Stability of spin-0 graviton and strong coupling in Horava-Lifshitz theory of gravity

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In this paper, we consider two different issues, stability and strong coupling, raised lately in the newly-proposed Horava-Lifshitz (HL) theory of quantum gravity with projectability condition. We find that all the scalar modes are stable in the de Sitter background, due to two different kinds of effects, one from high-order derivatives of the spacetime curvature, and the other from the exponential expansion of the de Sitter space. Combining these effects properly, one can make the instability found in the Minkowski background never appear even for small-scale modes, provided that the IR limit is sufficiently closed to the relativistic fixed point. At the fixed point, all the modes become stabilized. We also show that the instability of Minkowski spacetime can be cured by introducing mass to the spin-0 graviton. The strong coupling problem is investigated following the effective field theory approach, and found that it cannot be cured by the Blas-Pujolas-Sibiryakov mechanism, initially designed for the case without projectability condition, but might be circumvented by the Vainshtein mechanism, due to the non-linear effects. In fact, we construct a class of exact solutions, and show explicitly that it reduces smoothly to the de Sitter spacetime in the relativistic limit.

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I. INTRODUCTION

Properly formulating the theory of quantum gravity has been one of the main driving forces in gravitational physics over past several decades [1]. Although there are several very promising candidates, including Loop Quantum Gravity [2] and string/M theory [3], it is fair to see that our understanding of it is still very limited. Horava recently proposed another alternative [4], motivated by the Lifshitz theory in solid state physics [5], for which it is often referred to as the Horava-Lifshitz (HL) theory. It has various remarkable features, including its power-counting renormalizability [6], the divergence of its effective speed of light in the ultra-violet (UV), which could potentially resolve the horizon problem without invoking inflation [7]. Scale-invariant super-horizon curvature perturbations can also be produced without inflation [8–13], and dark matter and dark energy can have their geometric origins [14, 15]. Furthermore, bouncing universe can be easily constructed due to the high-order derivative terms of the spacetime curvature [16–18]. For detail, we refer readers to [19] and references therein.

Despite all of these attractive features, the theory plagues with two serious problems: the instability of the Minkowski background [4, 11, 20–22], and the strong coupling [23–27]. To solve these problems, various modifications were proposed [28, 29]. In particular, Blas, Pujolas and Sibiryakov (BPS) [30] found that inclusion of terms made of a_i, where

$$a_i = \partial_i \ln(N), \tag{1.1}$$

can cure the instability of the Minkowski background, where N is the lapse function. Of course, this is possible only for the version of the HL theory without projectability condition. Otherwise, N depends only on time, and a_i vanishes identically. By properly choosing the coupling constants, the strong coupling problem can be also addressed in such a setup [31]. The main idea is to introduce two energy scales, the UV cutoff scale M_* and the strong coupling scale Λ_k. If M_* is low enough so that

$$M_* \lesssim \Lambda_k, \tag{1.2}$$

then the linear perturbations become invalid before Λ_k is researched, so that the strong coupling problem does not show up at all [cf. Fig.5]. Applications of the BPS model to cosmology were studied recently in [32–34], while spherically symmetric spacetimes were investigated in [35]. However, a price to pay in such a setup is the enormous number of independent coupling constants. It can be shown that only the sixth-order derivative terms in the potential are more than 60 [27]. It should be also noted that giving up the projectability condition often causes the theory to suffer the inconsistence problem [36]. However, Kluson recently showed that the Hamiltonian formalism of the BPS model is very rich, and that the algebra of constraints is well-defined [37].

On the other hand, Sotiriou, Visser and Weinfurtner (SVW) generalized the original version of the HL theory to the most general form by giving up the detailed balance condition but still keeping the projectability one [21]. In the SVW setup, the inconsistence problem does not exist, and the gravitational sector contains totally ten coupling constants, G, Λ, ξ, g_i ($i = 2, 3, ..., 8$), where G and Λ denote, respectively, the 4-dimensional Newtonian and cosmological constants, ξ and g_n's are other coupling constants, due to the breaking of the Lorentz invariance

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of the theory. Although the Minkowski background is still not stable in such a setup [11, 21], de Sitter spacetime is [19]. Therefore, in such a setup, one may consider the latter as its legitimate background, similar to what happened in the massive gravity [38]. Moreover, the SVW setup also faces the strong coupling problem [25]. Recently, Mukohyama showed that this problem could be solved by the Vainshtein mechanism [39], at least as far as the spherically symmetric, static, vacuum spacetimes are concerned [40].

In this paper, our purposes are two-folds. We first generalize our studies of [19] to include high-order derivative terms, whereby we show explicitly that all the scalar modes, including the short-scale ones, are stable in the de Sitter spacetime, by properly combining two different kinds of effects, one from high-order derivatives of the spacetime curvature, and the other from the exponential expansion of the de Sitter space. This is done in Sec. II. Second, we systematically study the strong coupling problem by following the effective theory approach [41], and show clearly that the BPS mechanism for solving the strong coupling problem originally invented in the case without projectability condition cannot be applied to the SVW case with projectability condition. This is consistent with the results found by BPS using the Stückelberg trick [42, 43]. This is done in Sec. III. In Sec. IV, we construct a class of non-perturbative cosmological solutions, and show that it reduces smoothly to the de Sitter spacetime (with rotation) in the relativistic limit. This implies that the spin-0 graviton indeed decouples in the IR limit and does not cause additional problem, once nonlinear effects are included, a similar situation also happens other theories, such as the DGP model [44] [1]. This can be considered as the generalization of Mukohyama's analysis of the spherical case to the cosmological one. In Sec. V, we present our main conclusions, and shown that the Minkowski spacetime can also become stable by introducing mass to the spin-0 graviton.

II. STABILITY OF DE SITTER SPACETIME

To start with, in this section we shall first give a brief introduction to the SVW setup, and then consider linear perturbations in the de Sitter background.

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[1] It should be noted that, although in both theories it is the nonlinear interaction that makes the theories consistent with observations, there is a fundamental difference between them: the DGP model represents the modification of general relativity in the IR, while the HL theory modifies general relativity mainly in the UV. So, the coupling of the scalar graviton in the DGP model is of order one, and the non-linear effects help to screen its coupling to external sources. In the HL theory, on the other hand, the scalar graviton is self-interacting, and it is this interaction that leads to the strong coupling problem. See the analysis carried in Sec. III.

