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M. Hayakawa[a], K.-I. Ishikawa[b], Y. Osaki[b], S. Takeda[c], S. Uno[a], and N. Yamada[d,e*]

[a] *Department of Physics, Nagoya University, Nagoya 464-8602, Japan*

[b] *Department of Physics, Hiroshima University, Higashi-Hiroshima 739-8526, Japan*

[c] *School of Mathematics and Physics, College of Science and Engineering, Kanazawa university, Kakuma-machi, Kanazawa, Ishikawa 920-1192, Japan*

[d] *KEK Theory Center, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Tsukuba 305-0801, Japan*

[e] *School of High Energy Accelerator Science, The Graduate University for Advanced Studies (Sokendai), Tsukuba 305-0801, Japan*

Abstract

Walking technicolor theory attempts to realize electroweak symmetry breaking as the spontaneous chiral symmetry breakdown caused by the gauge dynamics with slowly varying gauge coupling constant and large mass anomalous dimension. Many-flavor QCD is one of the candidates owning these features. We focus on the SU(3) gauge theory with ten flavors of massless fermions in the fundamental representation, and compute the gauge coupling constant in the Schrödinger functional scheme. Numerical simulation is performed with $O(a)$-unimproved lattice action, and the continuum limit is taken in linear in lattice spacing. We observe evidence that this theory possesses an infrared fixed point.

*norikazu.yamada@kek.jp

I. INTRODUCTION

While the standard model has been established through a number of experiments, unnatural hierarchies are present between the electroweak scale and the Planck scale and also among the fermion masses. Large Hadron Collider (LHC) is expected to give a new insight into these hierarchies. Among various new physics models proposed so far, Technicolor (TC) model [1] is one of the most attractive ones in these regards, as it does not require any fundamental scalar particles, which cause the former hierarchy, and its extension, Extended TC model [2], has a possibility to generate the Yukawa hierarchy in a dynamical way. For recent review articles, see, for example, Refs. [3].

TC should be a strongly coupled vector-like gauge system, which triggers spontaneous chiral symmetry breaking (SχSB). It is widely known, however, that the simplest TC models obtained by rescaling ordinary QCD have already been ruled out by the S-parameter [4] and FCNC [5] constraints. Refs. [6] suggested a series of TC models to circumvent the FCNC problem. Those TC models appeal to the gauge dynamics in which the effective gauge coupling constant runs slowly (*i.e.* "walks") at a relatively large value over a wide range of energy scale above the SχSB scale, and in which the chiral condensate gets large anomalous dimension. Such TC is called *walking TC (WTC)*, and possible candidates have been enumerated through semi-quantitative analyses [7]. Since the dynamics that underlie WTC significantly differ from those of two or three-flavor QCD, the naive scaling argument in N_c or N_f to estimate the S-parameter would not work, and any quantitative predictions from WTC require solving nonperturbative dynamics explicitly. Lattice gauge theory provides a unique way to study this class of models from the first principles at present.

Search for candidate theories of WTC is frequently linked to the N_f-dependent phase structure of the gauge theories. Let us take SU(3) gauge theory with N_f flavors of fermions in the fundamental representation as an example. According to the analysis of the perturbative β-function, the system with large enough N_f ($N_f > 16.5$) is asymptotically non-free and trivial unless non-trivial ultraviolet fixed point exists. On the other hand, if N_f is sufficiently small ($N_f \leq 3$) the dynamics is QCD-like and thus in the chirally broken phase. It is believed that for the in-between N_f there exists a so-called conformal phase, where the coupling constant reaches an infrared fixed point (IRFP) without SχSB set in, but confinement may take place [8]. The range of N_f in which the conformal phase is realized is called conformal

window, and is represented by $N_f^{\rm crit} < N_f < 16.5$. It is then natural to speculate that the gauge dynamics slightly below $N_f^{\rm crit}$ exhibit the features required for WTC; slow running of the gauge coupling constant and SχSB. The first goal in the search for WTC is thus to identify $N_f^{\rm crit}$.

In the past years, many groups have used techniques of lattice simulations to search for $N_f^{\rm crit}$ and/or WTC through hadron spectrum, eigenvalue distribution of Dirac operator, the behavior of running coupling constant, or renormalization group analysis of candidate theories [9]. For non-lattice studies, see, for example, Refs. [10, 11]. Among various candidates, many flavor QCD [12–19], sextet QCD [20–25], and two-color adjoint QCD [26–31] have been intensively studied. In this work, we focus on many flavor QCD with $N_c = 3$ and fermions in the fundamental representation. In a seminal work [12], the running coupling constants were calculated for eight- and twelve-flavor QCD using the Schrödinger functional (SF) scheme on the lattice [32]. They concluded that twelve-flavor QCD has an IRFP at $g^2_{\rm SF} \sim 5$ while eight-flavor QCD does not. In practice, the study of the running coupling alone is supposed to be unable to fully exclude the possibility of a large IRFP because it requires lattice simulations at arbitrarily large coupling. Even worse, the unphysical, bulk first-order phase transition was found to occur in strong coupling regime of several gauge theories [23, 33, 34]. In such simulations, there exists an upper limit on the bare coupling at which lattice calculation is sensible. Nevertheless, because of the supports from the spectroscopy studies [14, 16, 18] the conclusion in Ref. [12] that the eight-flavor QCD is QCD-like, *i.e.* $N_f^{\rm crit} > 8$, seems to be established nowadays.

After the work of Ref. [12], one group [15] has presented an evidence of the conformality of twelve-flavor QCD. The opposite conclusion, however, has also been reported by the other groups [16, 18]. Therefore $N_f^{\rm crit} < 12$ is still under debate. Clearly the observed contradiction must be clarified before going further. While in the spectroscopy study of twelve-flavor QCD many sources of systematic uncertainties due to finite volume, taste breaking, chiral extrapolation, lack of continuum limit, *etc.*, remain to be quantified, the calculation of the SF coupling constant of Ref. [12] appears, at present, to be less ambiguous. In such a circumstance, we are tempted to explore the dynamics of ten-flavor QCD. In this paper, we investigate, as a first step, the running coupling constant of ten-flavor QCD on the lattice to see whether it shows conformal behavior. We find that the running slows down and observe evidence that this theory possesses an infrared fixed point.

The paper is organized as follows. In sec. II, we give remarks on how we identify IRFP on the lattice. Sec. III summarizes the coefficients relevant to the perturbative calculation of the running coupling constant for later use. In sec. IV, the simulation setup including the definition of the running coupling constant in the Schrödinger functional scheme is presented. In sec. V, we describe analysis method and present the numerical results. Sec. VI is devoted to the summary and outlook.

II. REMARKS ON SEARCHING FOR IRFP ON THE LATTICE

Since there exists a subtlety in proving the existence of IRFP with lattice gauge theory, in this section we briefly explain what is actually calculated and then give how to identify the existence of IRFP. Here we focus on the concept only. For further details of the calculational and analysis method that we take, see the following sections.

In this work, we calculate the renormalized coupling constant in Schrödinger functional scheme at two different length scales, L and $s \cdot L$. In practice, this is realized by repeating the calculation on two different volumes, l^4 and $(s \cdot l)^4$, at a common lattice bare coupling g_0^2, where $l = L/a$. We denote those couplings by u (or $g^2(g_0^2, l)$) and $g^2(g_0^2, s \cdot l)$, respectively. Using those, we define the discrete beta function (DBF) by $B(u, s, l) = 1/g^2(g_0^2, s \cdot l) - 1/u$, where the rescaling factor s is arbitrary but is fixed to 2. If the DBF is free from lattice discretization errors, the sign of this quantity may directly tell whether the coupling constant increases or decreases against the scale change by s at the scale L, which is implicitly set by the value of u that we can choose. Since discretization errors do exist, however, we need to take the continuum limit. The $a \to 0$ limit is taken for a fixed L, *i.e.* for a fixed u, by varying lattice spacing a. A series of the DBF thus obtained is then a function of l, and the $l = L/a \to \infty$ limit is expected to give the continuum limit. In summary, the DBF is constructed from a pair of lattice volumes (l^4, $(s \cdot l)^4$), and choice of larger l results in the DBF closer to the continuum limit.

In practice, lattice spacing is varied by changing the lattice bare coupling g_0^2. If $g^2(g_0^2, l_2)$ turns out to be always larger than $g^2(g_0^2, l_1)$ with $l_2 > l_1$, $B(u, s, l) < 0$ should hold for any l and $s > 1$. In this case, the bare coupling at which $g^2(g_0^2, l_1)$ is equal to a fixed value u becomes small as lattice size l_1 increases or one approaches the continuum limit. Thus the $a \to 0$ limit is realized in the $g_0^2 \to 0$ limit. This is the case for asymptotically free theories

with no IRFP such as ordinary QCD, and no subtlety is present. Even if an IRFP exists in such theories, the situation does not change as long as the input u is smaller than the IRFP, $g^2_{\rm IRFP}$. In other words, if the DBF extrapolated to $l \to \infty$ (or equivalently $1/l \to 0$) is negative, the limiting value is interpreted as the continuum limit and the possibility that an IRFP exists below u is excluded.

When the DBF extrapolated to $l \to \infty$ is positive, interpretation of numerical results becomes ambiguous. In this case, in the vicinity of $1/l = 0$, $g^2(g_0^2, s \cdot l) < g^2(g_0^2, l)$, *i.e.* $B(u, s, l) > 0$. Indeed, it happens below $\beta = 4.4$ in Fig. 4 of Ref. [13], for example. Then, one may expect that the $l \to \infty$ limit is realized by $g_0^2 \to \infty$ on first sight. However, recalling ϕ^4 theory, this expectation turns out to be too naive. In ϕ^4 theory, the continuum limit exists only in the trivial case unless the theory possesses a non-trivial UV fixed point. Since the situation is similar to this case, the most plausible interpretation is that, when $u > g^2_{\rm IRFP}$, the continuum limit does not exist unless a nontrivial UV fixed point exists. Since no nontrivial UV fixed points has been established so far, it is not suitable to call the extrapolated value the continuum limit when it is positive. Nevertheless, we can still infer that $u > g^2_{\rm IRFP}$ because no other possibility remains.

