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## Art of spin decomposition

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### Abstract

We analyze the problem of spin decomposition for an interacting system from a natural perspective of constructing angular momentum eigenstates. We split, from the total angular momentum operator, a proper part which can be separately conserved for a stationary state. This part commutes with the total Hamiltonian and thus specifies the quantum angular momentum. We first show how this can be done in a gauge-dependent way, by seeking a specific gauge in which part of the total angular momentum operator vanishes identically. We then construct a gauge-invariant operator with the desired property. Our analysis clarifies what is the most pertinent choice among the various proposals for decomposing the nucleon spin. A similar analysis is performed for extracting a proper part from the total Hamiltonian to construct energy eigenstates.

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How the nucleon spin originates from its internal quark-gluon dynamics appears to be a rather complicated and challenging problem. [1] A reflection of the complication is that even a schematic decomposition of the nucleon spin is not agreed upon. So far, quite a few proposals for decomposing the nucleon spin have appeared, from the early popular schemes of Jaffe-Manohar [2] and Ji [3], to the recent gauge-invariant schemes of Chen *et al.* [4–6], Wakamatsu [7], Cho *et al.* [8], and Leader [9] (to list just a few). This frequently causes great confusion, when one piece (say, gluon spin) from a certain scheme is examined together with another piece (say, quark orbital angular momentum) from another scheme, and makes people feel lost as to which decomposition is better.

To clarify the issue, we point out a seldom noticed fact that decomposing the total angular momentum of an interacting system is actually a common practice in quantum mechanics. The same occurs also for the Hamiltonian. These decompositions, however, indeed involve a rather delicate art, which apparently has never been properly addressed. Revealing this art provides a clear clue as to how to decompose the nucleon spin most naturally.

To see the point, let us look at the quantum-mechanical study of the Hydrogen atom (an interacting system of electron, proton, and electromagnetic field). The standard procedure is to construct Hamiltonian and angular momentum eigenstates of *only* the electron:

$$i\partial_t\psi_e = H_e\psi_e = E_e\psi_e \quad (1)$$

$$\vec{J}_e^2\psi_e = j(j+1)\psi_e, \quad J_e^z\psi_e = m\psi_e, \quad (2)$$

where

$$H_e = \vec{\alpha} \cdot \frac{1}{i}\vec{D}_e + M_e\beta + q_e A^0 \quad (3)$$

$$\vec{J}_e = \frac{1}{2}\vec{\Sigma} + \vec{x} \times \frac{1}{i}\vec{\partial} \quad (4)$$

are the electron Hamiltonian and angular momentum operators, respectively.  $\vec{D}_e = \vec{\partial} - iq_e\vec{A}$  is the covariant derivative for the electron, with  $q_e$  the electron charge. The *electron* quantum numbers  $E_e$ ,  $j$  and  $m$  are then used to label an *atomic* state. But one may seriously ask: how can  $E_e$ ,  $j$  and  $m$  represent the energy and angular momentum of the atom, when only

the following *total atomic* Hamiltonian and angular momentum are conserved?

$$\begin{aligned}
H_{\text{atom}} = & \int d^3x \psi_e^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_e + M_e \beta) \psi_e \\
& + \int d^3x \psi_p^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_p + M_p \beta) \psi_p \\
& + \int d^3x \frac{1}{2} (\vec{E}^2 + \vec{B}^2)
\end{aligned} \tag{5}$$

$$\begin{aligned}
\vec{J}_{\text{atom}} = & \int d^3x \psi_e^\dagger (\frac{1}{2} \vec{\Sigma} + \vec{x} \times \frac{1}{i} \vec{\partial}) \psi_e \\
& + \int d^3x \psi_p^\dagger (\frac{1}{2} \vec{\Sigma} + \vec{x} \times \frac{1}{i} \vec{\partial}) \psi_p \\
& + \int d^3x (\vec{E} \times \vec{A} + E_i \vec{x} \times \vec{\partial} A_i) \\
\equiv & \vec{J}_e + \vec{J}_p + \vec{J}_\gamma
\end{aligned} \tag{6}$$

(We use the same symbols for quantum-mechanical and quantum-field operators, since confusion can hardly arise.  $\psi_e$  is the electron field,  $\psi_p$  is the proton field.)

