Six-point next-to-maximally helicity-violating amplitude in maximally supersymmetric Yang-Mills theory

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The Six-Point NMHV amplitude in Maximally Supersymmetric Yang-Mills Theory

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Abstract

We present an integral representation for the parity-even part of the two-loop six-point planar NMHV amplitude in terms of Feynman integrals which have simple transformation properties under the dual conformal symmetry. We probe the dual conformal properties of the amplitude numerically, subtracting the known infrared divergences. We find that the subtracted amplitude is invariant under dual conformal transformations, confirming existing conjectures through two-loop order. We also discuss the all-loop structure of the six-point NMHV amplitude and give a parametrization whose dual conformal invariant building blocks have a simple physical interpretation.

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I. INTRODUCTION

What is the scattering matrix of a matter-coupled gauge theory? In general, this is a hard question involving both conceptual and technical subtleties. Nevertheless, scattering amplitudes enjoy a much simpler structure than implied by their expansion in terms of Feynman diagrams. For some theories, additional off-shell and on-shell symmetries simplify the amplitudes enormously. The further simplification exhibited in the planar (or infinite-color) limit may even allow a complete answer to the question.

The $\mathcal{N} = 4$ or maximally supersymmetric Yang–Mills theory (MSYM) may be such a theory. The simplifications inherent in the larger symmetry have already allowed explicit calculations of scattering amplitudes well beyond those for other theories. At weak coupling, advances in multi-loop and multi-leg calculations [1–7] have opened the possibility of probing the structure of the scattering matrix to high order in perturbation theory. The BDS conjecture [8] for the all-loop resummation of maximally helicity-violating (MHV) amplitudes (based on an earlier relation [9] linking one- and two-loop amplitudes) provides an example of possible structures that can emerge. At strong coupling, the leading expansion of scattering amplitudes has been computed using the AdS/CFT correspondence [10] by Alday, Gaiotto, Maldacena, Sever, and Vieira [11–14]. This two-sided approach, together with the recent developments in the evaluation of scattering amplitudes at strong coupling for any number of external legs and the realization that the relation between certain scattering amplitudes and null polygonal Wilson loops carries over from strong coupling [15] to the weak-coupling regime [16–20] offer circumstantial evidence that the $\mathcal{N} = 4$ super-Yang-Mills theory may ultimately be solvable in the planar limit.

Gluon MHV amplitudes, with two external legs of negative helicity and the rest of positive helicity, are the simplest amplitudes in a gauge theory. They are particularly simple in the planar limit of MSYM, where they are determined by a single helicity structure, accompanied by a function of scalar and pseudo-scalar momentum invariants, to all orders in perturbation theory. This simplicity is of course shared by their parity conjugates, the $\overline{\text{MHV}}$ amplitudes. The structure of the remaining non-MHV amplitudes is more complicated. At one loop, explicit expressions are known [21, 22] at arbitrary multiplicity for next-to-MHV (NMHV) gluon amplitudes, with three gluons of negative helicity from the rest. These are already more intricate, with the number of independent helicity structures growing cubicly with the number of external legs, each multiplied by an independent function of scalar and pseudo-scalar invariants. How does this structure generalize to higher loops? No explicit expressions are known to date for higher-loop non-MHV amplitudes. In this paper, we take a first step towards filling this gap, computing the parity-even part of the two-loop six-gluon NMHV amplitude. These results were first reported in ref. [23]. This amplitude, which comes with three inequivalent helicity configurations, is the simplest non-MHV amplitude.

General results on the structure of infrared divergences in massless gauge theories suggest on one hand, that the divergent terms have a simple iterative structure; and on the other, that all planar amplitudes with fixed number of external legs share the same structure of infrared-divergent terms. Together with the structure of the splitting amplitudes, this implies that a natural way to extract infrared-finite quantities from non-MHV amplitudes is to divide them by the MHV amplitudes with the same number of external legs. Drummond, Henn, Korchemsky, and Sokatchev (DHKS) showed to one-loop order that this ratio is not only finite but also dual conformal invariant for NMHV amplitudes [5, 24] and conjectured that the same holds to all orders for all non-MHV amplitudes [24]. Here we clarify and test this conjecture to two-loop order for the six-point amplitude. This test requires the use of Dixon and Schabinger's result [25] for the $\mathcal{O}(\epsilon)$ terms in the one-loop NMHV six-point amplitude.

As an intriguing consequence of the semiclassical approach of Alday and Maldacena [11], anticipated by the structure of flat space string theory scattering amplitudes at high energy and fixed angles [26], to leading order in the strong coupling expansion all scattering amplitudes are (in a certain sense) insensitive to the flavor and polarization of external legs. While quantum corrections are likely to alter this conclusion, this structure is surprising from the standpoint of the intricate analytic structure of the weak-coupling scattering matrix.

The arguments of Alday and Maldacena led to the identification [16, 17] of a surprising relation between one-loop MHV amplitudes and the one-loop expectation value of special null polygonal Wilson loops. This relation was shown to hold at two loops as well for four-, five- [18] and six-particle scattering amplitudes [19, 27]. Integral representations of higher-point two-loop MHV amplitudes are also known [28]; comparison with Wilson-loop expectation values [29] is hindered, however, by the complexity of evaluating the required higher-point two-loop Feynman integrals. Dual conformal symmetry [17, 18, 24, 30, 31] plays an important role in the relation between MHV scattering amplitudes and Wilson loops. This symmetry is manifest for the integrands of both MHV scattering amplitudes and Wilson loops, but it is broken by the dimensional regulator. Dual-conformal invariants can be constructed by using the general structure of divergent terms. A particular pattern of spontaneous breaking of the gauge group provides an alternative regularization in which this symmetry is restored through natural transformations of the regulator [32].

It would be interesting to understand whether non-MHV amplitudes also exhibit a similar presentation in terms of Wilson loops. A necessary condition is that they exhibit dual conformal invariance upon extraction of infrared divergences. It is possible to argue that, to all orders in the loop expansion, four-dimensional cuts of any planar scattering amplitude in $\mathcal{N} = 4$ SYM, in particular non-MHV amplitudes, have this symmetry. Hints in this direction also come from the Grassmannian interpretation of leading singularities; in that framework it was shown [33, 34] that leading singularities are dual conformally invariant. Whether this symmetry survives in the complete amplitude, in the presence of the terms not constructible from four-dimensional cuts, is an open question. Here we will see that the parity-even part of the six-point NMHV amplitudes can be expressed in terms of pseudo-conformal integrals, *i.e.* dimensionally regulated integrals that are invariant under dual conformal transformations when continued off-shell.

While the structure of collinear limits of non-MHV amplitudes is somewhat more intricate than those of MHV amplitudes, the former are governed by the same splitting amplitudes as the latter. The iteration relation for MHV amplitudes [9] suggests that one can capture both the infrared-divergent parts of non-MHV amplitudes, as well as the amplitudes' behavior under collinear limits, via an exponentiation ansatz for all the scalar functions that characterize them. This is similar in spirit to the BDS [8] exponentiation ansatz for MHV amplitudes. Such an ansatz is not expected to hold to all orders. Departures from it are characterized by dual conformal invariant functions which have properties analogous to the MHV remainder function [19, 27].

We perform the calculation using the generalized unitarity-based method, employing a variety of four-dimensional and D-dimensional cuts to express the amplitude in terms of six-point two-loop Feynman integrals. The four-dimensional cuts are evaluated in on-shell superspace [35]. This approach automatically takes into account supersymmetry relations between different components of cuts and also offers guidance in organizing the calculation. We find that the (appropriately defined) parity-even part of the six-point amplitude may be expressed as a sum of pseudo-conformal integrals [30], in close analogy with the four-point amplitude through five loops [8, 36–39] and the parity-even part of the five-point amplitude through two loops [40–43]. There are some additional integrals in the one- and two-loop six-point amplitudes, whose pseudo-conformal nature is less clear. Their integrands vanish as $D \to 4$, yet their integrals can be nonvanishing in this limit. We evaluate the integrals using the `AMBRE` [44] and `MB` [45] packages and compute the amplitude numerically at several kinematic points, related in pairs by dual conformal transformations. The infrared singularities of our expression have the structure expected from general considerations [46, 47]. We have tested numerically the dual-conformal properties of the various finite functions that can be constructed from the six-point NMHV amplitude.

The paper is organized as follows. We review the tree-level and one-loop six-point amplitudes in section II, along with their superspace presentation and their conjectured properties. Most importantly, we identify a canonical separation of the six-point NMHV amplitude into parity-even and parity-odd components. We expect this separation to extend to all orders in perturbation theory. In section III, we discuss the expected structure of the six-point NMHV amplitude to all loop orders, based on our calculation using generalized unitarity. We introduce certain finite functions that characterize the amplitude and are expected to be invariant under dual conformal transformations. In section IV, we describe some of the details of our calculation. We use a superspace version of the generalized unitarity method. We discuss some of the subtle points, and give details on the calculation of two important cuts. In section V, we present an integral representation of the even part of the two-loop six-point NMHV amplitude. For completeness we also list the even part of the two-loop six-point MHV amplitude in our notation. We proceed in section VI to analyze our analytic and numerical results for the amplitude, and to test the dual conformal-symmetry properties of the various functions that have been conjectured to be invariant under dual conformal transformations. We give our conclusions and a selection of open problems in section VII.

Note added: As the writing of this paper was being completed we received ref. [93] in which an alternative presentation of the six-point NMHV amplitude was proposed as a consequence of a generalization of the Grassmannian duality for leading singularities to the full amplitude. The result also contains a proposal for the parity-odd part of the amplitude. Unlike our result, it is expressed in terms of a basis of chiral, tensor integrals written in momentum-twistor space. Such integrals were used in [94] to simplify the presentation of the two-loop six-point MHV of [19].

II. REVIEW

The n-point L-loop planar (leading-color) contributions to scattering amplitudes of an $SU(N_c)$ gauge theory with fields in the adjoint representation may be written as[1]

$$\boldsymbol{A}_n^{(L)} = a^L \sum_{\rho\in S_n/\mathbb{Z}_n} \mathrm{Tr}[T^{a_{\rho(1)}} \dots T^{a_{\rho(n)}}] A_n^{(L)}(k_{\rho(1)}, \varepsilon_{\rho(1)}; \dots ; k_{\rho(n)}, \varepsilon_{\rho(n)}) \,, \tag{2.1}$$

where we follow the normalization conventions[2] of ref. [19] (which differ from those used in refs. [2, 24]). The loop expansion parameter a is,

$$a = (4\pi e^{-\gamma})^{-\epsilon} \frac{\lambda}{8\pi^2} = (4\pi e^{-\gamma})^{-\epsilon} \frac{g_{\rm YM}^2 N_c}{8\pi^2} \,. \tag{2.2}$$

Here λ is the 't Hooft coupling constant and γ is the Euler constant, $\gamma = -\Gamma'(1)$. The sum runs over all the noncyclic permutations of the external legs, each of which carries momentum k_i and a polarization vector ε_i.

Choosing a specific helicity and flavor configuration for the external legs reduces $A_n^{(L)}(k_{\rho(1)}, \varepsilon_{\rho(1)}; \dots ; k_{\rho(n)}, \varepsilon_{\rho(n)})$ to a color-ordered partial amplitude. Every partial amplitude can be decomposed into a sum of terms, each of which is a product of a function ensuring the correct transformation properties of the amplitude under Lorentz transformations (henceforth called "spin factor") and a (pseudo-)scalar function which may be written as a sum of L-loop Feynman integrals (the "loop factor"). The spin factor is a rational function of the momentum spinors λ_i and $\tilde\lambda_i$ associated to the external legs; the parity-even parts of the loop factor are functions of external Lorentz invariants alone, while the parity-odd parts also depend on Levi-Civita contractions of the external momenta. One could of course choose to re-express the Levi-Civita contractions in terms of spinor variables.

MHV amplitudes have two negative-helicity, and any number of positive-helicity, external legs. These amplitudes in MSYM have the simplest structure of all amplitudes: they have a single spin factor, which is equal to the tree-level scattering amplitude. Computing the

[1] We normalize the classical action so that the only coupling constant dependence is an overall factor of g_{YM}^{-2}.

[2] This definition of the loop expansion parameter extracts the complete dependence on the Euler constant from the momentum integrals.

L-loop MHV amplitude thus amounts to finding the ratio

$$M_n^{(L)} \equiv \frac{A_n^{(L),\mathrm{MHV}}}{A_n^{(0),\mathrm{MHV}}} \,. \tag{2.3}$$

CPT implies similar properties for the $\overline{\mathrm{MHV}}$ amplitudes; they also contain a single spin factor which is the tree-level amplitude and their scalar and pseudo-scalar factors are obtained from corresponding MHV amplitude by a parity transformation. (For alternative presentations of the parity-odd terms in M_n in terms of spinor variables, see refs. [48, 49].)

All-gluon NMHV amplitudes have three external legs of negative helicity, and any number of positive helicity. They are the next-simplest amplitudes after the MHV ones. The five-point NMHV amplitudes are $\overline{\mathrm{MHV}}$; the simplest distinct ones appear for six external legs. These have three independent helicity configurations. In contrast to the MHV amplitudes, NMHV amplitudes contain several distinct spin factors; their forms depend on the helicity configuration of the external legs. As a consequence of relations between spin factors, there are many possible presentations of the tree-level amplitudes. We can single out a canonical form by constructing the corresponding one-loop amplitude and taking the form that appears as the coefficient of the double pole in the dimensional-regularization parameter ϵ. This relation [50–53] is guaranteed by the general theorems governing the factorization of soft and collinear divergences. We will focus here on the the six-point amplitude.

A. The Six-Point Gluon Scattering Amplitude at One Loop

All six-gluon NMHV amplitudes may be obtained by applying cyclic permutations and reflections to the three independent helicity configurations $(+++---)$, $(++-+--)$ and $(+-+-+-)$. The one-loop amplitudes for these configurations were first obtained in ref. [2] through $\mathcal{O}(\epsilon^0)$ (see also ref. [22]). They can be expressed in terms of three different spin factors. The spin factors for the 'split-helicity' configuration $(+++---)$ are,

$$B_1 \;=\; i\frac{s_{123}^3}{\langle 12\rangle\,\langle 23\rangle\,\langle 1(2+3)4]\langle 3(1+2)6]\,[45]\,[56]}\,, \tag{2.4}$$

$$B_2 \;=\; i\frac{\langle 4(2+3)1]^3}{\langle 23\rangle\,\langle 34\rangle\,\langle 2(3+4)5]\,[56]\,[61]\,s_{234}} + i\frac{\langle 56\rangle^3\,[23]^3}{\langle 61\rangle\,\langle 1(2+3)4]\,\langle 5(3+4)2]\,[34]\,s_{234}}\,, \tag{2.5}$$

$$B_3 \;=\; i\frac{\langle 6(1+2)3]^3}{\langle 61\rangle\,\langle 12\rangle\,\langle 2(1+6)5]\,[34]\,[45]\,s_{345}} + i\frac{\langle 45\rangle^3\,[12]^3}{\langle 34\rangle\,\langle 3(1+2)6]\,\langle 5(1+6)2]\,[61]\,s_{345}}\,; \tag{2.6}$$

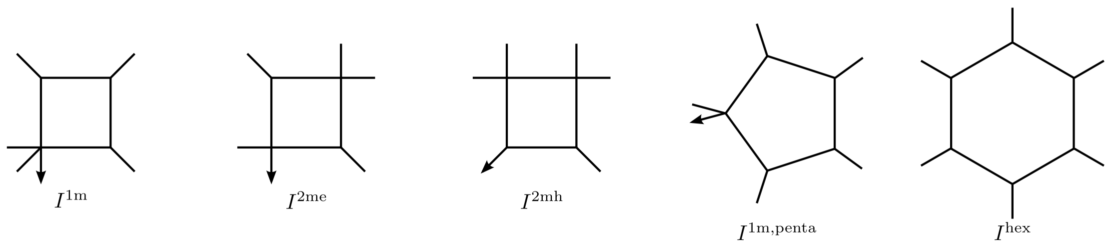

FIG. 1: The integrals contributing to the six-point one-loop MHV and NMHV amplitudes. An arrow marks the leg with momentum k_1; the remaining momenta follow clockwise. The one-mass box $I^{1\mathrm{m}}$ and two-mass easy $I^{2\mathrm{me}}$ integrals contribute to the MHV amplitude and the one-mass box $I^{1\mathrm{m}}$ and two-mass hard $I^{2\mathrm{mh}}$ integrals contribute to the NMHV amplitude. The one-mass pentagon $I^{1\mathrm{m,penta}}$ and the hexagon I^{hex} have numerator factors of μ^2 (the square of the (-2ϵ)-dimensional components of the loop momentum), and hence are finite. They contribute to both the MHV and NMHV amplitudes only at $\mathcal{O}(\epsilon)$ and higher (I^{hex} contributes to the even parts while $I^{1\mathrm{m,penta}}$ contributes to the odd parts).

we refer the reader to the original paper [2] for the spin factors of the other independent helicity configurations. In all cases, the spin factors are uniquely determined by cuts in three-particle invariants.

The six-point one-loop NMHV amplitude for the $(+++---)$ helicity configuration is given by,

$$A_6^{(1),\mathrm{NMHV}}(+++---) = \frac{1}{2}\left(B_1 W_1^{(1)} + B_2 W_2^{(1)} + B_3 W_3^{(1)}\right) + \mathcal{O}(\epsilon)\,, \tag{2.7}$$

where,

$$W_1^{(1)} = -\frac{1}{2}\sum_{\sigma\in\mathcal{S}_1}\left(\frac{1}{2}s_{45}s_{56}I^{1\mathrm{m}}(\sigma) + \frac{1}{2}s_{61}s_{123}I^{2\mathrm{mh}}(\sigma)\right) + \mathcal{O}(\epsilon)\,; \tag{2.8}$$

and the sum runs over the permutations,

$$\mathcal{S}_1 = \{(123456), (321654), (456123), (654321)\}\,. \tag{2.9}$$

All permutations in $\mathcal{S}_1$ leave the spin factor B_1 invariant. The integrals in eq. (2.8) are shown in fig. 1. The factors of $\frac{1}{2}$ in the summand in eq. (2.8) are symmetry factors needed to compensate for double counting in the summation over $\mathcal{S}_1$. The expressions (2.7) and (2.8) hold only through order $\mathcal{O}(\epsilon^0)$. At $\mathcal{O}(\epsilon)$ eq. (2.8) receives contributions from additional integrals while equation (2.7) receives contributions from additional spin factors. The terms of higher order in ϵ have been computed only recently [25].