We investigate the sign of the DBF, starting with the weak coupling regime $u \sim 1$ where the perturbative calculation is reliable and predicts a negative value. We keep monitoring the sign of the DBF with increasing u. The identification of the IRFP is then made by sign-flip of the DBF extrapolated to $l \to \infty$. Notice that, when the extrapolated value is positive, the extrapolation does not make sense and hence we do not insist that the continuum limit is determined.

III. PERTURBATIVE ANALYSIS

We start with defining the β function of an effective gauge coupling constant in a mass-independent renormalization scheme, which should have the following expansion in the perturbative regime

$$\beta(g^2(L)) \;=\; L\,\frac{\partial\, g^2(L)}{\partial L} = b_1\, g^4(L) + b_2\, g^6(L) + b_3\, g^8(L) + b_4\, g^{10}(L) + \cdots\,, \qquad (1)$$

where L denotes a length scale. The first two coefficients on the right hand side are scheme-independent, and given by

$$b_1 = \frac{2}{(4\pi)^2}\left[11 - \frac{2}{3}N_f\right], \qquad b_2 = \frac{2}{(4\pi)^4}\left[102 - \frac{38}{3}N_f\right]. \tag{2}$$

The remaining coefficients are scheme-dependent and known only in the limited schemes and orders. The third coefficient takes the following form in the Schrödinger functional scheme;

$$b_3^{\rm SF} \;=\; b_3^{\overline{\rm MS}} + \frac{b_2\, c_2^\theta}{2\pi} - \frac{b_1\,(c_3^\theta - c_2^{\theta\,2})}{8\pi^2}, \tag{3}$$

where $b_3^{\overline{\rm MS}}$ is a coefficient in the $\overline{\rm MS}$ scheme,

$$b_3^{\overline{\rm MS}} \;=\; \frac{2}{(4\pi)^6}\left[\frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2\right], \tag{4}$$

and the calculable quantities c_2^θ and c_3^θ depend on the spatial boundary condition imposed on the fermion fields in the SF setup, *i.e* so-called θ. Those for $\theta = \pi/5$ and c_2^θ for $\theta = 0$ are known to be [35]

$$c_2^{\theta=\pi/5} \;=\; 1.25563 + 0.039863 \times N_f, \tag{5}$$

$$c_3^{\theta=\pi/5} \;=\; (c_2^{\theta=\pi/5})^2 + 1.197(10) + 0.140(6) \times N_f - 0.0330(2) \times N_f^2, \tag{6}$$

$$c_2^{\theta=0} \;=\; 1.25563 + 0.022504 \times N_f, \tag{7}$$

but $c_3^{\theta=0}$ has not been calculated yet. Although $\theta = 0$ is chosen in our simulation as described in sec. IV, the coefficients for $\theta = \pi/5$ are used only to see the situation of conformal windows inferred just from the perturbative analysis, and the potential size of difference between the two- and three-loop calculations.

The perturbative estimates of the infrared fixed point (IRFP) for SU(3) gauge theory with N_f flavors of fundamental fermion are summarized in Tab. I. We note that in the three-loop perturbative analysis the existence of IRFP is determined only by the sign of b_3, which is always negative for the range of N_f shown in Tab. I. Therefore, the existence of IRFP as well as its value may be unstable against including higher orders. Nevertheless, for $N_f \geq 14$ the difference between the two- and three-loop results is reasonably small, and one may expect that higher order corrections do not spoil the existence of IRFP or even do not change its value by much for such a large N_f.

According to the analysis based on Schwinger-Dyson equation, SχSB is expected to occur when the running coupling constant reaches $g^2 \sim \pi^2$ in SU(3) gauge theories [36]. In spite

N_f	4	6	8	10	12	14	16
two-loop universal	-	-	-	27.74	9.47	3.49	0.52
three-loop SF with $\theta = \pi/5$	43.36	23.75	15.52	9.45	5.18	2.43	0.47

TABLE I: The perturbative IRFP obtained from the two-loop universal and the three-loop SF scheme analyses.

of the scheme-dependence of the running coupling constant and the value of IRFP, those results motivate us to speculate that ten-flavor QCD may exhibit strongly coupled walking dynamics, and thus deserves full nonperturbative calculation.

IV. SIMULATION DETAILS

A. Schrödinger functional

We employ the Schrödinger functional (SF) method [32] to study the scale dependence of the running coupling constant. Unimproved Wilson fermion action and the standard plaquette gauge action are used without any boundary counter terms as described below.

The SF on the lattice is defined on a four dimensional hypercubic lattice with a volume $(L/a)^3 \times (T/a)$ in the cylindrical geometry. Throughout this work, the temporal extent T/a is chosen to be equal to the spatial one L/a. Periodic boundary condition in the spatial directions with vanishing phase factor ($\theta = 0$) and Dirichlet one in the temporal direction are imposed for both gauge ($U_\mu(x)$) and fermion ($\psi(x)$ and $\bar{\psi}(x)$) fields. The boundary values for gauge and fermion fields are represented by three-by-three color matrices, C and C', and spinors, ρ, ρ', $\bar{\rho}$ and $\bar{\rho}'$, respectively. The partition function of this system is given by

$$Z_{\rm SF}(C', \bar{\rho}', \rho'\,; C, \bar{\rho}, \rho) = e^{-\Gamma(C', \bar{\rho}', \rho'\,; C, \bar{\rho}, \rho)} = \int D[U, \psi, \bar{\psi}] e^{-S[U, \psi, \bar{\psi}, C, C', \rho, \rho', \bar{\rho}, \bar{\rho}']}, \tag{8}$$

where Γ is the effective action, and

$$S[U, \psi, \bar{\psi}, C, C', \rho, \rho', \bar{\rho}, \bar{\rho}'] = S_g[U, C, C'] + S_q[U, \psi, \bar{\psi}, \rho, \rho', \bar{\rho}, \bar{\rho}']. \tag{9}$$

For the pure gauge part, we employ the plaquette action,

$$S_g[U, C, C'] = \frac{\beta}{6} \sum_x \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \bar{\delta}_{\mu,\nu} w_{\mu,\nu}(x_0)\ {\rm Tr}\left[1 - P_{\mu,\nu}(x)\right], \tag{10}$$

where $\beta = 6/g_0^2$ denotes the inverse of the bare coupling constant, $\bar{\delta}_{\mu,\nu}$=0 when $\mu = \nu$ otherwise 1, and $P_{\mu,\nu}(x)$ denotes a 1×1 Wilson loop on the μ-ν plane starting and ending at x. The spatial link variables on the boundaries, the hypersurfaces at $x_0 = 0$ and L/a, are all set to the diagonal, constant $SU(3)$ matrices as

$$U_k(x)|_{x_0=0} = \exp[C], \quad C = \frac{ia}{L}\begin{pmatrix} \eta - \frac{\pi}{3} & 0 & 0 \\ 0 & -\frac{1}{2}\eta & 0 \\ 0 & 0 & -\frac{1}{2}\eta + \frac{\pi}{3} \end{pmatrix}, \tag{11}$$

$$U_k(x)|_{x_0=L/a} = \exp[C'], \quad C' = \frac{ia}{L}\begin{pmatrix} -\eta - \pi & 0 & 0 \\ 0 & \frac{1}{2}\eta + \frac{\pi}{3} & 0 \\ 0 & 0 & \frac{1}{2}\eta + \frac{2\pi}{3} \end{pmatrix}, \tag{12}$$

where $k = 1$, 2, 3, and η is parameterizing the gauge boundary fields. The weight $w_{\mu,\nu}(x_0)$ in eq. (10) is given by

$$w_{\mu,\nu}(x_0) = \begin{cases} c_t & \text{for } (t = 0 \text{ or } t = (L/a) - 1) \text{ and } (\mu \text{ or } \nu{=}0) \\ 0 & \text{for } (t = (L/a)) \text{ and } (\mu \text{ or } \nu{=}0) \\ \frac{1}{2}c_s & \text{for } (t = 0 \text{ or } t = (L/a)) \text{ and } (\mu \neq 0 \text{ and } \nu \neq 0) \\ 1 & \text{for all the other cases} \end{cases}. \tag{13}$$

By tuning c_t, $O(a)$ errors induced from the boundaries in the time direction can be removed perturbatively, but in this work we simply take its tree level values, $c_t = 1$. With this setup, the value of c_s can be arbitrarily chosen because the spatial plaquettes on the boundaries do not contribute to the action. We thus set $c_s = 0$.