A careful reader would notice that  $H_e$  and  $\vec{J}_e$  are even gauge dependent and so have no definite contents at all! For these operators to be useful, a key role must be played by the choice of gauge. Indeed, we show by a careful examination that, under certain circumstance and in a special gauge, the use of electron quantum numbers  $E_e$ ,  $j$  and  $m$  for the total atom can be justified.

The circumstance we consider is a *stationary* system, namely, the electric current  $j^\mu = q_e \bar{\psi}_e \gamma^\mu \psi_e + q_p \bar{\psi}_p \gamma^\mu \psi_p$  and the electromagnetic fields  $F^{\mu\nu}$  are *time-independent*. This includes the typical case of solving for the quantum-mechanical eigenfunctions. Both  $j^\mu$  and  $F^{\mu\nu}$  are gauge-invariant quantities, so their time-independence has a definite physical meaning. In contrast, the electron wavefunction  $\psi_e$  and the electromagnetic vector potential  $A^\mu$  are gauge-dependent. Such a gauge-dependence can be both a disadvantage and an advantage, which are just two sides of the same coin. The disadvantage, as we remarked above, is that the individual angular momentum operators in Eq. (6),  $\vec{J}_e$ ,  $\vec{J}_p$ , and  $\vec{J}_\gamma$  are all gauge-dependent; therefore an electron angular momentum eigenstate may have no definite physical meaning. (A more serious and tricky problem is the gauge dependence of  $H_e$ , which we address shortly below.) The advantage, on the other hand, is that there must always exist a gauge in which one element of the electromagnetic angular momentum (say,  $J_\gamma^z$ ) vanishes, therefore  $J_e^z + J_p^z$  in this gauge equals the total  $J_{\text{atom}}^z$  and thus can be in an eigenstate. A non-trivial and remarkable feature, however, is that for a stationary system, a single gauge

condition can lead to the vanishing of all three components of  $\vec{J}_\gamma$ .

*Proof:* As  $\partial_t F^{\mu\nu} = 0$ , we have  $\vec{\partial} \times \vec{E} = -\partial_t \vec{B} = 0$ , hence  $\vec{E}$  must be a gradient, which we denote as  $-\vec{\partial}\phi$ . Then a little algebra shows that

$$\begin{aligned}\vec{J}_\gamma^{\text{stat}} &= \int d^3x (-\vec{\partial}\phi) \times \vec{A} + \int d^3x \vec{x} \times (-\partial_i \phi) \vec{\partial} A_i \\ &= \int d^3x \phi \vec{x} \times \vec{\partial}(\partial_i A_i).\end{aligned}\tag{7}$$

Thus,  $\vec{J}_\gamma^{\text{st}} \equiv \vec{0}$  in the Coulomb gauge  $\partial_i A_i = 0$ . In this gauge, therefore, not only  $J_p^z + J_e^z$ , but also  $(J_p^z + J_e^z)^2$ , can be in an eigenstate. Furthermore, if the proton (with a magnetic moment much smaller than that of the electron) is unpolarized, then the electron quantum numbers  $j$  and  $m$  do represent the angular momentum of the total atom.

The use of Coulomb gauge appears to be taken-for-granted in quantum mechanics. From our analysis, it is truly a very fortunate choice: Should other gauges be chosen, one would have to account for a (spuriously) non-zero electromagnetic angular momentum so as to obtain the correct total atomic spin. (Unfortunately, such good fortune is not always cherished: We will see soon that the study of nucleon spin structure involves a great deal of effort in exploring *spurious* gluon angular momentum.)