The other two scalar functions, $W_2^{(1)}$ and $W_3^{(1)}$, may be obtained from eq. (2.8) by replacing the set of permutations $\mathcal{S}_1$ by the sets $\mathcal{S}_2$ and $\mathcal{S}_3$, respectively, where

$$\mathcal{S}_2 = \{(234561), (432165), (561234), (165432)\}\,, \tag{2.10}$$
$$\mathcal{S}_3 = \{(345612), (543216), (612345), (216543)\}\,. \tag{2.11}$$

The elements of each of the permutations sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ leave invariant the spin factors B_1, B_2 and B_3, respectively. The union of these three permutations sets, $\mathcal{S}_0 = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, is the set of all cyclic permutations and their reflections; the MHV amplitude can be expressed as a sum over this larger set.

The one-loop scattering amplitudes for the other two independent helicity configurations have a structure similar to eq. (2.7); the scalar functions $W_i^{(1)}$ are unchanged while the spin factors B_1, B_2 and B_3 are replaced [2] by new spin factors D_1, D_2 and D_3 for the $(++-+--)$ helicity configuration, and by G_1, G_2 and G_3 for the $(+-+-+-)$ configuration[3]:

$$A_6^{(1),\mathrm{NMHV}}(++-+--) = \frac{1}{2}\left(D_1 W_1^{(1)} + D_2 W_2^{(1)} + D_3 W_3^{(1)}\right)\,, \tag{2.12}$$
$$A_6^{(1),\mathrm{NMHV}}(+-+-+-) = \frac{1}{2}\left(G_1 W_1^{(1)} + G_2 W_2^{(1)} + G_3 W_3^{(1)}\right)\,. \tag{2.13}$$

Infrared consistency then implies that the tree-level amplitudes for the corresponding helicity configurations are [2],

$$A_6^{(0),\mathrm{NMHV}}(+++---) = \frac{1}{2}\left(B_1 + B_2 + B_3\right)\,, \tag{2.14}$$
$$A_6^{(0),\mathrm{NMHV}}(++-+--) = \frac{1}{2}\left(D_1 + D_2 + D_3\right)\,, \tag{2.15}$$
$$A_6^{(0),\mathrm{NMHV}}(+-+-+-) = \frac{1}{2}\left(G_1 + G_2 + G_3\right)\,. \tag{2.16}$$

The classic expression for these amplitudes was derived in ref. [55]. In later sections we will see that the structure present in eqs. (2.7), (2.12) and (2.13) — in which only the spin factors change between various helicity configurations of the external lines — persists at higher loops as well.

[3] The notable difference between B_i and the non-split helicity spin factors D_i, G_i is that, while the former are rational functions of products of adjacent spinors, the latter also contain products of non-adjacent spinors. This obscures their transformation properties under the dual conformal symmetry [54], which become manifest only when the amplitudes are combined into a superamplitude [24].

B. Superspace and Superamplitudes

On-shell superspace provides a very convenient way of organizing amplitudes in $\mathcal{N} = 4$ SYM theory and making manifest supersymmetry relations between them. The bosonic part of this superspace is parametrized by the usual bosonic spinor variables $\lambda_i, \tilde{\lambda}_i$, related to the external momenta k_i by $k_{i\alpha\dot{\alpha}} = \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}$. The fermionic part is parametrized by Grassmann coordinates η_i^A, where $A = 1, \cdots, 4$ is an R-symmetry index. The on-shell fields of the $\mathcal{N} = 4$ theory are assembled into a superfield,

$$\Phi(\eta) = g_- + \eta^A\psi_A + \frac{1}{2!}\eta^A\eta^B\phi_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\psi^D + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}g_+ \quad . \tag{2.17}$$

A superamplitude is a generating function for the scattering amplitudes of component fields, which may be identified as the coefficients of the appropriate combinations of η_i variables.

The component amplitudes may be extracted by multiplying the superamplitude with the appropriate superfield and integrating over all Grassmann parameters:

$$A_n(k_1, h_1; \ldots; k_n, h_n) = \int \prod_{i=1}^{n} d^4\eta_i \prod \Phi_{h_i}(\eta)\ \mathcal{A}_n(k_1, \eta_1, \ldots, k_n, \eta_n)\,. \tag{2.18}$$

The superfields $\Phi_{h_i}(\eta)$ have a single nonvanishing term corresponding to the field with helicity h_i.

As an example, the n-point NMHV gluon scattering amplitudes appear inside the superamplitude as follows:

$$\begin{aligned}\mathcal{A}_n(k_1, \eta_1, \ldots, k_n, \eta_n) = \cdots \quad & +\eta_1^4\eta_2^4\eta_3^4 A_n(-,-,-,+,+,\ldots,+) \\ & +\eta_1^4\eta_2^4\eta_4^4 A_n(-,-,+,-,+,\ldots,+) + \cdots\,,\end{aligned} \tag{2.19}$$

where η^4 is the $SU(4)$-invariant expression $\frac{1}{4!}\epsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D$. In extracting these component amplitudes, the η variables corresponding to the positive-helicity gluons are supplied by the superfields (2.17) while those for the negative-helicity ones appear explicitly in the superamplitude. Because the half of the supersymmetries manifest in this on-shell superspace can be preserved at all stages of scattering amplitude calculations, eq. (2.19) holds to all orders in perturbation theory.

The dual superspace in which the superfield is given by,

$$\widetilde{\Phi}(\tilde{\eta}) = g_+ + \tilde{\eta}_A\psi^A + \frac{1}{2!}\tilde{\eta}_A\tilde{\eta}_B\phi^{AB} + \frac{1}{3!}\tilde{\eta}_A\tilde{\eta}_B\tilde{\eta}_C\epsilon^{ABCD}\psi_D + \frac{1}{4!}\tilde{\eta}_A\tilde{\eta}_B\tilde{\eta}_C\tilde{\eta}_D\epsilon^{ABCD}g_-\,, \tag{2.20}$$

has also been used, see for example refs. [5, 6, 24, 56]. While the expression for the superamplitude is unchanged, component amplitudes are extracted by differentiating with respect to selected superspace coordinates, eight for MHV amplitudes, twelve for NMHV ones, etc.:

$$A_n(k_1,h_1;\ldots;k_n,h_n) = \prod \widetilde{\Phi}_{h_i}\left(\frac{\partial}{\partial\eta}\right)\mathcal{A}_n(k_1,\eta_1,\ldots,k_n,\eta_n)\,. \tag{2.21}$$

For pure-gluon amplitudes, the differentiation is solely with respect to the Grassmann coordinates of the negative-helicity gluons. The structure of superfields is, however, unimportant for the computation of superamplitudes.

In general, n-point tree-level scattering amplitudes can be written as follows [24],

$$\mathcal{A}_n^{(0)} = \frac{i\,\delta^{(4)}(\sum_{i=1}^n \lambda_i\tilde{\lambda}_i)\delta^{(8)}(\sum_{i=1}^n \lambda_i\eta_i^A)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\sum_{k=0}^{n-4}\mathcal{P}_n^k, \tag{2.22}$$

where $\mathcal{P}_n^k$ are polynomials in the Grassmann variables η_i of degree $4k$. Invariance under R-symmetry implies that $\mathcal{P}_n^k$ are invariant under $SU(4)$ rotations of the Grassmann variables η^A. The lowest-order term in the η expansion has Grassmann weight 8, while the highest-order term has Grassmann weight $4n-8$. CPT conjugation exchanges weight $4k+8$ with weight $4n-4k-8$. The $k=0$ term in eq. (2.22) has $\mathcal{P}_n^0=1$ and contains all the MHV amplitudes. The NMHV amplitudes are contained in the $k=1$ term. Similarly to equation (2.19) and for the same reason, equation (2.22) is expected to hold to all orders in perturbation theory. Higher-order corrections can alter only the coefficients of the polynomials $\mathcal{P}_n^k$, *i.e.* the component amplitudes. Throughout the paper, four-fold bosonic momentum-conserving delta functions will appear, products of delta functions over the four components whose indices (a vector index μ or a pair $(\alpha,\dot\alpha)$ of spinorial indices) we suppress. A variety of four-fold Grassmann delta functions, products of delta functions taken over the $SU(4)$ index A, and eight-fold Grassmann delta functions, products of delta functions taken over a pair of a spinor index α and an $SU(4)$ index A, will also appear. In these delta functions, we will suppress the (bosonic) spinor index, but display the (Grassmann) $SU(4)$ index explicitly.

The tree-level MHV superamplitude was written down long ago by Nair [35],

$$\mathcal{A}_n^{(0),\mathrm{MHV}} = \frac{i\,\delta^{(4)}\left(\sum_{i=1}^n \lambda_i\tilde{\lambda}_i\right)\delta^{(8)}\left(\sum_{i=1}^n \lambda_i\eta_i^A\right)}{\langle 12\rangle\cdots\langle n1\rangle}\,. \tag{2.23}$$

The $\overline{\mathrm{MHV}}$ amplitude has an equally simple form in the conjugate superspace, whose coordinates are the conjugate spinors $\tilde\lambda_i$ and the Fourier-conjugate $\tilde\eta$ of the Grassmann variables

η. Fourier-transforming to the same superspace as the MHV amplitude implies [5] that the $\overline{\text{MHV}}$ superamplitude is

$$\mathcal{A}_n^{(0),\overline{\text{MHV}}} = i\frac{\delta^{(4)}(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i)}{[12]\cdots[n1]} \int d^8\omega \prod_{i=1}^n \delta^{(4)}(\eta_i^A - \tilde{\lambda}_i^{\dot{\alpha}} \omega_{\dot{\alpha}}^A). \tag{2.24}$$

Manifestly supersymmetric expressions for non-MHV amplitudes could be obtained [57] through a supersymmetric generalization of the MHV vertex expansion [58–60]. The expressions obtained this way do not *a priori* exhibit any special properties. DHKS presented [24] a special form for $\mathcal{P}_6^1$, and showed that it enjoys an extended symmetry, so-called dual superconformal symmetry. Explicit expressions for all the $\mathcal{P}_n^k$ polynomials were given by Drummond and Henn [54], using a supersymmetric form [61, 62] of the Britto, Cachazo, Feng, and Witten (BCFW) on-shell recursion relations [7].

On-shell superspace encodes the relations between amplitudes that are implied by supersymmetry, but does not identify the basic, irreducible components from which all others can be obtained. Identifying such basic amplitudes, from which all others can be obtained via supersymmetry transformations (along with the required sequence of transformations) is in general a difficult problem. Not all corrections to the coefficients in the polynomials $\mathcal{P}_n^k$ are independent; as these coefficients are nothing but component amplitudes, they are related by supersymmetry Ward identities. Elvang, Freedman, and Kiermaier have provided a solution [63] to this class of questions.

Apart from clarifying the structure of tree-level amplitudes, knowledge of tree-level superamplitudes allows us to perform manifestly-supercovariant higher-loop calculations using generalized unitarity. The one-loop calculation of ref. [24] generalizes the result of ref. [2] for the NMHV six-gluon amplitudes to a manifestly supersymmetric expression encompassing all possible external states. In refs. [6, 64] superamplitudes were used to evaluate the sum over all the particles crossing generalized unitarity cuts for n-point MHV amplitudes at any loop order. In the section IV we will describe in detail the steps needed for evaluating two- and higher-loop superamplitudes for any number of external legs and Grassmann weight, and elucidate the subtleties that arise in such evaluations.

C. The six-point NMHV superamplitude

As our focus in later sections will be on the two- and higher-loop six-point NMHV (super)amplitude, we first review and extend the supersymmetric results of ref. [24] for the tree-level and one-loop expressions for this amplitude.

As is true for the component amplitudes, relations between rational functions of bosonic spinor products and Grassmann variables allow the tree-level superamplitude to be expressed in several equivalent forms. We may identify a canonical form, which will also be useful for higher-loop calculations, by starting from the ϵ-pole terms in the one-loop superamplitude. This superamplitude is given by the supersymmetrization [24] of eqs. (2.7), (2.12) and (2.13),

$$\mathcal{A}_6^{(1),\mathrm{NMHV}} = \frac{a}{2}\mathcal{A}_6^{(0),\mathrm{MHV}}\big((R_{413}+R_{146})W_1^{(1)} + (R_{524}+R_{251})W_2^{(1)} + (R_{635}+R_{362})W_3^{(1)} + \mathcal{O}(\epsilon)\big)\,, \tag{2.25}$$

where $A_6^{(0),\mathrm{MHV}}$ is the tree-level MHV superamplitude, the loop expansion parameter a is defined in eq. (2.2) and the products $A^{(0),\mathrm{MHV}}R_{j,j+3,j+5}$ with $j=1,\ldots,6$ (all indices understood mod 6) are,

$$\begin{aligned}\mathcal{A}^{(0),\mathrm{MHV}}R_{j,j+3,j+5} = &\\ &\frac{\delta^{(8)}\big(\sum\lambda_i\eta_i^A\big)}{\langle j\,(j{+}1)\rangle\,\langle (j{+}1)\,(j{+}2)\rangle\,[(j{+}3)\,(j{+}4)]\,[(j{+}4)\,(j{+}5)]}\\ &\times\frac{\delta^{(4)}\big(\eta_{j+3}^A\,[(j{+}4)\,(j{+}5)] + \eta_{j+4}^A\,[(j{+}5)\,(j{+}3)] + \eta_{j+5}^A\,[(j{+}3)\,(j{+}4)]\big)}{\langle j|\,K_{j+1,j+2}\,|(j{+}3)]\,\langle (j{+}2)|\,K_{j+3,j+4}\,|(j{+}5)]\,s_{j,j+1,j+2}}\,.\end{aligned} \tag{2.26}$$

This product is covariant under dual inversion, with the same weight as the tree-level MHV superamplitude. For generic momentum configurations (that is, away from soft or collinear configurations), the superfunctions $R_{j,j+3,j+5}$ are thus invariant under dual superconformal transformations.

The functions $W_i^{(1)}$ have identical poles in the dimensional regularization parameter ϵ; this reflects the universality of infrared divergences. A canonical expression for the tree-level six-point NMHV superamplitude is then simply,

$$\mathcal{A}_6^{(0),\mathrm{NMHV}} = \frac{1}{2}\mathcal{A}_6^{(0),\mathrm{MHV}}\big(R_{146}+R_{251}+R_{362}+R_{413}+R_{524}+R_{635}\big)\,. \tag{2.27}$$

The R invariants are not all independent; in the presence of the super-momentum conservation constraint they obey the linear six-term relation,

$$\mathcal{A}_6^{(0),\mathrm{MHV}}\big(R_{146}-R_{251}+R_{362}-R_{413}+R_{524}-R_{635}\big) = 0\quad. \tag{2.28}$$

This relation, akin to relations derived from the Grassmannian formulation of tree-level amplitudes [65], leads to two apparently different presentations of the six-point NMHV superamplitude:

$$\mathcal{A}_6^{(0),\text{NMHV}} = \mathcal{A}_6^{(0),\text{MHV}}\big(R_{146} + R_{362} + R_{524}\big) = \mathcal{A}_6^{(0),\text{MHV}}\big(R_{251} + R_{413} + R_{635}\big) \ . \tag{2.29}$$

A proof of eq. (2.28) amounts to showing that the first expression in eq. (2.29) can be derived from the BCFW recursion relation [7] with a supersymmetric shift [62] while the second expression follows from the cyclic symmetry of the superamplitude. At higher loops the identity (2.28) is crucial for ensuring the consistency of unitarity cuts. It should also play a role in reconstructing scattering amplitudes from their leading singularities [65].

The six-point NMHV amplitude is special among NMHV amplitudes as it exhibits a discrete invariance related to parity transformations. We will discuss this symmetry and its consequences in the following. A similar discussion generalizes to the $2n$-point N^{n-2}MHV amplitudes. As mentioned previously, (CPT) conjugation of superamplitudes amounts to Fourier-transforming the Grassmann coordinates (reversing the helicities of all component fields) and exchanging spinors and conjugate spinors, $\lambda_i \leftrightarrow \tilde\lambda_i$. It is easy to check that this sequence of transformations maps the products $\mathcal{A}_6^{(0),\text{MHV}} R_{ijk}$ into themselves up to a cyclic permutation by three units:

$$\mathcal{A}_6^{(0),\text{MHV}} R_{146} \rightarrow \mathcal{A}_6^{(0),\text{MHV}} R_{413}, \quad \text{etc.} \tag{2.30}$$

This is the supersymmetric generalization of an obvious invariance of the six-gluon NMHV scattering amplitudes. Invariance of the six-point superamplitude under this transformation in turn requires that the functions $W_i^{(1)}$ in equation (2.25) be invariant under conjugation.

Apart from terms proportional to the sum of R invariants, the $\mathcal{O}(\epsilon)$ part of the one-loop amplitude also contain terms which are proportional to differences of R invariants. They have been computed directly in a one-loop calculation [25], and their existence may also be inferred from the two-loop calculation we will describe in later sections. As such differences are odd under conjugation, they must be accompanied by parity-odd (pseudo-scalar) functions $\widetilde{W}_i^{(1)}$.

D. Dual Conformal Invariance and the Six-Point Superamplitude

As mentioned above, DHKS showed [24] that tree-level amplitudes are covariant, with weight (-1), under dual superconformal symmetry. This property extends to the rational functions $\mathcal{A}_6^{(0),\mathrm{MHV}} R_{ijk}$. To what extent does the symmetry extend to the full one-loop amplitude?

The dual conformal and dual superconformal symmetries are only defined in four dimensions. One possible extension is the notion of pseudo-conformality: were we to regulate the integral functions $W_i^{(1)}$ by off-shell continuation, they would become dual conformal invariant, as they are sums of box integrals with the appropriate prefactors. Additional evidence towards a kind of dual conformal invariance comes from the observation [33, 34] that leading singularities are dual conformal invariant.

We can do better than this. DHKS noticed [24] that the ratio[4] of the six-point NMHV to MHV superamplitudes, each taken through one-loop order, is invariant under dual conformal transformations. That is, the ratio is invariant under transformations that preserve the cross-ratios

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \qquad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \qquad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}\,. \tag{2.31}$$

The ratio of superamplitudes is a natural quantity, as it is infrared finite.

In gauge theories, the structure of infrared divergences in dimensional regularization is independent of the helicity configuration [46, 47]. At one loop, for example, the pole terms are proportional to the tree amplitudes. This makes the ratio of any helicity amplitude to the MHV amplitude infrared finite.

The finiteness of such ratios makes it possible to take the four-dimensional limit, and to inquire about their properties under dual (super)conformal transformations. Of course, finiteness does not guarantee dual conformal invariance. Indeed, the relation between these two properties has been investigated in ref. [5] with the conclusion that, in dimensional regularization, there exist infrared-finite combinations of pseudo-conformal integrals which are not dual conformal invariant.

