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Cosmology in nonrelativistic general covariant theory of gravity

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Horava and Melby-Thompson recently proposed a new version of the Horava-Lifshitz theory of gravity, in which the spin-0 graviton is eliminated by introducing a Newtonian pre-potential φ and a local $U(1)$ gauge field A . In this paper, we first derive the corresponding Hamiltonian, super-momentum constraints, the dynamical equations, and the equations for φ and A , in the presence of matter fields. Then, we apply the theory to cosmology, and obtain the modified Friedmann equation and the conservation law of energy, in addition to the equations for φ and A . When the spatial curvature is different from zero, terms behaving like dark radiation and stiff-fluid exist, from which, among other possibilities, bouncing universe can be constructed. We also study linear perturbations of the FRW universe with any given spatial curvature k , and derive the most general formulas for scalar perturbations. The vector and tensor perturbations are the same as those recently given by one of the present authors [A. Wang, Phys. Rev. D **82**, 124063 (2010)] in the setup of Sotiriou, Visser and Weinfurtner. Applying these formulas to the Minkowski background, we have shown explicitly that the scalar and vector perturbations of the metric indeed vanish, and the only remaining modes are the massless spin-2 gravitons.

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I. INTRODUCTION

Recently, Horava proposed a quantum gravity theory [1], motivated by the Lifshitz theory in solid state physics [2], for which the theory is often referred to as the Horava-Lifshitz (HL) theory. It is non-relativistic and power-counting ultraviolet (UV)-renormalizable, and was expected to recover general relativity (GR) in the infrared (IR) limit. HL theory has attracted a great deal of attention due to its several remarkable features, such as the divergence of its effective speed of light in the UV, which could potentially resolve the horizon problem without invoking inflation [3]. Scale-invariant super-horizon curvature perturbations can also be produced without inflation [4–11], and dark matter and dark energy can have their geometric origins [12, 13]. Furthermore, bouncing universe can be easily constructed due to the high-order derivative terms of the spacetime curvature [14–16]. For detail, we refer readers to [17] and references therein.

The HL theory is based on the perspective that Lorentz symmetry should appear as an emergent symmetry at long distances, but can be fundamentally absent at high energies [18]. With this in mind, Horava considered systems whose scaling at short distances exhibits a strong anisotropy between space and time,

$$\mathbf{x} \rightarrow \ell \mathbf{x}, \quad t \rightarrow \ell^z t. \quad (1.1)$$

In $(d+1)$ -dimensional spacetimes, in order for the theory to be power-counting renormalizable, it requires $z \geq d$. At low energies, the theory is expected to flow to $z = 1$, whereby the Lorentz invariance is “accidentally restored.” Such an anisotropy between time and space

can be easily realized when one writes the metric in the Arnowitt-Deser-Misner (ADM) form [19],

$$ds^2 = -N^2 c^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (i, j = 1, 2, 3). \quad (1.2)$$

Under the rescaling (1.1) with $z = d = 3$, a condition we shall assume in the rest of this paper, the dynamical variables N , N^i and g_{ij} scale, respectively, as,

$$N \rightarrow N, \quad N^i \rightarrow \ell^{-2} N^i, \quad g_{ij} \rightarrow g_{ij}. \quad (1.3)$$

The gauge symmetry of the system are the foliation-preserving diffeomorphisms $\text{Diff}(M, \mathcal{F})$,

$$\tilde{t} = t - f(t), \quad \tilde{x}^i = x^i - \zeta^i(t, \mathbf{x}), \quad (1.4)$$

for which the dynamical variables change as

$$\begin{aligned} \delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + \dot{f} g_{ij}, \\ \delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \\ \delta N &= \zeta^k \nabla_k N + \dot{N} f + N \dot{f}, \end{aligned} \quad (1.5)$$

where $\dot{f} \equiv df/dt$, ∇_i denotes the covariant derivative with respect to the 3-metric g_{ij} , and $N_i = g_{ik} N^k$, etc. From these expressions one can see that the lapse function N and the shift vector N_i play the role of gauge fields of the $\text{Diff}(M, \mathcal{F})$ symmetry. Therefore, it is natural to assume that N and N_i inherit the same dependence on spacetime as the corresponding generators, in addition to the fact that the dynamical variables g_{ij} should in general depend on both time and space, that is,

$$N = N(t), \quad N_i = N_i(t, x), \quad g_{ij} = g_{ij}(t, x), \quad (1.6)$$

which is clearly preserved by the $\text{Diff}(M, \mathcal{F})$, and often referred to as the projectability condition.

Due to the restricted diffeomorphisms (1.4), one more degree of freedom appears in the gravitational sector -

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a spin-0 graviton. This is potentially dangerous, and needs to decouple in the IR regime, in order to be consistent with observations. Unfortunately, it was shown that this might not be the case. In particular, the spin-0 mode is not stable in the original version of the HL theory [1] as well as in the Sotiriou, Visser and Weinfurter (SVW) generalization [7, 8]. Note that in both of these two versions, it was all assumed the projectability condition. In addition, these instabilities are all found in the Minkowski background. Recently, it was found that the de Sitter spacetime is stable in the SVW setup [17]. So, one can take the latter as its legitimate background, similar to what happened in the massive gravity [20]. However, the strong coupling problem still exists [21, 22], although it might be circumvented by the Vainshtein mechanism [23], as recently showed in the spherical [24] and cosmological [22] cases.