One can further appreciate the good fortune in choosing Coulomb gauge by considering the Hamiltonian. In Eq. (5), we have written the total Hamiltonian in a most familiar, explicitly gauge-invariant form, where  $\frac{1}{2}(\vec{E}^2 + \vec{B}^2)$  is the usual expression for energy of the electromagnetic field in classical electrodynamics. But the form in Eq. (5) is neither useful nor illuminating in quantum mechanics. First of all,  $H_e$  does not show up explicitly in Eq. (5). Moreover, it is not clear from Eq. (5) whether the parts other than  $H_e$  (i.e.,  $H_{\text{atom}} - H_e$ ) can be ignored for an atom. For the sake of justifying the “electron energy”,  $E_e$ , as the pertinent label of an atomic energy level, it is better to put the total atomic Hamiltonian in the canonical form (with gauge-variant densities):

$$\begin{aligned}H_{\text{atom}} &= \int d^3x (\psi_e^\dagger i \partial_t \psi_e + \psi_p^\dagger i \partial_t \psi_p - E^i \partial_t A^i - \mathcal{L}) \\ &= \int d^3x \psi_e^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_e + M_e \beta + q_e A^0) \psi_e \\ &\quad + \int d^3x \psi_p^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_p + M_p \beta + q_p A^0) \psi_p \\ &\quad - \int d^3x [E^i \partial_t A^i + \frac{1}{2}(\vec{E}^2 - \vec{B}^2)],\end{aligned}\tag{8}$$

where the Lagrangian is

$$\mathcal{L} = \bar{\psi}_e(i\gamma_\mu D_e^\mu - M_e)\psi_e + \bar{\psi}_p(i\gamma_\mu D_p^\mu - M_p)\psi_p - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (9)$$

By some careful algebra, we obtain

$$\begin{aligned} H_{\text{atom}} = & \int d^3x \psi_e^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_e + M_e \beta + q_e A^0) \psi_e \\ & + \int d^3x \psi_p^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_p + M_p \beta) \psi_p \\ & + \int d^3x \frac{1}{2} [\vec{E}_\perp^2 - \vec{A}_\perp \cdot \partial_t^2 \vec{A}_\perp + (\vec{j}_e - \vec{j}_p) \cdot \frac{1}{\partial^2} \partial_t^2 \vec{A}_\perp] \\ & - \vec{j}_p \cdot \vec{\partial} \frac{1}{\partial^2} (\vec{\partial} \cdot \vec{A}) + j_e^0 \partial_t \frac{1}{\partial^2} (\vec{\partial} \cdot \vec{A}) \end{aligned} \quad (10)$$

Here we have omitted self-energy terms which belong to the issue of radiative corrections and the Lamb shift. In Eq. (10), the first line is the usual electron Hamiltonian  $H_e$  in an electromagnetic field, the second line is now a *free* proton Hamiltonian. It just gives the proton rest mass if the proton were regarded as being infinitely heavy compared to electron. The third line (where  $\vec{E}_\perp = -\partial_t \vec{A}_\perp$ ) is the dynamic part of the electromagnetic field, which vanishes for a stationary state (note that  $\vec{A}_\perp = -\frac{1}{\partial^2} \vec{\partial} \times \vec{B}$  is gauge invariant, and is time-independent as  $\vec{B}$  is). The fourth line contains the gauge-dependent terms, which vanish only in the Coulomb gauge. This is what we have intended to show: in the stationary approximation, the “electron energy”,  $E_e$ , computed in (*and only in*) the Coulomb gauge can label the total atomic energy (except for the trivial proton mass term).

Beyond the stationary approximation, i.e., when considering quantum fluctuations of the electromagnetic field,  $E_e$  is no longer a precise measure of atomic energy (as reflected by the Lamb shift). Analogously,  $\vec{J}_\gamma$  will be non-zero in an atom, and  $\vec{J}_p + \vec{J}_e$  receives radiative corrections as well. Then, one may begin to consider “atomic spin structure”. In fact, Eq. (6) is just the atomic version of the Jaffe-Manohar scheme of separating the nucleon spin components. (The quark and gluon angular momentum operators take the same forms as in Eq. (6), but with implicit color indices summed over.) We see that from the perspective of constructing a stationary angular-momentum eigenstate, the Jaffe-Manohar scheme can indeed be useful, though limited to Coulomb gauge.