Explicit calculations show that such subtleties do not arise here and, through one-loop

[4] This ratio is sensible because in chiral on-shell superspace any superamplitude is proportional to the supermomentum conservation constraint $\delta^{(8)}(\sum_{i=1}^{n}\lambda_i\eta_i^A)$, which contains the entire Grassmann-dependent factor in the MHV amplitude.

order, the six-point NMHV superamplitude has the factorized form [24]

$$\mathcal{A}_6^{\rm NMHV} = \frac{1}{2}\mathcal{A}_6^{\rm MHV}\left[R_{146}\left(1+aC_{146}^{(1)}\right) + \text{cyclic} + \mathcal{O}(a^2)\right], \tag{2.32}$$

with the functions $C_{i,i+3,i+5}$ manifestly expressed in terms of the dual conformal ratios (2.31):

$$C_{146}^{(1)} = -\ln u_1 \ln u_2 + \frac{1}{2}\sum_{k=1}^{3}\left(\ln u_k \ln u_{k+1} + \mathrm{Li}_2(1-u_k)\right) - \frac{\pi^2}{6} + \mathcal{O}(\epsilon), \qquad \text{etc.} \tag{2.33}$$

This function differs from $V^{(1)}$ as defined in ref. [24] by $-\pi^2/6$, due to differences in normalization of amplitudes and finite differences between the Wilson loop expression and the one-loop amplitude. It also differs in including $\mathcal{O}(\epsilon)$ terms; its ϵ-independent part agrees with the function $V^{(1)}$ defined in ref. [5].

For completeness we record [1] the integral representation of the one-loop six-point MHV amplitude through $\mathcal{O}(\epsilon^0)$:

$$\begin{aligned}
\mathcal{A}_6^{(1),\rm MHV} &= \mathcal{A}_6^{(0),\rm MHV} M_6^{(1)}, \\
M_6^{(1)} &= -\frac{1}{8}\sum_{\sigma\in\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{S}_3}\left(s_{12}s_{23}I^{\rm 1m}(\sigma) + \frac{1}{2}(s_{234}s_{345} - s_{61}s_{34})I^{\rm 2me}(\sigma)\right).
\end{aligned} \tag{2.34}$$

In writing eq. (2.33) we used the convention that $u_{i+3} = u_i$. The ratio function is thus manifestly dual conformal invariant through one loop. It does not have the full dual superconformal invariance, dual supersymmetry being broken by a holomorphic anomaly [66].

DHKS conjectured [24] that the main features of eq. (2.32) survive higher-loop corrections: that the six-point NMHV superamplitude may be factorized as

$$\mathcal{A}_6^{\rm NMHV} = \frac{1}{2}\mathcal{A}_6^{\rm MHV}\left[R_6^{\rm NMHV} + \mathcal{O}(\epsilon)\right]; \tag{2.35}$$

and that the functions $R_6^{\rm NMHV}$ have no further ϵ dependence, are well-defined in four dimensions and, to all loop orders, are dual conformal invariant. The conjecture does not specify the structure of the $\mathcal{O}(\epsilon)$ terms or of the spin factors that enter the functions $R_6^{\rm NMHV}$ beyond one-loop level. At one-loop, the $\mathcal{O}(\epsilon)$ terms are irrelevant to any 'physical' quantity. However, these terms will contribute nontrivially to both the divergent and finite parts of the $\mathcal{O}(a^2)$ terms in the product on the right-hand side of eq. (2.35). Our calculation will clarify the meaning of these one-loop terms for that part of the amplitude dependent on parity-even combinations of R invariants. We will show that they are determined by the $\mathcal{O}(\epsilon)$ terms in the one-loop NMHV amplitude, which have been calculated recently by Dixon and Schabinger [25].

Before proceeding to describe our calculation, we will discuss in the next section the structure of our result as well as the expected properties of the resummed six-point NMHV amplitude.

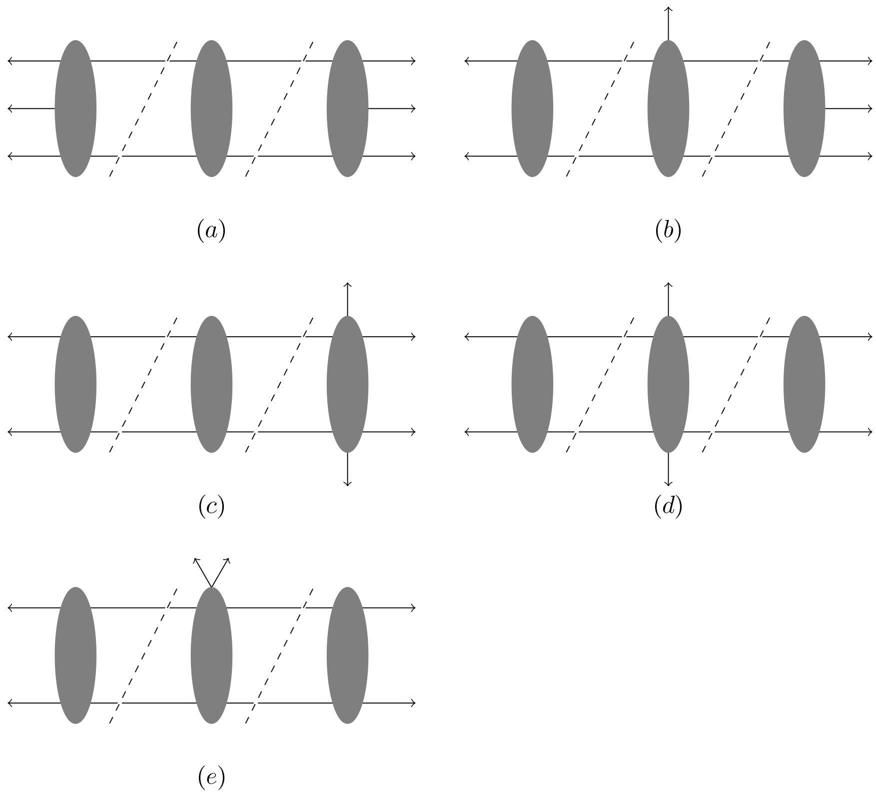

FIG. 2: Generalized cuts required to determine the two-loop NMHV amplitude: (a) the 'double-pentagon' cut (b) the 'turtle' cut (c) the 'hexabox' cut (d) the 'flying-squirrel' cut (e) the 'rabbit-ears' cut. Unlike the MHV calculation, all permutations of the external legs must be considered.

III. STRUCTURE OF THE SIX-POINT NMHV AMPLITUDE

In order to obtain the six-point NMHV amplitude to a given loop order, we must determine all spin factors that appear, and construct the functions of external momenta and coupling multiplying each one of them. In the next section we will show explicitly that, through two-loop order and through $\mathcal{O}(\epsilon^0)$, the R invariants are the only spin factors that appear in the superamplitude. The transformation of the R invariants under conjugation (2.30) then implies that, through two-loop order, the superamplitude can be written as follows,

$$\mathcal{A}_6^{\rm NMHV} = \frac{1}{2}\mathcal{A}_6^{(0),\rm MHV}\Big[(R_{413}+R_{146})W_1(a)+(R_{524}+R_{251})W_2(a)+(R_{635}+R_{362})W_3(a)$$
$$+(R_{413}-R_{146})\widetilde{W}_1(a)+(R_{524}-R_{251})\widetilde{W}_2(a)+(R_{635}-R_{362})\widetilde{W}_3(a)+\mathcal{O}(\epsilon)\Big]\,, \quad (3.1)$$

where the $W_i(a)$ are scalar functions and the $\widetilde{W}_i(a)$ are pseudoscalar functions. We present the calculation of the two-loop six-point NMHV superamplitude, computing explicitly the terms depending on parity-even combinations of R invariants. We will find that the four-dimensional cut-constructible part of the parity-even functions $W_i^{(2)}$ can be expressed as a sum of pseudo-conformal integrals. We will also confirm that, unlike their one-loop counterparts, the pseudoscalar functions $\widetilde{W}_i^{(2)}$ have nonvanishing divergent and finite parts in the ϵ expansion. We will not compute these functions explicitly, but the general infrared structure of gauge theories divergences requires that they have at most simple $(1/\epsilon)$ poles, as both the tree and one-loop amplitudes [through $\mathcal{O}(\epsilon^0)$] are free of such terms. In this section, we describe the expected general structure of the NMHV amplitude, and the structure of its collinear limits.

A. Beyond Two Loops

We expect the pseudo-conformality of the coefficient functions to continue to all loop orders. To see this, consider a four-dimensional generalized unitarity cut that decomposes an L-loop superamplitude into a product of k tree-level superamplitudes

$$\mathcal{A}_n^{(L)}\Big|_{\rm cut} = \prod \mathcal{A}_1 \ldots \mathcal{A}_k\,. \quad (3.2)$$

As mentioned earlier and shown in [24], each superamplitude has weight (-1) under dual inversion. Because a cut propagator simply identifies the Grassmann variables and momenta of the legs that are sewn, it has weight $(+2)$ under this transformation. Thus, the product above together with the cut propagators has vanishing weight for the cut legs and weight (-1) for the external legs. This implies that these cuts can all be saturated by cuts of pseudo-conformal integrals.

Unlike the scalar functions $W_i^{(2)}$, the pseudo-scalar functions $\widetilde{W}_i^{(2)}$ are not uniquely defined. Indeed, the identity (2.28) implies that it is possible to uniformly add an arbitrary

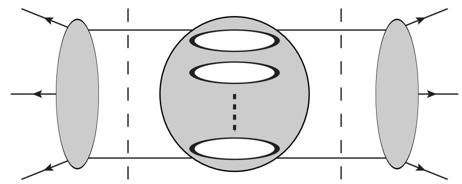

FIG. 3: A cut of an L-loop six-point amplitude isolating an $(L-2)$-loop four-point amplitude with no external legs. The cut is proportional to a lone R invariant.

pseudoscalar function to the $\widetilde{W}_i^{(2)}$ without affecting the value of the amplitude. In particular, we could set any one of these functions to zero. This ambiguity can be partly eliminated by requiring that the superamplitude be manifestly invariant under cyclic permutations of external legs:

$$\widetilde{W}_i^{(2)} = \mathbb{P}\widetilde{W}_{i-1}^{(2)}, \tag{3.3}$$

where $\mathbb{P}$ is the operation of permutation to the right by one unit:

$$\mathbb{P}f[k_1, k_2, k_3, k_4, k_5, k_6] = f[k_2, k_3, k_4, k_5, k_6, k_1]\,. \tag{3.4}$$

The corresponding equation for the $W_i^{(2)}$ functions,

$$W_i^{(2)} = \mathbb{P}W_{i-1}^{(2)}\,, \tag{3.5}$$

follows from the symmetry of the superamplitude.

Requiring cyclic symmetry does not completely fix the ambiguity in the pseudoscalar functions $\widetilde{W}_i^{(2)}$, as parity-odd cyclicly symmetric functions do exist. An example of such a function is,

$$f = \epsilon_{1234}f_1 + \epsilon_{2345}f_2 + \epsilon_{3456}f_3 + \epsilon_{4561}f_4 + \epsilon_{5612}f_5 + \epsilon_{6123}f_6, \tag{3.6}$$

where f_i are parity-even functions of external momenta k_j related by the action of the shift operator $f_i = \mathbb{P}f_{i-1}$ and $\epsilon_{ijmn} = \epsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_m^\rho k_n^\sigma$.

The generalized-unitarity argument above does not reveal the complete set of spin factors that appear at higher loops in the six-point NMHV amplitude. The structure of leading singularities suggests [33] that new structures beyond the R invariants of one and two loops will be generated at three loops for amplitudes with ten or more external legs, but that no new structures will appear beyond that order. It also suggests that no new invariants should appear beyond two loops for amplitudes with seven or more, but fewer than ten, external legs; and that no new invariants will appear beyond one loop for the six-point amplitude.

We can, however, argue that the spin factors present at one and two loops will appear to all loop orders. As we will see in the next section, all tree-level R invariants appear in double two-particle cuts in a channel carrying a three-particle invariant. Such a double cut, shown in fig. 2(a), isolates a tree-level four-point amplitude with no external legs attached to it. An all-loop generalization of this cut is shown in fig. 3. The well-known property of the four-point amplitude at any loop order, that its spin factor is the same as at tree level, implies that this cut will generate exactly the same spin factors as at two loops. This argument extends trivially to all higher-loop contributions to the six-point amplitude that have double two-particle cuts and isolate four- and five-point amplitudes inside them. It can be thought of as a direct superspace generalization of the box substitution rule [39]. At three loops and beyond, however, it is easy to construct cuts that are outside this class. Such cut-based arguments thus cannot rule out spin factors beyond the R invariants seen to date.

Apart from terms containing such new spin factors which may start at three loops, the organization of the six-point NMHV amplitude in (3.1) holds to all orders in perturbation theory. It is therefore interesting to discuss the properties of the parity-even and parity-odd functions,

$$W_i(a) = 1 + aW_i^{(1)} + a^2 W_i^{(2)} + \ldots \quad \text{and} \quad \widetilde{W}_i(a) = 1 + a\widetilde{W}_i^{(1)} + a^2\widetilde{W}_i^{(2)} + \ldots \tag{3.7}$$

in equation (3.1) and of the finite functions

$$C_{i,i+3,i+5}(a) = 1 + aC^{(1)}_{i,i+3,i+5} + a^2 C^{(2)}_{i,i+3,i+5} + \ldots \tag{3.8}$$

that appear in the ratio $A_6^{\text{NMHV}}/A_6^{\text{MHV}}$. The functions $C_{i,i+3,i+5}(a)$ will not have definite parity. This is due to their relation to linear combinations of functions with differing parity properties, to wit $(W_i(a) \pm \widetilde{W}_i(a))$, as well as to division by the MHV amplitude which does not have definite parity properties. For later convenience let us introduce the combinations $C_i(a)$ and $\widetilde{C}_i(a)$,

$$\frac{1}{2}(C_{i+3,i,i+2} + C_{i,i+3,i+5}) \equiv C_i(a) \ = \ \frac{W_i(a)}{M_6(a)}\,, \tag{3.9}$$

$$\frac{1}{2}(C_{i+3,i,i+2} - C_{i,i+3,i+5}) \equiv \widetilde{C}_i(a) \ = \ \frac{\widetilde{W}_i(a)}{M_6(a)}\,. \tag{3.10}$$

The properties of $M_6(a)$ together with the universality of infrared divergences implies, that through two loops, these functions have definite parity up to corrections that vanish in the $\epsilon \to 0$ limit.[5]

B. Collinear Limits

The scalar and pseudo-scalar functions $W_i(a)$ and $\widetilde{W}_i(a)$ have specific properties dictated by the behavior of the amplitude in collinear limits [67]:

$$A_6^{(L)}(\ldots, i^{\lambda_i}, (i+1)^{\lambda_{i+1}}, \ldots) \to \sum_{\lambda=\pm} \sum_{s=0}^{L} \text{Split}_{-\lambda}^{(s)}(z; i^{\lambda_i}, (i+1)^{\lambda_{i+1}}) A_5^{(L-s)}(\ldots, k^{\lambda}, \ldots)\,, \tag{3.11}$$

where $k = k_i + k_{i+1}$ and z is the collinear momentum fraction, $k_i \simeq zk$. We can rewrite this equation for the all-orders amplitude,

$$A_6(\ldots, i^{\lambda_i}, (i+1)^{\lambda_{i+1}}, \ldots) \to \sum_{\lambda=\pm} \text{Split}_{-\lambda}(z; i^{\lambda_i}, (i+1)^{\lambda_{i+1}}) A_5(\ldots, k^{\lambda}, \ldots)\,. \tag{3.12}$$

The properties of $C_i(a)$ and $\widetilde{C}_i(a)$ are more intricate as they involve additional contributions from $M_6(a)$.

We will find it easiest to discuss the collinear limits in components. Because W_i and $\widetilde{W}_i$ do not depend on the precise helicity assignment to the external legs, it suffices to discuss the split-helicity configuration. In the three independent collinear limits, the spin factors B_i in eqs. (2.4), (2.5) and (2.6) behave as follows:

$$\begin{aligned}
1 \parallel 2: \quad & B_{1,3} \to \text{Split}_{-}^{\text{tree}}(1^+, 2^+, k^-) A_5^{(0)}(k^+, 3^+, 4^-, 5^-, 6^-)\,, \quad B_2 \to 0 \\
5 \parallel 6: \quad & B_{1,2} \to \text{Split}_{+}^{\text{tree}}(5^-, 6^-, k^+) A_5^{(0)}(1^+, 2^+, 3^+, 4^-, k^-)\,, \quad B_3 \to 0 \\
3 \parallel 4: \quad & B_{2,3} \to \text{Split}_{+}^{\text{tree}}(3^+, 4^-, k^+) A_5^{(0)}(1^+, 2^+, k^-, 5^-, 6^-) \\
& \qquad + \text{Split}_{-}^{\text{tree}}(3^+, 4^-, k^-) A_5^{(0)}(1^+, 2^+, k^+, 5^-, 6^-)\,, \quad B_1 \to 0
\end{aligned} \tag{3.13}$$

The collinear limits of the parity-odd coefficients $\widetilde{B}_i$, which are contained in the parity-odd combinations of R invariants, are similar except that the relative sign between the two terms in the $3 \parallel 4$ limit is reversed. Combining these limits with the overall behavior of the amplitude (3.12), we find,

$$\begin{aligned}
1 \parallel 2: \quad & W_1 + W_3 \to r_-(z; 1^+, 2^+)\, \mathbb{E}(M_5^{\overline{\text{MHV}}}(k, 3, 4, 5, 6)) \\
& \widetilde{W}_1 + \widetilde{W}_3 \to r_-(z; 1^+, 2^+)\, \mathbb{O}(M_5^{\overline{\text{MHV}}}(k, 3, 4, 5, 6))
\end{aligned} \tag{3.14}$$

---

[5] To guarantee that the functions $C_i(a)$ and $\widetilde{C}_i(a)$ have definite parity to all orders in perturbation theory it is necessary to divide only by the parity-even part of $M_6(a)$.

$$
\begin{aligned}
5 \parallel 6: \quad & W_1 + W_2 \to r_+(z; 1^-, 2^-)\, \mathbb{E}(M_5^{\mathrm{MHV}}(1,2,3,4,k)) \\
& \widetilde{W}_1 + \widetilde{W}_2 \to r_+(z; 5^-, 6^-)\, \mathbb{O}(M_5^{\mathrm{MHV}}(1,2,3,4,k)) \\
3 \parallel 4: \quad & W_2 + W_3 \to r_+(z; 3^+, 4^-)\, \mathbb{E}(M_5^{\overline{\mathrm{MHV}}}(1,2,k,5,6)) + r_-(z; 3^+, 4^-)\, \mathbb{E}(M_5^{\mathrm{MHV}}(1,2,k,5,6)) \\
& \widetilde{W}_2 + \widetilde{W}_3 \to r_+(z; 3^+, 4^-)\, \mathbb{O}(M_5^{\overline{\mathrm{MHV}}}(1,2,k,5,6)) - r_-(z; 3^+, 4^-)\, \mathbb{O}(M_5^{\mathrm{MHV}}(1,2,k,5,6))
\end{aligned}
$$

in which,

$$
r_{-\lambda}(z; i^{\lambda_i}, (i+1)^{\lambda_{i+1}}) = \frac{\mathrm{Split}_{-\lambda}(z; i^{\lambda_i}, (i+1)^{\lambda_{i+1}})}{\mathrm{Split}_{-}^{\mathrm{tree}}(z; i^{\lambda_i}, (i+1)^{\lambda_{i+1}})}\,; \tag{3.15}
$$

M_5^{MHV} and $M_5^{\overline{\mathrm{MHV}}}$ are the ratios of the resummed five-point MHV and $\overline{\mathrm{MHV}}$ amplitudes to their tree-level counterparts; and $\mathbb{E}$ and $\mathbb{O}$ denote projection operators onto the parity-even and parity-odd components. Functions not explicitly mentioned are unconstrained.