The fermion fields are described by the unimproved Wilson fermion action,

$$S_q[U, \psi, \bar{\psi}] = N_f \sum_{x,y} \bar{\psi}(x) D(x, y; U) \psi(y) = N_f \sum_{x,y} \bar{\psi}^{\text{lat}}(x) D^{\text{lat}}(x, y; U) \psi^{\text{lat}}(y), \tag{14}$$

$$D^{\text{lat}}(x, y; U) = \delta_{xy} - \kappa \sum_{\mu} \left\{ (1 - \gamma_\mu) U_\mu(x) \delta_{x+\hat{\mu},y} + (1 + \gamma_\mu) U_\mu^\dagger(x - \hat{\mu}) \delta_{x-\hat{\mu},y} \right\}, \tag{15}$$

where

$$\psi^{\text{lat}}(x) = \frac{1}{\sqrt{2\kappa}} \psi(x), \quad \bar{\psi}^{\text{lat}}(x) = \frac{1}{\sqrt{2\kappa}} \bar{\psi}(x), \quad D^{\text{lat}}(x, y; U) = 2\kappa\, D(x, y; U). \tag{16}$$

The hopping parameter κ is related to the bare mass m_0 through $2\,\kappa = 1/(am_0 + 4)$. The dynamical degrees of freedom of the fermion field $\psi(x)$ and anti-fermion fields $\bar{\psi}(x)$ reside

on the lattice sites x with $0 < x_0 < T$. On both boundaries ($x_0 = 0$ and T), the half of the Dirac components are set to zero and the remaining components are fixed to some prescribed values, ρ, $\bar{\rho}$, ρ' and $\bar{\rho}'$, as

$$P_+\psi(x)|_{x_0=0} = \rho(\mathbf{x}), \quad P_-\psi(x)|_{x_0=0} = 0, \tag{17}$$

$$P_-\psi(x)|_{x_0=T} = \rho'(\mathbf{x}), \quad P_+\psi(x)|_{x_0=T} = 0, \tag{18}$$

$$\bar{\psi}(x)P_-\big|_{x_0=0} = \bar{\rho}(\mathbf{x}), \quad \bar{\psi}(x)P_+\big|_{x_0=0} = 0, \tag{19}$$

$$\bar{\psi}(x)P_+\big|_{x_0=T} = \bar{\rho}'(\mathbf{x}), \quad \bar{\psi}(x)P_-\big|_{x_0=T} = 0, \tag{20}$$

where $P_\pm = (1 \pm \gamma_0)/2$. In this work, the boundary values for the fermion fields are set to zero, *i.e.*

$$\rho = \rho' = \bar{\rho} = \bar{\rho}' = 0. \tag{21}$$

B. Definition of the running coupling

With the gauge boundary conditions (11) and (12), the absolute minimum of the action is given by a color-electric background field denoted by $B(x)$. Then, the effective action can be defined as a function of B by

$$\Gamma[B] = -\ln Z_{\rm SF}(C', \bar{\rho}', \rho'\,; C, \bar{\rho}, \rho), \tag{22}$$

which has the following perturbative expansion in the bare coupling constant,

$$\Gamma = \frac{1}{g_0^2}\,\Gamma_0 + \Gamma_1 + O(g_0^2)\,, \tag{23}$$

and, in particular, the lowest-order term

$$\Gamma_0 = \left[g_0^2\, S_g[B]\right]_{g_0=0}\,, \tag{24}$$

is exactly the classical action of the induced background field. The SF scheme coupling is then defined in the massless limit for fermions by

$$\left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=0} = \frac{1}{g_{\rm SF}^2(g_0^2,\, l = L/a)} \left.\frac{\partial\Gamma_0}{\partial\eta}\right|_{\eta=0} = \frac{k}{g_{\rm SF}^2(g_0^2,\, l)}, \tag{25}$$

where the normalization constant k is determined such that $g_{\rm SF}^2 = g_0^2$ holds in the leading order of the perturbative expansion, and is found to be

$$k = \left.\frac{\partial\Gamma_0}{\partial\eta}\right|_{\eta=0} = 12\left(\frac{L}{a}\right)^2 \left[\sin(2\gamma) + \sin(\gamma)\right] = k \quad \text{with } \gamma = \frac{\pi}{3}\left(\frac{a}{L}\right)^2. \tag{26}$$

Because of the absence of the clover term, only the η-derivative of the gauge action contributes to $1/g^2_{\rm SF}(g_0^2,\, l)$.

C. Parameters

The simulation was performed on the lattice sizes of $l^4 = (L/a)^4 = 4^4$, 6^4, 8^4, 12^4, and 16^4 with the inverse of bare gauge coupling constant $\beta = 6/g_0^2$ in the range, $4.4 \leq \beta \leq 96.0$. However, the data from $l = 4$ lattices are not used in the following analysis because it was found that they have large discretization errors. We calculated the SF coupling on 18^4 lattice with a single β (β=4.55), and the result is used to check the scaling violation at a specific value of $g^2_{\rm SF}$.

The algorithm to generate the gauge configuration follows the standard HMC with five pseudo-fermion fields introduced to simulate the ten flavors of dynamical fermions. The numerical simulations were carried out on several different architectures including GPGPU, PC cluster and supercomputers. In order to achieve high performance on each architecture, the HMC code, especially the fermion solver part, were optimized depending on each architecture. In particular, mixed precision solver using multiple GPUs enables us to obtain high statistics on $g^2_{\rm SF}$ at $l^4 = 12^4$ and 16^4 [37]. Acceptance ratio is kept to around 80 % by adjusting the molecular dynamics step size ($\delta\tau$).

Since the Wilson fermion explicitly breaks chiral symmetry, the value of κ is tuned, for every pair of ($\beta,\ L/a$), to its critical value κ_c realizing the massless fermion by monitoring the corresponding PCAC mass. The values of β, κ, the number of trajectories, $\delta\tau$ and the results for $l = L/a$=6, 8, 12, 16, and 18 lattices are tabulated in Tabs. II-VI, respectively.

D. Comment on $O(a)$-unimprovement

In our pilot study, we employed the $O(a)$-improved fermion action with the perturbatively determined counter terms. With this setup, we encountered a sudden change of the plaquette and the PCAC mass at l=6 and β=3.6 when κ was decreased from 0.1517, and we could not realize the vanishing PCAC mass. The expected SF coupling constant is about $3 \sim 4$ there. The same phenomenon also occurs on $l = 4$ lattices at almost the same value of bare coupling constant. Since the observed behavior looks similar to those reported in Refs. [23, 33, 34], we

infer that this is a bulk, first order phase transition. In order to cover the region $g^2_{\rm SF} \sim O(10)$, we omitted any $O(a)$ improvements. Thus the leading discretization error in our result is linear in lattice spacing.

Even without $O(a)$ improvements, the bulk, first order phase transition is observed for $\beta = 6/g_0^2 \sim 4.4$. However, this time it happens at the renormalized coupling constant greater than the $O(a)$-improved case, typically $g^2_{\rm SF} \sim O(10)$. Since this bulk phase transition is considered as a lattice artifact, whenever this happens we discard the gauge configurations at such β. Thus the position of the critical β (~ 4.4) sets the lower limit on our exploration of β.

V. ANALYSIS METHOD AND RESULTS

A. Raw data

The SF coupling constant ($g^2_{\rm SF}$) and the PCAC mass (M) obtained on each (β, κ, l) are shown in Tabs. II-VI. $g_0^2/g^2_{\rm SF}$ is plotted as a function of the bare coupling constant g_0^2 in Fig. 1. The figure shows that $g^2_{\rm SF}$ increases with $l = L/a$ at a fixed g_0^2, but the change between the data from $l = 12$ and $l = 16$ is tiny. For later use, we fit the data of $g_0^2/g^2_{\rm SF}$ to an interpolating formula as a function of the bare coupling constant g_0^2. Among various functional forms we examined, the following form

$$\frac{g_0^2}{g^2_{\rm SF}(g_0^2, l)} = \frac{1 - a_{l,1}\, g_0^4}{1 + p_{1,l} \times g_0^2 + \sum_{n=2}^{N} a_{l,n} \times {g_0^2}^n}, \tag{27}$$

turned out to give the minimum χ^2/dof for a fixed number of free parameters, N. We thus employ eq. (27). In eq. (27), $p_{1,l}$ is the l-dependent coefficient and we have calculated them perturbatively in the SF scheme

$$p_{1,l} = \begin{cases} 0.4477107831 & \text{for } l = 6 \\ 0.4624813408 & \text{for } l = 8 \\ 0.4756888260 & \text{for } l = 12 \\ 0.4833079203 & \text{for } l = 16 \\ 0.4864767958 & \text{for } l = 18 \end{cases} . \tag{28}$$

The other coefficients $a_{l,n}$'s are determined for each l independently. We optimize the degree of polynomial N in the denominator of eq. (27) by monitoring χ^2/dof, and take $N = 5$ for

$l = 6$ and 12, and $N = 4$ for $l = 8$ and 16. Tab. VII shows the fit results for the coefficients in eq. (27). The fit results are also shown as the region sandwiched by a pair of solid curves in Fig. 1.

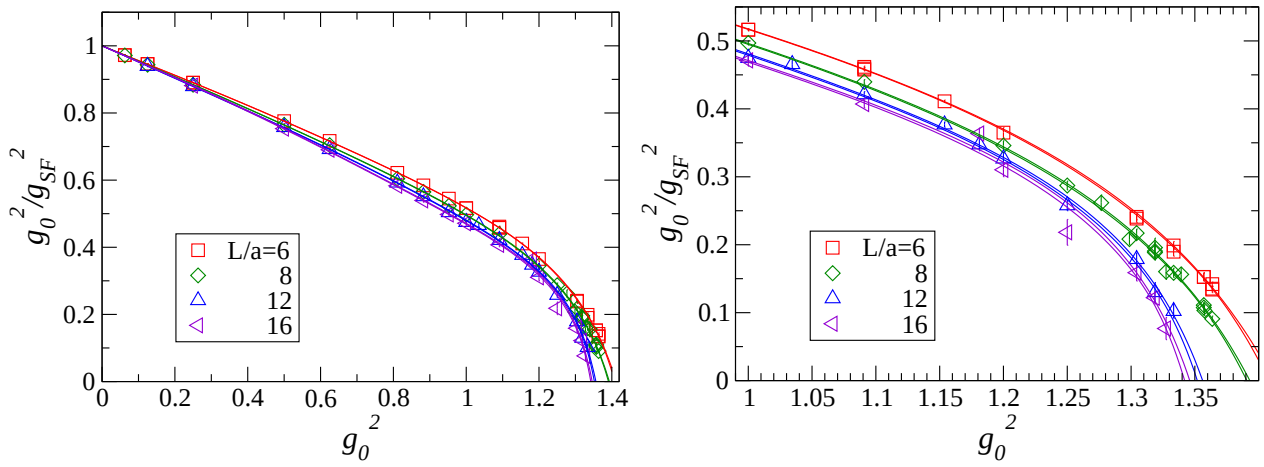

FIG. 1: g_0^2 dependence of $g_0^2/g_{\rm SF}^2$ for $l = L/a$=6, 8, 12 and 16. The right panel magnifies the region of $g_0^2 \in [1.1,\ 1.40]$.

Hereafter we denote the SF coupling obtained at a bare coupling constant g_0^2 and at a lattice length of l by $g_{\rm SF}^2(g_0^2, l)$ and its continuum counterpart by $g_{\rm SF}^2(L)$.