The unsatisfactory aspect of the Jaffe-Manohar scheme, of course, is that gauge dependence obscures the physical meaning of  $\vec{J}_e$  and  $\vec{J}_p$ . The Ji scheme [3] is intended as an

improvement regarding gauge invariance. Its atomic version is

$$\begin{aligned}
\vec{J}_{\text{atom}} &= \int d^3x \psi_e^\dagger \left( \frac{1}{2} \vec{\Sigma} + \vec{x} \times \frac{1}{i} \vec{D}_e \right) \psi_e \\
&+ \int d^3x \psi_p^\dagger \left( \frac{1}{2} \vec{\Sigma} + \vec{x} \times \frac{1}{i} \vec{D}_p \right) \psi_p \\
&+ \int d^3x \vec{x} \times (\vec{E} \times \vec{B}) \\
&\equiv \vec{J}_e + \vec{J}_p + \vec{J}_\gamma
\end{aligned} \tag{11}$$

The gauge invariance of  $\vec{J}_e$ ,  $\vec{J}_p$  and  $\vec{J}_\gamma$  is evident by the use of the gauge-covariant derivative in  $\vec{J}_e$ ,  $\vec{J}_p$  and the Poynting vector in  $\vec{J}_\gamma$ ; therefore  $\vec{J}_e$ ,  $\vec{J}_p$  and  $\vec{J}_\gamma$  have well-defined contents. But relative to the art of spin decomposition as elaborated above, (namely, to be able to choose a proper part which can specify the quantum number of the whole atom,) the gauge-invariance of  $\vec{J}_e$ ,  $\vec{J}_p$  and  $\vec{J}_\gamma$  can also be a danger! The point is that without any adjustable gauge variation to use, one can only hope that  $\vec{J}_e + \vec{J}_p$  *intrinsically* describe the atomic angular momentum. That hope, however, cannot be realized. In fact,  $\vec{J}_e$  and  $\vec{J}_p$  are not angular-momentum operators at all:  $\vec{J}_{e,p} \times \vec{J}_{e,p} \neq i\vec{J}_{e,p}$ , thus  $\vec{J}_e + \vec{J}_p$  cannot possibly equal the proper angular momentum operator,  $\vec{J}_{\text{atom}}$ , except in the trivial case of neglecting magnetic interaction. As a cross-check, one can see that  $\vec{J}_\gamma$  is not an angular-momentum operator either ( $\vec{J}_\gamma \times \vec{J}_\gamma \neq i\vec{J}_\gamma$ ), and it does not vanish even for a stationary system.  $\vec{J}_\gamma$  relates to  $\vec{J}_\gamma$  by

$$\vec{J}_\gamma = \vec{J}_\gamma + \int d^3x \vec{x} \times \rho \vec{A}. \tag{12}$$

where  $\rho$  is the total charge density. We have shown that, for a stationary system,  $\vec{J}_\gamma = 0$  in the Coulomb gauge, while in this gauge  $\vec{A} = -\frac{1}{\partial^2} \vec{j}$  is not zero. Therefore, despite being gauge-invariant,  $\vec{J}_e$  and  $\vec{J}_p$  are not useful in atomic physics (with regard to construction of angular-momentum eigenstates), and  $\vec{J}_\gamma$  (more precisely,  $\vec{x} \times \rho \vec{A}$ ) represents a spurious angular momentum of the electromagnetic field. If one were to regard  $\vec{J}_\gamma$  as the electromagnetic angular momentum, then, even without considering quantum fluctuations, the atomic spin would exhibit a non-trivial structure: both the electron and photon would contribute to the atomic spin and neither would be in a well-defined angular momentum eigenstate. Such a structure, however, is just an artificial complication.