The collinear properties of the functions $C_{i,i+3,i+5}$ and $C_{i+3,i,i+2}$ can be easily found by combining equations (3.9), (3.14) and the collinear properties of the MHV ratio $M_6 = A_6^{\mathrm{MHV}}/A_6^{(0),\mathrm{MHV}}$. In particular, they contain the Levi-Civita tensors necessary to transform, for example in the $1 \parallel 2$ limit, the MHV five-point amplitude factor into an $\overline{\mathrm{MHV}}$ five-point amplitude.

The iteration relation for the rescaled splitting amplitude (3.15) [9] suggests another natural organization of the functions W_i, which is similar in spirit to the BDS ansatz [8]:

$$
\ln W_i = \sum_{l=1}^{\infty} a^l \left[f_l(\epsilon) W_i^{(1)}(l\epsilon) + C_l + R_{6;i}^{(l)} + \mathcal{O}(\epsilon) \right] \quad . \tag{3.16}
$$

The structure of infrared singularities and the collinear behavior require that $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ terms in the functions $f_l(\epsilon)$ be the same as for the six-point MHV amplitude. The functions $R_{6;i}^{(l)}$, are similar in spirit to the remainder function $R_6^{(l)}$ of the six-point MHV amplitude. They are closely related to the functions $C_i(a)$ introduced in equation (3.9):

$$
C_i(a) = \exp\left[\gamma_K(a) \left(W_i^{(1)} - M_6^{(1)} \right)\right] \exp\left[R_{6;i}(a) - R_6(a)\right] + \mathcal{O}(\epsilon)\,, \tag{3.17}
$$

where $\gamma_K(a)$ is the cusp anomalous dimension and $R_{6;i}(a) = \sum_{l=2}^{\infty} a^l R_{6;i}^{(l)}$, etc. A natural consequence of the conjecture that $C_i(a)$ are invariant under dual conformal transformations is that the remainder-like functions $R_{6;i}(a)$ are also invariant. We will see that this is indeed so.

The expectation [11, 68] that to leading order in the strong coupling limit, all amplitudes with the same number of external legs are identical (or, equivalently, that $\lim_{a\to\infty} \ln C_i(a) =$

$\mathcal{O}(a^0)$ rather than $\mathcal{O}(\sqrt{a})$) predicts a simple relation between the remainder functions $R_{6;i}$ and the MHV remainder function R_6 to this order. Indeed, using the one-loop relation

$$W_i^{(1)} - M^{(1)} = C_i^{(1)} \tag{3.18}$$

and the known value of the strong coupling expansion of the cusp anomaly [69–72], it follows that

$$\frac{R_6 - R_{6;i}}{C_i^{(1)}} = \frac{\sqrt{\lambda}}{\pi}\,, \tag{3.19}$$

with $C_i^{(1)}$ given in eq. (2.33). Using the numerical results presented in later sections one may check that the weak-coupling expansion of the ratio appearing on the left-hand side depends on the spin factor labeled by i; it seems therefore that a relation of this type may hold only in the strong-coupling limit.

C. Triple-collinear limits

Multi-collinear limits provide a richer set of constraints on amplitudes with at least six external legs. Unlike the collinear limits discussed in the previous section, they probe the detailed structure of the dual-conformal invariant functions unrelated to the infrared structure of the amplitude. In the case of the six-point MHV amplitude, they provided a physical interpretation of the remainder function [19]. The most detailed limit we can consider with six external legs involves three adjacent external momenta becoming collinear,

$$k_a = z_1 P\,, \quad k_b = z_2 P\,, \quad k_c = z_3 P\,, \quad z_1 + z_2 + z_3 = 1\,, \quad 0 \le z_i \le 1\,, \quad P^2 \to 0\,. \tag{3.20}$$

Let us understand what such limits imply about the six-point NMHV amplitude and, in particular, about the remainder-like functions $R_{6;i}(a)$.

An L-loop n-point amplitude factorizes as follows [67]:

$$A_n^{(L)}(k_1, \ldots, k_{n-2}, k_{n-1}, k_n) \mapsto \sum_{\lambda=\pm} \sum_{s=0}^{L} A_n^{(L-s)}(k_1, \ldots, P^\lambda)\, \mathrm{Split}_{-\lambda}^{(s)}(k_{n-2} k_{n-1} k_n; P)\,. \tag{3.21}$$

Taking into account parity and reflection symmetries, there are six independent triple-collinear splitting amplitudes [19]:

$$\mathrm{Split}_{+}(k_a^+ k_b^+ k_c^+; P), \tag{3.22}$$

$$\mathrm{Split}_{-\lambda_P}(k_a^{\lambda_a} k_b^{\lambda_b} k_c^{\lambda_c}; P), \qquad \lambda_a + \lambda_b + \lambda_c - \lambda_P = 2\,, \tag{3.23}$$

$$\mathrm{Split}_{-\lambda_P}(k_a^{\lambda_a} k_b^{\lambda_b} k_c^{\lambda_c}; P), \qquad \lambda_a + \lambda_b + \lambda_c - \lambda_P = 0\,. \tag{3.24}$$

The first one (3.22) vanishes in any supersymmetric theory. The three triple-collinear splitting amplitudes of the second type (3.23), an example of which is $\lambda_a = \lambda_b = \lambda_c = \lambda_P = 1$, appear in limits of MHV amplitudes. The $\mathcal{N} = 4$ supersymmetry Ward identities for MHV amplitudes imply that their rescaled forms[6] are all equal,

$$\frac{\mathrm{Split}^{(l)}_{\mp}(k_a^{\pm} k_b^{+} k_c^{+}; P)}{\mathrm{Split}^{(0)}_{\mp}(k_a^{\pm} k_b^{+} k_c^{+}; P^{\mp})} = \frac{\mathrm{Split}^{(l)}_{\mp}(k_a^{+} k_b^{\pm} k_c^{+}; P)}{\mathrm{Split}^{(0)}_{\mp}(k_a^{+} k_b^{\pm} k_c^{+}; P)} = r_S^{(l)}\left(\tfrac{s_{ab}}{s_{abc}}, \tfrac{s_{bc}}{s_{abc}}, z_1, z_3\right). \tag{3.25}$$

These splitting amplitudes are relevant only for NMHV amplitudes with at least seven external legs. They do not arise in the factorization of six-point amplitudes, because the four-point amplitude entering the factorization (3.21) vanishes identically.

The two splitting amplitudes of the third kind (3.24) arise only in limits of NMHV amplitudes and do not have a simple factorized form similar to (3.25)[7]. They are however the only splitting amplitudes that can appear in the triple-collinear limit of the six-point NMHV amplitude.

As is true for the tree-level NMHV amplitudes, the splitting amplitudes (3.24) have several different presentations related by potentially nontrivial spinor identities. A canonical one, that is useful for our purpose, is obtained from the triple-collinear limit of the six-point tree-level amplitude in equations (2.14)-(2.16).

As the functions W_i are independent of the helicity assignment of the external legs, we again discuss only the split helicity configuration. Up to conjugation and relabeling the only non-trivial limit is $2 \parallel 3 \parallel 4$. With the momentum fractions $k_2 = z_1 P$, $k_3 = z_2 P$, $k_4 = z_3 P$, the spin factors B_i become:

$$\begin{aligned}
b_1 &= \frac{B_1}{A_4^{(0)}(1^+ P^+ 5^- 6^-)} \mapsto \frac{(1-z_3)^2}{\sqrt{z_1 z_2 z_3}\langle 23\rangle(\sqrt{z_1}[24] + \sqrt{z_2}[34])} \\
b_2 &= \frac{B_2}{A_4^{(0)}(1^+ P^+ 5^- 6^-)} \mapsto -\frac{(\sqrt{z_1}\langle 24\rangle + \sqrt{z_2}\langle 34\rangle)^3}{s_{234}\langle 23\rangle\langle 34\rangle(\sqrt{z_2}\langle 23\rangle + \sqrt{z_3}\langle 24\rangle)} \\
&\qquad + \frac{[23]^3}{s_{234}[34](\sqrt{z_1}[24] + \sqrt{z_2}[34])(\sqrt{z_2}[23] + \sqrt{z_3}[24])} \\
b_3 &= \frac{B_3}{A_4^{(0)}(1^+ P^+ 5^- 6^-)} \mapsto \frac{z_2^{3/2}}{\sqrt{z_1 z_3}(1-z_1)[34](\sqrt{z_2}\langle 23\rangle + \sqrt{z_3}\langle 24\rangle)}
\end{aligned} \tag{3.26}$$

[6] We omit a trivial dimensional dependence on s_{abc} from the argument list of $r_S^{(l)}$.

[7] The spin-averaged absolute values squared of tree-level triple-collinear splitting amplitudes were computed in ref. [73]; without spin-averaging, in refs. [74]. The tree-level triple (and higher) collinear splitting amplitudes themselves were computed in ref. [75] using the MHV rules [76]. The one-loop correction to the $q \to q\bar{Q}Q$ triple-collinear splitting amplitude in QCD was computed in ref. [77].

$$+\frac{(z_1z_3)^{3/2}}{\sqrt{z_2}\langle 34\rangle(\sqrt{z_2}[23]+\sqrt{z_3}[24])}\quad .$$

The tree-level splitting amplitude is simply

$$\mathrm{Split}_-^{(0)}(k_2^+k_3^+k_4^-;P)=\frac{1}{2}(b_1+b_2+b_3)\quad . \tag{3.27}$$

Thus, while these splitting amplitudes do not have a simple factorized form similar to that for splitting amplitudes of the second type (3.25), we see that the structure of the six-point amplitude (3.1) implies that to this order each component b_i is dressed at higher loops by scalar functions of momenta,

$$\mathrm{Split}_-(k_2^+k_3^+k_4^-;P)=\frac{1}{2}\left(b_1w_1(a)+b_2w_2(a)+b_3w_3(a)\right)\quad . \tag{3.28}$$

The parity-odd spin factors $\widetilde{B}_i$ also have nontrivial triple-collinear limits. Their coefficients $\widetilde{W}_i$, though, must contain Levi-Civita tensors and thus naively vanish in this limit. The triple-collinear limits of additional spin factors that may appear beyond two loop order must be considered separately.

As was true for the limit discussed in ref. [19], none of the conformal cross-ratios (2.31) vanish as $2 \parallel 3 \parallel 4$; they become

$$\bar{u}_1=\frac{z_1z_3}{(1-z_1)(1-z_3)}\,,\qquad \bar{u}_2=\frac{s_{23}}{s_{234}}\,\frac{1}{1-z_3}\,,\qquad \bar{u}_3=\frac{s_{34}}{s_{234}}\,\frac{1}{1-z_1}\,. \tag{3.29}$$

Thanks to their expected dual conformal invariance (which we will confirm in later sections), the remainder-like functions $R_{6;i}(a)$ retain their complete kinematic content, and may be read off the two-loop triple-collinear splitting amplitude (3.24) by subtracting the triple-collinear limit of the two-loop iteration of the one-loop functions $W_i^{(1)}$.

IV. CONSTRUCTING THE EVEN PART OF THE TWO-LOOP AMPLITUDE

We will construct the even part of the two-loop six-point NMHV amplitude using a superspace form [5, 6] of the generalized unitarity method [1–4, 21, 22, 78]. On general grounds, the result will be expressed as a sum of planar two-loop Feynman integrals with coefficients that are rational functions of the spinor variables. At this order, one-loop calculations suggest that it is possible to exclude integrals with triangle or bubble subintegrals.

Similarly to the two-loop MHV amplitude, we will find that neither $W_i^{(2)}$ nor $\widetilde{W}_i^{(2)}$ can be completely determined by four-dimensional cuts. Rather, they receive both divergent and finite nontrivial contributions from integrals whose integrand is proportional to the (-2ϵ) components of the loop momenta. It is quite nontrivial that these latter contributions can be organized in terms of the same R invariants as the four-dimensional cut-constructible terms.

The generalized cuts that determine the amplitude are then the ones shown in fig. 2, which are the same ones that determine the MHV amplitude [19]. Unlike the calculation of the MHV amplitude, however, here it is necessary to evaluate cuts with all external helicity configurations, as each yields information about different spin factors.

In any supersymmetric theory the improved power-counting ensures that at one-loop order and through $\mathcal{O}(\epsilon^0)$ all terms can be detected in four-dimensional cuts. Beyond one loop this is no longer true generically; for example, the six-point MHV amplitude at two loops receives nontrivial contributions from integrals whose integrand vanishes identically when evaluated in four dimensions. Four-point amplitudes in the $\mathcal{N} = 4$ SYM theory are an exception: through five loops they appear to be determined solely by four-dimensional cuts. We therefore decompose the functions $W_i^{(2)}$ in eqs. (3.1) and (3.7) into a four-dimensional cut-constructible part and a part that requires D-dimensional calculations,

$$W_i^{(2)} = W_i^{(2),D=4} + W_i^{(2),\mu} \quad . \tag{4.1}$$

For the former, powerful helicity and supersymmetry methods can be employed. The latter part of the amplitude is determined by comparing the result of D-dimensional and four-dimensional calculations and is expressed in terms of "μ-integrals" — nontrivial integrals whose integrand vanishes identically in four dimensions.

While all cuts may be evaluated easily, separating their contributions to each one of the functions $W_i^{(2),D=4}$ is not always straightforward. As mentioned previously, (multiple) cuts in channels carrying three-particle invariants capture a single even and odd spin factor at a time and thus determine terms in a single $W_i^{(2),D=4}$ and $\widetilde{W}_i^{(2),D=4}$, with the index i determined by the helicity configuration of external legs. This is the case for cuts (a) and (b) in fig. 2. In contrast, (multiple) cuts in channels carrying only two-particle invariants contribute simultaneously to several spin structures and thus to several $W_i^{(2),D=4}$ and $\widetilde{W}_i^{(2),D=4}$ functions. This feature is already present in the cut construction of the one-loop amplitude; in that case however, cuts in channels carrying three-particle invariants suffice to completely determine the amplitude [2]. The similarity between the expression for cut (c) of fig. 2 and a cut of the one-loop amplitude makes it possible to disentangle it. The cuts of fig. 2(d) and (e) however seem intractable in a component approach.

The component approach also fails to incorporate in a transparent way the constraints imposed by supersymmetry. On-shell superspace provides the additional structure necessary for identifying the contributions of the remaining cuts to each of the $W_i^{(2),D=4}$. We shall therefore formulate the entire calculation of the four-dimensional cut-constructible part of the amplitude in on-shell superspace. After a brief overview of the structure of supercuts and of the techniques necessary to disentangle them, we will discuss cut (a), and then proceed to a more detailed analysis of the challenging cuts (c), (d) and (e). For the latter cuts we shall use a superspace generalization of the maximal cut method [39, 79].

A. Unitarity in Superspace: General Features and Techniques

Generalized cuts may be classified following the number of cut conditions they impose. The same is true for generalized supercuts. At L loops in four dimensions it is possible to impose at most $4L$ cut conditions; based on one- and two-loop information, it it likely that their solutions generically form a discrete set. This type of cut has been considered in the maximal-unitarity approach as well as in the leading-singularity approach. Maximal cuts, *i.e.* cuts with the maximal number of cut propagators, are typically insufficient to completely determine an amplitude. For example, at two loops one frequently encounters double box integrals, which cannot be detected by cutting eight propagators. Near-maximal cuts, obtained by successively relaxing cut condition in maximal cuts, provide an algorithmic way of identifying these contributions. Near-maximal cuts exhibit additional propagator-like singularities which are exploited in the leading singularity approach to reduce the one-parameter family of solutions to the cut conditions to a discrete set.

The two-loop six-point NMHV amplitude can in principle be determined entirely from the iterated two-particle cuts shown in fig. 2. The Feynman integrals that contribute only to cuts (c), (d) and (e) are also detected by certain near-maximal cuts. We have used them instead to check that the resulting amplitude correctly reproduces cuts (c), (d) and (e), supplemented by an additional cut condition isolating terms in one of the tree amplitudes.

General supercuts are constructed [5, 6, 56] by multiplying together superamplitudes, identifying the η parameters of the lines that are sewn together and integrating over the common values of the internal η variables. The structure and properties of general supercuts have been analyzed in detail in ref. [6] where it was shown that, upon use of a supersymmetric generalization of the MHV vertex rules [76], their building blocks are generalized supercuts constructed only out of MHV and $\overline{\text{MHV}}$ tree-level amplitudes.