B. Discrete β function

In order to see the scale dependence of the SF coupling constant, we analyze the discrete β function (DBF) introduced in Refs. [20, 23]. The whole procedure is described below.

First, we choose an initial value of the running coupling constant, denoted by u. This implicitly sets the initial length scale L_0 through $g_{\rm SF}^2(L_0) = u$. Using the interpolating formula (27) for the lattice size l $(= L/a)$, the bare coupling constant g_0^* is numerically obtained by solving the equation $g_{\rm SF}^2(g_0^{*2}, l) = u$. l is identified with L_0/a, so that the lattice spacing at g_0^{*2} is found to be $a(g_0^{*2}, l) = L_0/l$. Now we choose a rescaling factor, s. The lattice step scaling function $\Sigma_0(u, s, l)$ is then defined as the SF coupling for $l' = s \cdot l$ at the same bare coupling g_0^{*2}, *i.e.*

$$\Sigma_0(u, s, l) \equiv g_{\rm SF}^2(g_0^{*2}, s \cdot l)\big|_{g_{\rm SF}^2(g_0^{*2}, l) = u}\,. \tag{29}$$

The meaning of the subscript "0" becomes clear soon. Of course, both l and $s \cdot l$ must be equal to one of 6, 8, 12 and 16, and hence the possible values for the rescaling factor s are

β	κ	Trajs.	plq.	$\delta\tau$	Acc.	$g^2_{\rm SF}$	M
96.0000	0.1267030	39,700	0.979268(0.000002)	0.0076	0.827(0.002)	0.06431(0.00006)	0.00012(0.00003)
96.0000	0.1267070	49,900	0.979267(0.000002)	0.0076	0.826(0.002)	0.06428(0.00005)	−0.00004(0.00003)
48.0000	0.1276060	40,100	0.958852(0.000005)	0.0098	0.857(0.002)	0.13221(0.00010)	−0.00013(0.00002)
48.0000	0.1276100	41,100	0.958846(0.000004)	0.0098	0.857(0.002)	0.13209(0.00010)	−0.00016(0.00002)
24.0000	0.1295180	24,700	0.917566(0.000009)	0.0149	0.848(0.002)	0.28079(0.00015)	0.00006(0.00002)
24.0000	0.1295200	60,300	0.917562(0.000005)	0.0152	0.838(0.002)	0.28086(0.00010)	0.00006(0.00001)
12.0000	0.1339640	48,700	0.833056(0.000014)	0.0250	0.812(0.002)	0.64450(0.00076)	−0.00004(0.00005)
9.6000	0.1365680	160,300	0.789765(0.000012)	0.0256	0.826(0.001)	0.87189(0.00068)	0.00002(0.00003)
7.4000	0.1410690	120,500	0.724148(0.000015)	0.0270	0.854(0.001)	1.30413(0.00194)	0.00006(0.00006)
6.8000	0.1430520	120,300	0.698517(0.000018)	0.0270	0.870(0.001)	1.51024(0.00244)	−0.00025(0.00008)
6.3000	0.1451400	17,400	0.673231(0.000076)	0.0333	0.817(0.001)	1.74788(0.00505)	0.00015(0.00019)
6.0000	0.1466380	33,600	0.655993(0.000034)	0.0333	0.837(0.002)	1.93684(0.00691)	0.00044(0.00021)
6.0000	0.1466410	80,300	0.655981(0.000029)	0.0333	0.833(0.001)	1.93605(0.00362)	0.00004(0.00010)
5.5000	0.1497590	50,300	0.622923(0.000025)	0.0370	0.817(0.002)	2.38340(0.01092)	0.00042(0.00020)
5.5000	0.1497610	36,000	0.622942(0.000027)	0.0357	0.827(0.002)	2.36232(0.00634)	−0.00018(0.00022)
5.5000	0.1497620	140,300	0.622977(0.000023)	0.0357	0.831(0.001)	2.37542(0.00963)	0.00023(0.00014)
5.2000	0.1521330	220,300	0.600097(0.000019)	0.0380	0.812(0.001)	2.80668(0.01246)	−0.00015(0.00014)
5.0000	0.1539800	59,900	0.583463(0.000049)	0.0400	0.806(0.002)	3.28837(0.06618)	−0.00005(0.00046)
4.6000	0.1585140	33,800	0.545776(0.000055)	0.0400	0.813(0.002)	5.47008(0.13064)	0.00092(0.00043)
4.6000	0.1585150	150,000	0.545680(0.000041)	0.0400	0.813(0.001)	5.41263(0.09891)	0.00123(0.00042)
4.5000	0.1599020	100,300	0.535280(0.000061)	0.0400	0.813(0.001)	7.02516(0.24479)	0.00111(0.00069)
4.5000	0.1599030	100,300	0.535305(0.000066)	0.0400	0.813(0.002)	6.70575(0.19622)	0.00033(0.00061)
4.4215	0.1610680	105,900	0.526537(0.000087)	0.0385	0.825(0.001)	8.88882(0.36944)	0.00238(0.00097)
4.4215	0.1610820	92,400	0.526692(0.000066)	0.0385	0.826(0.001)	8.90139(0.32355)	0.00073(0.00075)
4.4000	0.1614210	249,500	0.524331(0.000060)	0.0400	0.811(0.001)	9.60163(0.19661)	0.00051(0.00050)
4.4000	0.1614220	182,500	0.524342(0.000091)	0.0400	0.812(0.001)	10.17980(0.33990)	0.00119(0.00073)
4.4000	0.1614230	250,500	0.524387(0.000062)	0.0400	0.811(0.001)	10.07713(0.25379)	0.00049(0.00053)

TABLE II: Simulation parameters and results obtained at L/a=6.

β	κ	Trajs.	plq.	$\delta\tau$	Acc.	$g^2_{\rm SF}$	M
96.0000	0.1263270	22,500	0.979420(0.000002)	0.0056	0.811(0.004)	0.06434(0.00004)	0.00001(0.00001)
48.0000	0.1272250	18,300	0.958843(0.000003)	0.0100	0.818(0.007)	0.13247(0.00017)	0.00002(0.00002)
24.0000	0.1291450	42,300	0.917260(0.000004)	0.0125	0.804(0.003)	0.28282(0.00023)	−0.00004(0.00002)
12.0000	0.1335850	68,500	0.832266(0.000007)	0.0167	0.828(0.005)	0.65380(0.00071)	−0.00010(0.00003)
9.6000	0.1361800	21,820	0.788830(0.000013)	0.0200	0.828(0.006)	0.88751(0.00287)	−0.00008(0.00005)
7.4000	0.1406600	63,330	0.723081(0.000010)	0.0250	0.818(0.003)	1.34182(0.00417)	−0.00004(0.00016)
6.8000	0.1426200	41,500	0.697409(0.000013)	0.0250	0.797(0.002)	1.56232(0.00662)	0.00012(0.00011)
6.3000	0.1447000	28,000	0.672208(0.000021)	0.0250	0.816(0.003)	1.81987(0.01036)	−0.00034(0.00014)
6.0000	0.1462000	47,000	0.654999(0.000012)	0.0250	0.820(0.003)	2.01248(0.01258)	−0.00042(0.00011)
5.5000	0.1492700	35,900	0.622016(0.000021)	0.0286	0.797(0.003)	2.48139(0.01969)	−0.00021(0.00015)
5.0000	0.1533600	27,900	0.582458(0.000038)	0.0250	0.825(0.004)	3.46930(0.07238)	0.00094(0.00034)
4.8000	0.1554270	114,500	0.564464(0.000020)	0.0250	0.860(0.001)	4.35348(0.09845)	0.00026(0.00024)
4.7000	0.1565500	35,400	0.554789(0.000040)	0.0256	0.854(0.002)	4.87595(0.21035)	0.00027(0.00051)
4.6200	0.1575500	86,300	0.546856(0.000030)	0.0312	0.783(0.001)	6.23744(0.25321)	−0.00023(0.00033)
4.6000	0.1577800	149,300	0.544695(0.000027)	0.0250	0.852(0.002)	6.01108(0.17093)	−0.00007(0.00028)
4.5500	0.1584200	24,500	0.539428(0.000090)	0.0278	0.833(0.003)	6.92022(0.46491)	0.00087(0.00088)
4.5500	0.1584270	93,300	0.539336(0.000033)	0.0278	0.831(0.002)	6.99432(0.31873)	0.00135(0.00041)
4.5500	0.1584500	25,700	0.539683(0.000064)	0.0278	0.832(0.004)	6.74187(0.46970)	−0.00163(0.00071)
4.5200	0.1588500	56,570	0.536316(0.000058)	0.0278	0.827(0.002)	8.28029(0.57687)	−0.00010(0.00059)
4.5000	0.1591300	107,100	0.534108(0.000036)	0.0250	0.859(0.001)	8.40630(0.37369)	−0.00007(0.00038)
4.4800	0.1594000	41,555	0.531781(0.000085)	0.0250	0.827(0.002)	8.57214(0.57202)	0.00027(0.00070)
4.4215	0.1602640	160,900	0.525143(0.000050)	0.0263	0.837(0.001)	12.21877(0.49625)	−0.00012(0.00041)
4.4215	0.1602700	127,500	0.525149(0.000058)	0.0250	0.861(0.001)	12.62365(0.68980)	−0.00059(0.00048)
4.4200	0.1602700	29,700	0.524651(0.000132)	0.0278	0.828(0.002)	13.15085(0.99774)	0.00214(0.00075)
4.4000	0.1606000	229,500	0.522502(0.000057)	0.0278	0.819(0.002)	15.00764(0.69115)	0.00020(0.00042)

TABLE III: Simulation parameters and results obtained at L/a=8.