On the other hand, giving up the projectability condition, that is, assuming that the lapse function N depends on both time and spatial coordinates, Blas, Pujolas and Sibiryakov (BPS) [25] found that inclusion of terms made of a_i ,

$$a_i = \partial_i \ln(N), \quad (1.7)$$

can cure the instability of the Minkowski spacetime. By properly choosing the coupling constants, the strong coupling problem [26–29] can be also addressed [30]. However, a price to pay is the enormous number of independent coupling constants: only the sixth-order derivative terms in the potential are more than 60 [29]. It should be also noted that giving up the projectability condition often causes the theory to suffer the inconsistency problem [31]. Kluson recently showed that the Hamiltonian formalism of the BPS model is very rich, and the corresponding algebra of constraints is well-defined [32].

To cure the instability problem, another very attractive way is to eliminate the spin-0 graviton from the theory, so that the resulting one has as many generators per space-time point as GR does. This is done recently by Horava and Melby-Thompson (HMT) [33] (with the assumption of the projectability condition (1.6)) by extending the foliation-preserving-diffeomorphisms, $\text{Diff}(M, \mathcal{F})$, to include a local $U(1)$ symmetry,

$$U(1) \ltimes \text{Diff}(M, \mathcal{F}). \quad (1.8)$$

Effectively, the spatial diffeomorphism symmetries of GR are kept intact, but its time reparametrization symmetry is linearized and the corresponding algebra is contracted to a local gauge symmetry [34]. The restoration of general covariance, characterized by Eq.(1.8), nicely maintains the special status of time, so that the anisotropic scaling (1.1) with $z > 1$ can still be realized.

A remarkable by-product of this “non-relativistic general covariant” setup is that it forces the coupling constant λ , introduced originally to characterize the deviation of the kinetic part of the action from GR [1], to take exactly its relativistic value $\lambda = 1$. Note that in GR

the spacetime diffeomorphism symmetry, $\text{Diff}(M)$,

$$\tilde{x}^\mu = x^\mu - \zeta^\mu(t, \mathbf{x}), \quad (\mu = 0, 1, 2, 3) \quad (1.9)$$

also forces $\lambda = 1$ and protects this value from quantum corrections.

At short distances, the theory exhibits a high anisotropy between time and space. As a result, the UV behavior of the theory is dramatically improved. At long distances, the theory is driven to an IR regime, where it shares many features with GR. In particular, under the influences of the relevant terms, the scaling is naturally isotropic with the relativistic value $z = 1$. Moreover, since the extended symmetry forces $\lambda = 1$, in the IR limit the action will be dominated exactly by the Einstein-Hilbert terms in the ADM decomposition [19].

In this paper, we investigate this new version of the HL theory. Specifically, in Sec. II we first give a brief review of it, and then derive the corresponding Hamiltonian, super-momentum constraints, the dynamical equations, and the field equations for the Newtonian pre-potential φ and the $U(1)$ gauge field A , in the presence of matter fields. The potential used in this paper is the one constructed by SVW [7], which represents the most general form, subjected to the assumptions that it respects the parity and its highest order of the spatial derivatives is six, the minimal requirement to have the theory be power-counting renormalizable [1]. In Sec. III, we apply the theory to cosmology, and obtain the modified Friedmann equation and conservation law of energy, in addition to the equations for φ and A . In Sec. IV we study the linear perturbations of the FRW universe with any given spatial curvature k , and present the general formulas for scalar perturbations. The vector and tensor perturbations are the same as those given by one of the current authors in [11] in the SVW setup [7], because the gauge field and the Newtonian pre-potential have no contributions to these parts. Applying these formulas to the Minkowski background in Sec. V, we show explicitly that the scalar and vector perturbations of the metric vanish identically. The only non-vanishing dynamical variables are the traceless and divergence-free tensor H_{ij} , which describes the massless spin-2 graviton, a situation that is precisely the same as in GR. These results are consistent with the ones obtained earlier in [33]. In Sec. VI, we present our main conclusions.

II. NON-RELATIVISTIC GENERAL COVARIANT HL THEORY

In order to limit the spin-0 graviton, HMT introduced two new fields, the $U(1)$ gauge field A and the Newtonian pre-potential φ , where in general both of them depend on space and time,

$$A = A(t, x^k), \quad \varphi = \varphi(t, x^k). \quad (2.1)$$

Note that the notations used in this paper are slightly different from those adopted in [33]¹. Under the $\text{Diff}(M, \mathcal{F})$, these fields transfer as,

$$\begin{aligned}\delta A &= \zeta^i \partial_i A + \dot{f} A + f \dot{A}, \\ \delta \varphi &= f \dot{\varphi} + \zeta^i \partial_i \varphi,\end{aligned}\quad (2.2)$$

while under the local $U(1)$, they, together with g_{ij} , transfer as

$$\begin{aligned}\delta_\alpha A &= \dot{\alpha} - N^i \nabla_i \alpha, \quad \delta_\alpha \varphi = -\alpha, \\ \delta_\alpha N_i &= N \nabla_i \alpha, \quad \delta_\alpha g_{ij} = 0 = \delta_\alpha N,\end{aligned}\quad (2.3)$$

where α is the generator of the local $U(1)$ gauge symmetry. For the detail, we refer readers to [33].