The recent proposal of Chen *et al* is a reconciliation of gauge-invariance and construction

of angular-momentum eigenstates. Its atomic version is [4–6]:

$$\begin{aligned}
\vec{J}_{\text{atom}} &= \int d^3x \psi_e^\dagger \left[ \frac{1}{2} \vec{\Sigma} + \vec{x} \times \frac{1}{i} (\vec{\partial} - iq_e \vec{A}_\parallel) \right] \psi_e \\
&+ \int d^3x \psi_p^\dagger \left[ \frac{1}{2} \vec{\Sigma} + \vec{x} \times \frac{1}{i} (\vec{\partial} - iq_p \vec{A}_\parallel) \right] \psi_p \\
&+ \int d^3x [\vec{E}_\perp \times \vec{A}_\perp + E_\perp^i \vec{x} \times \vec{\partial} A_\perp^i] \\
&\equiv \mathbf{J}_e + \mathbf{J}_p + \mathbf{J}_\gamma
\end{aligned} \tag{13}$$

Here the longitudinal field  $\vec{A}_\parallel = \vec{\partial} \frac{1}{\vec{\partial}^2} (\vec{\partial} \cdot \vec{A})$  is the pure-gauge part of  $\vec{A}$ , and vanishes in the Coulomb gauge. Its gauge transformation is the same as that of the full  $\vec{A}$ , therefore  $\vec{\partial} - iq\vec{A}_\parallel$  is a (pure-gauge) covariant derivative. Consequently,  $\mathbf{J}_e$  and  $\mathbf{J}_p$  are gauge invariant, and in Coulomb gauge,  $\mathbf{J}_e = \vec{J}_e$  and  $\mathbf{J}_p = \vec{J}_p$ . Analogously,  $\mathbf{J}_\gamma$  is gauge invariant, and is equal to  $\vec{J}_\gamma$  in Coulomb gauge.  $\vec{E}_\perp = -\partial_t \vec{A}_\perp$  is the gauge invariant, dynamical (transverse) part of the electric field. It is evident from Eq. (13) that  $\mathbf{J}_\gamma$  has the nice feature of vanishing identically for a stationary configuration (while the gauge-dependent  $\vec{J}_\gamma$  does so in Coulomb gauge only). Regarding the labeling of atomic states,  $\mathbf{J}_e$  and  $\mathbf{J}_p$  are therefore the pertinent and satisfactory operators to use (in any gauge), just as  $\vec{J}_e$  and  $\vec{J}_p$  are in Coulomb gauge.

A gauge-invariant expression similar to Eq. (13) can be derived for the Hamiltonian, and indeed, for the whole energy-momentum tensor, which can be put into the gauge-invariant, canonical form:

$$T^{\mu\nu} = \bar{\psi}_e \gamma^\mu i \bar{D}_e^\nu \psi_e + \bar{\psi}_p \gamma^\mu i \bar{D}_p^\nu \psi_p + F^{\rho\mu} \partial^\nu \hat{A}_\rho - \eta^{\mu\nu} \mathcal{L} \tag{14}$$

Here  $\bar{D}^\mu = \partial^\mu + iq\bar{A}^\mu$  is the pure-gauge covariant derivative.  $\bar{A}^\mu = -\partial^\mu \frac{1}{\vec{\partial}^2} (\vec{\partial} \cdot \vec{A})$  is the pure-gauge part of  $A^\mu$ . Its spatial component,  $\vec{\bar{A}}$ , is just  $\vec{A}_\parallel$ .  $\hat{A}_\rho = \frac{1}{\vec{\partial}^2} \partial_i F_{i\rho} = A_\rho - \bar{A}_\rho$  is the (gauge-invariant) physical part of  $A_\rho$ , with spatial component  $\vec{\hat{A}} = \vec{A}_\perp$ .