β	κ	Trajs.	plq.	$\delta\tau$	Acc.	$g^2_{\rm SF}$	M
48.0000	0.1269700	11,200	0.958648(0.000002)	0.0056	0.815(0.003)	0.13304(0.00033)	−0.00014(0.00003)
24.0000	0.1288929	54,620	0.916777(0.000002)	0.0083	0.798(0.002)	0.28432(0.00036)	−0.00008(0.00002)
12.0000	0.1333359	68,955	0.831306(0.000003)	0.0125	0.808(0.002)	0.66007(0.00119)	−0.00012(0.00003)
9.6000	0.1359350	86,700	0.787681(0.000003)	0.0133	0.806(0.002)	0.90325(0.00233)	−0.00001(0.00003)
7.4000	0.1404060	106,050	0.721824(0.000004)	0.0154	0.795(0.004)	1.36896(0.00543)	−0.00001(0.00004)
6.8000	0.1423250	45,150	0.696157(0.000006)	0.0167	0.819(0.002)	1.59998(0.00983)	0.00091(0.00006)
6.3000	0.1444050	23,500	0.671015(0.000014)	0.0182	0.767(0.002)	1.89012(0.01692)	0.00013(0.00012)
6.0000	0.1459000	43,296	0.653900(0.000008)	0.0182	0.820(0.007)	2.10612(0.02202)	0.00011(0.00007)
5.8000	0.1470200	43,400	0.641479(0.000007)	0.0182	0.799(0.002)	2.22171(0.02802)	−0.00009(0.00007)
5.5000	0.1489400	45,200	0.621162(0.000012)	0.0167	0.842(0.003)	2.58933(0.02496)	0.00017(0.00017)
5.2000	0.1512000	68,000	0.598557(0.000009)	0.0200	0.781(0.002)	3.06212(0.03210)	0.00056(0.00009)
5.0800	0.1522350	25,480	0.588819(0.000013)	0.0167	0.848(0.002)	3.39969(0.08924)	0.00015(0.00013)
5.0000	0.1529700	90,921	0.582135(0.000019)	0.0167	0.826(0.002)	3.67356(0.08908)	−0.00044(0.00020)
4.8000	0.1549700	194,300	0.564213(0.000006)	0.0182	0.810(0.002)	4.84805(0.14509)	0.00009(0.00011)
4.6000	0.1572300	127,922	0.544423(0.000012)	0.0182	0.818(0.002)	7.29885(0.38589)	0.00075(0.00017)
4.5500	0.1578500	57,260	0.539025(0.000021)	0.0192	0.802(0.002)	10.15231(1.11827)	0.00125(0.00027)
4.5000	0.1585500	104,570	0.534014(0.000049)	0.0250	0.701(0.007)	13.03915(1.33994)	−0.00237(0.00039)

TABLE IV: Simulation parameters and results obtained at L/a=12.

limited. The difference between $\Sigma_0(u,s,l)$ and u gives the scale dependence through the scale change from L to $s\cdot L$, up to lattice artifacts.

Since the raw data of $1/g^2_{\rm SF}(g_0^2,l)$ fluctuate around zero in the strong coupling region, converting from $1/g^2_{\rm SF}(g_0^2,l)$ to $g^2_{\rm SF}(g_0^2,l)$ sometimes induces huge statistical uncertainty. To avoid this we treat the inverse coupling constant, $1/g^2_{\rm SF}(g_0^2,l)$, directly. Then, to see the scale dependence of the inverse coupling constant, we introduce the lattice DBF [20, 23] by

$$B_0(u,s,l)=\frac{1}{\Sigma_0(u,s,l)}-\frac{1}{u}. \tag{30}$$

We calculate the continuum limit of this function for various initial values of the coupling constant, u. If the sign of the DBF in the continuum limit turns out to flip at a certain renormalized coupling constant u, it indicates the existence of IRFP.

β	κ	Trajs.	plq.	$\delta\tau$	Acc.	$g^2_{\rm SF}$	M
24.0000	0.1288000	22,890	0.916505(0.000002)	0.0067	0.796(0.003)	0.28368(0.00057)	0.00011(0.00002)
12.0000	0.1332590	40,950	0.830799(0.000003)	0.0091	0.718(0.011)	0.66384(0.00293)	−0.00016(0.00002)
9.6000	0.1358600	30,900	0.787096(0.000002)	0.0080	0.807(0.003)	0.90538(0.00570)	−0.00023(0.00003)
7.4000	0.1403250	62,300	0.721190(0.000003)	0.0111	0.812(0.002)	1.39094(0.00834)	−0.00002(0.00003)
6.8000	0.1422900	39,796	0.695613(0.000004)	0.0133	0.787(0.002)	1.63562(0.01649)	−0.00042(0.00005)
6.3000	0.1443400	69,000	0.670503(0.000004)	0.0133	0.798(0.002)	1.91412(0.01628)	−0.00036(0.00004)
6.0000	0.1457950	18,900	0.653363(0.000007)	0.0156	0.712(0.004)	2.12147(0.03887)	0.00043(0.00010)
5.5000	0.1488500	50,330	0.620911(0.000007)	0.0143	0.782(0.002)	2.67936(0.03897)	−0.00047(0.00011)
5.0800	0.1521310	23,760	0.588803(0.000007)	0.0139	0.804(0.003)	3.24742(0.07271)	−0.00003(0.00012)
5.0000	0.1528550	71,954	0.582121(0.000004)	0.0143	0.797(0.002)	3.86709(0.12622)	−0.00004(0.00009)
4.8000	0.1548310	46,000	0.564445(0.000008)	0.0156	0.755(0.002)	5.72911(0.49013)	0.00003(0.00014)
4.6000	0.1570500	83,705	0.544764(0.000008)	0.0143	0.794(0.001)	8.21243(0.63114)	0.00128(0.00012)
4.5500	0.1576750	107,069	0.539609(0.000010)	0.0139	0.809(0.002)	10.81452(0.80073)	0.00011(0.00011)
4.5200	0.1580650	42,400	0.536387(0.000021)	0.0156	0.754(0.002)	17.34193(3.72829)	−0.00030(0.00021)

TABLE V: Simulation parameters and results obtained at L/a=16.

β	κ	Trajs.	plq.	$\delta\tau$	Acc.	$g^2_{\rm SF}$	M
4.5500	0.1576500	32,309	0.540093(0.000014)	0.0143	0.785(0.003)	11.13131(1.41381)	−0.00124(0.00018)

TABLE VI: Simulation parameters and results obtained at L/a=18.

C. improving discretization errors

Since $O(a)$ discretization errors are not improved at all in the lattice actions, it is important to remove the scaling violation as much as possible. To do this, we perform the following improvements on the step scaling function and the DBF before taking the continuum limit.

First let $\sigma(u,s)$ be the continuum limit of $\Sigma_0(u,s,l)$, *i.e.* $\sigma(u,s) = g^2_{\rm SF}(sL)$ with $u =$

L/a	N	χ^2/dof	$a_{L/a,1}$	$a_{L/a,2}$	$a_{L/a,3}$	$a_{L/a,4}$	$a_{L/a,5}$
6	3	9.0(1.3)	0.4906(0.0025)	-0.2749(0.0105)	-0.1897(0.0151)		
6	4	1.4(0.5)	0.5048(0.0014)	-0.3993(0.0119)	0.1136(0.0283)	-0.2042(0.0184)	
6	5	1.3(0.5)	0.5015(0.0032)	-0.4240(0.0256)	0.2538(0.1301)	-0.4043(0.1815)	0.0899(0.0808)
8	3	2.1(0.7)	0.5068(0.0018)	-0.2308(0.0104)	-0.2412(0.0150)		
8	4	0.6(0.3)	0.5153(0.0019)	-0.3410(0.0260)	0.0405(0.0629)	-0.1852(0.0390)	
8	5	0.6(0.8)	0.5153(0.0051)	-0.3419(0.1697)	0.0444(0.7500)	-0.1904(0.9672)	0.0021(0.3906)
12	3	3.0(1.0)	0.5239(0.0047)	-0.1923(0.0118)	-0.3019(0.0198)		
12	4	1.1(0.6)	0.5400(0.0038)	-0.3614(0.0376)	0.1063(0.0884)	-0.2671(0.0550)	
12	5	1.0(0.6)	0.5438(0.0039)	-0.2783(0.0726)	-0.2779(0.2977)	0.2457(0.3815)	-0.2165(0.1582)
16	3	4.9(1.4)	0.5308(0.0055)	-0.1881(0.0266)	-0.3057(0.0375)		
16	4	1.8(0.8)	0.5520(0.0039)	-0.4948(0.0663)	0.4387(0.1516)	-0.4762(0.0903)	
16	5	1.9(0.9)	0.5538(0.0050)	-0.4403(0.1324)	0.1801(0.5648)	-0.1283(0.7332)	-0.1457(0.3025)

TABLE VII: The results for the coefficients in the fit function (27)

$g^2_{\rm SF}(L)$. Its perturbative expression is given by

$$\sigma(u,s) = u + s_0 u^2 + s_1 u^3 + s_2 u^4 + \cdots , \tag{31}$$

$$s_0 = b_1 \ln(s), \tag{32}$$

$$s_1 = \ln(s)\left({b_1}^2 \ln(s) + b_2\right), \tag{33}$$

$$s_2 = \ln(s)\left({b_1}^3 \ln^2(s) + \frac{5}{2} b_1 b_2 \ln(s) + b_3\right), \tag{34}$$

where b_i's are the coefficients of the β-function introduced in sec. III. Recalling the parametric form of the discretization error [35], the error normalized by $\sigma(u,s)$, denoted by $\delta_0(u,s,l)$, is written as

$$\delta_0(u,s,l) = \frac{\Sigma_0(u,s,l) - \sigma(u,s)}{\sigma(u,s)} = \delta^{(1)}(s,l)\, u + \delta^{(2)}(s,l)\, u^2 + O(u^3). \tag{35}$$

With eq. (31), the discretization error at the lowest order in u is found to be

$$\delta^{(1)}(s,l) = \Big(p_{1,s\cdot l} - b_1 \ln(s\cdot l)\Big) - \Big(p_{1,l} - b_1 \ln(l)\Big) = p_{1,s\cdot l} - p_{1,l} - b_1 \ln(s). \tag{36}$$

Now by replacing $\Sigma_0(u,s,l)$ in eq. (35) with $\Sigma_1(u,s,l) = \Sigma_0(u,s,l)/(1+\delta^{(1)}(s,l)\,u)$, the discretization error reduces to $O(u^2)$. Using $\Sigma_1(u,s,l)$, the one-loop improved DBF is defined

by

$$B_1(u,s,l) \;=\; \frac{1}{\Sigma_1(u,s,l)} - \frac{1}{u}. \tag{37}$$

This completes the one-loop improvement.