A. The SVW Setup

Sotiriou, Visser and Weinfurtner (SVW) formulated the most general HL theory with projectability condition but without the detailed balance [21]. Writing the 4-dimensional metric in the ADM form,

$$ds^2 = -N^2c^2dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right),$$
$$(i, j = 1, 2, 3), \qquad (2.1)$$

the projectability condition requires that

$$N = N(t), \quad N^i = N^i(t,x), \quad g_{ij} = g_{ij}(t,x). \qquad (2.2)$$

Note that in [11, 13, 45], the constant c, representing the speed of light, was absorbed into N. The ADM form (2.1) is preserved by the types of coordinate transformations,

$$t \rightarrow f(t), \quad x^i \rightarrow \zeta^i(t, \mathbf{x}). \qquad (2.3)$$

Due to these restricted diffeomorphisms, one more degree of freedom appears in the gravitational sector - a spin-0 graviton. This is potentially dangerous, and needs to decouple in the Infrared (IR), in order to be consistent with observations. Similar problems are also found in other modified theories, such as the massive gravity [43].

Then, it can be shown that the most general action, which preserves the parity, is given by [21],

$$S = \zeta^2 \int dt d^3x N\sqrt{g}\left(\mathcal{L}_K - \mathcal{L}_V + \zeta^{-2}\mathcal{L}_M\right), \qquad (2.4)$$

where $g = \det g_{ij}$, $\mathcal{L}_M$ denotes the matter Lagrangian density, and

$$\mathcal{L}_K = K_{ij}K^{ij} - (1-\xi)K^2,$$
$$\mathcal{L}_V = 2\Lambda - R + \frac{1}{\zeta^2}\left(g_2 R^2 + g_3 R_{ij}R^{ij}\right)$$
$$+ \frac{1}{\zeta^4}\left(g_4 R^3 + g_5 R\, R_{ij}R^{ij} + g_6 R^i_j R^j_k R^k_i\right)$$
$$+ \frac{1}{\zeta^4}\left[g_7 R\nabla^2 R + g_8\left(\nabla_i R_{jk}\right)\left(\nabla^i R^{jk}\right)\right], (2.5)$$

where $\zeta^2 = 1/16\pi G$, and the covariant derivatives and Ricci and Riemann terms are all constructed from the three-metric g_{ij}, while K_{ij} is the extrinsic curvature,

$$K_{ij} = \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right), \qquad (2.6)$$

where $N_i = g_{ij}N^j$. In the IR limit, all the high order curvature terms (with coefficients g_i, $i = 2, ..., 8$) drop out, and the total action reduces when $\xi = 0$ to the Einstein-Hilbert action.

B. Linear Perturbations in de Sitter Background

With the conformal time η, the de Sitter spacetime is given by $ds^2 = a^2(\eta)\left(-d\eta^2 + \delta_{ij}dx^i dx^j\right)$, where $a(\eta) = -1/(H\eta) = e^{Ht}$, and t denotes the cosmic time.

Linear scalar perturbations of the metric are given by

$$\delta g_{ij} = a^2(\eta)\left(-2\psi\delta_{ij} + 2E_{,ij}\right),$$
$$\delta N_i = a^2(\eta)B_{,i} \quad \delta N = a(\eta)\phi(\eta). \quad (2.7)$$

Choosing the quasi-longitudinal gauge [11],

$$\phi = 0 = E, \quad (2.8)$$

we find that the two gauge-invariant quantities defined in [11] reduce to,

$$\Phi = \mathcal{H}B + B', \quad \Psi = \psi - \mathcal{H}B, \quad (2.9)$$

where $\mathcal{H} = a'/a = -1/\eta$, and ψ and B are given by [19]

$$(2 - 3\xi)\,\psi_k{}' = -\xi k^2 B, \quad (2.10)$$
$$\psi_k{}'' + 2\mathcal{H}\psi_k{}' + \omega_k^2\psi_k = 0, \quad (2.11)$$

in the momentum space, where

$$\omega_k^2 = |c_\psi|^2\,k^2\left(-1 + \frac{k^2}{M_A^2 a^2} + \frac{k^4}{M_B^4 a^4}\right), \quad (2.12)$$

with $c_\psi^2 \equiv \xi/(2 - 3\xi)$ and

$$M_A \equiv \frac{M_{pl}}{\left[2(8g_2 + 3g_3)\right]^{1/2}},$$
$$M_B \equiv \frac{M_{pl}}{\left[4(8g_7 - 3g_8)\right]^{1/4}}. \quad (2.13)$$

Clearly, to have M_A and M_B real, we must assume that

$$8g_2 + 3g_3 \geq 0, \quad 8g_7 - 3g_8 \geq 0, \quad (2.14)$$

conditions we shall take for granted in the rest of this paper. Note that in writing the above expressions, we had assumed that $\xi \leq 0$. When $\xi = 0$ the corresponding solutions are stable, as shown in [19], so in the following we shall not consider this case any further, and concentrate ourselves only to the case $\xi < 0$. Then, from the above one can see that the studies of stability of the de Sitter spacetime reduces to the studies of the master equation (2.11). Once ψ_k is known, from Eq. (2.10) one can find B_k. Then, the gauge-invariant quantities Φ_k and Ψ_k can be read off from Eq.(2.9). From the latter one can see that the properties of Φ_k and Ψ_k are uniquely determined by ψ_k. In particular, if ψ_k is not singular, so are Φ_k and Ψ_k. Therefore, in the following we shall concentrate ourselves only on ψ_k.

To study the perturbations further, we notice that Eq.(2.11) is quite similar to an oscillator with a dissipative force $\mathcal{F}$ [46],

$$\ddot{x} + \mathcal{F}\dot{x} + \omega^2 x = 0, \quad (2.15)$$

which has the general solution,

$$x = Ae^{-\mathcal{F}t/2}e^{-i\omega t}, \quad (2.16)$$

where A is a constant. When $\mathcal{F} > 0$, from the above expression we can see that the free modes ω is exponentially damped.

In the Minkowski background, we have $a = $ Constant. Without loss of generality, we can set $a = 1$. Then, we find that $\mathcal{F} = 0$, and

$$\omega_k^2 = -|c_\psi|^2\,k^2\left(1 - \frac{k^2}{M_A^2} - \frac{k^4}{M_B^4}\right), \quad (a = 1). \quad (2.17)$$

Therefore, if the scale of a mode is large enough so that ω_k^2 becomes negative, this mode is unstable. In particular, without the high-order corrections, all the modes are unstable [11]. This is quite similar to the Jeans instability [47], for which there exists a characteristic Jeans length $\lambda_J = 1/k_J$, where when scales are smaller than the Jeans length, the modes are stable. When scales are larger than the Jeans length, they become unstable. The largest instability occurs at

$$k_M^2 = \frac{M_B^2}{\sqrt{r^4 + 3} + r^2}, \quad (2.18)$$

for which we have

$$\omega_k(k_M) = i\frac{|c_\psi|\,M_B}{\mathcal{B}^{3/2}}\left(r^4 + r^2\sqrt{r^4 + 3} + 2\right)^{1/2}$$
$$\equiv i\Gamma, \quad (2.19)$$

where

$$\mathcal{B} \equiv \sqrt{r^4 + 3} + r^2,$$
$$r \equiv \frac{M_B}{M_A} = \left(\frac{(8g_2 + 3g_3)^2}{8g_7 - 3g_8}\right)^{1/4}. \quad (2.20)$$

The instability will grow significantly during a time $t \geq t_\Gamma \equiv \Gamma^{-1}$, or in other words, for any given time t_0 of interest, only when $t_0 < t_\Gamma$, the growth of the instability during t_0 can be neglected.