From Eq. (14), the conserved four-momentum,  $P^\nu = \int d^3x T^{0\nu}$ , is

$$P^\nu = \int d^3x (\psi_e^\dagger i \bar{D}_e^\nu \psi_e + \psi_p^\dagger i \bar{D}_p^\nu \psi_p - E_\perp^i \partial^\nu A_\perp^i - \eta^{0\nu} \mathcal{L}) \tag{15}$$

Here we have used the fact that  $\int d^3x E_\parallel^i \partial^\nu A_\perp^i = 0$ , where  $\vec{E}_\parallel = -\vec{\partial} \hat{A}^0$  is the gauge-invariant longitudinal part of the electric field. In particular, the spatial three-momentum is

$$\vec{P} = \int d^3x (\psi_e^\dagger \frac{1}{i} \vec{D}_e \psi_e + \psi_p^\dagger \frac{1}{i} \vec{D}_p \psi_p + E_\perp^i \vec{\partial} A_\perp^i), \tag{16}$$



and the Hamiltonian is

$$\begin{aligned}
H_{\text{atom}} &= \int d^3x (\psi_e^\dagger i \vec{D}_e^0 \psi_e + \psi_p^\dagger i \vec{D}_p^0 \psi_p - E_\perp^i \partial_t A_\perp^i - \mathcal{L}) \\
&= \int d^3x \psi_e^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_e + M_e \beta + q_e \hat{A}^0) \psi_e \\
&\quad + \int d^3x \psi_p^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_p + M_p \beta + q_p \hat{A}^0) \psi_p \\
&\quad + \int d^3x \frac{1}{2} (\vec{E}_\perp^2 + \vec{B}^2 - \vec{E}_\parallel^2)
\end{aligned} \tag{17}$$

The second line is the operator we used to replace  $H_e$  for computing the atomic energy. [10] It is gauge invariant, and equals  $H_e$  in Coulomb gauge. Therefore, the rest (also gauge-invariant) must intrinsically be irrelevant for a stationary configuration. This property is displayed by

$$\begin{aligned}
H_{\text{atom}} &= \int d^3x \psi_e^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_e + M_e \beta + q_e \hat{A}^0) \psi_e \\
&\quad + \int d^3x \psi_p^\dagger (\vec{\alpha} \cdot \frac{1}{i} \vec{D}_p + M_p \beta) \psi_p \\
&\quad + \int d^3x \frac{1}{2} [\vec{E}_\perp^2 - \vec{A}_\perp \cdot \partial_t^2 \vec{A}_\perp + (\vec{j}_e - \vec{j}_p) \cdot \frac{1}{\partial^2} \partial_t^2 \vec{A}_\perp],
\end{aligned} \tag{18}$$

where the second line is a gauge-invariant free proton part, and the third line vanishes for a static field. In deriving this expression, we have rewritten  $\vec{B}^2$  as

$$\begin{aligned}
\int d^3x \vec{B}^2 &= - \int d^3x A_\perp^i \partial^2 A_\perp^i = \int d^3x A_\perp^i (j_\perp^i - \partial_t^2 A_\perp^i) \\
&= \int d^3x [j_\perp^i \frac{1}{\partial^2} (\partial_t^2 A_\perp^i - j_\perp^i) - A_\perp^i \partial_t^2 A_\perp^i],
\end{aligned} \tag{19}$$

and self-energy terms have been discarded as above.

By examining the familiar and unambiguous examples in atomic physics, we display clearly the art of spin decomposition: A good decomposition should give a simple structure and physical picture, and should not give arise to spurious complications.

We now turn to the hadronic sector. A decomposition of QCD angular momentum operator that respects the above art precisely mimics the atomic expression in Eq. (13) [6]:

$$\begin{aligned}
\vec{J}_{\text{QCD}} &= \int d^3x \psi_q^\dagger \frac{1}{2} \vec{\Sigma} \psi_q + \int d^3x \vec{x} \times \psi_q^\dagger \frac{1}{i} \vec{D} \psi_q \\
&\quad + \int d^3x (-\mathcal{D}_t \vec{A}) \times \vec{A} + \int d^3x \vec{x} \times (-\mathcal{D}_t \hat{A}_i) \vec{D} \hat{A}_i \\
&\equiv \mathbf{S}_q + \mathbf{L}_q + \mathbf{S}_g + \mathbf{L}_g.
\end{aligned} \tag{20}$$