The total action is given by,

$$S = \zeta^2 \int dt d^3x N \sqrt{g} \left(\mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_\varphi + \mathcal{L}_A + \frac{1}{\zeta^2} \mathcal{L}_M \right), \quad (2.4)$$

where $g = \det g_{ij}$, and

$$\begin{aligned}\mathcal{L}_K &= K_{ij} K^{ij} - K^2, \\ \mathcal{L}_\varphi &= \varphi \mathcal{G}^{ij} \left(2K_{ij} + \nabla_i \nabla_j \varphi \right), \\ \mathcal{L}_A &= \frac{A}{N} \left(2\Lambda_g - R \right).\end{aligned}\quad (2.5)$$

Here the coupling constant Λ_g , acting like a 3-dimensional cosmological constant, has the dimension of $(\text{length})^{-2}$. The Ricci and Riemann terms all refer to the three-metric g_{ij} , K_{ij} is the extrinsic curvature, and \mathcal{G}_{ij} is the 3-dimensional “generalized” Einstein tensor, defined, respectively, by

$$\begin{aligned}K_{ij} &= \frac{1}{2N} (-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i), \\ \mathcal{G}_{ij} &= R_{ij} - \frac{1}{2} g_{ij} R + \Lambda_g g_{ij}.\end{aligned}\quad (2.6)$$

\mathcal{L}_M is the matter Lagrangian density, which in general is a function of all the dynamical variables, $U(1)$ gauge field, and the Newtonian prepotential,

$$\mathcal{L}_M = \mathcal{L}_M(N, N_i, g_{ij}, \varphi, A; \chi), \quad (2.7)$$

where χ denotes collectively the matter fields. \mathcal{L}_V is an arbitrary $\text{Diff}(\Sigma)$ -invariant local scalar functional built out of the spatial metric, its Riemann tensor and spatial covariant derivatives, without the use of time derivatives. In the original approach of Horava [1], the detailed balance condition was imposed, in order to limit the number

of the coupling constants. With this condition, \mathcal{L}_V takes the simple form,

$$\mathcal{L}_V = w^2 C_{ij} C^{ij}, \quad (2.8)$$

where w is a coupling constant, and C_{ij} denotes the Cotton tensor, defined by

$$C^{ij} = \epsilon^{ikl} \nabla_k \left(R_l^j - \frac{1}{4} R \delta_l^j \right). \quad (2.9)$$

In [7], by assuming that the highest order derivatives are six and that the theory respects the parity, SVW constructed the most general form of \mathcal{L}_V , given by

$$\begin{aligned}\mathcal{L}_V &= \zeta^2 g_0 + g_1 R + \frac{1}{\zeta^2} (g_2 R^2 + g_3 R_{ij} R^{ij}) \\ &+ \frac{1}{\zeta^4} \left(g_4 R^3 + g_5 R R_{ij} R^{ij} + g_6 R_j^i R_k^j R_i^k \right) \\ &+ \frac{1}{\zeta^4} [g_7 R \nabla^2 R + g_8 (\nabla_i R_{jk}) (\nabla^i R^{jk})],\end{aligned}\quad (2.10)$$

where the coupling constants g_s ($s = 0, 1, 2, \dots, 8$) are all dimensionless. The relativistic limit in the IR requires $g_1 = -1$ and $\zeta^2 = 1/(16\pi G)$ [7]. In this paper, we shall be concerned only with this potential, and our formulas to be obtained below can be easily generalized to other forms of the potential, including the one given by Eq.(2.8), and the $f(R)$ term studied in [13].

Variation with respect to the lapse function $N(t)$ yields the Hamiltonian constraint,

$$\int d^3x \sqrt{g} (\mathcal{L}_K + \mathcal{L}_V - \varphi \mathcal{G}^{ij} \nabla_i \nabla_j \varphi) = 8\pi G \int d^3x \sqrt{g} J^t, \quad (2.11)$$

where

$$J^t = 2 \frac{\delta(N \mathcal{L}_M)}{\delta N}. \quad (2.12)$$

Variation with respect to the shift N^i yields the super-momentum constraint,

$$\nabla_j (\pi^{ij} - \varphi \mathcal{G}^{ij}) = 8\pi G J^i, \quad (2.13)$$

where the super-momentum π^{ij} and matter current J^i are defined as

$$\begin{aligned}\pi^{ij} &\equiv \frac{\delta(N \mathcal{L}_K)}{\delta \dot{g}_{ij}} = -K^{ij} + K g^{ij}, \\ J^i &\equiv -N \frac{\delta \mathcal{L}_M}{\delta N_i}.\end{aligned}\quad (2.14)$$

Similarly, variations of the action with respect to φ and A yield,

$$\mathcal{G}^{ij} (K_{ij} + \nabla_i \nabla_j \varphi) = 8\pi G J_\varphi, \quad (2.15)$$

$$R = 2\Lambda_g + 8\pi G J_A, \quad (2.16)$$

¹ In particular, we have $\varphi = -\nu^{HMT}$, $K_{ij} = -K_{ij}^{HMT}$, $\Lambda_g = \Omega^{HMT}$, $\mathcal{G}_{ij} = \Theta_{ij}^{HMT}$, where quantities with super indice “HMT” are the ones used in [33].