The above procedure can be repeated to an arbitrarily higher order in u, but it requires the perturbative coefficients like $p_{1,l}$ and the perturbative expression of $\sigma(u,s)$ to the corresponding order in u. All the coefficients necessary for the two-loop improvement are not available at this moment. Instead, we follow an alternative prescription proposed in Ref. [38]. After the one-loop improvement, the scaling violation is written as

$$\delta_1(u,s,l) = \frac{\Sigma_1(u,s,l) - \sigma(u,s)}{\sigma(u,s)} = \delta^{(2)}(s,l)\,u^2 + O(u^3). \tag{38}$$

If one can somehow know $\delta^{(2)}(s,l)$, the scaling violation can be reduced to $O(u^3)$ by replacing $\Sigma_0(u,s,l)$ in eq. (35) with

$$\Sigma_2(u,s,l) = \Sigma_0(u,s,l)/(1 + \delta^{(1)}(s,l)\,u + \delta^{(2)}(s,l)\,u^2). \tag{39}$$

$\delta^{(2)}(s,l)$ can be determined by fitting our data for $\delta_1(u,s,l)$ in eq. (38) to the function quadratic in u. Notice that in order for this fitting to make sense, the perturbative series of $\sigma(u,s)$ must be known through $O(u^3)$. Since the first two coefficients, b_1 and b_2, are available, the correct value of $\sigma(u,s)$ can be calculated to $O(u^3)$ as seen from eq. (31).

$\delta_1(u,s,l)$ is fitted to the form of eq. (38), neglecting $O(u^3)$ or higher order terms, for all possible pairs of (s,l) as shown in Fig. 2. The fit has to be performed in a weak coupling region where the perturbative expansion is reliable. We examine two fit ranges, $0 \le u \le 1.6$ and $0 \le u \le 2.0$ to see the fit range dependence. The extracted values for $\delta^{(2)}(s,l)$ are tabulated in Tab. VIII together with $\delta^{(1)}(s,l)$ defined in eq. (36).

The table shows that the values of $\delta^{(1)}(s,l)$ and $\delta^{(2)}(s,l)$ lie between 10^{-2} and 10^{-3}, and $\delta^{(2)}(s,l)$ turns out not to depend on the fit range. In the following analysis, we employ $\delta^{(2)}(s,l)$ from the shorter fit range. It is also seen from the table that generally the coefficients for $(s,l) = (4/3, 12)$ are the smallest among others. This is anticipated because the improvement coefficient vanish as s approaches to unity or l becomes large. An exception is the one-loop coefficient $\delta^{(1)}(4/3,6)$. Since two-loop coefficient $\delta^{(2)}(4/3,6)$ is, however, much larger than $\delta^{(1)}(4/3,6)$, the smallness of $\delta^{(1)}(4/3,6)$ is probably by accident. In the data sets we have, the data with $(s,\, l) = (4/3,\, 6)$ is the coarsest one. As we will show in the

(s,l)	$(4/3,6)$	$(2,6)$	$(8/3,6)$	$(3/2,8)$	$(2,8)$	$(4/3,12)$
$\delta^{(1)}(s,l)$	-0.00102	-0.0101	-0.0182	-0.00905	-0.0172	-0.00817
$\delta^{(2)}(s,l)$ [0, 1.60]	0.0075(12)	0.0108(15)	0.0123(26)	0.0035(14)	0.0054(24)	0.0026(23)
χ^2/dof	2.2	3.0	0.1	1.0	0.7	1.7
$\delta^{(2)}(s,l)$ [0, 2.0]	0.0061(9)	0.0106(12)	0.0112(18)	0.0038(13)	0.0053(18)	0.0014(17)
χ^2/dof	2.2	2.1	0.2	0.9	0.6	1.4

TABLE VIII: Coefficients for perturbative correction, $\delta^{(1)}(s,l)$ and $\delta^{(2)}(s,l)$, for each pair of (s,l). The square brackets in the first column indicate the fit range in u.

following subsections, this data turns out to suffer from non-linear scaling violation larger than the linear one in the strong coupling region. Thus, we omit this data point throughout the analysis. Using $\delta^{(2)}(s,l)$ thus obtained, we define the two-loop improved step scaling function $\Sigma_2(u,s,l)$ in eq. (39), and in turn the two-loop improved DBF

$$B_2(u,s,l) \;=\; \frac{1}{\Sigma_2(u,s,l)} - \frac{1}{u}\,. \tag{40}$$

D. strategy

The continuum limit is taken for a fixed rescaling factor s and a fixed input length scale L varying a lattice spacing a $(= L/l)$. As described in the preceding subsections, an input length scale is fixed by choosing a particular value of input coupling u. However, for a given s the number of data sets with different a in this work is, at most, two; $(s,l) = (2,6)$ and (2,8) for $s = 2$. While it is still possible to employ these two sets of data to evaluate the continuum limit, the validity of the linear extrapolation can not be tested. Alternatively, we may supplement a data set with a desired s by interpolating or extrapolating data of $g^2_{\rm SF}(g_0^2,l)$ in l. However, a lack of guiding principles in the interpolation or extrapolation may cause a systematic uncertainty. In this work, we use the two available data sets, $(s,l) = (2,6)$ and (2,8) to evaluate the continuum limit by linear extrapolation, and the other data sets are used to monitor the validity of the linear extrapolation. For this purpose, we introduce a relation which approximately converts the DBF for s' into that for s, as follows.

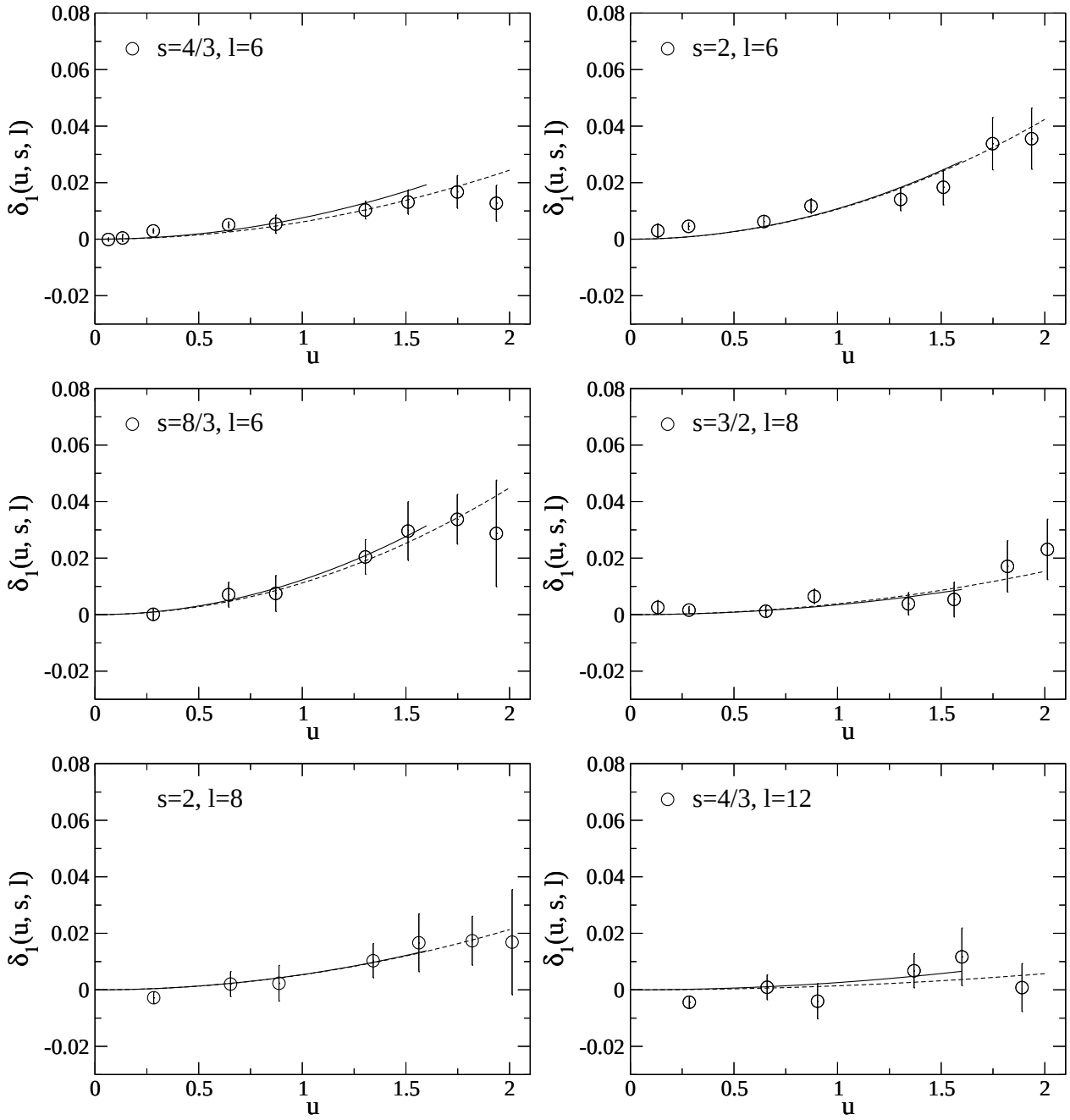

FIG. 2: Fit of δ_1 to a quadratic function of u. The solid and dashed curves show the fit results and the fit ranges.