These expressions are more complicated than Eq. (13) due to color structure and non-linear terms.  $\hat{A}^\mu$  is physical part of the non-Abelian gluon field.  $\hat{A}^\mu$  transforms in the same gauge-covariant manner as  $F^{\mu\nu}$ , and therefore also requires covariant derivatives. Note that in  $\mathbf{S}_g$  and  $\mathbf{L}_g$  the pure-gauge field  $\vec{A}$  is used in  $\vec{\mathcal{D}} \equiv \vec{\partial} - ig[\vec{A}, \ ]$ , while  $\mathcal{D}_t \equiv \partial_t + ig[A^0, \ ]$  involves the full  $A^0$ . As a result,  $\mathbf{S}_g$  and  $\mathbf{L}_g$  show a key difference from  $\mathbf{S}_\gamma$  and  $\mathbf{L}_\gamma$ , namely that they may survive in a stationary configuration. This property may potentially be crucial at the non-perturbative low-energy scale, and lead to a sizable gluon contribution to the nucleon spin. In the perturbative regime, however, the non-linear terms in  $\mathbf{S}_g$  and  $\mathbf{L}_g$  are of higher order, and the leading-order terms still vanish for a stationary configuration. For example, if we consider a hadron made entirely of heavy quarks (so that perturbative QCD applies), at the order of one-gluon exchange,  $\mathbf{S}_g = \mathbf{L}_g = 0$ , and the hadron spin comes solely from quarks. This picture is also true if the Coulomb gauge is adopted for the (gauge-dependent) Jaffe-Manohar scheme [2]:

$$\begin{aligned} \vec{J}_{\text{QCD}} &= \int d^3x \psi_q^\dagger \frac{1}{2} \vec{\Sigma} \psi_q + \int d^3x \vec{x} \times \psi_q^\dagger \frac{1}{i} \vec{\partial} \psi_q \\ &+ \int d^3x \vec{E} \times \vec{A} + \int d^3x \vec{x} \times E_i \vec{\partial} A_i \\ &\equiv \vec{S}_q + \vec{L}_q + \vec{S}_g + \vec{L}_g. \end{aligned} \quad (21)$$

In a gauge other than Coulomb, however,  $\vec{S}_g + \vec{L}_g$  develops a leading non-zero term of

$$\begin{aligned} &\int d^3x [(-\vec{\partial} A^0) \times \vec{A} + \vec{x} \times (-\partial_i A^0) \vec{\partial} A_i] \\ &= \int d^3x A^0 \vec{x} \times \vec{\partial} (\partial_i A_i), \end{aligned} \quad (22)$$

which can lead to another type of spurious gluon angular momentum in a hadron. In this regard, it is somewhat awkward that many of the theoretical techniques developed so far are for exploration of  $\vec{S}_g$  in the light-cone gauge, which greatly simplifies the expression of polarized gluon distribution function [11]. In a very recent paper [12], the formalism of gauge-field decomposition in Ref. [4–6] is adopted to construct a gauge-invariant gluon spin which agrees with  $\vec{S}_g$  in light-cone gauge. While such an  $\vec{S}_g$  may still reveal some gluon dynamics in the nucleon, one must be very cautious in using the data, since a sizable  $\vec{S}_g$  so obtained does not necessarily imply a significantly non-trivial gluonic content of the nucleon.

Analogously, if a sizable  $\vec{J}_g$  were found in the Ji scheme [3]:

$$\begin{aligned}\vec{J}_{\text{QCD}} &= \int d^3x \psi_q^\dagger \frac{1}{2} \vec{\Sigma} \psi_q + \int d^3x \vec{x} \times \psi_q^\dagger \frac{1}{i} \vec{D} \psi_q \\ &+ \int d^3x \vec{x} \times (\vec{E} \times \vec{B}) \\ &\equiv \vec{S}_q + \vec{L}'_q + \vec{J}_g,\end{aligned}\tag{23}$$

it may actually come from the spurious gluon angular momentum  $\int d^3x \vec{x} \times \psi_q^\dagger \vec{A} \psi_q$ , and so may also not imply a significant gluon content in the nucleon.

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