However, it is well-known that Jeans instability can be removed by Hubble friction in an expanding universe [47]. In the following we shall show that this is also true in the HL theory. In particular, in the de Sitter background, two important modifications occur: (a) For any given k, ω_k^2 is always positive at sufficiently early time, due to high-order corrections, as one can see from Eq.(2.12). (b) The damping force $\mathcal{F}$ [$= -2/\eta$] is strictly non-negative and independent of H. When $\eta \to 0^-$ it becomes infinitely large. For short-scale waves, although the spacetime can be considered as locally flat, the high-order derivatives can kick in at a very early time, if the UV cutoff scale is very low. As time increases, the damping force becomes more and more important, and will finally become dominant. Therefore, if the UV cutoff is sufficiently low, by combining these two kinds of effects, one works in the IR ($\eta \simeq 0^-$) and the other works in the UV ($|\eta| \gg 1$), one might be able to stabilize the modes of both the short- and large-scales. To see that this is indeed possible here in the HL theory, we first notice that,

as the universe expands, a becomes larger and larger, and there exists a moment, say, η_c, at which $\omega_k^2(\eta_c) = 0$, where

$$\eta_c(k) = -\frac{\sqrt{2}M_B}{Hk\left(r^2 + \sqrt{r^4+4}\right)^{1/2}}. \qquad (2.21)$$

From this moment on, the instability starts to develop until $\eta = 0^-$, at which we have $\omega_k^2(0^-) = -|c_\psi|^2 k^2$. Note that for the modes with $k \gtrsim k_{stable}$, we have $|\eta_c(H_0)/\eta_0| \lesssim 1$, that is, the instability of these modes has not occurred within the age of our universe, where $\eta_0 \simeq -H_0^{-1}$ denotes the current conformal time of our universe, and

$$k_{stable} \equiv \left(r^2 + \sqrt{r^4+4}\right)^{1/2} M_B. \qquad (2.22)$$

Therefore, the only possible unstable modes that occur within the age of our universe are those with their wavelengths $\lambda > \lambda_{stable}$, where $\lambda_{stable} \equiv 1/k_{stable}$. However, if the UV cutoff scale M_* is low enough, so that the exponentially damping force kicks in before these modes become unstable, that is, if

$$\mathcal{F}(\eta) - 2\,|\omega_k(\eta)| \geq 0, \ \ (\eta > \eta_c), \qquad (2.23)$$

then these unstable modes will be stabilized, and never show up, where

$$M_* = min\{M_A, \ M_B\}. \qquad (2.24)$$

Setting

$$X \equiv H^2\eta^2 + \frac{M_B^4}{3k^2 M_A^2}, \qquad (2.25)$$

the condition (2.23) can be written as

$$D(X) \equiv X^3 - 3bX + 2d \geq 0, \qquad (2.26)$$

where

$$b \equiv \frac{M_A^4 r^4}{9k^4}\left(r^4 + 3\right),$$

$$d \equiv \frac{M_B^6}{27k^6}\left(r^6 + \frac{9}{2}r^2 + \frac{27H^2}{2\,|c_\psi|^2\,M_B^2}\right). \qquad (2.27)$$

Fig. 1 schematically shows the function $D(X)$, from which we can see that the condition (2.26) holds when

$$D\left(X_m\right) = \frac{2M_B^{12}}{3^6\left(d + b^{3/2}\right)k^{12}}\left[-\left(r^4 + 3\right)^3\right.$$
$$\left. + \left(r^6 + \frac{9}{2}r^2 + \frac{27H^2}{2\,|c_\psi|^2\,M_B^2}\right)^2\right] \geq 0, \ (2.28)$$

where $X_m = \sqrt{b}$. This yields

$$M_B \leq \Lambda_{stable}, \qquad (2.29)$$

where

$$\Lambda_{stable} \equiv \frac{H}{|c_\psi|}\left\{\frac{2}{r^4+4}\left[\left(r^4+3\right)^{3/2} + r^2\left(r^4 + \frac{9}{2}\right)\right]\right\}^{1/2}. \qquad (2.30)$$

It is remarkable to note that the condition (2.29) does not depend on k. As a result, it is valid for any scale of modes. In particular, once it is satisfied, the short-scale modes become stabilized, too. Thus, for any given M_A and M_B, if ξ is sufficiently closed to its fixed point $\xi = 0$ (at which one has $c_\psi = 0$) [2], Λ_{stable} becomes large, and the condition (2.29) can be easily satisfied. At the fixed point $\xi = 0$, we have $\Lambda_{stable} = \infty$, that is, now for any given g_i (or equivalently for any given M_A and M_B), all the modes, of large- and small-scales, are stable.

The above can be further seen from the following limiting cases. First, when $r \ll 1$, we have

$$\Lambda_{stable} = \left(\frac{27}{4}\right)^{1/4}\frac{H}{|c_\psi|}, \ (r \ll 1). \qquad (2.31)$$

Thus, even H is taken to be the current Hubble constant H_0, Λ_{stable} can still be large, as longer as c_ψ is sufficiently closed to its fixed point $c_\psi = 0$.

When $r \simeq 1$, on the other hand, we have

$$\Lambda_{stable} = \sqrt{\frac{27}{5}}\frac{H}{|c_\psi|}, \ (r \simeq 1). \qquad (2.32)$$

Once again, if ξ is sufficiently closed to $\xi = 0$, Λ_{stable} will be large, and the condition $M_B \leq \Lambda_{stable}$ can be satisfied for a given non-zero H.

When $r \gg 1$, Eq.(2.29) reduces to

$$M_A \leq \frac{2H}{|c_\psi|}, \ (r \gg 1). \qquad (2.33)$$

Taking $H = H_0$, one can see that the conditions (2.31)-(2.33) can be written as

$$M_* \lesssim \mathcal{O}(1)\frac{H_0}{|c_\psi|}, \qquad (2.34)$$

which is equivalent to the condition that the instability found in the Minkowski background does not happen within the age of our universe [25, 31].

