ij} = 0$ is just what we find in Ref. [5] the "true radiation gauge for gravity". It is straightforward to verify that the spherical coordinate is not pertinent even in a flat space-time. This explains why it gives unreasonable gravitational energy by the traditional pseudo-tensors.

We note that the pertinency condition is fairly non-trivial. E.g., while the Cartesian coordinate in flat space-time gives $\Gamma^{\rho}_{\mu\nu} \equiv 0$ and is clearly pertinent, the quasi-Cartesian coordinate in a curved space-time is not necessary pertinent, e.g., the simplest Schwarzschild solution: $ds^2 = \left(\frac{1-MG/2r}{1+MG/2r}\right)^2 dt^2 - \left(1+\frac{MG}{2r}\right)^4 d\vec{r}^2$. Moreover, it is not trivial to convert this coordinate to a pertinent one, except at linear order [5]. It is exactly the non-triviality of the pertinency condition that calls for our field-decomposition approach, which works straightforwardly in any coordinate, and can pick out the true gravitational content of the metric up to moderate strength.

We are now in the position to explain how to calculate a physically meaningful energy density of the gravitational field, for any given $g_{\mu\nu}$ of a finite and not-too-strong gravitating system. The metric $g_{\mu\nu}$ may either be obtained by solving the Einstein equation directly, or may just be worked out with some guessing, or even be the experimentally measured result. First, the metric is put into the pertinency test: If one finds $g^{ij}\Gamma^{\rho}_{ij} = 0$, it means that this $g_{\mu\nu}$ contains no spurious gravitational effect, thus can be used directly in the traditional pseudo-tensors to compute the energy density. If, instead, $g^{ij}\Gamma^{\rho}_{ij} \neq 0$, it means that this $g_{\mu\nu}$ does contain spurious gravitational effect, and one should revise a pseudo-tensor by replacing the quantities in it with their corresponding physical counterparts, which are obtained by the field-decomposition approach we just presented. This would give a concrete gravitational energy as physical as that in the pertinent coordinate.

Discussion.—(i) Various pseudo-tensors show a high degeneracy concerning the total energy of a gravitating body. It would be interesting to examine whether such degeneracy persists to the level of a meaningful density.

(ii) In gauge theories, gauge transformation and Lorentz transformation are two different manipulations. Therefore, in Eq. (3b)/(5b), $\hat{A}_\mu$ is gauge invariant/covariant so as to make the equation gauge invariant/covariant. However, to make the equation hold in any Lorentz frame, the physical field $\hat{A}_\mu$ must not transform as a four-vector. This is an inevitable physical feature of a massless particle with spin-1 or higher [9]. In general relativity, however, gauge transformation and coordinate transformation mean the same thing. Therefore, to make Eq. (1b) hold in any coordinate, the physical term $\hat{\Gamma}^{\rho}_{\sigma\mu}$ must not transform covariantly under four-dimensional transformations, even linear (Lorentz) ones. This manifests the masslessness of the gravitational field. But by our construction $\hat{\Gamma}^{\rho}_{\sigma\mu}$ is indeed a true tensor under spatial transformations, following the same line as in proving the non-Abelian transformations in Eq. (6).

(iii) At leading order, $h^{(1)}_{\mu\nu}$ is essentially the field defined in the pertinent coordinate as we discuss in Ref. [5], where we have derived the second expression in Eq. (16) by a method of gauge transformation. Moreover, the expression mimics exactly the form of the "transverse" part of the matter stress-energy tensor, derive in Ref. [5] by yet another method:

$$\hat{S}_{ij} = S_{ij} - \frac{1}{\vec{\partial}^2}(\partial_i\partial_k S^k{}_j + \partial_j\partial_k S^k{}_i - \partial_i\partial_j S^k{}_k), \tag{18}$$

where $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\rho{}_\rho$. This "coincidence" is actually profound and reveals that our tensor-separation is a unique extension of the usual vector-separation by curl-free and divergence-free conditions: Riemann curvature is the unique covariant "curl" of a tensor, hence comes Eq. (1a). The uniqueness of the expression in Eq. (1b) is explained in [5].

(iv) Arnowitt, Deser and Misner (ADM) discussed a linear orthogonal separation of a symmetric spatial tensor [10]: $h_{ij} = h^{TT}_{ij} + h^T_{ij} + h^L_{ij}$, where h^{TT}_{ij} is transverse and traceless, h^T_{ij} is transverse, and h^L_{ij} is longitudinal; all expressed uniquely via h_{ij}:

$$\begin{aligned} h^L_{ij} &= f_{i,j} + f_{j,i},\ f_i = \frac{1}{\vec{\partial}^2}(h_{ik,k} - \frac{1}{2}\frac{1}{\vec{\partial}^2}h_{kl,kli}) & \text{(19a)} \\ h^T_{ij} &= \frac{1}{2}(\delta_{ij}f^T - \frac{1}{\vec{\partial}^2}f^T_{,ij}),\ f^T = h_{kk} - \frac{1}{\vec{\partial}^2}h_{kl,kl} & \text{(19b)} \\ h^{TT}_{ij} &= h_{ij} - h^T_{ij} - h^L_{ij}. & \text{(19c)} \end{aligned}$$