When evaluating a supercut one encounters the situation that on one side of the cut a momentum is outgoing and on the other side it is incoming. In order to write the tree-level amplitude and in particular the argument of their delta functions, it is necessary to define the spinors $|-p\rangle$ and $|-p]$ corresponding to the incoming momentum $(-p)$. We use the analytic continuation rule [56] that the change in sign of the momentum is realized by a change of sign of the holomorphic spinor

$$\begin{aligned} p \mapsto -p \;\leftrightarrow \quad & \lambda_p \mapsto -\lambda_p\,, \qquad \tilde\lambda_p \mapsto +\tilde\lambda_p\,; \\ \leftrightarrow \quad & |-p\rangle \mapsto -|p\rangle\,, \qquad |-p] \mapsto |p]\,. \end{aligned} \tag{4.2}$$

Let us discuss in detail the building blocks we require, supercuts constructed only out of MHV and $\overline{\text{MHV}}$ tree-level amplitudes. Their evaluation requires the evaluation of integrals of products of delta functions with arguments linear in Grassmann parameters, see eqs. (2.23) and (2.24). For a p-particle cut of an N^{k}MHV amplitude this product contains $(8+4(k+p))$ delta function factors of which $(8+4k)$ remain upon integration. As discussed in refs. [5, 6, 56], the integration over the internal η parameters realizes the sum over the states crossing the (generalized) supercut. For any supercut, eight of these delta functions can always be singled out: they enforce the super-momentum conservation of the amplitude,

$$\delta^{(8)}\big(\sum_{i\in\mathcal{E}} \lambda_i \eta_i^A\big) \tag{4.3}$$

where $\mathcal{E}$ denotes the set of external lines. These delta functions may be thought of as the supersymmetric generalization of the usual momentum conservation constraint. They may be extracted without carrying out any Grassmann integrations, by taking suitable linear combinations of the arguments of all delta functions. If the supercut contains at least one MHV superamplitude factor, the Jacobian of this transformation is unity. Their extraction also makes manifest the invariance of the amplitude under half of the maximal supersymmetry. Invariance under the other half of the supersymmetry, generated by

$$\bar q_A^{\dot\alpha} = \sum_{i=1}^{n} \tilde\lambda_i^{\dot\alpha} \frac{\partial}{\partial \eta_i^A} \quad , \tag{4.4}$$

is not manifest, but can in principle be checked at the level of the Grassmann integrand.

The delta functions (4.3) represent the complete Grassmann parameter dependence of a supercut of an MHV amplitude. The Grassmann integrals simply yield the determinant of the system of linear equations which are the arguments of the other $4p$ delta functions, where p is the number of cut lines [6].

For cuts of an N^{k}MHV amplitude there is a certain amount of freedom in evaluating the internal Grassmann integrals. In general, however, the resulting $4k$ delta functions have many undesirable features. The essential ones are that (1) their arguments may depend on loop momenta (if the cut conditions do not completely freeze the momentum integrals) and (2) they may not make the symmetries of the amplitude manifest. We wish to express these Grassmann delta functions in terms of structures that appeared at lower-loop order; in the case of the six-point NMHV amplitude; these are the dual superconformal R invariants. This is a non-trivial operation, and we have but a limited set of tools available.

Given a set of $4k$ delta functions

$$\prod_{i=1}^{k} \delta^{(4)}(e_i(\eta, \lambda)) \,, \tag{4.5}$$

it may be possible to construct linear combinations of their arguments $M_{ij}(\lambda)e_j(\eta, \lambda)$ which factorize into products of the desired combinations of spinors and Grassmann variables upon use of momentum and super-momentum conservation, cut conditions, and the fact that a Grassmann delta function equals its argument. For $k = 1$, which is the case of interest to us, no linear combinations can be constructed.

A possible strategy for eliminating the dependence of the delta functions on loop momenta is to make use of the fact that a Grassmann delta function equals its argument. This observation replaces a cut carrying a Grassmann delta function with a sum of cuts of tensor integrals with Grassmann-valued coefficients. Albeit nontrivial due to their high rank, the tensor integrals may then be reduced following the standard strategy of integral reduction. While indeed successful in eliminating the loop momentum dependence from the Grassmann delta functions, this strategy is likely to lead to rather unwieldy expressions. We will not pursue this direction.

An alternate approach to reorganizing Grassmann delta functions is to use the Lagrange interpolation formula, which is most efficient when applied to next-to-maximal cuts, which impose $(4L - 1)$ on-shell conditions. Let y be the variable that parametrizes the solution to these cut conditions. The product of the $4k$ Grassmann delta functions is then just a polynomial $P_d(y)$ of degree $d = 4k$ with Grassmann-valued coefficients. Any such polynomial may be written as

$$P_d(y) = \sum_{i=1}^{d+1} \prod_{\substack{j=1 \\ j \neq i}}^{d+1} \frac{y - y_j}{y_i - y_j} P_d(y_i) , \tag{4.6}$$

where the values y_i are arbitrary. This equation simply encodes the fact that a polynomial of degree d is determined by its values at $d + 1$ points.

Choosing the points y_i can be regarded as freezing the momentum component unfixed by the cut condition; from this perspective it is akin to the leading-singularity method which uses additional cut-like conditions for the same purpose. The Lagrange interpolation formula (4.6) provides a different strategy, as the points y_i need not be chosen following the leading-singularity prescription. If the two approaches are to agree, the residue of the leading singularity must be proportional to the Grassmann delta functions appearing in dual superconformal invariants. Evidence that this is indeed true has been presented in ref. [33]. In general, however, in order to use the interpolation formula (4.6), there must exist more y_i such that $P_d(y_i)$ is (proportional to) a dual superconformal invariant than are given by the leading-singularity approach.

In the next subsection we will use this strategy to analyze certain seven-particle cuts of the six-point two-loop NMHV superamplitude. As we will see, with judiciously chosen points y_i it is possible to have $P_4(y_i)$ be proportional to the delta functions appearing in the R dual superconformal invariants.

Because of the arbitrariness in the choice of the y_i, the decomposition of $P_d(y)$ in a linear combination of "good" Grassmann delta functions, such as the delta functions appearing in the dual superconformal invariants, is not unique. This signals the existence of linear relations between the dual superconformal invariants. For six-point amplitudes, an identity arising this way is eq. (2.28), which was already required for the consistency of the various possible presentations of the tree-level amplitude. It is conceivable that at higher points and/or higher loops, new relations arise, beyond those that can be obtained from tree-level considerations.

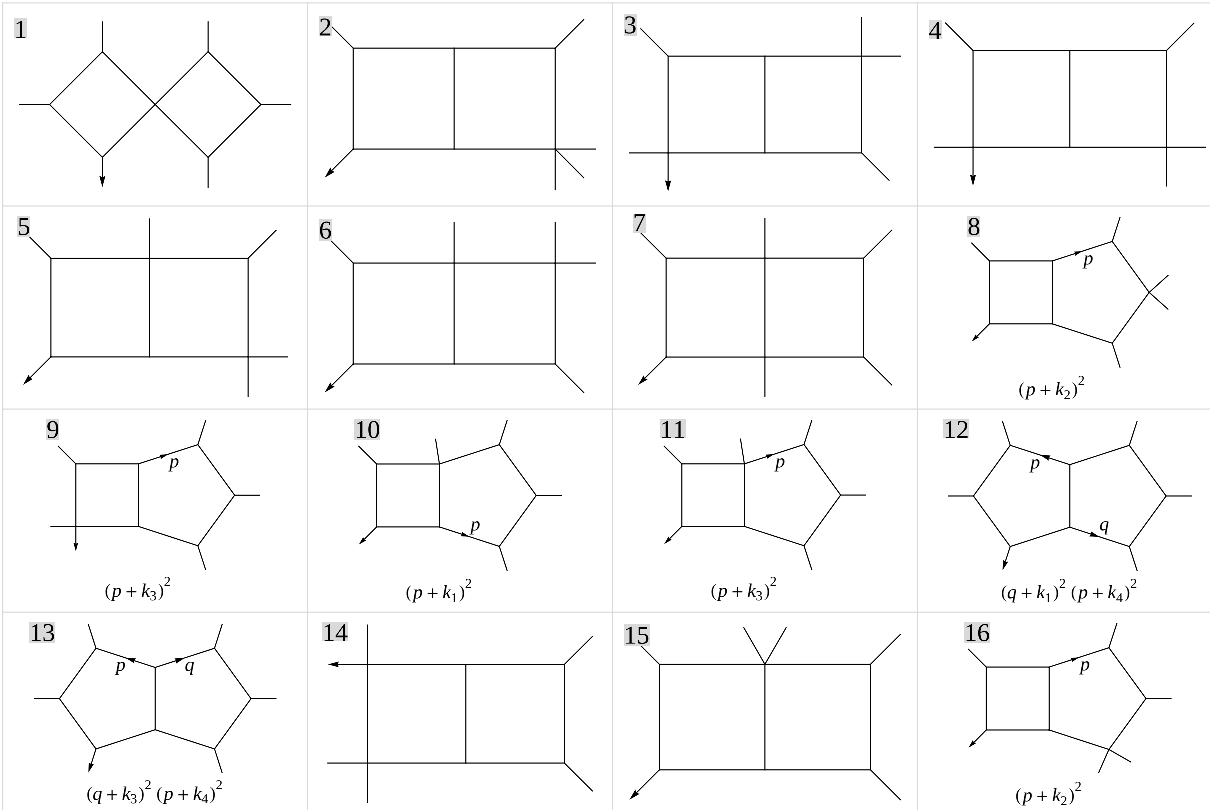

FIG. 4: Two-loop topologies entering the two-loop six-point amplitudes. The arrow on the external line indicates leg number 1.

B. Supercut Example: the Double-Pentagon Cut

Let us illustrate the general strategy outlined in the previous section, by examining in some detail two cuts that are essential for the construction of the six-point NMHV super-amplitude. We begin with the 'double-pentagon' cut, shown in fig. 2(a), which isolates the double-pentagon integrals $I^{(12)}$ and $I^{(13)}$ (shown in fig. 4) from a wide class of other integrals (hence its name).

This cut provides two distinct contributions to the coefficients of the NMHV amplitude, depending on which of the two five-point tree-level factors is an MHV or an $\overline{\text{MHV}}$ super-amplitude. Each of the contributions is closed under supersymmetry transformations, so we will call them supersectors. (They were called "holomorphicity configurations" in ref. [6].)

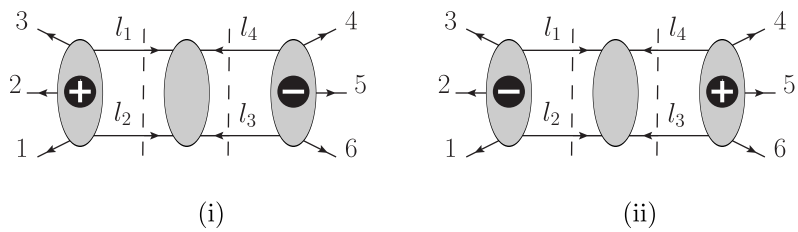

FIG. 5: The two contributions to the 'double-pentagon' supercut 2(a). The circled + and − denote MHV and $\overline{\text{MHV}}$ superamplitudes, respectively. The middle amplitude may be chosen to be either of MHV or of $\overline{\text{MHV}}$ type. Here we choose to present it as an MHV superamplitude.

These two supersectors are shown in fig. 5; their values are,

$$\mathcal{C}^{\text{dp}}_{5(i)} = \int d^4\eta_{l_1} d^4\eta_{l_2} d^4\eta_{l_4} d^4\eta_{l_3} d^8\omega \\ \times \frac{\delta^{(8)}(q^A_{123} + \lambda_{l_1}\eta^A_{l_1} + \lambda_{l_2}\eta^A_{l_2})}{\langle 12\rangle\langle 23\rangle\langle 3l_1\rangle\langle l_1 l_2\rangle\langle l_2 1\rangle} \frac{\delta^{(8)}(\lambda_{l_1}\eta^A_{l_1} + \lambda_{l_2}\eta^A_{l_2} + \lambda_{l_4}\eta^A_{l_4} + \lambda_{l_3}\eta^A_{l_3})}{\langle l_2 l_1\rangle\langle l_1 l_4\rangle\langle l_4 l_3\rangle\langle l_3 l_2\rangle} \\ \times \frac{\delta^{(4)}(\eta^A_{l_4} - \tilde\lambda^{\dot\alpha}_{l_4}\omega^A_{\dot\alpha})\delta^{(4)}(\eta^A_{l_3} - \tilde\lambda^{\dot\alpha}_{l_3}\omega^A_{\dot\alpha})}{[45][56][6l_3][l_3 l_4][l_4 4]} \prod_{i=4}^{6} \delta^{(4)}(\eta^A_i - \tilde\lambda^{\dot\alpha}_i \omega^A_{\dot\alpha}) \tag{4.7}$$

and

$$\mathcal{C}^{\text{dp}}_{5(ii)} = \int d^4\eta_{l_1} d^4\eta_{l_2} d^4\eta_{l_4} d^4\eta_{l_3} d^8\omega \\ \times \frac{\delta^{(8)}(q^A_{456} + \lambda_{l_4}\eta^A_{l_4} + \lambda_{l_3}\eta^A_{l_3})}{\langle 45\rangle\langle 56\rangle\langle 6l_3\rangle\langle l_3 l_4\rangle\langle l_4 4\rangle} \frac{\delta^{(8)}(\lambda_{l_1}\eta^A_{l_1} + \lambda_{l_2}\eta^A_{l_2} + \lambda_{l_4}\eta^A_{l_4} + \lambda_{l_3}\eta^A_{l_3})}{\langle l_2 l_1\rangle\langle l_1 l_4\rangle\langle l_4 l_3\rangle\langle l_3 l_2\rangle} \\ \times \frac{\delta^{(4)}(\eta^A_{l_1} - \tilde\lambda^{\dot\alpha}_{l_1}\omega^A_{\dot\alpha})\delta^{(4)}(\eta^A_{l_2} - \tilde\lambda^{\dot\alpha}_{l_2}\omega^A_{\dot\alpha})}{[12][23][3l_1][l_1 l_2][l_2 1]} \prod_{i=1}^{3} \delta^{(4)}(\eta^A_i - \tilde\lambda^{\dot\alpha}_i \omega^A_{\dot\alpha}) \tag{4.8}$$

where

$$q^{\alpha A}_{123} = \sum_{i=1}^{3} \lambda^\alpha_i \eta^A_i \quad \text{and} \quad q^{\alpha A}_{456} = \sum_{i=4}^{6} \lambda^\alpha_i \eta^A_i \,. \tag{4.9}$$

By taking appropriate linear combinations of the arguments of the delta functions in these equations it is easy to extract the overall super-momentum conservation delta function, $\delta^{(8)}(q^A_{123} + q^A_{456})$. Carrying out the Grassmann integrals we then find

$$\mathcal{C}^{\text{dp}}_{5(i)} = \mathcal{A}^{(0),\text{MHV}} R_{146} \frac{s_{123}\langle 1|2+3|4]\langle 3|4+5|6]\langle l_1 l_2\rangle^4}{(\langle 3l_1\rangle\langle l_1 l_2\rangle\langle l_2 1\rangle)\,(\langle l_2 l_1\rangle\langle l_1 l_4\rangle\langle l_4 l_3\rangle\langle l_3 l_2\rangle)\,([6l_3][l_3 l_4][l_4 4])}\,, \tag{4.10}$$

$$\mathcal{C}^{\text{dp}}_{5(ii)} = \mathcal{A}^{(0),\text{MHV}} R_{413} \frac{s_{123}\langle 4|2+3|1]\langle 6|4+5|3]\,\langle l_4 l_3\rangle^4}{(\langle 6l_3\rangle\langle l_3 l_4\rangle\langle l_4 4\rangle)\,(\langle l_2 l_1\rangle\langle l_1 l_4\rangle\langle l_4 l_3\rangle\langle l_3 l_2\rangle)\,([3l_1][l_1 l_2][l_2 1])}\,. \tag{4.11}$$

We can reorganize the contributions into even and odd components,

$$\mathcal{C}^{\rm dp} = \mathcal{C}^{\rm dp}_{5(i)} + \mathcal{C}^{\rm dp}_{5(ii)} = \mathcal{A}^{(0),{\rm MHV}}\, \mathcal{C}^{\rm dp}_{+}\, (R_{413} + R_{146}) + \mathcal{A}^{(0),{\rm MHV}}\, \mathcal{C}^{\rm dp}_{-}\, (R_{413} - R_{146})\,. \quad (4.12)$$

The two functions of vanishing weight in equations (4.10) and (4.11) may be identified as the contribution of gluon intermediate states in a component approach. They can be decomposed by standard means, by reconstructing propagators and organizing the numerator into a single trace.[8] For example,

$$\begin{aligned}
\mathcal{C}^{\rm dp}_{+} \;=\; & \frac{s_{123}}{(\langle l_2 l_1\rangle\langle l_1 l_4\rangle\langle l_4 l_3\rangle\langle l_3 l_2\rangle)}\Bigg(\frac{\langle 1|2+3|4]\langle 3|4+5|6]\langle l_1 l_2\rangle^4}{(\langle 3 l_1\rangle\langle l_1 l_2\rangle\langle l_2 1\rangle)\,([6 l_3][l_3 l_4][l_4 4])} \\
& \qquad + \frac{\langle 4|2+3|1]\langle 6|4+5|3]\,\langle l_4 l_3\rangle^4}{(\langle 6 l_3\rangle\langle l_3 l_4\rangle\langle l_4 4\rangle)\,([3 l_1][l_1 l_2][l_2 1])}\Bigg) \\
=\; & \frac{1}{4}\Bigg[\frac{s_{123}(s_{123}s_{345} - s_{12}s_{45})}{(l_1+k_3)^2(l_2+l_3)^2(l_3+k_6)^2} + \frac{s_{61}s_{123}^2}{(l_2+k_1)^2(l_2+l_3)^2(l_3+k_6)^2} \\
& + \frac{s_{34}s_{123}^2}{(l_1+k_3)^2(l_2+l_3)^2(l_4+k_4)^2} + \frac{s_{123}(s_{123}s_{234} - s_{23}s_{56})}{(l_2+k_1)^2(l_2+l_3)^2(l_4+k_4)^2} \\
& + \frac{s_{12}s_{23}s_{123}(k_6-l_2)^2}{(l_2+k_1)^2(l_1+k_3)^2(l_2+l_3)^2(l_3+k_6)^2} + \frac{s_{12}s_{23}s_{123}(k_4-l_1)^2}{(l_2+k_1)^2(l_1+k_3)^2(l_2+l_3)^2(l_4+k_4)^2} \\
& + \frac{s_{45}s_{56}s_{123}(k_3-l_4)^2}{(l_1+k_3)^2(l_2+l_3)^2(l_4+k_4)^2(l_3+k_6)^2} + \frac{s_{45}s_{56}s_{123}(k_1-l_3)^2}{(l_2+k_1)^2(l_2+l_3)^2(l_4+k_4)^2(l_3+k_6)^2} \\
& + \frac{1}{(l_2+k_1)^2(l_1+k_3)^2(l_2+l_3)^2(l_4+k_4)^2(l_3+k_6)^2} \\
& \quad \times s_{123}\Big((s_{12}s_{45} - s_{123}s_{345})(k_1-l_3)^2(k_4-l_1)^2 + s_{34}s_{123}(k_1-l_3)^2(k_6-l_2)^2 \\
& \qquad + s_{61}s_{123}(k_3-l_4)^2(k_4-l_1)^2 + (s_{23}s_{56} - s_{123}s_{234})(k_3-l_4)^2(k_6-l_2)^2\Big) \\
& + \frac{1}{(l_2+k_1)^2(l_1+k_3)^2(l_4+k_4)^2(l_3+k_6)^2} \\
& \quad \times \Big(2s_{12}s_{23}s_{45}s_{56} - s_{123}\big(s_{61}s_{34}s_{123} + s_{12}s_{45}s_{234} + s_{23}s_{56}s_{345} - s_{123}s_{234}s_{345}\big)\Big)\Bigg]
\end{aligned} \quad (4.13)$$

From this expression we can easily read off the coefficients of all integrals in fig. 4 that have a double cut in the s_{123} channel. Some integrals appear multiple times, corresponding to different cyclic permutations of external legs that have such a cut. The numerator factors in the expression above are precisely those required to render the integrals invariant under dual inversion. As we did not need to specify the helicity labels of the external legs, all cuts with this topology can be obtained by simple cyclic relabeling.