Note that, since the damping force always dominates when $|\eta| \ll 0$, even the instability develops, it always occur within the period, $\eta_1 < \eta < \eta_2$, and will be finally stabilized by $\mathcal{F}$, where $\eta_c < \eta_1 \leq \eta_2 < 0$ [as can be seen from Fig. 2], where $\eta_{1,2}$ are the two real and positive roots of $D(X) = 0$. Thus, for a given time interval of interest, if the instability happens in a sufficient short

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[2] It should be noted that c_ψ cannot be too closed to zero. Otherwise, Cherenkov radiation will impose severe constraints. We thank Thomas Sotiriou for pointing out this to us.
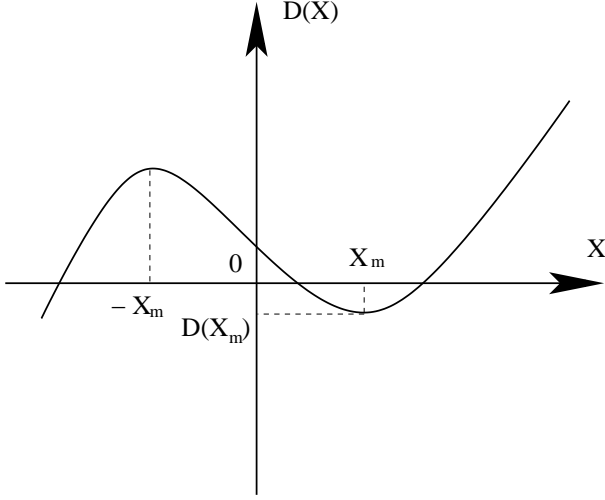
FIG. 1: The function $D(X)$ defined by Eq.(2.28). Note that only the half plane $X > 0$ is valid, as one can see from Eq.(2.25).
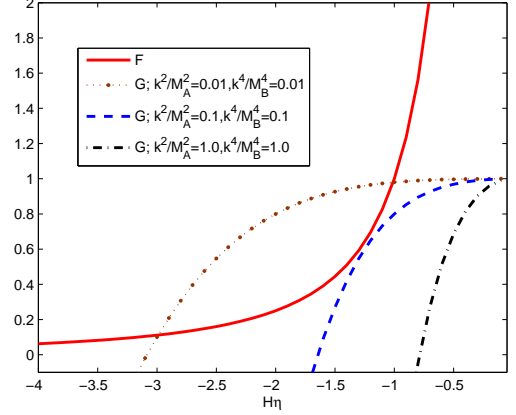

FIG. 2: The functions $F \equiv \mathcal{F}^2/(4H^2)$ and $G \equiv -\omega_k^2/H^2$ for different choices of k^2/M_A^2 and k^4/M_B^4, where $\omega_0 \equiv |c_\psi| k/H$ has been sent to one in all the cases. When k^2/M_A^2 and k^4/M_B^4 are small, $F - G = 0$ has two solutions η_1 and η_2 where $\eta_c < \eta_1 < \eta_2 < 0$, as shown by the doted line. When k^2/M_A^2 and k^4/M_B^4 are large, F is always greater than G, and $F - G = 0$ has no solution, as shown by the dash-dot line. The dashed line represents the case where $F - G = 0$ has only one solution.
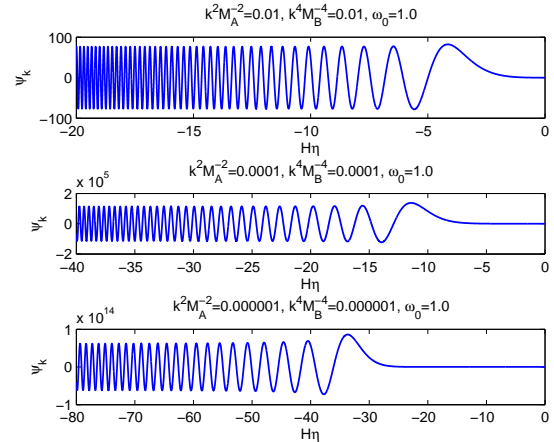
period and will not grow much, before the damping force takes over, then such an instability is still acceptable. However, the following results show that this is possible only for modes of short-scales. In fact, it can be shown that

$$\Delta \eta \equiv \eta_2 - \eta_1$$
$$= \frac{2M_A^2 r^2}{k^2} \sqrt{\frac{r^4 + 3}{3}} \, \cos\left(\frac{2\theta + \pi}{6}\right), \quad (2.35)$$

where $\theta \in [\pi/2, \, \pi]$ and is defined as

$$\cos\theta = -\frac{1}{\left(r^4 + 3\right)^{3/2}} \left(r^6 + \frac{9}{2}r^2 + \frac{27H^2}{2\left|c_\psi\right|^2 M_B^2}\right). \tag{2.36}$$

Clearly, to have a real θ, the denominator of Eq.(2.36) has to be greater or at least equal to the nominator, which is equivalent to $M_B > \Lambda_{stable}$, where Λ_{stable} is defined in Eq.(2.29). Since $\Delta\eta \propto k^{-2}$, one can see that, for any given ξ, g_i and H, $\Delta\eta \to \infty$ when $k^2 \to 0$. Thus, to limit the instability completely for any k, one needs to require that the condition Eq.(2.29) hold strictly.

Fig. 3 shows the case where $M_B > \Lambda_{stable}$ with a finite and non-zero k, from which we can see that the mode is oscillating all the way down to $H\eta = -\mathcal{O}(10)$, and then grows a little bit, before it starts to decay. Since the decaying rate is very large (inversely proportional to $-\eta$, as one can see from Eq.(2.16) where $\mathcal{F} = -2/\eta$), it dies away rapidly afterwards.

Our above analytical analysis is further supported by the following numerical calculations. Let us first notice that $\omega^2 \to -|c_\psi|^2 k^2$ as $H\eta \to 0^-$. Then, the asymptotical solution of Eq.(2.11) satisfies the equation,

$$\psi_k'' - \frac{2}{\eta}\psi_k' - |c_\psi|^2 k^2 \psi_k = 0, \tag{2.37}$$

FIG. 3: The metric perturbation ψ_k, in the cases where both k^2/M_A^2 and k^4/M_B^4 are very small, so that $F - G = 0$ has two solutions η_1 and η_2 where $\eta_c < \eta_1 < \eta_2 < 0$, as shown in Fig. 2. It can be seen that ψ_k oscillates with almost a constant amplitude when $H\eta \ll 0$, and then increases a bit, before it decays rapidly to zero, where we had chosen $c_1 = c_2$.