where

$$J_\varphi \equiv -\frac{\delta \mathcal{L}_M}{\delta \varphi}, \quad J_A \equiv 2 \frac{\delta (N \mathcal{L}_M)}{\delta A}. \quad (2.17)$$

On the other hand, variation with respect to g_{ij} leads to the dynamical equations,

$$\begin{aligned} \frac{1}{N\sqrt{g}} [\sqrt{g} (\pi^{ij} - \varphi \mathcal{G}^{ij})]_{,t} &= -2 (K^2)^{ij} + 2K K^{ij} \\ &+ \frac{1}{N} \nabla_k [N^k \pi^{ij} - 2\pi^{k(i} N^{j)}] \\ &+ \frac{1}{2} (\mathcal{L}_K + \mathcal{L}_\varphi + \mathcal{L}_A) g^{ij} \\ &+ F^{ij} + F_\varphi^{ij} + F_A^{ij} + 8\pi G \tau^{ij}, \end{aligned} \quad (2.18)$$

where $(K^2)^{ij} \equiv K^{il} K_l^j$, $f_{(ij)} \equiv (f_{ij} + f_{ji})/2$, and

$$\begin{aligned} F^{ij} &\equiv \frac{1}{\sqrt{g}} \frac{\delta (-\sqrt{g} \mathcal{L}_V)}{\delta g_{ij}} = \sum_{s=0}^8 g_s \zeta^{n_s} (F_s)^{ij}, \\ F_\varphi^{ij} &= \sum_{n=1}^3 F_{(\varphi,n)}^{ij}, \\ F_A^{ij} &= \frac{1}{N} [A R^{ij} - (\nabla^i \nabla^j - g^{ij} \nabla^2) A], \end{aligned} \quad (2.19)$$

with $n_s = (2, 0, -2, -2, -4, -4, -4, -4, -4)$. The stress 3-tensor τ^{ij} is defined as

$$\tau^{ij} = \frac{2}{\sqrt{g}} \frac{\delta (\sqrt{g} \mathcal{L}_M)}{\delta g_{ij}}, \quad (2.20)$$

and the geometric 3-tensors $(F_s)_{ij}$ and $F_{(\varphi,n)}^{ij}$ are defined as,

$$\begin{aligned} (F_0)_{ij} &= -\frac{1}{2} g_{ij}, \\ (F_1)_{ij} &= R_{ij} - \frac{1}{2} R g_{ij}, \\ (F_2)_{ij} &= 2(R_{ij} - \nabla_i \nabla_j) R - \frac{1}{2} g_{ij} (R - 4\nabla^2) R, \\ (F_3)_{ij} &= \nabla^2 R_{ij} - (\nabla_i \nabla_j - 3R_{ij}) R - 4(R^2)_{ij} \\ &\quad + \frac{1}{2} g_{ij} (3R_{kl} R^{kl} + \nabla^2 R - 2R^2), \\ (F_4)_{ij} &= 3(R_{ij} - \nabla_i \nabla_j) R^2 - \frac{1}{2} g_{ij} (R - 6\nabla^2) R^2, \\ (F_5)_{ij} &= (R_{ij} + \nabla_i \nabla_j) (R_{kl} R^{kl}) + 2R (R^2)_{ij} \\ &\quad + \nabla^2 (R R_{ij}) - \nabla^k [\nabla_i (R R_{jk}) + \nabla_j (R R_{ik})] \\ &\quad - \frac{1}{2} g_{ij} [(R - 2\nabla^2) (R_{kl} R^{kl}) \\ &\quad - 2\nabla_k \nabla_l (R R^{kl})], \\ (F_6)_{ij} &= 3(R^3)_{ij} + \frac{3}{2} [\nabla^2 (R^2)_{ij} \\ &\quad - \nabla^k (\nabla_i (R^2)_{jk} + \nabla_j (R^2)_{ik})] \end{aligned}$$

$$\begin{aligned} & -\frac{1}{2} g_{ij} [R_l^k R_m^l R_k^m - 3\nabla_k \nabla_l (R^2)^{kl}], \\ (F_7)_{ij} &= 2\nabla_i \nabla_j (\nabla^2 R) - 2(\nabla^2 R) R_{ij} \\ &\quad + (\nabla_i R) (\nabla_j R) - \frac{1}{2} g_{ij} [(\nabla R)^2 + 4\nabla^4 R], \\ (F_8)_{ij} &= \nabla^4 R_{ij} - \nabla_k (\nabla_i \nabla^2 R_j^k + \nabla_j \nabla^2 R_i^k) \\ &\quad - (\nabla_i R_l^k) (\nabla_j R_k^l) - 2(\nabla^k R_l^i) (\nabla_k R_{jl}) \\ &\quad - \frac{1}{2} g_{ij} [(\nabla_k R_{lm})^2 - 2(\nabla_k \nabla_l \nabla^2 R^{kl})], \quad (2.21) \\ F_{(\varphi,1)}^{ij} &= \frac{1}{2} \varphi \left\{ (2K + \nabla^2 \varphi) R^{ij} - 2(2K_k^j + \nabla^j \nabla_k \varphi) R^{ik} \right. \\ &\quad \left. - 2(2K_k^i + \nabla^i \nabla_k \varphi) R^{jk} \right. \\ &\quad \left. - (2\Lambda_g - R) (2K^{ij} + \nabla^i \nabla^j \varphi) \right\}, \\ F_{(\varphi,2)}^{ij} &= \frac{1}{2} \nabla_k \left\{ \varphi \mathcal{G}^{ik} \left(\frac{2N^j}{N} + \nabla^j \varphi \right) \right. \\ &\quad \left. + \varphi \mathcal{G}^{jk} \left(\frac{2N^i}{N} + \nabla^i \varphi \right) - \varphi \mathcal{G}^{ij} \left(\frac{2N^k}{N} + \nabla^k \varphi \right) \right\}, \\ F_{(\varphi,3)}^{ij} &= \frac{1}{2} \left\{ 2\nabla_k \nabla^i f_\varphi^{(j)k} - \nabla^2 f_\varphi^{ij} - (\nabla_k \nabla_l f_\varphi^{kl}) g^{ij} \right\}, \end{aligned} \quad (2.22)$$