We start with a closer look at the discretization error. The discretization error of the lattice DBF, *i.e.* $B_i(u,s,l) - B(u,s)$ (i=0, 1, 2), can be expressed in terms of an asymptotic expansion in $1/l$ [35] as

$$B_i(u,s,l) - B(u,s) \;=\; \left(\frac{1}{l} - \frac{1}{s\,l}\right) e_i(u) + O(l^{-2}), \tag{41}$$

where $e_i(u)$ is an unknown coefficient of $O(a)$ error and is a function of u. We then define the *rescaled* lattice DBF by

$$B_i'(u,s,l,s') \;=\; \frac{\ln(s)}{\ln(s')} B_i(u,s',l). \tag{42}$$

In addition, using the continuum counterpart of eq. (42), we define

$$\delta B(u,s,s') = B(u,s) - \frac{\ln(s)}{\ln(s')}\, B(u,s'), \tag{43}$$

which represents the difference between the true continuum DBF and the *rescaled* continuum DBF. Combining eqs. (41), (42) and (43) together and introducing

$$\xi(s,l,s') = \frac{\ln(s)}{\ln(s')} \left(\frac{1}{l} - \frac{1}{s'l} \right), \tag{44}$$

we arrive at

$$B_i'(u,s,l,s') = B(u,s) + \xi(s,l,s')\; e_i(u) - \delta B(u,s,s') + O(l^{-2}). \tag{45}$$

Therefore, if $\delta B(u,s,s')$ and $O(l^{-2})$ (or higher order) discretization errors are negligible compared to the statistical error of $B_i'(u,s,l,s')$, the numerical data of $B_i'(u,s,l,s')$ plotted against ξ will line up on a single line, and even two unknown coefficients in eq. (45), $B(u,s)$ and $e_i(u)$, for given u and s can be extracted from that behavior. Instead, if both or one of them is large, the data will not align. Thus, whether $B_i'(u,s,l,s')$ plotted against ξ aligns or not tests the validity of the linear extrapolation within the statistical uncertainty.

We comment on the size of $\delta B(u,s,s')$. Solving eq. (1) perturbatively, the continuum DBF is found to be

$$\begin{aligned} B(u,s) \;=\; & -\ln(s) \left\{ \frac{\beta(u)}{u^2} + u^2 \ln(s) \frac{1}{2} b_1 b_2 + u^3 \ln(s) \left(\frac{1}{3} {b_1}^2 b_2 \ln(s) + b_1 b_3 + \frac{1}{2} {b_2}^2 \right) \right\} \\ & + O\left(u^4 \ln^2(s) \right), \end{aligned} \tag{46}$$

and thus the perturbative expression of $\delta B(u,s,s')$ is

$$\begin{aligned} \delta B(u,s,s') \;=\; & u^2 \ln(s) \ln\left(\frac{s}{s'}\right) \left[-\frac{1}{2} b_1 b_2 + u \Big\{ -\frac{1}{3} {b_1}^2 b_2 \ln(ss') - \left(b_1 b_3 + \frac{1}{2} {b_2}^2 \right) \Big\} \right] \\ & + O\left(u^4 \, \ln(s) \, \ln(s/s') \right). \end{aligned} \tag{47}$$

Since the numerical values of b_i's are small, *e.g.* $b_1 \sim 0.055$, $b_2 \sim -0.002$, $b_3^{\rm SF} \sim O(10^{-4})$, $\delta B(u,s,s')$ is also small in the perturbative regime as $10^{-5} \times u^2\,(1.5 + 0.6\,u)$, $10^{-5} \times u^2\,(1.1 +$

$0.4\,u$) and $10^{-5}\times u^2\,(-1.1-0.5\,u)$ for $(s,\,s')=(2,\,16/12)$, $(2,\,12/8)$ and $(s,\,s')=(2,\,16/6)$, respectively. As u becomes large, $\delta B(u,s,s')$ may become sizable and at some point exceed the statistical error of $B_i'(u,s,l,s')$. Then, the alignment will be deformed. Notice that, the smaller $B(u,s')$ is, the smaller $\delta B(u,s,s')$ is, and in particular, when $B(u,s')=0$ for a certain s', $\delta B(u,s,s')=0$ holds exactly.

We extract the continuum DBF $B(u,s)$ as follows. First, we assume linear scaling violation and calculate $B(u,s)$ for s=2 by extrapolating the two data sets, $(s',l)=(2,6)$ and (2,8), to $\xi=0$. Since $s'=s$, $B_i'(u,s,l,s')=B_i(u,s',l)$ and $\delta B(u,s,s')=0$ by construction. Thus we do not have to rely on the smallness of $\delta B(u,s,s')$. Then, to test the linearity of the scaling violation, we calculate the rescaled lattice DBF $B_i'(u,s,l,s')$ with s=2 from the other data sets and plot them as a function of $\xi(s,l,s')$. If the data align within the statistical error of $B_i'(u,2,l,s')$, the assumption of the linear scaling violation is valid, $\delta B(u,s,s')$ is negligible and then the value of $B(u,s)$ thus obtained is reliable. Alternatively, once the linearity is confirmed, we can even determine the continuum limit by taking the linear extrapolation of $B_i'(u,2,l,s')$. Since $\delta B(u,s,s')$ is negligible in perturbative regime, the linearity can be tested more rigorously in such a regime. When the data do not align, either or both of the linear violation dominance and small $\delta B(u,s,s')$ are invalid and the result for $B(u,s)$ becomes uncertain.

E. extraction of the continuum DBF

Extrapolation to the continuum limit described in the following is carried out for every jack-knife ensemble, and the statistical error in the continuum limit is estimated by the single elimination jack-knife method.

We begin with analysis at relatively weak coupling. Figure 3 shows the continuum limit of $B_i'(u,s,l,s')$ for $s=2$ (i=1, 2) at the four representative values of $1/u$ corresponding to $u=1.0, 2.0, 10/3, 5.0$, where the data with $s'=2$ are shown in filled symbols and the other in open symbols and two data of the one-loop improved lattice DBF (B_1') with $s'=2$ (filled squares) are linearly extrapolated to $\xi=0$. The two-loop improvement described in sec. V C is equivalent to tuning the improvement coefficients such that the resulting DBF reproduces the perturbative DBF in the region $0<u<1.6$. Indeed, the constant fit of the two-loop improved DBF with $s'=2$ (filled diamonds) gives the value consistent with

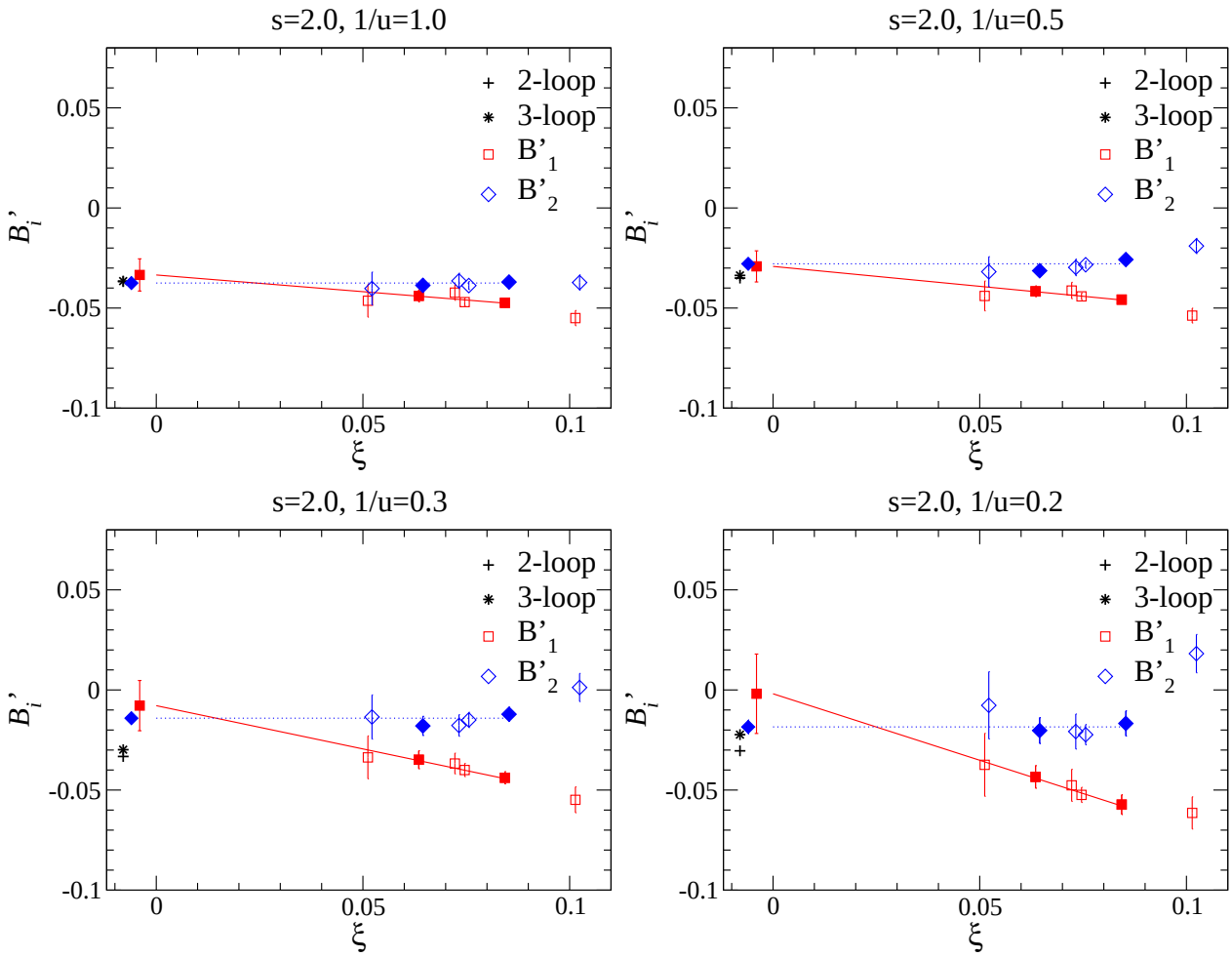

FIG. 3: Linear extrapolation of B_1 (filled squares) and constant fit of B_2' (filled diamonds) to the continuum limit. The extrapolation and fit use the data with $s' = 2$ (filled symbols). The data with $s' \neq 2$ (open symbols) are also shown to see whether they align or not. The values of (s', l) of the data shown are (4/3, 6), (2, 6), (8/3, 6), (3/2, 8), (2, 8), and (4/3, 12) from right to left. The data points are slightly shifted in horizontal direction for clarity. The perturbative predictions including the 2-loop (plus) and 3-loop (star) effects are also shown.

the perturbative prediction when $1/u = 1.0$ and 0.5 as seen in the figure. In the same region ($1/u \gtrsim 0.5$), the data of the one-loop improved DBF align and the linear extrapolation reproduces the perturbative DBF as well. Importantly, the extracted continuum DBF is clearly negative in this region.