which has the general solution [19],

$$\psi_k = c_1(z-1)e^z + c_2(z+1)e^{-z},$$
$$B_k = \frac{(3\xi - 2)z}{\xi k^2}\left(c_1 e^z - c_2 e^{-z}\right), \tag{2.38}$$

where $z \equiv |c_\psi| k\eta = -(|c_\psi| k/H)e^{-Ht}$. Clearly, they are all finite as $H\eta \to 0^-$ (or $t \to \infty$). In particular, $B_k \to 0$ and $\psi_k \to c_2 - c_1$ in the IR limit $H\eta \to 0^-$. Note the slit difference between the two constants $c_{1,2}$ defined here and the ones used in [19]. Fig. 4 shows the function $\psi_k(\eta)$
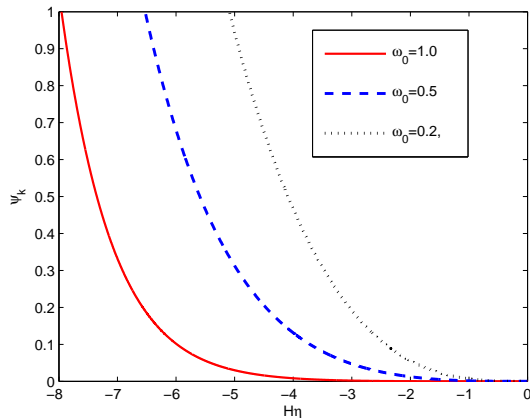
FIG. 4: The metric perturbation ψ_k, defined by Eq.(2.7) in the quasi-longitudinal gauge for different choices of ω_0, with $k^2/M_A^2 = 0$ and $k^4/M_B^4 = 0$. In all the three cases we have set $c_1 = c_2$.

with different choices of $\omega_0[\equiv |c_\psi|\, k/H]$, from which we can see that the larger ω_0 is, the faster ψ_k decays. That is, small-scale modes always decay faster than large-scale ones.

To study the effects of the high-order curvatures, we gradually turn on the fourth- and sixth- order corrections. In particular, Fig. 5 shows the case where $\omega_0 \equiv |c_\psi|\, k/H = 1$ and $k^4/M_B^4 = 0$, with three different values of the suppressed scale, M_A. From there one can see that, when M_A is small, the perturbation oscillates many times before it starts to decay. As M_A increases, the oscillating times becomes less and less. The same characteristics persist even for small values of ω_0, as shown by Figs. 6 and 7. The effects of the sixth-order term k^4/M_B^4 are shown in Fig. 8.

In all the cases, the perturbations will finally decay exponentially for any given k, as the damping force $\mathcal{F}$ is independent of k and $\mathcal{F} + 2i\omega_k(\eta) \simeq \mathcal{F} - 2\omega_0 \gg 1$ as $\eta \to 0^-$, since $\mathcal{F}(0^-) = \infty$. Therefore, for any given k the perturbations always decay exponentially as $\eta \simeq 0^-$ or $(t \to \infty)$.

III. STRONG COUPLING

To understand the strong coupling problem, we shall restrict ourselves mainly to the perturbations in the Minkowski background, because such obtained results can be easily generalized to the de Sitter background [25]. Such a study also helps us to clarify some differences regarding to the strength of strong couplings, obtained recently in [25–27, 31]. In addition, the treatment in this section can be applied to the HL theory both with and without the projectability condition. So, the conclusions obtained in this section are applicable to both cases.

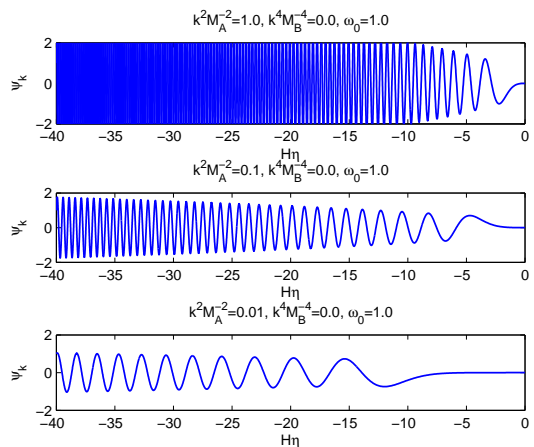When the background is Minkowski, without loss of

FIG. 5: The function ψ_k, defined by Eq.(2.7) for different choices of k^2/M_A^2 with $\omega_0 = 1$ and $k^4/M_B^4 = 0$.
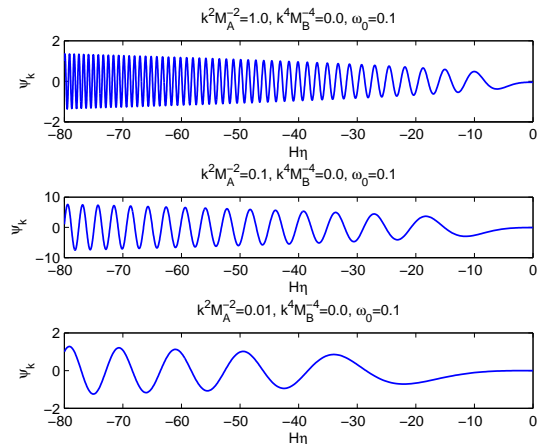

FIG. 6: The function ψ_k, defined by Eq.(2.7) for different choices of k^2/M_A^2 with $\omega_0 = 0.1$ and $k^4/M_B^4 = 0$.
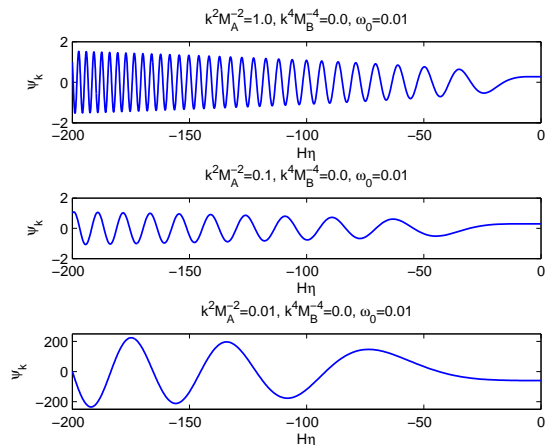

FIG. 7: The function ψ_k, defined by Eq.(2.7) for different choices of k^2/M_A^2 with $\omega_0 = 0.01$ and $k^4/M_B^4 = 0$.
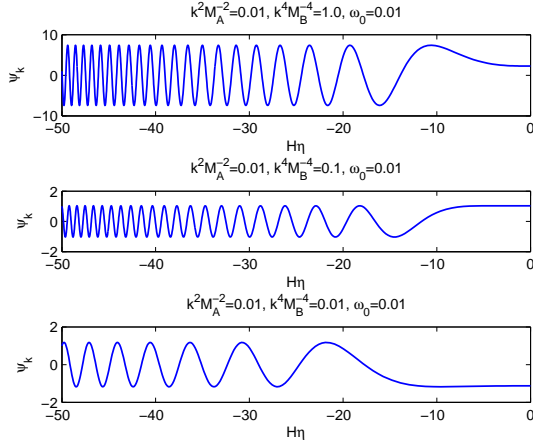
FIG. 8: The function ψ_k, defined by Eq.(2.7) for different choices of k^4/M_B^4 with $\omega_0 = 0.01$ and $k^2/M_A^2 = 0.01$.