[8] This last step is important to avoid the appearance of parity-even terms which are a product of two parity-odd factors.

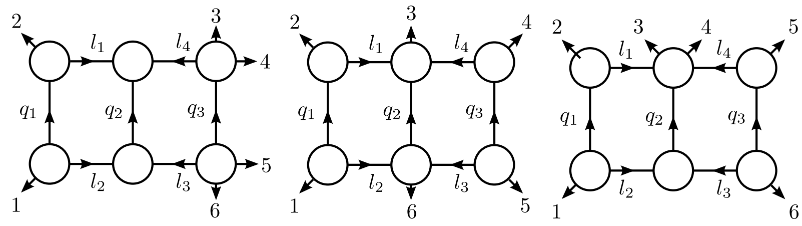

FIG. 6: Next-to-maximal cuts that detect the integrals not easily isolated by the iterated two-particle cuts: from left to right, next-to-maximal cuts for the 'hexabox,' 'flying-squirrel,' and 'rabbit-ears' cuts of fig. 2(c), (d), and (e), respectively.

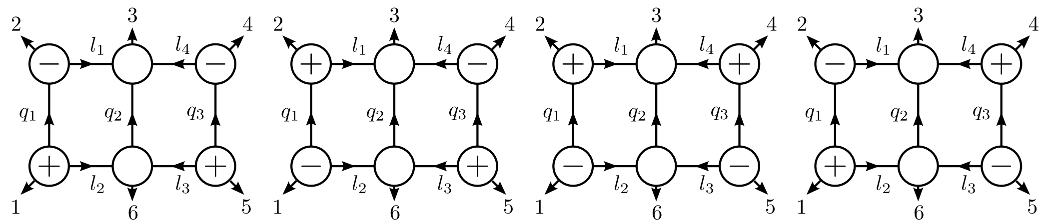

FIG. 7: The four possible assignments of internal helicities for the next-to-maximal 'flying-squirrel' cut of fig. 6(ii). The '$\ominus$' vertices denote three-point $\overline{\text{MHV}}$ amplitudes while the '$\oplus$' vertices denote three-point MHV amplitudes.

The 'turtle' cut shown in fig. $2(b)$ can be computed in a similar way, and also contributes a lone R invariant. These two cuts determine the coefficients of all integral topologies in fig. 4 except $I^{(7)}$, $I^{(14)}$, and $I^{(15)}$. Other cuts are necessary to determine these contributions. An efficient strategy, which makes use of the results obtained from the double-pentagon cut of fig. $2(a)$ and the turtle cut of fig. $2(b)$, is to analyze the relevant next-to-maximal cuts and find the remaining integrals one at a time.

C. Supercut Example: A Contribution to the Flying-Squirrel Cut

Following this strategy, we present one contribution to the next-to-maximal cut, shown in fig. 6(ii), that imposes additional cut constraints beyond the 'flying-squirrel' cut of fig. $2(d)$. It serves to isolate one of our target integrals, $I^{(7)}$, and allows us to determine its coefficient. As explained above, we impose the additional cut conditions because of difficulties in organizing the results for cuts like the 'flying-squirrel' cut in terms of dual superconformal R

invariants. A superspace calculation, combined with the reduced set of Feynman integrals isolated by the next-to-maximal cut conditions, reduces the ambiguity in such a reorganization by enforcing supersymmetry relations between the reduced number of contributions to these cuts.

The next-to-maximal 'flying-squirrel' cut of fig. 6(ii) has four supersectors, corresponding to three-point amplitudes at the corners being of MHV or $\overline{\text{MHV}}$ type. The supersectors are shown in fig. 7. We discuss in detail the configuration in fig. 7(ii), quote the result for the configuration in fig. 7(i), and explain how to construct the other two components by relabeling.

The product of the tree superamplitudes entering the supersector shown in fig. 7(ii) is

$$\begin{aligned}
\mathcal{C}_{7(ii)} = \int d^4\eta_{l_1} d^4\eta_{l_2} d^4\eta_{l_3} d^4\eta_{l_4} d^4\eta_{q_1} d^4\eta_{q_2} d^4\eta_{q_3} & \frac{\delta^{(4)}([1q_1]\eta_{l_2}^A + [q_1 l_2]\eta_1^A + [l_2 1]\eta_{q_1}^A)}{[1q_1][q_1 l_2][l_2 1]} \\
\times \frac{\delta^{(8)}(-\lambda_{q_1}\eta_{q_1}^A + \lambda_2\eta_2^A + \lambda_{l_1}\eta_{l_1}^A)}{\langle q_1 2\rangle\langle 2 l_1\rangle\langle l_1 q_1\rangle} & \frac{\delta^{(8)}(-\lambda_{l_1}\eta_{l_1}^A + \lambda_3\eta_3^A - \lambda_{l_4}\eta_{l_4}^A - \lambda_{q_2}\eta_{q_2}^A)}{\langle l_1 3\rangle\langle 3 l_4\rangle\langle l_4 q_2\rangle\langle q_2 l_1\rangle} \\
\times \frac{\delta^{(4)}([l_4 4]\eta_{q_3}^A + [4 q_3]\eta_{l_4}^A + [q_3 l_4]\eta_4^A)}{[l_4 4][4 q_3][q_3 l_4]} & \frac{\delta^{(8)}(\lambda_{q_3}\eta_{q_3}^A + \lambda_5\eta_5^A + \lambda_{l_3}\eta_{l_3}^A)}{\langle q_3 5\rangle\langle 5 l_3\rangle\langle l_3 q_3\rangle} \\
& \times \frac{\delta^{(8)}(\lambda_6\eta_6^A - \lambda_{l_2}\eta_{l_2}^A + \lambda_{q_2}\eta_{q_2}^A - \lambda_{l_3}\eta_{l_3}^A)}{\langle 6 l_2\rangle\langle l_2 q_2\rangle\langle q_2 l_3\rangle\langle l_3 6\rangle}. \qquad (4.14)
\end{aligned}$$

The expected overall super-momentum conservation constraint may be extracted by adding the arguments of all the eightfold delta functions $\delta^{(8)}$ to the last such function, and then using momentum conservation to eliminate $\lambda_{q_1}\eta_{q_1}^A$ and $\lambda_{l_4}\eta_{l_4}^A$. These transformations have unit Jacobian.

The remaining Grassmann integrals can be computed easily; we obtain:

$$\begin{aligned}
\mathcal{N} = [1q_1]^4 \delta^{(4)}\big(&\langle q_1 2\rangle\langle q_2 l_1\rangle\langle q_3 l_3\rangle[4q_3]\eta_2^A + \langle q_1 l_1\rangle\langle q_2 3\rangle\langle q_3 l_3\rangle[4q_3]\eta_3^A \\
&+\langle q_1 l_1\rangle\langle q_2 l_4\rangle\langle q_3 l_3\rangle[q_3 l_4]\eta_4^A + \langle q_1 l_1\rangle\langle q_2 l_4\rangle\langle l_3 5\rangle[l_4 4]\eta_5^A\big) \ . \qquad (4.15)
\end{aligned}$$

The contribution from the supersector in fig. 7(ii) is obtained by dividing $\mathcal{N}$ by the explicit denominators in equation (4.14). This expression can be simplified in several different ways; we proceed by solving the cut conditions. The internal spinors (except for those associated to q_2) may be expressed conveniently in terms of two variables y and z:

$$\begin{aligned}
\lambda_{l_1} &= y\lambda_1 - \lambda_2\,, & \tilde\lambda_{l_1} &= \tilde\lambda_2\,, & \lambda_{q_1} &= y\lambda_1\,,\\
\lambda_{l_2} &= -\lambda_1\,, & \tilde\lambda_{l_2} &= \tilde\lambda_1 + y\tilde\lambda_2\,, & \tilde\lambda_{q_1} &= \tilde\lambda_2\,,\\
\lambda_{l_3} &= -z\lambda_4 - \lambda_5\,, & \tilde\lambda_{l_3} &= \tilde\lambda_5\,, & \lambda_{q_3} &= z\lambda_4\,,\\
\lambda_{l_4} &= \lambda_4\,, & \tilde\lambda_{l_4} &= -\tilde\lambda_4 + z\tilde\lambda_5\,, & \tilde\lambda_{q_3} &= \tilde\lambda_5\,.
\end{aligned} \tag{4.16}$$

The momentum q_2 can be determined through momentum conservation; the condition that it be on shell relates the two parameters y and z. These relations imply that all spinor products in eq. (4.15) that do not contain the holomorphic spinor $|q_2\rangle$ are monomials in y, z and spinor products of external momenta. The remaining holomorphic spinor products, which do contain $|q_2\rangle$, can be converted into functions of y and external spinor products by multiplying and dividing by $[q_2 5]^4$ and using the identities,

$$\begin{aligned}
\langle l_1 q_2\rangle[q_2 5] &= -\langle 2(3+4)5] + y\langle 1(3+4)5]\,,\\
\langle l_2 q_2\rangle[q_2 5] &= \langle 16\rangle[65]\,,\\
\langle l_3 q_2\rangle[q_2 5] &= \langle 16\rangle([16] + y[26])\,,\\
\langle l_4 q_2\rangle[q_2 5] &= \langle 4(2+3)5] + y\langle 14\rangle[25]\,.
\end{aligned} \tag{4.17}$$

The numerator factor $\mathcal{N}$ in eq. (4.15) is then,

$$\begin{aligned}
\mathcal{N} = \frac{s_{12}^4 s_{45}^4 y^4 z^4}{[q_2 5]^4}\delta^{(4)}\Big(&\eta_2^A(\langle 2(1+6)5] - y\langle 1(2+6)5]) + \eta_3^A(\langle 3(1+6)5] - y\langle 13\rangle[25])\\
&+ \eta_4^A(\langle 4(1+6)5] - y\langle 14\rangle[25]) + \eta_5^A(s_{234} - y\langle 1(3+4)2])\Big).
\end{aligned} \tag{4.18}$$

This expression is invariant, though not manifestly, under the action of the supersymmetry generators $\bar q = \sum \tilde\lambda \partial_\eta$.

Overall super-momentum conservation provides the means to further simplify $\mathcal{N}$. By subtracting $\sum_i \eta_i^A\langle i(1+6)5] = 0$, and adding $\sum_i y\eta_i^A\langle 1i\rangle[25] = 0$ to the argument of the delta function we find

$$\delta^{(8)}(\sum_i \lambda_i\eta_i)\mathcal{N} = s_{12}^4 s_{45}^4 y^4 z^4 \frac{\langle 16\rangle^4}{[q_2 5]^4}\delta^{(4)}(\eta_1^A[56] + \eta_5^A[61] + \eta_6^A[15] + y(\eta_2^A[56] + \eta_5^A[62] + \eta_6^A[25]))\ . \tag{4.19}$$

This superspace expression has the two unwanted features already mentioned in section IV A: on the one hand, the argument of the delta function depends on the internal momenta through the variable y; on the other, it is not manifestly a function only of the same superspace structures as the tree-level amplitude (3.1). We wish to reorganize it in terms of R invariants, and at the same time remove the dependence on internal momenta from the arguments of the delta function by using the Lagrange interpolation formula (4.6) on the degree four polynomial,

$$\begin{aligned} P_4(y) &= \delta^{(4)}(\eta_1^A[56] + \eta_5^A[61] + \eta_6^A[15] + y(\eta_2^A[56] + \eta_5^A[62] + \eta_6^A[25])) \\ &= \sum_{i=1}^{5} \prod_{\substack{j=1 \\ j\neq i}}^{5} \frac{y - y_j}{y_i - y_j} P_4(y_i)\,. \end{aligned} \tag{4.20}$$

For this to be possible, as explained earlier it is necessary that there exist at least five values y_i such that $P_4(y_i)$ is proportional to an R invariant. It turns out that there are at least six such values:

$$P_4\left(\frac{\langle 2(3+4)5]}{\langle 1(3+4)5]}\right) = \left(\frac{\langle 34\rangle[56]}{\langle 1(3+4)5]}\right)^4 \delta^{(4)}(\eta_3^A[45] + \eta_4^A[53] + \eta_5^A[34]) \propto R_{635}\,, \tag{4.21a}$$

$$P_4\left(-\frac{[16]}{[26]}\right) = \left(\frac{[56]}{[26]}\right)^4 \delta^{(4)}(\eta_6^A[12] + \eta_1^A[26] + \eta_2^A[61]) \propto R_{362}\,, \tag{4.21b}$$

$$P_4\left(-\frac{\langle 4(5+6)1]}{\langle 4(5+6)2]}\right) = \left(\frac{\langle 34\rangle[56]}{\langle 4(5+6)2]}\right)^4 \delta^{(4)}(\eta_1^A[23] + \eta_2^A[31] + \eta_3^A[12]) \propto R_{413}\,, \tag{4.21c}$$

$$P_4\left(\frac{\langle 23\rangle}{\langle 13\rangle}\right) = \left(\frac{\langle 34\rangle}{\langle 13\rangle}\right)^4 \delta^{(4)}(\eta_4^A[56] + \eta_5^A[64] + \eta_6^A[45]) \propto R_{146}\,, \tag{4.21d}$$

$$P_4\left(\frac{s_{234}}{\langle 1(3+4)2]}\right) = \left(\frac{\langle 34\rangle[56]}{\langle 1(3+4)2]}\right)^4 \delta^{(4)}(\eta_2^A[34] + \eta_3^A[42] + \eta_4^A[23]) \propto R_{524}\,, \tag{4.21e}$$

$$P_4(0) = \delta^{(4)}(\eta_5^A[61] + \eta_6^A[15] + \eta_1^A[56]) \propto R_{251}\,. \tag{4.21f}$$

For some of these cases, we have used overall super-momentum conservation constraint as well as nontrivial spinor identities to transform the argument of $\delta^{(4)}$. As we will see shortly, only the first four values of y correspond to leading singularities.

We can use the Lagrange interpolation formula for any five of the six special values

$$\{y_1, y_2, y_3, y_4, y_5, y_6\} = \left\{\frac{\langle 2(3+4)5]}{\langle 1(3+4)5]}, -\frac{[16]}{[26]}, -\frac{\langle 4(5+6)1]}{\langle 4(5+6)2]}, \frac{\langle 23\rangle}{\langle 13\rangle}, \frac{s_{234}}{\langle 1(3+4)2]}, 0\right\}\,. \tag{4.22}$$

Clearly, the decomposition obtained this way is not unique as there are six different possibilities. Let us denote them by $\mathcal{L}_i$, where i is the index of the missing root. In general we can construct a five-parameter decomposition

$$P_4(y) = \sum_{i=1}^{6} \alpha_i \mathcal{L}_i(y)\,, \tag{4.23}$$

with $\sum \alpha_i = 1$. The remaining parameters α_i may be constrained by requiring that the superamplitude have additional manifest symmetries; for example, one may require that the parity of the superamplitude be manifest. We impose such a requirement in the following.

All the presentations of the cut obtained for different possible choices of five values y_i are physically equivalent. However, they contain different R invariants; the existence of more than five values y_i is equivalent to the existence of nontrivial relations between R invariants. These relations hold only in the presence of the overall super-momentum conservation constraint.

We are now in position to assemble the result $\mathcal{C}_{7(ii)}$ for the supersector of fig. 7(ii). Further use of the identities (4.17) implies that it is given by

$$\frac{-\mathcal{A}_6^{(0),\mathrm{MHV}}\, s_{12}s_{45}\, P_4(y)}{\langle 34\rangle[56](y\langle 13\rangle - \langle 23\rangle)(y\langle 1(3+4)5] - \langle 2(3+4)5])(\langle 4(5+6)1] + y\langle 4(5+6)2])([16]+y[26])}\,. \tag{4.24}$$

Inspecting the denominator of this expression, we see that the first four points y_i in eq. (4.22) correspond to poles. They are in fact positions of leading singularities, as all of them arise from the $[q_2 5]^{-4}$ factor in eq. (4.19) which is the Jacobian arising from solving the cut conditions.

Choosing the five points y_i to be $\{y_1, y_2, y_3, y_4, y_5\}$, we obtain,

$$\begin{aligned}\mathcal{C}_{7(ii)} = s_{12}s_{45}s_{234}\mathcal{A}_6^{(0),\mathrm{MHV}} \Bigg(& \frac{y_4(y-y_5)}{y_5(y-y_4)} R_{146} + \frac{y_2(y-y_5)}{y_5(y-y_2)} R_{362} \\ & - \frac{y_3(y-y_5)}{y_5(y-y_3)} R_{413} - \frac{y_1(y-y_5)}{y_5(y-y_1)} R_{635} + R_{524}\Bigg)\,. \end{aligned} \tag{4.25}$$

Each of the denominators appearing in this expression may be identified with a propagator evaluated on the kinematic configuration (4.16).

Thus, the contribution of this supersector depends only on R invariants. We can decompose it in even and odd invariants ($R_{i,i+3,i+5} \pm R_{i+3,i,i+2}$), following the form (3.1) of the superamplitude. To identify the part of $\mathcal{C}_{7(ii)}$ that receives contributions from the missing integral $I^{(7)}$ we need to subtract from it the contribution of all the other integrals in fig. 4, determined from the cuts of fig. 2(a) and (b). These cuts can contribute only terms proportional to the invariants R_{146}, R_{362}, R_{413} or R_{635}. Thus, we can conclude immediately that R_{524} arises solely as a coefficient of $I^{(7)}$, whose coefficient must therefore be,

$$\frac{1}{2}\mathcal{A}_6^{(0),\mathrm{MHV}}\, s_{12}s_{45}s_{234}(R_{251}+R_{524})\,. \tag{4.26}$$

Indeed, it is intuitively clear that because of its topology, $I^{(7)}$ can appear in the coefficient (4.25) only in terms that have no additional propagators.