where

$$f_\varphi^{ij} = \varphi \left\{ (2K^{ij} + \nabla^i \nabla^j \varphi) - \frac{1}{2} (2K + \nabla^2 \varphi) g^{ij} \right\}. \quad (2.23)$$

The matter quantities $(J^t, J^i, J_\varphi, J_A, \tau^{ij})$ satisfy the conservation laws,

$$\begin{aligned} \int d^3x \sqrt{g} \left[\dot{g}_{kl} \tau^{kl} - \frac{1}{\sqrt{g}} (\sqrt{g} J^t)_{,t} + \frac{2N_k}{N\sqrt{g}} (\sqrt{g} J^k)_{,t} \right. \\ \left. - 2\dot{\varphi} J_\varphi - \frac{A}{N\sqrt{g}} (\sqrt{g} J_A)_{,t} \right] = 0, \end{aligned} \quad (2.24)$$

$$\begin{aligned} \nabla^k \tau_{ik} - \frac{1}{N\sqrt{g}} (\sqrt{g} J_i)_{,t} - \frac{J^k}{N} (\nabla_k N_i - \nabla_i N_k) \\ - \frac{N_i}{N} \nabla_k J^k + J_\varphi \nabla_i \varphi - \frac{J_A}{2N} \nabla_i A = 0. \end{aligned} \quad (2.25)$$

III. COSMOLOGICAL MODELS

The homogeneous and isotropic universe is described by,

$$N = 1, \quad N_i = 0, \quad g_{ij} = a^2(t) \gamma_{ij}, \quad (3.1)$$

where

$$\gamma_{ij} = \frac{\delta_{ij}}{(1 + \frac{1}{4} k r^2)^2}, \quad (3.2)$$

with $r^2 \equiv x^2 + y^2 + z^2$ and $k = 0, \pm 1$. Using the $U(1)$ gauge, on the other hand, we can set

$$\varphi = 0, \quad (3.3)$$

without loss of generality. Then, we find that

$$K_{ij} = -a^2 H \gamma_{ij}, \quad R_{ij} = 2k \gamma_{ij}, \quad (3.4)$$

where $H = \dot{a}/a$. Thus, we obtain

$$\begin{aligned} \mathcal{L}_K &= -6H^2, \quad \mathcal{L}_\varphi = 0, \\ \mathcal{L}_A &= 2A \left(\Lambda_g - \frac{3k}{a^2} \right), \\ \mathcal{L}_V &= 2\Lambda + \frac{6kg_1}{a^2} + \frac{12\beta_1 k^2}{a^4} + \frac{24\beta_2 k^3}{a^6}, \end{aligned} \quad (3.5)$$

where $\Lambda \equiv \zeta^2 g_0/2$, and

$$\beta_1 \equiv \frac{3g_2 + g_3}{\zeta^2}, \quad \beta_2 \equiv \frac{9g_4 + 3g_5 + g_6}{\zeta^4}. \quad (3.6)$$

The matter components are

$$J^t = -2\rho, \quad J^i = 0, \quad \tau_{ij} = p g_{ij}, \quad (3.7)$$

where ρ and p are the total density and pressure of the matter fields. Then the Hamiltonian constraint (2.11) reduces to the super-Hamiltonian constraint,

$$\mathcal{L}_K(t) + \mathcal{L}_V(t) = 8\pi G J^t(t),$$

which leads to the modified Friedmann equation,

$$H^2 - \frac{g_1 k}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} + \frac{2\beta_1 k^2}{a^4} + \frac{4\beta_2 k^3}{a^6}. \quad (3.8)$$

From Eqs. (2.14) and (2.19) we also find that

$$\begin{aligned} F^{ij} &= \left(-\Lambda - \frac{g_1 k}{a^2} + \frac{2\beta_1 k^2}{a^4} + \frac{12\beta_2 k^3}{a^6} \right) g^{ij}, \\ \pi^{ij} &= -2H g^{ij}. \end{aligned} \quad (3.9)$$

Hence, the dynamical equation (2.18) reduces to [8]

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{1}{3}\Lambda - \frac{A}{2} \left(\Lambda_g - \frac{k}{a^2} \right) \\ &\quad - \frac{2\beta_1 k^2}{a^4} - \frac{8\beta_2 k^3}{a^6}. \end{aligned} \quad (3.10)$$

Similar to GR, the super-momentum constraint (2.13) is then satisfied identically, since $J^i = 0$ and, from Eq. (3.6), $\vec{\nabla}_j \pi^{ij} \equiv \pi^{ij}|_{,j} = 0$, where $\vec{\nabla}_i$ denotes the covariant derivative with respect to γ_{ij} . Using Eqs. (3.8) and (3.10), it follows that in the FRW background the matter satisfies the conservation law,

$$\dot{\rho} + 3H(\rho + p) = A J_\varphi. \quad (3.11)$$

Thus, due to the interaction between the gauge field and the fluid, its energy in general is not conserved.