generality, we consider the metric perturbations [26],

$$N = 1, \quad g_{ij} = e^{2\zeta(t,x)}\delta_{ij}, \quad N_i = \partial_i\beta(t,x), \qquad (3.1)$$

which lead, respectively, to the following second- and third-order actions,

$$S^{(2)} = M_{pl}^2 \int dt d^3x \left(-\frac{\dot{\zeta}^2}{c_\psi^2} + (\partial\zeta)^2\right), \qquad (3.2)$$

$$S^{(3)} = M_{pl}^2 \int dt d^3x \left\{ \zeta(\partial\zeta)^2 - \frac{3\left(2c_\psi^2 + 1\right)}{2c_\psi^4}\zeta\dot{\zeta}^2 \right.$$
$$\left. -\frac{2}{c_\psi^4}\dot{\zeta}\partial_i\zeta\frac{\partial^i}{\Delta}\dot{\zeta} + \frac{3\zeta}{c_\psi^4}\left(\frac{\partial_i\partial_j}{\Delta}\dot{\zeta}\right)^2 \right\}, \qquad (3.3)$$

where $\Delta \equiv \partial^i\partial_i$, and

$$\Delta\beta = -\frac{1}{c_\psi^2}\dot{\zeta}. \qquad (3.4)$$

Note that the above expressions can be easily obtained from the limit $\eta \to \infty$, where η is the coupling constant of the gradient term, $a_i a^i$, introduced in [30], which should not be confused with the conformal time, used in the last section. In addition, in the Minkowski background the conformal time is identical to the cosmic time t. Then, comparing Eqs.(3.1) and (2.7), we find that $\zeta = -\psi$ and $\beta = a^2 B$ to the linear order of perturbations for $a = 1$.

To consider the strong coupling problem, we first write the quadratic action $S^{(2)}$ in a canonical form with unity coupling constants. This can be done by the coordinate transformations,

$$t = \alpha\hat{t}, \quad x^i = \beta\hat{x}^i, \qquad (3.5)$$

which are allowed by the gauge freedom (2.3), where α and β are arbitrary constants. Choosing

$$\zeta = \frac{\hat{\zeta}}{M_{pl}|c_\psi|^{1/2}\alpha}, \quad \beta = |c_\psi|\alpha, \qquad (3.6)$$

one finds that $S^{(2)}$ given by Eq.(3.2) can be written as,

$$S^{(2)} = \int d\hat{t}d^3\hat{x}\left(\hat{\zeta}^{*2} + (\hat{\partial}\hat{\zeta})^2\right), \qquad (3.7)$$

where $\hat{\zeta}^* \equiv d\hat{\zeta}/d\hat{t}$, while the cubic action $S^{(3)}$ takes the form,

$$S^{(3)} = \frac{1}{\Lambda_{SC}} \int d\hat{t}d^3\hat{x}\left\{\frac{2|c_\psi|^2}{3}\hat{\zeta}(\hat{\partial}\hat{\zeta})^2 + \hat{\zeta}\left(\frac{\hat{\partial}_i\hat{\partial}_j}{\hat{\Delta}}\hat{\zeta}^*\right)^2 \right.$$
$$\left. -\frac{4}{3}\hat{\zeta}^*\hat{\partial}_i\hat{\zeta}\frac{\hat{\partial}^i}{\hat{\Delta}}\hat{\zeta}^* - \left(1 - 2|c_\psi|^2\right)\hat{\zeta}\hat{\zeta}^{*2}\right\}, \qquad (3.8)$$

where

$$\Lambda_{SC} \equiv \frac{2}{3}M_{pl}|c_\psi|^{5/2}\alpha. \qquad (3.9)$$

Clearly, if one chooses $\alpha \propto |c_\psi|^{-5/2}$, one finds that Λ_{SC} will remain finite when $c_\psi \to 0$. In the following we shall choose $\alpha = 3|c_\psi|^{-5/2}/2$, so that $\Lambda_{SC} = M_{pl}$.

Requiring that the quadratic action $S^{(2)}$ be invariant under the rescaling [41],

$$\hat{t} \to s^{-\gamma_1}\hat{t}, \quad \hat{x} \to s^{-\gamma_2}\hat{x}, \quad \hat{\zeta} \to s^{\gamma_3}\hat{\zeta}, \qquad (3.10)$$

we find that $\gamma_1 = \gamma_2 = \gamma_3$. Without loss of generality, we can always choose $\gamma_i = 1$ $(i = 1, 2, 3)$, so that Eq.(3.10) is identical to the relativistic scaling. Then, it can be shown that all the terms in the cubic action (3.8) scale as s^1, which means that these terms are irrelevant in the low energy limit, but diverge in the UV, so they are not renormalizable [41]. This indicates that the perturbations break down when the coupling coefficients greatly exceed units. To calculate these coefficients, let us consider a process at the energy scale E, then we find that all the terms in the cubic action has the same magnitude as E, for example,

$$\int d\hat{t}d^3\hat{x}\hat{\zeta}(\hat{\partial}\hat{\zeta})^2 \simeq E. \qquad (3.11)$$

Since the action is dimensionless, all the coefficients in (3.8) must have the dimension E^{-1}. Writing them in the form,

$$\lambda_i = \frac{\hat{\lambda}_i}{\Lambda_i}, \qquad (3.12)$$

where $\hat{\lambda}_i$ is a dimensionless parameter of order one, we find that the lowest scale of Λ_i's is given by the last three terms in Eq.(3.8) and is of the order of Λ_{SC}. Translating it back to the coordinates t and x, the corresponding energy and momentum scales are,

$$\Lambda_\omega = \frac{\Lambda_{SC}}{\alpha} \simeq |c_\psi|^{5/2}M_{pl},$$
$$\Lambda_k = \frac{\Lambda_{SC}}{\beta} \simeq |c_\psi|^{3/2}M_{pl}, \qquad (3.13)$$
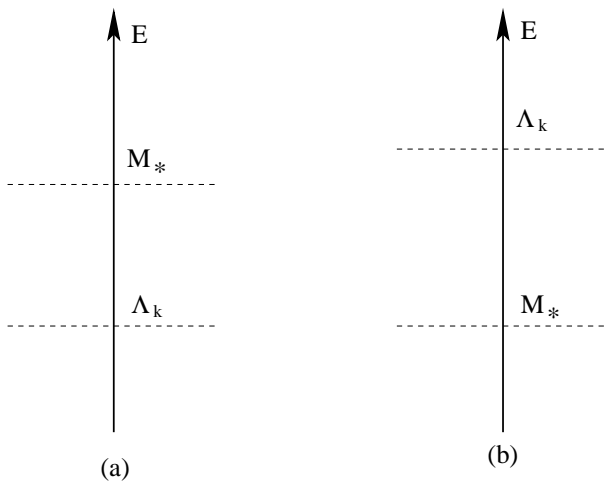
FIG. 9: The energy scales: (a) $\Lambda_k \lesssim M_*$; and (b) $\Lambda_k \gtrsim M_*$.

which are consistent with the results obtained in [30, 31] by using the Stückelberg trick (See also [25]), but slightly different from that given in [26].