ADM regard h^{TT}_{ij} as the physical part of the gravitational field. At linear order, both h^{TT}_{ij} and our $\hat{h}_{\mu\nu}$ are gauge invariant. But a key difference is that in our method the rest part $\bar{h}_{\mu\nu}$ is a pure gauge, while in the ADM method h^T_{ij} is also gauge invariant and only h^L_{ij} is a pure gauge. This implies that h^{TT}_{ij} does not contain all physical content of h_{ij}, and is not as pertinent as $\hat{h}_{\mu\nu}$. Since at linear order $\hat{h}_{\mu\nu}$, h^{TT}_{ij}, and h^T_{ij} are all gauge-invariant, we can expect some relations among them. Remarkably, indeed, a little algebra shows

$$\begin{aligned} f^T &= \frac{1}{2}\hat{h}^{(1)}_{kk} = h_{kk} - \frac{1}{\vec{\partial}^2}h_{kl,kl}, & \text{(20a)} \\ h^{TT}_{ij} &= \hat{h}^{(1)}_{ij} - \frac{1}{4}(\delta_{ij}\hat{h}^{(1)}_{kk} + \frac{1}{\vec{\partial}^2}\hat{h}^{(1)}_{kk,ij}). & \text{(20b)} \end{aligned}$$

Thus, the relation of h^{TT}_{ij} and $\hat{h}_{ij}$ is similar to that of the TT gauge and our pertinency condition: They agree for pure waves without matter source, but disagree otherwise [5].

(v) York has proposed a different extraction of TT component from a symmetric tensor: $h^{TT}_{ij} \equiv h_{ij} - \tilde{h}^L_{ij} - \frac{1}{3}\delta_{ij}h_{kk}$, with $\tilde{h}^L_{ij}$ another longitudinal part and $\frac{1}{3}\delta_{ij}h_{kk}$ a trace part. [11]

At linear order, the explicit expression is:

$$\tilde{h}^L_{ij} = W_{i,j} + W_{j,i} - \frac{2}{3}\delta_{ij}W_{k,k} \tag{21a}$$

$$W_i = \frac{1}{\vec{\partial}^2}(h_{ik,k} - \frac{1}{4}h_{kk,i} - \frac{1}{4}\frac{1}{\vec{\partial}^2}h_{kl,kli}) \tag{21b}$$

It can be checked that at linear order h^{TT}_{ij} defined by York equals that of ADM. Moreover, all gauge dependence is contained in the pure-gauge part $W_{i,j}+W_{j,i}$ in $\tilde{h}^L_{ij}$, while the $-\frac{2}{3}\delta_{ij}W_{k,k}$ term in $\tilde{h}^L_{ij}$ can join $\frac{1}{3}\delta_{ij}h_{kk}$ to make a gauge-invariant combination:

$$\frac{1}{3}(h_{kk} - 2W_{k,k}) = \frac{1}{2}(h_{kk} - \frac{1}{\vec{\partial}^2}h_{kl,kl}) = \frac{1}{2}f^T. \tag{22}$$

It must be noted, however, that the W_i of York differs from the f_i of ADM, and the pure-gauge terms defined by York and ADM are different: $W_{i,j}+W_{j,i} \neq f_{i,j}+f_{j,i}$. They are both much more complicated than our pure-gauge term in Eq. (16):

$$\begin{aligned}\bar{h}^{(1)}_{\mu\nu} &= \frac{1}{\vec{\partial}^2}(h_{\mu i,i\nu} + h_{\nu i,i\mu} - h_{ii,\mu\nu}) \\ &= \epsilon_{\mu,\nu} + \epsilon_{\nu,\mu},\ \epsilon_\mu = \frac{1}{\vec{\partial}^2}(h_{\mu i,i} - \frac{1}{2}h_{ii,\mu}).\end{aligned} \tag{23}$$

The relations between our decomposition and that of ADM and York, especially beyond the linear order, will be further explored elsewhere. [12]

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[6] Greek indices run from 0 to 3, Latin indices run from 1 to 3, and repeated indices are summed over (even when they both appear upstairs or downstairs).

[7] For an infinite system, it is not obvious to specify a natural boundary condition. Though an infinite system is hardly relevant for particle physics, in gravity the universe does present a real example. Here we restrict out attention to finite systems. Note that this does not exclude the radiating system, which can be made spatially finite by letting the radiation occur in a given period.

[8] This may easily lead to a confusion that $\hat{A}_\mu$ is just the field $A_\mu^{\rm Rad}$ defined in radiation gauge $\partial_i A_i^{\rm Rad} = 0$. It should be clarified that $A_\mu^{\rm Rad}$ so defined is a *fixed* (in this sense gauge-*invariant*) quantity, while in non-Abelian theories $\hat{A}_\mu$ is gauge-*covariant*. Only in radiation gauge we have $\hat{A}_\mu = A_\mu^{\rm Rad}$. In other gauges $\hat{A}_\mu \neq A_\mu^{\rm Rad}$.

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