On the other hand, Eqs.(2.15) and (2.16) yield, respectively,

$$H \left(\Lambda_g - \frac{k}{a^2} \right) = -\frac{8\pi G}{3} J_\varphi, \quad (3.12)$$

$$\frac{3k}{a^2} - \Lambda_g = 4\pi G J_A. \quad (3.13)$$

When matter is not present, we have $J_\varphi = 0 = J_A$. Then, Eqs.(3.12) and (3.13) implies that $\Lambda_g = 0 = k$, while Eq.(3.8) yields $a(t) = e^{Ht}$, where $H \equiv \sqrt{\Lambda/3}$, which is the de Sitter spacetime. It is interesting to note that the de Sitter space can be also obtained from $\rho = p = k = 0$ and $J_A = -\Lambda_g/(4\pi G)$, $J_\varphi = -3H\Lambda_g/(8\pi G)$.

It should be noted that the energy conservation (3.11) can be also obtained from Eq. (2.24), while the conservation law of momentum, Eq. (2.25), is satisfied identically. When $\beta_1 \beta_2 \neq 0$, the corresponding terms act like a dark radiation and a stiff-fluid, respectively. Due to the presence of these terms, one can easily construct bouncing universe in the early epoch of the universe [14–16].

In addition, in deriving Eq. (3.8) we followed the usual assumption that the whole FRW universe is homogeneous and isotropic. In [12], it was argued that such an assumption might be too strong. If one relaxes the assumption and requires that only the observed patch of our universe is homogeneous and isotropic, one can introduce the notion of “dark matter as an integration constant” of the Hamiltonian constraint (2.11): $\rho(t)$ in Eqs. (3.8) and (3.11) can be replaced by $\rho(t) + \mathcal{E}(t)$ in the observable patch, where $\mathcal{E}(t) = \text{const}/a^3$ in the IR limit [12, 35]. Beyond the observable patch, \mathcal{E} is necessarily inhomogeneous. In order to analyze perturbations on an FRW background, one needs to restrict the perturbations to the observable patch, which then raises issues about matching across the boundary of the observable patch. In our approach, the background is a homogeneous FRW spacetime, so that $\mathcal{E} = 0$ in the background.

IV. COSMOLOGICAL SCALAR PERTURBATIONS

In this section, we consider linear scalar perturbation of the FRW universe studied in the last section. We shall closely follow [8] and use the notations adopted there without further explanations. However, in order to have the present paper as independent as possible, it is difficult to avoid repeating the same materials, although we shall try to limit it to its minimum.

In the quasi-longitudinal gauge [8],

$$\phi = 0 = E, \quad (4.1)$$

the metric scalar perturbations are given by

$$ds^2 = a^2 [-d\eta^2 + 2B_{|i} dx^i d\eta + (1 - 2\psi) \gamma_{ij} dx^i dx^j]. \quad (4.2)$$

The gauge-invariant quantities are now given by

$$\Phi = \mathcal{H}B + B', \quad \Psi = \psi - \mathcal{H}B, \quad (4.3)$$

where $\mathcal{H} \equiv a'/a$ and a prime denotes the ordinary derivative with respect to η . Using the $U(1)$ gauge, we can further set

$$\delta\varphi = 0, \quad (4.4)$$

so that $\mathcal{L}_\varphi = 0$. Following [8], we use quantities with over-bars as the ones calculated in the background. Then we find that

$$\begin{aligned} K_{ij} &= -a\mathcal{H}\gamma_{ij} + a[B_{|ij} + (\psi' + 2\mathcal{H}\psi)\gamma_{ij}], \\ R_{ij} &= 2k\gamma_{ij} + \psi_{|ij} + \vec{\nabla}^2\psi\gamma_{ij}, \\ \mathcal{L}_K &= -\frac{6\mathcal{H}^2}{a^2} + \frac{4\mathcal{H}}{a^2}(\vec{\nabla}^2 B + 3\psi'), \\ \mathcal{L}_A &= \bar{\mathcal{L}}_A + \frac{2}{a}\left[\left(\Lambda_g - \frac{3k}{a^2}\right)\delta A - \frac{2\bar{A}}{a^2}(\vec{\nabla}^2\psi + 3k\psi)\right], \\ \mathcal{L}_V &= \bar{\mathcal{L}}_V + \frac{4}{a^2}\left(g_1 + \frac{4\beta_1 k}{a^2}\right)(\vec{\nabla}^2 + 3k)\psi \\ &\quad + \frac{48\beta_2 k^2}{a^6}(\vec{\nabla}^2 + 3k)\psi \\ &\quad + \frac{24g_7 k}{\zeta^4 a^6}\vec{\nabla}^2(\vec{\nabla}^2 + 3k)\psi, \end{aligned} \quad (4.5)$$

where $\bar{\mathcal{L}}_V$ denotes the potential of the background given by Eq.(3.5), and $B_{|i} \equiv \vec{\nabla}_i$, with $\vec{\nabla}_i$ being the covariant derivative with respect to γ_{ij} and $\vec{\nabla}^2 \equiv \gamma^{ij}\vec{\nabla}_i\vec{\nabla}_j$.