As $c_\psi \to 0$, these scales vanish, indicating that strong coupling happens when c_ψ is very small. For processes with momentum $k \gtrsim \Lambda_k$, the problem becomes strong coupling, and non-linear effects are important and must be taken into account. Mukohyama recently showed that these effects make the spin-0 graviton finally decoupled, and the relativistic limit $\xi \to 0$ in the IR exists for spherically symmetric, static, vacuum spacetimes [40].

It must be noted that the above analysis is valid only for $M_* \gtrsim \Lambda_k$ [cf. Fig.9(a)]. If

$$M_* \lesssim \Lambda_k, \qquad (3.14)$$

which is the precise condition for the BPS mechanism to work [cf. Eq.(1.2)], then the high-order derivative terms become important before the strong coupling energy scale Λ_k reaches, and the above analysis is no longer valid [cf. Fig.9(b)]. Including the high order derivatives, one finds that the quadratic action becomes,

$$S^{(2)} = M_{pl}^2 \int dt d^3x \left(-\frac{\dot{\zeta}^2}{c_\psi^2} + (\partial\zeta)^2 - \frac{1}{M_A^2}\zeta\partial^4\zeta \right.$$
$$\left. + \frac{1}{M_B^4}\zeta\partial^6\zeta \right). \qquad (3.15)$$

Depending on whether $M_B < M_A$ or $M_B > M_A$, the low energy behavior will be different. In the following, let us consider them separately.

A. $M_B < M_A$

In this case, we have $M_* = M_B$. Then, we can see that the sixth-order derivative term will dominate the fourth-order one. If we consider a process at the momentum scale $k \gtrsim M_B$, then the first and last terms in $S^{(2)}$ will be dominant. Using the coordinate transformation (3.5) and the rescaling of $\zeta = \gamma\hat{\zeta}$, we first transform these terms to the ones with unit coefficients. It can be shown that this can be realized by choosing

$$\alpha = \frac{M_B}{|c_\psi|}\beta^3, \quad \gamma = \frac{M_B}{M_{pl}}|c_\psi|^{1/2}, \qquad (3.16)$$

where β is arbitrary. Then, we obtain that

$$S^{(2)} = \int d\hat{t}d^3\hat{x}\left(\hat{\zeta}^{*2} + \beta^4 M_B^4\left(\hat{\partial}\hat{\zeta}\right)^2 \right.$$
$$\left. - \frac{M_B^4\beta^2}{M_A^2}\hat{\zeta}\hat{\partial}^4\hat{\zeta} + \hat{\zeta}\hat{\partial}^6\hat{\zeta}\right), \qquad (3.17)$$

and

$$S^{(3)} = \frac{1}{\Lambda_{SC}^{(B)}}\int d\hat{t}d^3\hat{x}\left\{\beta^4 M_B^4\hat{\zeta}\left(\hat{\partial}\hat{\zeta}\right)^2 \right.$$
$$-\frac{3}{2}\left(1-2|c_\psi|^2\right)\hat{\zeta}\hat{\zeta}^{*2} - 2\hat{\zeta}^*\hat{\partial}_i\hat{\zeta}\frac{\hat{\partial}^i}{\hat{\Delta}}\hat{\zeta}^*$$
$$\left. +3\hat{\zeta}\left(\frac{\hat{\partial}_i\hat{\partial}_j}{\hat{\Delta}}\hat{\zeta}^*\right)^2 + ...\right\}, \qquad (3.18)$$

with

$$\Lambda_{SC}^{(B)} = \frac{M_{pl}}{M_B}|c_\psi|^{3/2}. \qquad (3.19)$$

The "..." in $S^{(3)}$ represents the cubic terms coming from the high-order derivative corrections, such as $f_1\zeta^2\partial^4\zeta$ and $f_2\zeta^2\partial^4\zeta$, where f_1 and f_2 are independent of c_ψ and functions of the coupling constants g_i only. As a result, the limit, $c_\psi \to 0$, of these terms always finite, and have no contributions to the strong coupling problem. In fact, it can be shown that these terms are either relevant or marginal (cf. the following analysis.). So, in the following, without loss of generality we shall ignore them. Then, under the re-scaling,

$$t \to s^{-3}t, \quad x \to s^{-1}x, \quad \zeta \to s^0\zeta, \qquad (3.20)$$

the first and last terms in the right-hand side of Eq.(3.17) are unchanged, while the second and third terms scale like s^{-4} and s^{-2}, respectively. Therefore, these terms are relevant and super-renormalizable. Similarly, the first term in the cubic action $S^{(3)}$ of Eq.(3.18) scales as s^{-4}, while all the rest scales as s^0, that is, the term $\hat{\zeta}\left(\hat{\partial}\hat{\zeta}\right)^2$ is relevant, while the rest, the second, third and fourth in $S^{(3)}$, are all marginal and strictly renormalizable. Thus, as the energy scale of the system changes, the amplitude of these latter terms remain the same. That is, these terms are equally important at all scales of energy, provided that the condition (3.14) holds. Since they are all suppressed by the dimensionless quantity $\Lambda_{SC}^{(B)}$, we can see that in the present case the problem becomes strong coupling when c_ψ is very small, unless $\Lambda_{SC}^{(B)} \gtrsim 1$, which is equivalent to,

$$M_B \lesssim M_{pl}|c_\psi|^{3/2}. \qquad (3.21)$$

B. $M_B > M_A$

When $M_B > M_A$, we have $M_* = M_A$. Then, the fourth-order derivative term in the quadratic action $S^{(2)}$ given by Eq.(3.15) will dominate the sixth-order term. Then, the rescaling (3.5) and $\zeta = \gamma\hat{\zeta}$ with

$$\alpha = \frac{1}{|c_\psi| M_A}, \quad \beta = \frac{1}{M_A}, \quad \gamma = \frac{M_A}{M_{pl}} |c_\psi|^{1/2}, \quad (3.22)$$