Carefully repeating this analysis for the other even invariants implies that the complete contribution of this supersector to the even part of $I^{(7)}$'s coefficient is,

$$\begin{aligned}
&-\frac{1}{4}\mathcal{A}_6^{(0),\mathrm{MHV}}(R_{146}+R_{413})s_{123}(s_{234}s_{345}-s_{61}s_{34})\\
&-\frac{1}{4}\mathcal{A}_6^{(0),\mathrm{MHV}}(R_{362}+R_{635})s_{345}(s_{123}s_{234}-s_{23}s_{56})\\
&+\frac{1}{2}\mathcal{A}_6^{(0),\mathrm{MHV}}(R_{251}+R_{524})s_{12}s_{45}s_{234}\,.
\end{aligned} \tag{4.27}$$

This conclusion must be checked against the other configurations in fig. 7. Fig. 7(iv) is the parity conjugate of fig. 7(ii) and should therefore yield the same result for the even R invariants (and its negative for the odd-parity ones).

The configurations in figs. 7(i) and 7(iii) and are parity conjugates of each other. Evaluating them following the same steps yields,

$$\mathcal{C}_{7(i)} = \mathcal{A}_6^{(0),\mathrm{MHV}}\frac{s_{12}s_{45}\delta^{(4)}([61]\eta_5^A+[15]\eta_6^A+[56]\eta_1^A)}{\langle 23\rangle\langle 34\rangle[56][61]\langle 4(2+3)1]\langle 2(1+6)5]} = \mathcal{A}_6^{(0),\mathrm{MHV}}R_{251}s_{12}s_{45}s_{234}\,, \tag{4.28}$$

$$\mathcal{C}_{7(iii)} = \mathcal{A}_6^{(0),\mathrm{MHV}}\frac{s_{12}s_{45}\delta^{(4)}([34]\eta_2^A+[42]\eta_3^A+[23]\eta_4^A)}{\langle 16\rangle\langle 65\rangle[23][34]\langle 5(1+6)2]\langle 1(2+3)4]} = \mathcal{A}_6^{(0),\mathrm{MHV}}R_{524}s_{12}s_{45}s_{234}\,. \tag{4.29}$$

Unlike the configuration in fig. 7(ii), the loop-momentum dependence here cancels completely after integration over the internal Grassmann variables. One may verify that evaluating (4.27) on the relevant internal kinematic configuration reproduces eqs. (4.28) and (4.29).

The component of the flying-squirrel cut of fig. 2(d) that we evaluated shows that this cut contributes to all two-loop scalar functions $W_i^{(2)}$. The same is true for the other cuts with this topology but with cyclicly-permuted external legs. The sum over cyclic permutations may be reorganized in terms of permutations of a single "even" spin coefficient $R_{146}+R_{413}$ which multiplies three different integrals of the type $I^{(7)}$ with different assignments of external legs. We will use this presentation in the following section.

V. THE TWO-LOOP SIX-POINT SUPERAMPLITUDE

We determined the four-dimensional cut-constructible even parts $W_i^{(2),D=4}$ of the two-loop six-point NMHV superamplitude,

$$W_i^{(2),\mathrm{NMHV}} = W_i^{(2),D=4} + W_i^{(2),\mu}, \tag{5.1}$$

as explained in the previous section, by analyzing the cuts shown in fig. 2 or next-to-maximal versions of them. We obtained an explicit expression for the remaining part, $W_i^{(2),\mu}$, cut-constructible only in D-dimensions, by comparing the results of D-dimensional and four-dimensional cut calculations. We have carried out the calculation without assuming a specific (possibly overcomplete) basis of two-loop integrals and found that the integrals listed in figs. 4 and 8 are necessary and sufficient to saturate the cuts of the even part of the amplitude through $\mathcal{O}(\epsilon^0)$. For comparison, and because of changes in the labeling of these integrals with respect to the original calculations of the two-loop six-point MHV amplitude [19], we also present it in our labeling.

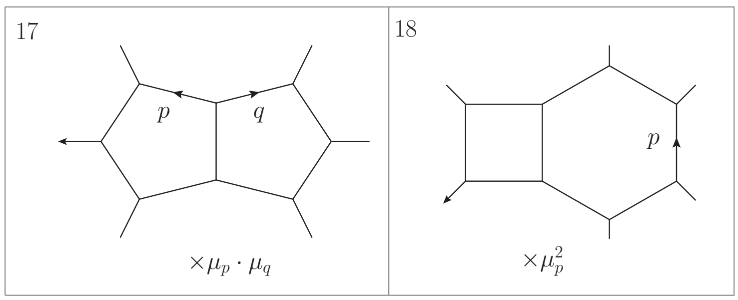

FIG. 8: μ-integrals entering the two-loop six-point amplitudes. The arrow on the external line indicates leg number 1.

A. The NMHV amplitude

The four-dimensional cut-constructible even part of all six-point two-loop amplitudes is built out of a sum of the 16 integrals shown in fig. 4

$$\begin{aligned} S^{(2),D=4}(\mathbb{1}) \;=\;& \frac{1}{4}c_1 I^{(1)}(\epsilon) + c_2 I^{(2)}(\epsilon) + \frac{1}{2}c_3 I^{(3)}(\epsilon) + \frac{1}{2}c_4 I^{(4)}(\epsilon) + c_5 I^{(5)}(\epsilon) + c_6 I^{(6)}(\epsilon) \\ &+ \frac{1}{4}\left(c_{7a}\mathbb{P}^{-2}I^{(7)}(\epsilon) + c_{7b}\mathbb{P}^{-1}I^{(7)}(\epsilon) + c_{7c}I^{(7)}(\epsilon)\right) + \frac{1}{2}c_8 I^{(8)}(\epsilon) + c_9 I^{(9)}(\epsilon) \\ &+ c_{10} I^{(10)}(\epsilon) + c_{11} I^{(11)}(\epsilon) + \frac{1}{2}c_{12} I^{(12)}(\epsilon) + \frac{1}{2}c_{13} I^{(13)}(\epsilon) \\ &+ \frac{1}{2}c_{14} I^{(14)}(\epsilon) + \frac{1}{2}c_{15} I^{(15)}(\epsilon) + c_{16} I^{16}(\epsilon)\,. \end{aligned} \quad (5.2)$$

The coefficients c_i, which differ between the MHV and the NMHV amplitudes, are functions of external momenta and the numerical coefficients are symmetry factors reflecting the symmetries of each integral under cyclic permutations of external legs.

The functions $W_i^{(2),D=4}$ are constructed by summing $S^{(2),D=4}$ over the sets of permutations $\mathcal{S}_i$ in eqs. (2.9), (2.10) and (2.11) that map each superinvariant $(R_{i+3,i,i+2} + R_{i,i+i,i+5})$ into itself:

$$W_i^{(2),D=4} = \frac{1}{8} \sum_{\sigma \in \mathcal{S}_i} S^{(2),D=4}(\sigma) + \mathcal{O}(\epsilon)\,. \tag{5.3}$$

Of the overall factor of 1/8, a factor of 1/4 emerges from the calculation of the unitarity cuts and a factor of 1/2 is due to our choice of normalization.

The coefficients c_j in the identity permutation entering $S^{(2),D=4}$ in eq. (5.2) are:

$$\begin{aligned}
c_1 &= -s_{123}^2 s_{34} s_{61} + s_{123}^2 s_{234} s_{345} & c_2 &= 2 s_{12}^2 s_{23} \\
&\quad - s_{123} s_{234} s_{12} s_{45} - s_{123} s_{345} s_{23} s_{56} + 2 s_{12} s_{23} s_{45} s_{56} & & \\
c_3 &= s_{123}(s_{123} s_{345} - s_{12} s_{45}) & c_4 &= s_{123}^2 s_{34} \\
c_5 &= -s_{12} s_{123} s_{234} & c_6 &= s_{61} s_{12} s_{123} \\
c_{7a} &= -s_{123}(s_{345} s_{234} - s_{61} s_{34}) & c_{7b} &= 2 s_{123} s_{34} s_{61} \\
c_{7c} &= -s_{123}(s_{234} s_{345} - s_{61} s_{34}) & c_8 &= 0 \\
c_9 &= s_{123} s_{45} s_{56} & c_{10} &= s_{56} s_{123} s_{345} \\
c_{11} &= -s_{56} s_{61} s_{123} & c_{12} &= -s_{123}(s_{123} s_{345} - s_{12} s_{45}) \\
c_{13} &= s_{123}^2 s_{61} & c_{14} &= 2 s_{34}^2 s_{123} \\
c_{15} &= 0 & c_{16} &= 2 s_{12} s_{34} s_{123} \\
c_{17} &= \text{not necessary} & c_{18} &= \tfrac{1}{6} s_{123}(2 s_{34} s_{61} - s_{234} s_{345})
\end{aligned} \tag{5.4}$$

In dimensional regularization, the six-point two-loop (and quite likely all higher-point higher-loop) amplitudes receive contributions from integrals — collectively referred to as "μ-integrals" — whose integrand vanishes identically when evaluated in four dimensions. The integrals shown in fig. 8 are of this type, where μ_p and μ_q denote the (-2ϵ) components of the loop momenta. As noted in ref. [19], the integral $I^{(17)}$ vanishes identically as $\epsilon \to 0$; we will therefore ignore it in the following. To determine the contributions of such integrals we compare the result of the four-dimensional cut calculation with that of the D-dimensional cuts and find that the even part of the amplitude also contains the terms,

$$W_i^{(2),\mu} = \left(\sum_{\sigma \in \mathcal{S}_i} \frac{1}{4} c_{18} \right) \sum_{\sigma \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3} \frac{1}{2} s_{12} I^{(18)}(\sigma)\,. \tag{5.5}$$

The coefficients c_j bear certain similarities to the corresponding coefficients in the MHV amplitude.

B. The MHV amplitude

For completeness, and because of differences of notation from ref. [19], we also present the integrand of the even part of the MHV amplitude. The four-dimensional cut-constructible part is given by

$$M^{(2),D=4} = \frac{1}{16} \sum_{\sigma \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3} S^{(2),D=4}(\sigma) + \mathcal{O}(\epsilon) \tag{5.6}$$

where the coefficients c_j in the identity permutation are given by

$$\begin{aligned}
c_1 &= s_{123}\big(s_{12}s_{45}s_{234} + s_{23}s_{56}s_{345} & c_2 &= 2s_{23}s_{12}^2 \\
&\quad + s_{123}(s_{34}s_{61} - s_{234}s_{345})\big) & & \\
c_3 &= s_{123}(s_{345}s_{123} - s_{45}s_{12}) & c_4 &= s_{34}s_{123}^2 \\
c_5 &= s_{12}(s_{234}s_{123} - 2s_{23}s_{56}) & c_6 &= -s_{61}s_{12}s_{123} \\
c_{7a} &= s_{123}(s_{234}s_{345} - s_{34}s_{61}) & c_{7b} &= -4s_{34}s_{61}s_{123} \\
c_{7c} &= s_{123}(s_{234}s_{345} - s_{34}s_{61}) & c_8 &= 2s_{12}(s_{345}s_{123} - s_{12}s_{45}) \\
c_9 &= s_{45}s_{56}s_{123} & c_{10} &= s_{56}(2s_{12}s_{45} - s_{123}s_{345}) \\
c_{11} &= s_{61}s_{56}s_{123} & c_{12} &= s_{123}(s_{345}s_{123} - s_{12}s_{45}) \\
c_{13} &= -s_{123}^2 s_{61} & c_{14} &= 0 \\
c_{15} &= 0 & c_{16} &= 0 \\
c_{17} &= -2s_{123}s_{345}(s_{234}s_{345} - s_{61}s_{34}) & c_{18} &= 2s_{12}(s_{123}s_{234}s_{345} - s_{12}s_{45}s_{234}) \\
&\quad + 2s_{345}(s_{12}s_{45}s_{234} + s_{23}s_{56}s_{345}) & &\quad - 2s_{12}(s_{23}s_{56}s_{345} + s_{34}s_{61}s_{123})
\end{aligned} \tag{5.7}$$

The μ-integral contribution is

$$M_6^{(2),\mu} = \frac{1}{16} \sum_{\sigma \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3} \left[\frac{1}{4} c_{17} I^{(17)}(\sigma) + \frac{1}{2} c_{18} I^{(18)}(\sigma) \right]. \tag{5.8}$$

As mentioned previously, $I^{(17)}$ starts at $\mathcal{O}(\epsilon)$ [19] and thus does not contribute through $\mathcal{O}(\epsilon^0)$.

C. A Comparison of the MHV and NMHV Amplitudes

A direct inspection of the integrals in fig. 4 and of their coefficients in eq. (5.4) reveals that the even part of the two-loop six-point NMHV amplitude is a sum of pseudo-conformal integrals. This is similar to the MHV amplitude, for which the four-dimensional cut-constructible part has a similar property [19], as may also be seen by directly inspecting the coefficients listed in eq. (5.7).

This is perhaps not completely surprising in light of the argument presented in section III that all four-dimensional cuts can be reproduced by cuts of pseudo-conformal integrals. This structure does not guarantee, however, that the even part of the amplitude is dual conformal invariant, even after infrared divergences are removed appropriately. We return to this point in the next section.

The structure of the NMHV amplitude is quite similar to that of the MHV amplitude, with only subtle differences in the values of the coefficients. Two of the integrals that did not contribute to the MHV amplitude — $I^{(14)}(\epsilon)$ and $I^{(16)}(\epsilon)$ — enter in the NMHV amplitude with nonvanishing coefficient; similarly, a topology that exists in both amplitudes — $I^{(1)}(\epsilon)$ — appears in the NMHV amplitude with an additional pseudo-conformal numerator. Moreover, an integral that contributes to the MHV amplitude — $I^{(8)}(\epsilon)$ — disappears from the NMHV one. We note also that a perfectly valid integral — $I^{(15)}(\epsilon)$ — appears in neither the MHV nor NMHV amplitudes. It would be interesting to understand the significance of this observation.

The properties of the MHV and NMHV amplitudes differ from those observed in the four-point amplitudes through five loops:

- All pseudo-conformal integrals appear with relative weights of ± 1 or 0 [9, 37–39].
- An integral appears with coefficient zero if and only if the integral is unregulated after taking its external legs off shell and taking $\epsilon \to 0$ [16].
- It has been proposed that the signs ± 1 of the contributing integrals can be understood by the requirement of cancelling unphysical singularities [80].

It would undoubtedly be interesting to understand the generalization of these features to higher-multiplicity amplitudes.

Although the amplitudes have very similar structures, the ratio of the NMHV six-point superamplitude to the MHV one does not appear to exhibit a transparent organization. This is due to the different structures of the permutation sums that contribute to the independent factors in the amplitudes. Indeed, each function $W_i^{(1),D=4}$ and $W_i^{(2),D=4}$ contains only a sum over the permutations in the set $\mathcal{S}_i$ while the functions $M^{(1),D=4}$ and $M^{(2),D=4}$ contain sums over all twelve permutations $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.

VI. DUAL CONFORMAL INVARIANCE

The explicit calculation in section IV of the two-loop NMHV six-point superamplitude shows that it indeed has the structure anticipated in section III. We obtained explicit integral representations for the scalar functions $W_i^{(2)}$, summarized in the previous section.

As it is the case with all massless theories in four dimensions, the amplitude is infrared divergent; to examine the dual conformal properties of the amplitude it is necessary to isolate these divergences. Based on the universality of infrared singularities and their exponentiation DHKS proposed [24] that these divergences be removed by simply dividing by the MHV amplitude. This ratio (2.35) was conjectured to be dual conformally invariant. An alternative method was described in section III, see eq. (3.16).

Since the functions $W_i^{(l)}$ have a natural decomposition into four-dimensional and D-dimensional cut-constructible contributions, the functions $C_i^{(l)}$ introduced in equation (3.9) inherit a similar decomposition

$$W_i^{(l)} = W_i^{(l),D=4} + W_i^{(l),\mu} \quad \longrightarrow \quad C_i^{(l)} = C_i^{(l),D=4} + C_i^{(l),\mu} \qquad l = 1, 2 \ . \tag{6.1}$$

At one loop, the μ-integral contribution $C_i^{(1),\mu}$ vanishes in the limit $\epsilon \to 0$. Because of infrared divergences, they nevertheless give rise to nontrivial contributions at two loops in the ratio with the MHV amplitude. The functions $C_i^{(l)}$ contain terms that vanish as $\epsilon \to 0$:

$$\begin{aligned} C_i^{(1)}(\epsilon) &= W_i^{(1)}(\epsilon) - M^{(1)}(\epsilon) \\ C_i^{(2)}(\epsilon) &= W_i^{(2)}(\epsilon) - M^{(2)}(\epsilon) - M^{(1)}(\epsilon)(W_i^{(1)}(\epsilon) - M^{(1)}(\epsilon)) \ . \end{aligned} \tag{6.2}$$

As explained in sect. II C, in the limit $\epsilon \to 0$, $C_i^{(1)}(\epsilon)$ reduces to dual conformal invariant functions very closely related to the $V^{(i)}$ defined in ref. [24]. The higher-order terms in ϵ were not considered in ref. [24]; we take them as determined by the amplitude through the relation (3.9) between $W_i^{(l)}$ and $C_i^{(l)}$.

The infrared-divergent terms in both $W_i^{(1)}(\epsilon)$ and $M^{(1)}(\epsilon)$ have the usual form

$$\mathrm{Div}_6^{(1)} = -\frac{1}{2\epsilon^2} \sum_{j=1}^{6} (-s_{j,j+1})^{-\epsilon} \ . \tag{6.3}$$

Thus, $C_i^{(1)}(\epsilon)$ is manifestly finite; moreover, the only divergent contribution arising from the last term in $C_i^{(2)}(\epsilon)$ is due to the overall factor of $M^{(1)}(\epsilon)$.

A. Terms Requiring D-dimensional Cuts

At one loop, the μ-integrals, requiring consideration of D-dimensional cuts, yield only terms of $\mathcal{O}(\epsilon)$ in both $W_i^{(1)}(\epsilon)$ and $M^{(1)}(\epsilon)$. This allows us to isolate the μ-integral contribution in eq. (6.2):

$$C_i^{(2),\mu} = W_i^{(2),\mu} - W_i^{(1),\mu}\mathrm{Div}_6^{(1)} - M^{(2),\mu} + M^{(1),\mu}\mathrm{Div}_6^{(1)} + \mathcal{O}(\epsilon)\,, \tag{6.4}$$

where we used the universality of the one-loop infrared divergences (6.3) and kept only the terms that have nontrivial divergent and finite parts. For example, we dropped the terms in eq. (6.2) coming from the finite part of the overall factor $M^{(1)}(\epsilon)$ in the last term.