To first-order the Hamiltonian constraint (2.11) takes the form,

$$\begin{aligned} \int \sqrt{\gamma} d^3x \left[(\vec{\nabla}^2 + 3k)\psi - \frac{(2-3\xi)\mathcal{H}}{2}(\vec{\nabla}^2 B + 3\psi') \right. \\ \left. - 2k\left(\frac{2\beta_1}{a^2} + \frac{6\beta_2 k}{a^4} + \frac{3g_7}{\zeta^4 a^4}\vec{\nabla}^2\right)(\vec{\nabla}^2 + 3k)\psi \right. \\ \left. - 4\pi G a^2 \delta\mu \right] = 0, \end{aligned} \quad (4.6)$$

which is the same as that given in the SVW setup [8], where $\delta\mu \equiv -\delta J^t/2$. Eq.(4.6) represents a generalization of the Poisson equation of GR [36].

To the first-order the supermomentum constraint (2.13) takes the form,

$$2\psi' - 2kB = 8\pi G a q, \quad (4.7)$$

which is the generalization of the GR $0i$ constraint [36], where $\delta J^i \equiv a^{-2}q^i$.

On the other hand, the linearized equations (2.15) and (2.16) for the Newtonian pre-potential and the gauge field reduce, respectively, to,

$$\begin{aligned} \left(\Lambda_g - \frac{k}{a^2}\right)[\vec{\nabla}^2 B + 3(\psi' + 2\mathcal{H}\psi)] \\ + \frac{2\mathcal{H}}{a^2}[\vec{\nabla}^2\psi + 3(2k - a^2\Lambda_g)\psi] = 8\pi G a \delta J_\varphi, \end{aligned} \quad (4.8)$$

$$\vec{\nabla}^2\psi + 3k\psi = 2\pi G a^2 \delta J_A. \quad (4.9)$$

The linearly perturbed dynamical equations require the calculations of the perturbed $(F_s)_{ij}$ of Eq. (2.21), which were given by Eq.(A1) in [8]. To avoid repeating, we shall not write them down here, and refer readers directly to that paper. Then, we find that the trace part

is given by

$$\begin{aligned} \psi'' + 2\mathcal{H}\psi' - \mathcal{F}\psi - \frac{1}{6}\gamma^{ij}\delta F_{ij} + \frac{1}{3}(\vec{\nabla}^2 B' + 2\mathcal{H}\vec{\nabla}^2 B) \\ + \frac{\bar{A}}{3a}(\vec{\nabla}^2 - 3\Lambda_g a^2 + 6k)\psi \\ - \frac{1}{6a}(2\vec{\nabla}^2 + 3\Lambda_g a^2 - 3k)\delta A = 4\pi G a^2 \delta \mathcal{P}, \end{aligned} \quad (4.10)$$

where $\delta F_{ij} = \sum g_s \zeta^{n_s} \delta(F_s)_{ij}$, with $\delta(F_s)_{ij}$ given by Eq. (A1) in [8], and

$$\begin{aligned} \mathcal{F} &= a^2\left(-\Lambda - \frac{g_1 k}{a^2} + \frac{2\beta_1 k^2}{a^4} + \frac{12\beta_2 k^3}{a^6}\right), \\ \delta\tau^{ij} &= \frac{1}{a^2}\left[(\delta\mathcal{P} + 2\bar{p}\psi)\gamma^{ij} + \Pi^{[ij]}\right], \end{aligned} \quad (4.11)$$

where the angled brackets on indices define the trace-free part:

$$f_{[ij]} \equiv f_{ij} - \frac{1}{3}\gamma_{ij}f_k{}^k. \quad (4.12)$$

The trace-free part of the dynamical equations is

$$\begin{aligned} B'_{[ij]} + 2\mathcal{H}B_{[ij]} + \delta F_{[ij]} - \frac{1}{a}(\delta A - \bar{A}\psi)_{[ij]} \\ = -8\pi G a^2 \Pi_{[ij]}. \end{aligned} \quad (4.13)$$

Eqs.(4.10) and (4.13) generalize the GR ij perturbed field equations [36].

The perturbed parts of the conservation laws (2.24) and (2.25) give

$$\begin{aligned} \int \sqrt{\gamma} d^3x \left\{ \delta\mu' + 3\mathcal{H}(\delta\mathcal{P} + \delta\mu) - 3(\bar{p} + \bar{p})\psi' \right. \\ \left. + \frac{1}{2a^4}\left[\left(a^3\bar{J}_A\right)'\delta A + \bar{A}\left(a^3(\delta A - 3\bar{A}\psi)\right)'\right] \right\} = 0, \end{aligned} \quad (4.14)$$

$$q' + 3\mathcal{H}q - a\delta\mathcal{P} - \frac{2a}{3}(\vec{\nabla}^2 + 3k)\Pi + \frac{1}{2}\bar{J}_A\delta A = 0, \quad (4.15)$$

where \bar{J}_A and \bar{J}_φ are given by Eqs.(3.12) and (3.13). The energy conservation equation is an integrated generalization of the GR energy equation, and the momentum equation generalizes the GR momentum equation [36].