The deviation from the perturbative prediction appears at $1/u = 0.3$, where the linear extrapolation gives the value closer to and consistent with zero. It is important to note that the data of the one-loop improved DBF align down to $1/u = 0.2$ with a slope increasing with

u. From this observation, we conclude that, in the region $1/u \geq 0.2$ ($u \leq 5$), $\delta B(u, s, s')$ is small, the scaling violation is linear for the one-loop improved DBF and hence the extracted continuum limit is reliable.

Next let us move on to the result at a stronger coupling shown in Fig. 4. As seen from the figure, first the data of B_1' except for the one with (s', l)=(4/3, 6) (right-most point) remains to align within the statistical uncertainty. Thus, the linear extrapolation of B_1 is reliable at $1/u = 0.15$. Secondly, the linear extrapolation of B_1 and the constant fit of B_2 lead to different continuum DBF. It appears that the constant fit of B_2 is no longer valid and the linear fit appears to be more reasonable. Indeed, the linear fit of B_2 (solid line and open diamond at $\xi = 0$) turns out to give the consistent limit as shown in the figure.

From the alignment of B_1', we infer that both δB and non-linear scaling violation remain small. This is consistent with the fact that the continuum DBF obtained by linear extrapolation of B_1 is consistent with zero and thus δB should be small as well.

The deviation of the coarsest data from the linear behavior indicates that the linear discretization error no longer dominates others in the data with (s', l)=(4/3, 6). Since in general non-linear scaling violation can be large for small l, the data with $l = 6$ may suffer from this though it is not visible in the figure. To evaluate the potential uncertainty due to the $O(l^{-2})$ discretization error, we performed a linear fit without the $l = 6$ data. The fit result is shown as open square at $\xi = 0$ and the dashed line in Fig. 4. The result is consistent with that using the $s' = 2$ data only.

From Figs. 3 and 4, it turns out that for $1/u \lesssim 0.3$ the extracted continuum DBF is consistent with zero. This indicates that in this region the running coupling constant reaches an infrared fixed point or, at least, the running appreciably slows down. In order to further investigate the existence of the infrared fixed point, we include the data obtained from l=18 lattice at $\beta = 4.55$ into analysis. This data is combined with the data with $l = 12$ to construct B_1' with (s', l)=(3/2, 12). At $\beta = 4.55$, the inverse SF coupling for l=12 turns out to be $1/u = 0.107$. On l=6 lattice, this value of $1/u$ is realized at $\beta \sim 4.4$. In such a small β, the SF couplings are not calculated on l=12, 16 lattices, and hence the following analysis is carried out without the data from l=6 lattices.

B_1' constructed from the l=18 data is shown in Fig. 5 (filled circle). Since the four data points shown align well, we take the linear extrapolation using all of them and obtain the positive value in $\xi = 0$. Interpretation of this result needs care as mentioned sec. II. The most

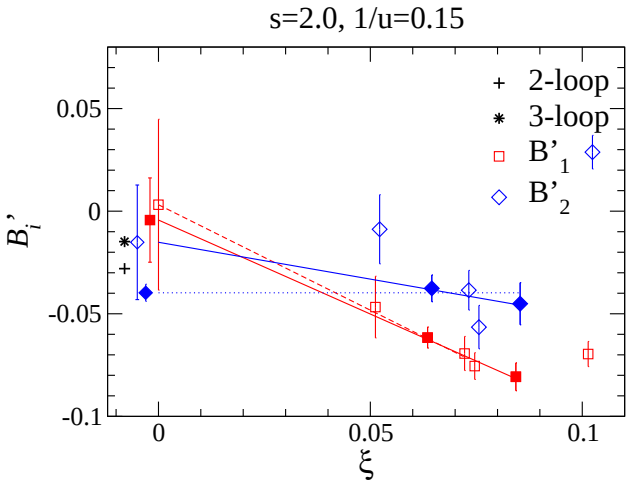

FIG. 4: Same as Fig. 3, but the result at the stronger coupling is plotted. Solid lines denote the linear extrapolation using the data with $s' = 2$ (filled symbols). Dashed line shows the linear extrapolation using the data with the three smallest ξ.

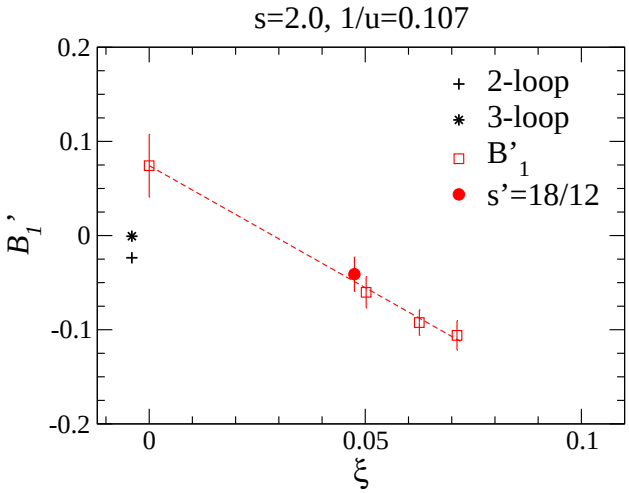

FIG. 5: Same as Fig. 4 but the data point obtained with $(s',\ l)$=(3/2, 12) (filled circle) is included in the analysis at $1/u$=0.107. The dashed line and the open square at $\xi = 0$ are the result of the linear fit.

plausible explanation for this observation is that an IRFP exists in $u_{\rm IRFP} < 1.0/0.107 = 9.35$.

Figure 6 shows the $1/u$ dependence of the continuum DBF, where the results are compared with the perturbative calculations. It is seen that the running starts to slow down at around $1/u \sim 0.5$, and eventually the coupling constant reaches a fixed point in the range of $0.107 < 1/u \lesssim 0.3$. When the DBF is positive, it is non-trivial for the continuum limit

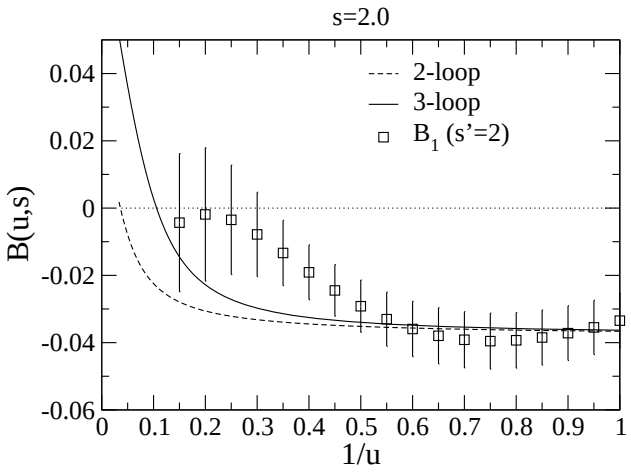

FIG. 6: $1/u$ dependence of $B(u,s)$ with s=2 obtained from the linear extrapolation of the data with $s' = 2$. Two- and three-loop perturbative predictions are shown by dashed and solid line, respectively.

to exist. Thus we omit the positive DBF data from the figure.

VI. SUMMARY AND OUTLOOK

In this work, the running coupling constant of ten-flavor QCD is numerically investigated using lattice technique. The extrapolation of the DBF to the continuum limit is taken linearly assuming that the $O(a)$ scaling violation dominates the higher order ones. The DBF extrapolated approaches zero from below as the SF coupling constant u increases and when $u \gtrsim 10/3$ the DBF becomes consistent with zero. Further investigation at one particular strong coupling $u = 9.3$ ($1/u = 0.107$) is made using the data from the large lattice ($l = 18$), and suggests that the continuum DBF at this coupling is not negative. This indicates the existence of the infrared fixed point $10/3 \lesssim g^2_{\rm IRFP} \lesssim 9.3$. The linear extrapolation is reasonably justified within the statistical error, but further rigorous check is clearly preferable. Combining our result with that of Ref. [12], the critical number of flavors which separates the conformal phase and the broken phase is $8 < N_f^{\rm crit} < 10$.

In order to confirm the existence of IRFP or even determine the value of the fixed point more precisely, data from larger lattices with high statistics are necessary. It is, however, difficult to do with machines currently available to us, and probably more efficient methods or different approaches are necessary to go further. As mentioned in sec. I, the conformal

window can also be studied by looking at hadrons' spectroscopy or renormalization group analysis on the lattice. Currently the conclusions based on various methods are not consistent among them. In order to pin down $N_f^{\rm crit}$, these contradictions must be clarified with further studies.

What is really important in the context of the WTC is the anomalous dimension of the $\bar{\psi}\psi$ operator. The calculation of the anomalous dimension in ten-flavor QCD is on-going. The result will be published elsewhere.

Once one has fixed an attractive candidate for WTC, the next important step would be the calculation of the S-parameter. The calculational method has been established in Ref. [39], where the QCD S-parameter is calculated on the lattice for the first time and is correctly reproduced. Later, the method was applied to three-flavor QCD [40], sextet QCD [41] and six-flavor QCD [42]. In Ref. [42], the evidence of the reduction of S-parameter is reported. Another important quantity which should be calculated is obviously the mass spectrum of the candidate theory, including vector and scalar resonances, the decay constant of the NG boson and the chiral condensate. Although the precise determinations of these quantities are challenging, the direct comparison with the upcoming LHC results is extremely interesting and hence we believe that such calculations are worth a lot of efforts.

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