The last two terms in the equation above contain information already available in the MHV amplitude. Indeed, this exact combination appears in the iteration of the μ-integrals for this amplitude [19]:

$$M^{(2),\mu} = M^{(1),\mu}\mathrm{Div}_6^{(1)}\,. \tag{6.5}$$

Thus, $C_i^{(2),\mu}$ is given by,

$$C_i^{(2),\mu} = W_i^{(2),\mu} - W_i^{(1),\mu}\mathrm{Div}_6^{(1)}\,, \tag{6.6}$$

with $W_i^{(2),\mu}$ given by eq. (5.5).

This expression for $W_i^{(2),\mu}$ may be further simplified by making use of the special properties of the hexabox integral discussed in section IV.A of ref. [19], in particular eq. (4.6):

$$I^{(18)}[\mu^2] = -\frac{1}{\epsilon^2}(-s_{12})^{-1-\epsilon} I^{\mathrm{hex}}[\mu^2] \quad . \tag{6.7}$$

Thus, $W_i^{(2),\mu}$ can be expressed exactly in terms of one-loop integrals, albeit in six dimensions. Moreover, using the fact that the one-loop hexagon integral is invariant under cyclic permutations of external legs, $W_i^{(2),\mu}$ can be expressed in terms of the massless six-dimensional hexagon integral with a coefficient given by the universal divergent part of one-loop amplitudes:

$$W_i^{(2),\mu} = -\frac{1}{12}\mathrm{Div}_6^{(1)} I^{\mathrm{hex}}[\mu^2] \sum_{\sigma\in\mathcal{S}_i} s_{123}(2 s_{34} s_{61} - s_{234} s_{345}) + \mathcal{O}(\epsilon) \quad . \tag{6.8}$$

The four terms in each sum are in fact equal.

The μ-integral contribution to the one-loop six-point NMHV amplitude is not yet available in the literature. Information on its structure may be obtained by analyzing a two-particle cut of the μ-integral contribution to the two-loop NMHV superamplitude. Dixon and Schabinger [25] have evaluated such a cut directly; quite surprisingly, they find that it can be organized in terms of the same R invariants as the four-dimensional cut-constructible terms. The μ-integrals' contribution to the even part of the one-loop six-point NMHV amplitude is,

$$W_i^{(1),\mu} = -\frac{1}{12} I^{\text{hex}}[\mu^2] \sum_{\sigma \in \mathcal{S}_i} s_{123}(2 s_{34} s_{61} - s_{234} s_{345}) \quad . \tag{6.9}$$

Combining this with eqs. (6.8) and (6.6) immediately shows that

$$\lim_{\epsilon \to 0} C_i^{(2),\mu} = 0 \quad . \tag{6.10}$$

In other words, the complete μ-integral contribution to the six-point two-loop NMHV amplitude is completely accounted for by extracting an overall factor of the MHV superamplitude.

Through similar manipulations it is possible to show that the remainder-like functions $R_{6;i}^{(2)}$ introduced in eq. (3.16) do not receive any μ-integral contributions. Indeed, directly expanding eq. (3.16) to $\mathcal{O}(a^2)$ we find that,

$$R_{6;i}^{(2)} = W_i^{(2)}(\epsilon) - \left[\frac{1}{2} \left(W_i^{(1)}(\epsilon) \right)^2 + f_2(\epsilon) W_i^{(1)}(2\epsilon) \right] \quad . \tag{6.11}$$

Identifying the μ-integral contributions to each of the terms on the right hand side and using the universality of infrared divergences implies that

$$R_{6;i}^{(2),\mu} = W_i^{(2),\mu}(\epsilon) - \text{Div}_6^{(1)} W_i^{(1)}(\epsilon) = C_i^{(2),\mu} + \mathcal{O}(\epsilon) \quad . \tag{6.12}$$

It therefore follows from equations (6.10) that $R_{6;i}^{(2),\mu}$ does not receive any finite μ-integral contributions.

The same, however, cannot be said about the μ-integral contribution to the odd part of the amplitude. Indeed, repeating the steps that lead to equations (6.2) we find that the coefficients of the parity-odd quantities $(R_{i+3,i,i+2} - R_{i,i+3,i+5})$ are,

$$\widetilde{C}_i^{(2)} = \widetilde{W}_i^{(2)} - \text{Div}_6^{(1)}\, \widetilde{W}_i^{(1)} \quad . \tag{6.13}$$

While the μ-integral contributions to $\widetilde{W}_i^{(2)}$ are given in terms of the hexabox integral or, equivalently in terms of the six-dimensional hexagon integral, their contributions to $\widetilde{W}_i^{(1)}$ are given in terms of a restricted set of the one-mass pentagon integrals [25]. This suggests that, for the odd part of the superamplitude, the μ integrals cannot be cleanly separated from the four-dimensional cut-constructible terms.

B. Numerical Evaluation of the Amplitude

In order to further analyze the properties of the two-loop six-point NMHV amplitude, we turn to a numerical evaluation of the two-loop integrals. Thanks to the results described in the previous section, we may focus on the four-dimensional cut-constructible part of the amplitude. The task of evaluating the integrals is simplified substantially by the fact that all of them have already been evaluated at several distinct kinematic points in [19]. We have evaluated additional kinematic points using the package `MB` [45] and the same Mellin–Barnes parametrization of integrals that was used in the calculation of the MHV amplitude. Apart from testing the symmetry properties of the amplitude, this calculation also verifies the expected universality of two-loop infrared divergences. Viewed differently, a successful test of the universality of the infrared divergences is a strong indication of the completeness of the cut construction described in previous sections.

We choose Euclidean kinematics for all configurations of external momenta. As in the calculation of the MHV amplitude, the symmetries of the momentum configuration,

$$K^{(0)}: \qquad s_{i,i+1} = -1\,,\ s_{i,i+1,i+2} = -2\,, \tag{6.14}$$

make it particularly useful, as all cyclic permutations or external legs yield the same value for all integrals. This implies that all functions $W_i^{(2),D=4}$ are equal for all $i = 1, 2, 3$. Using the values of the integrals collected in the Appendix B of [19] we find

$$\begin{aligned} W_i^{D=4}(K^{(0)}) &= 1 + a(-\frac{3}{\epsilon^2} + 5.27682 + 8.73314\epsilon + 8.11147\epsilon^2) \\ &\quad + a^2\Big(\frac{9}{2\epsilon^4} - \frac{14.5967}{\epsilon^2} + \frac{25.3014 \pm 0.0043}{\epsilon} - 21.064 \pm 0.002\Big) + \mathcal{O}(a^3). \end{aligned} \tag{6.15}$$

Where they are not explicitly included, the errors do not affect the last quoted digit. We have used the error estimated reported by `CUBA` [81]. In general we found the errors to be reliable, giving an accurate measure of the number of trustworthy digits. In some contributions, however, we found them to be underestimated, invariably in the presence of small integrals with a fast-varying integrand. In such cases, when `CUBA` reports a large χ^2, we take the average value of the integrals to be the central value and quote the variation of the integral under changes of sampling points as the error estimate. The issue presumably involves integration regions missed because of special properties of the integrand.

As discussed previously, the construction of the functions $C_i^{(2)}$ requires keeping higher orders in the small-ϵ expansion of the one-loop amplitude. For the MHV amplitude, we use the expression in terms of the iterated one-loop amplitude and the remainder function $R_6^{(2)}$. For the point $K^{(0)}$ we find

$$\begin{aligned} C_i(a,\epsilon,K^{(0)}) \;=\; & 1 + a(0.783676 + 1.10087\epsilon + 0.07507\epsilon^2) \\ & + a^2\Big(-\frac{0.0036 \pm 0.0043}{\epsilon} - (2.412 \pm 0.002) - R_6^{(2)}(K^{(0)})\Big) + \mathcal{O}(a^3). \end{aligned} \tag{6.16}$$

We note that the residue of the simple pole in ϵ vanishes within errors, as it should. We have confirmed this property for all the other kinematic points[9].

From equation (6.15) we can also find the value of the remainder-like functions $R_{6;i}^{(2)}$ introduced in equation (3.16) at the point $K^{(0)}$:

$$R_{6;i}^{(2)} = -1.430 \pm 0.002 \quad . \tag{6.17}$$

Similarly to the error quoted for $C_i(a,\epsilon,K^{(0)})$, the error of $R_{6;i}^{(2)}$ is completely inherited from that of $W_i^{(2)}$.

We have evaluated the amplitude at another kinematic point (denoted by $K^{(1)}$ in [19]) related to $K^{(0)}$ by dual conformal transformations as well as two other points related to each other but unrelated to $K^{(0)}$:

$$\begin{aligned} K^{(1)}\ :\ & s_{12} = -0.7236200, \quad s_{23} = -0.9213500, \quad s_{34} = -0.2723200, \quad s_{45} = -0.3582300, \\ & s_{56} = -0.4235500, \quad s_{61} = -0.3218573, \quad s_{123} = -2.1486192, \quad s_{234} = -0.7264904, \\ & s_{345} = -0.4825841, \\ K^{(3)}\ :\ & s_{i,i+1} = -1, \quad s_{123} = -1/2, \quad s_{234} = -5/8, \quad s_{345} = -17/14, \\ K^{(6)}\ :\ & s_{12} = -2, \quad s_{23} = -4, \quad s_{34} = -2, \quad s_{45} = -14/17, \quad s_{56} = -4/5, \quad s_{61} = -56/85, \\ & s_{i,i+1,i+2} = -1, \end{aligned} \tag{6.18}$$

In listing the kinematic points we attempted to preserve the notation for the points used in ref. [19]. We have collected our results for the values of $C_i^{(2)}$ and $R_{6;i}^{(2)}$ in tables I and II. The three dual conformal ratios

$$(u_1,u_2,u_3) = \Big(\frac{s_{12}s_{45}}{s_{123}s_{345}}, \frac{s_{23}s_{56}}{s_{234}s_{456}}, \frac{s_{34}s_{61}}{s_{345}s_{561}}\Big) \tag{6.19}$$

for the kinematic points are listed in the second column of these tables.

[9] We have also verified analytically the cancellation of infrared singular terms through $\mathcal{O}(\epsilon^{-2})$. The complete cancellation of infrared-singular terms was shown analytically by G. Korchemsky (private communication).

TABLE I: Comparison of conformally-related kinematic points. $C_i^{(2)}$ are the finite parts of the ratios $C_i = W_i/M_6$ at two-loops. $R_6^{(2)}$ is the two-loop remainder function of the six-point MHV amplitude.

kinematic pt.	(u_1, u_2, u_3)	$C_1^{(2)} + R_6^{(2)}$	$C_2^{(2)} + R_6^{(2)}$	$C_3^{(2)} + R_6^{(2)}$
$K^{(0)}$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	-2.413 ± 0.002	-2.413 ± 0.002	-2.413 ± 0.002
$K^{(1)}$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	-2.359 ± 0.048	-2.375 ± 0.025	-2.380 ± 0.033
$K^{(3)}$	$(\frac{28}{17}, \frac{16}{5}, \frac{112}{85})$	14.426 ± 0.003	12.614 ± 0.004	11.697 ± 0.009
$K^{(6)}$	$(\frac{28}{17}, \frac{16}{5}, \frac{112}{85})$	14.439 ± 0.078	12.614 ± 0.035	11.727 ± 0.145

TABLE II: Comparison of the remainder-like functions $R_{6;i}^{(2)}$ at conformally-related kinematic points.

kinematic pt.	(u_1, u_2, u_3)	$R_{6;1}^{(2)}$	$R_{6;2}^{(2)}$	$R_{6;3}^{(2)}$
$K^{(0)}$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	-1.431 ± 0.002	-1.431 ± 0.002	-1.431 ± 0.002
$K^{(1)}$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	-1.377 ± 0.048	-1.393 ± 0.025	-1.397 ± 0.033
$K^{(3)}$	$(\frac{28}{17}, \frac{16}{5}, \frac{112}{85})$	5.413 ± 0.003	4.749 ± 0.004	4.602 ± 0.009
$K^{(6)}$	$(\frac{28}{17}, \frac{16}{5}, \frac{112}{85})$	5.427 ± 0.078	4.749 ± 0.035	4.633 ± 0.145

As mentioned previously, we left the MHV remainder function $R_6^{(2)}$ unevaluated. Its dual conformal invariance [19]

$$R_6^{(2)}(K^{(0)}) = R_6^{(2)}(K^{(1)}) \qquad R_6^{(2)}(K^{(3)}) = R_6^{(2)}(K^{(6)}) \tag{6.20}$$

implies that the equality within errors of the relevant entries of table I extends to an equality of the functions $C_i^{(2)}$. Alternatively, we could have evaluated the remainder function from the analytic expression found in ref. [82], the integral representation in ref. [83], or the simplified form in ref. [84]. The results we obtain thus suggest that $C_i^{(2)}$ and $R_{6;i}^{(2)}$ are functions solely of the conformal cross-ratios, that is, that they are indeed invariant under dual conformal transformations.

VII. SUMMARY, CONCLUSIONS AND SOME OPEN QUESTIONS

The maximally supersymmetric gauge theory in four dimensions is an ideal testing ground for probing the properties of gauge theories at both weak and strong coupling. The large degree of symmetry makes perturbative calculations tractable to relatively high orders while its string-theory dual provides powerful tools for understanding its strong-coupling behavior. Its hidden symmetries yield additional constraints that go beyond their initial connection to the integrability of the dilatation operator of the theory.

In this paper we have computed the parity-even part of the two-loop six-point NMHV amplitude using generalized unitarity in superspace. We showed that the result is invariant under dual conformal transformations, after removal of universal infrared divergences (including terms arising from $\mathcal{O}(\epsilon)$ contributions at one loop, computed by Dixon and Schabinger [25]). The dual conformal invariant content may be organized in several different ways. The exponentiation of both the infrared divergences and of the collinear splitting amplitudes suggest the introduction of certain remainder-like functions which, similarly to the remainder function for MHV amplitudes, are functions only of the conformal cross ratios. We have shown that, to all orders in perturbation theory, it should be possible to reconstruct the remainder-like functions by evaluating certain triple-collinear splitting amplitudes.

Several interesting issues related to the calculation described here, and to the structure of the perturbative expansion of the theory and its strong coupling expansion remain to be clarified.

Through the AdS/CFT correspondence, Alday and Maldacena [11] argued that, to leading order in their strong-coupling expansion, all planar scattering amplitudes with fixed number of external legs are essentially identical up to perhaps a rational function of momenta and polarization vectors. Our calculation and arguments show that the weak-coupling structure of the six point amplitude involves at least six different spin factors dressed with scalar and pseudo-scalar functions to all orders in the weak-coupling expansion. Reconciling this structure with the AdS/CFT considerations remains an important open problem, which appears to require that, in the strong-coupling limit, the remainder-like functions introduced in section III have a very simple relation to the MHV remainder function.

Inspired by strong-coupling considerations, several groups showed that MHV amplitudes have a close relation to certain light-like polygonal Wilson loops order-by-order in weak-coupling perturbation theory, first in explicit calculations [16–19], and very recently, cast in a more general setting [20]. A similar relation for non-MHV amplitudes remains an open question; our numerical results provide check points for future calculations in this direction. The Wilson-loop formulation of the six-point MHV amplitude led to the analytic evaluation [82–84] of the remainder function at two loops. It seems likely that a Wilson-loop formulation of NMHV amplitudes will allow a similar evaluation of the remainder-like functions characterizing this amplitude.

A direct comparison of the integrands of the six-point MHV and NMHV amplitudes at two loops reveals that certain integrals appear in one but not the other, while the contributing integrals enter with numerical coefficients that do not conform with the effective rules inferred from four-point amplitudes. Moreover, one perfectly valid pseudo-conformal integral does not appear in either one of these amplitudes. It would be interesting to develop a better understanding of this pattern of numerical coefficients. Evaluation of higher-point NMHV amplitudes at two loops may help in this direction.

In our calculation dual conformal invariance is obscured by the dimensional regulator; removal of infrared divergences is a crucial step in studying the dual conformal properties of scattering amplitudes. Using four and five-point amplitudes as testing ground, it was shown [32, 85] that regulating the infrared divergences by a particular symmetry breaking of the gauge group makes dual conformal invariance more transparent. It would be interesting to repeat the calculation described in this paper as well as that of the two-loop six-point MHV amplitude in this framework. Apart from a better understanding of dual conformal invariance, such an endeavor would clarify the interpretation of μ-integrals as well as that of the parity-odd component of amplitudes. While we did not compute the parity-odd part of the six-point NMHV amplitude explicitly, its μ-integral contributions make it quite different from the parity-odd terms in MHV amplitudes with up to six external legs.

A very interesting question relates to symmetries of scattering amplitudes. Drummond, Henn and Plefka showed that tree-level scattering amplitudes are invariant under a Yangian which is the closure of conformal and dual conformal transformations [86]. Up to anomalies introduced by the regulator, it is expected that dual conformal transformations leave scattering amplitudes invariant to all orders in perturbation theory. Again apart from anomalies due to the presence of a regulator, ordinary conformal invariance exhibits additional anomalies of a holomorphic type [87], related to singular momentum configurations, already at tree level. An appropriate definition of a generating function for superamplitudes with variable number of external legs [88] allows this anomaly to be circumvented, and to be realized on scattering amplitudes through the one-loop level. Korchemsky and Sokatchev have recently given [34] a general construction of conformal and dual conformal invariants (and hence of Yangian invariants). Other approaches to the construction of Yangian invariants were discussed by Mason and Skinner [89] and by Drummond and Ferro [90]. These invariants have expressions that were conjectured [65] by Arkani-Hamed *et al.* to represent the leading singularities of scattering amplitudes to all orders in perturbation theory. All-order leading singularities have been derived directly by Bullimore, Mason, and Skinner [33]. While the form of subleading singularities is not yet clear, this structure suggests that it may be possible to realize both symmetries (and their closure) at higher loops. The extended algebra does not uniquely determine the S-matrix of $\mathcal{N} = 4$ super-Yang-Mills theory, however [34]. Unraveling its constraints on scattering amplitudes should prove fruitful.

The study of non-MHV amplitudes at higher loops is only in its early stages. Our explicit calculation provides an example of such an amplitude. The structure of these amplitudes is substantially richer than that of MHV amplitudes. It seems likely that new and exciting properties as well as new calculational techniques are waiting to be discovered.

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