V. STABILITY OF THE MINKOWSKI SPACETIME

It can be shown that the Minkowski spacetime,

$$a = 1, \quad \bar{A} = \bar{\varphi} = k = 0, \quad (5.1)$$

is a solution of the HMT theory, provided that

$$\Lambda_g = \Lambda = \bar{J}_A = \bar{J}_\varphi = \bar{\rho} = \bar{p} = 0. \quad (5.2)$$

Then, the linearized Hamiltonian constraint (4.6) and the field equation (4.8) for $\delta\varphi$ are satisfied identically, while Eqs.(4.7) and (4.9) yield

$$\dot{\psi} = 0 = \partial^2\psi, \quad (5.3)$$

where $\partial^2 \equiv \delta^{ij}\partial_i\partial_j$. These are the same as the ones obtained in GR, and lead to $\psi = 0$ with proper boundary conditions. It is interesting to note that in GR the equation $\partial^2\psi = 0$ is obtained from the local Hamiltonian constraint, while in the present setup it is obtained from the variation of the gauge field A . From this analysis, one can see clearly the reason why A is needed in order to eliminate the spin-0 graviton. On the other hand, the trace and traceless parts of the dynamical equations (4.10) and (4.13) yield

$$\dot{B} = \delta A. \quad (5.4)$$

Using the $U(1)$ gauge freedom (2.3), without loss of generality, we can set

$$\delta A = 0. \quad (5.5)$$

Note that this gauge choice is consistent with our quasi-longitudinal gauge $\phi = E = 0$ [8], because under this $U(1)$ gauge transformation, E and ϕ remain the same, as one can see from Eq.(2.3). Then, Eq.(5.4) yields $B = B(x)$, and the gauge-invariant quantities Ψ and Φ defined by Eq.(4.3) are zero,

$$\Psi = \Phi = 0. \quad (5.6)$$

Therefore, the scalar perturbations of the metric vanish identically in the Minkowski background. Hence, the spin-0 graviton is completely eliminated in the HMT setup [33].

It should be noted that $\delta\varphi$ is undetermined in the present case. However, since $\bar{\varphi} = 0$, it is quite reasonable to assume that its linear perturbation also vanishes in the Minkowski background.

VI. CONCLUSIONS

Recently, Horava and Melby-Thompson [33] proposed a new version of the HL theory of gravity, in which the spin-0 graviton, appearing in all the previous versions of the HL theory, is eliminated by introducing a Newtonian pre-potential φ and a local $U(1)$ gauge field A . Due to such an elimination, the dynamical coupling constant λ , which characterizes the deviation of the kinetic part of the action from that of the Einstein-Hilbert, is forced to take its relativistic value $\lambda = 1$. As a result, the theory in the IR regime exhibit many features that are quite similar to those given in GR.

Motivated by these remarkable features, in this paper we have studied the theory in some detail by assuming the presence of matter fields. The potential of the action has been taken to be the one constructed by SVW [7], which represents the most general potential, which respects the parity and its highest order of the spatial derivatives is six. We have first derived the Hamiltonian and super-momentum constraints, given, respectively, by Eqs.(2.11) and (2.13), and then the field equations (2.15) and (2.16), respectively, for the Newtonian pre-potential φ and the local $U(1)$ gauge field A . The dynamical equations are given by Eq.(2.18), while the conservation laws of energy and momentum are given, respectively, by Eqs.(2.24) and (2.25).

Applying the above general formulas to cosmology, we have obtained the general modified Friedmann equation (3.8) and the equation (3.10) for the acceleration \ddot{a} . It is remarkable that these equations give precisely the conservation law of energy, which takes the same form as that given in GR and can be also obtained from the conservation law (2.24), despite the fact that J_φ and J_A are non-vanishing, and given, respectively, by Eqs.(3.12) and (3.13). When the spatial curvature is different from zero, terms acting as dark radiation and stiff-fluid are present, and bouncing universe can be easily constructed from these terms.

We have also studied the scalar perturbations of the FRW universe with any given spatial curvature, and the linearized Hamiltonian, momentum constraints, the equations for the Newtonian pre-potential φ and the gauge field A , the trace and traceless parts of the dynamical equations are given, respectively, by Eqs.(4.6), (4.7), (4.8), (4.9), (4.10), and (4.13), while the conservation laws of energy and momentum are given, respectively, by Eqs.(4.14) and (4.15).

Applying these formulas to the Minkowski background, we have shown explicitly that the metric scalar perturbations vanish identically, that is, the spin-0 graviton appearing in all the previous versions of the HL theory is eliminated in the current HMT setup.

Since the Newtonian pre-potential φ and the gauge field A have no contributions to the vector and tensor perturbations, the corresponding linear perturbations are given precisely by the same equations as those recently presented in [11] in the SVW setup. In particular, in the Minkowski background vector perturbations also vanish, although the tensor perturbations in general do not [11]. These two non-vanishing components represent the massless spin-2 gravitons, which are exactly the same as those found in GR.

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