will bring the quadratic action (3.15) to the form,

$$S^{(2)} = \int d\hat{t}d^3\hat{x}\Big(\hat{\zeta}^{*2} + \big(\hat{\partial}\hat{\zeta}\big)^2$$
$$-\hat{\zeta}\hat{\partial}^4\hat{\zeta} + \Big(\frac{M_A}{M_B}\Big)^4 \hat{\zeta}\hat{\partial}^6\hat{\zeta}\Big), \quad (3.23)$$

while the cubic action takes the form,

$$S^{(3)} = \frac{1}{\Lambda_{SC}^{(A)}} \int d\hat{t}d^3\hat{x}\Bigg\{ |c_\psi|^2 \hat{\zeta}\big(\hat{\partial}\hat{\zeta}\big)^2$$
$$-\frac{3}{2}\Big(1 - 2\,|c_\psi|^2\Big)\hat{\zeta}\hat{\zeta}^{*2} - 2\hat{\zeta}^*\hat{\partial}_i\hat{\zeta}\frac{\hat{\partial}^i}{\hat{\Delta}}\hat{\zeta}^*$$
$$+3\hat{\zeta}\Big(\frac{\hat{\partial}_i\hat{\partial}_j}{\hat{\Delta}}\hat{\zeta}^*\Big)^2 + ...\Bigg\}, \quad (3.24)$$

where

$$\Lambda_{SC}^{(A)} \equiv \frac{M_{pl}}{M_A} |c_\psi|^{3/2}. \quad (3.25)$$

Similar to those given in Eq.(3.18), the "..." are cubic terms of the forms $f_1(g_s)\zeta^2\partial^4\zeta$ and $f_2(g_s)\zeta^2\partial^4\zeta$, which are all finite in the limit $\xi \to 0$, so they are irrelevant to the strong coupling problem.

Then, we find that the first and third terms in the right-hand side of Eq.(3.23) are unchanged, under the rescaling,

$$t \to s^{-2}t, \quad x \to s^{-1}x, \quad \zeta \to s^{1/2}\zeta, \quad (3.26)$$

for which the first term in the right-hand side of the cubic action $S^{(3)}$ given by Eq.(3.24) scales as $s^{-3/2}$. Thus, this term is relevant and super-renormalizable. The second, third and fourth terms, on the other hand, are all scale as $s^{1/2}$, so they are all irrelevant and non-renormalizable. Then, if we consider processes at the energy scale E, we find that $\int d\hat{t}d^3\hat{x}\hat{\zeta}\hat{\zeta}^{*2} \simeq E^{1/4}$, so that the second term in $S^{(3)}$ is suppressed by,

$$\Lambda_{\hat{\omega}} = \Big(\frac{M_{pl}}{M_A}\Big)^4 |c_\psi|^6. \quad (3.27)$$

It can be shown that the third and fourth terms are suppressed by the same factor. Transforming it back to the (t, x^i)-coordinates, we find that the energy and momentum are suppressed, respectively, by

$$\Lambda_\omega = \frac{\Lambda_{\hat{\omega}}}{\alpha} = \Big(\frac{M_{pl} |c_\psi|^{7/4}}{M_A}\Big)^4 M_A,$$
$$\Lambda_k = \frac{(\Lambda_{\hat{\omega}})^{1/2}}{\beta} = \Big(\frac{M_{pl} |c_\psi|^{3/2}}{M_A}\Big)^2 M_A. \quad (3.28)$$

Then, the condition (3.14) implies that $M_A \lesssim M_{pl} |c_\psi|^{3/2}$, which, together with Eq.(3.21), can be written as

$$M_* \lesssim M_{pl} |c_\psi|^{3/2}. \quad (3.29)$$

If one takes the Minkowski spacetime as the legitimate background, as shown in [11, 21], it is not stable in the SVW setup, and one would require that the instability should not show up within the age of the universe,

$$|c_\psi| \lesssim \frac{H_0}{M_*}. \quad (3.30)$$

BPS found that this, together with the condition (3.29), implies $M_* \lesssim (100\, m)^{-1}$, or equivalent to

$$|\xi| \lesssim \Big(\frac{H_0}{M_*}\Big)^{2/5} \simeq 10^{-24}. \quad (3.31)$$

Clearly, this raises the fine-tuning problem, as a natural value of ξ in the UV is expected to be order of one [4]. It is unclear by which kind of mechanism it can be driven so closed to its relativistic value $\xi = 0$ [31] (See also [25]). Following [25], it can be easily generalized the above studies to the de Sitter background, and similar conclusions will be obtained: (a) When $M_* \gtrsim \Lambda_k = M_{pl} |c_\psi|^{3/2}$, the theory becomes strong coupling for processes with energies $E \simeq \Lambda_k$ [See Eq.(3.13)]. (b) When $M_* \lesssim \Lambda_k$, the strong coupling problem does not exist. However, if one considers the studies of perturbations given in Sec. II as in the current universe, namely, $H = H_0$, then the stability condition is that of Eq.(2.34), which is the same as Eq.(3.30). Hence, the results obtained above also apply to the de Sitter background with $H = H_0$. Therefore, it is concluded that *the mechanism, $M_* \leq \Lambda_k$, of solving the strong coupling problem invented in [30, 31] for the HL theory without projectability condition, cannot be applied to the case with projectability condition.*

Thus, in the SVW setup one may choose the de Sitter spacetime as its legitimate background in order to avoid the instability problem. In order to have a reasonable UV cutoff scale M_*, where now M_* must satisfy the conditions,

$$M_{pl} |c_\psi|^{3/2} \lesssim M_* \lesssim \frac{H_0}{|c_\psi|}, \quad (3.32)$$

its IR limit has to be very closed to, if not precisely at, the GR fixed point. Of course, with such a choice, the

theory is strong coupling. This will not be a problem, if the relativistic limit can be obtained after the non-linear effects are taken into account. In the spherically symmetric, static, vacuum spacetimes, Mukohyama showed that this is indeed the case [40]. In the following section, we shall present a class of exact solutions of the theory, from which one can show clearly that the relativistic limit exists, and the limited spacetime is exactly the (rotating) de Sitter spacetime of GR.

IV. NON-PERTURBATIVE COSMOLOGICAL SOLUTIONS

In the DGP model of branes [48], Newtonian approximations lead to a Friedmann equation with a constant $\tilde{G}$ that is different from the Newtonian constant G by a factor $4/3$ [49], while the non-perturbative equation in the flat FRW universe with zero-cosmological constant takes the form,

$$H^2 = \frac{8\pi G}{3}\rho - m_c H,$$

where m_c is the graviton mass [44]. Clearly, when $m_c \to 0$, it reduces precisely to the Friedmann equation in GR. This shows clearly that the spin-0 massive graviton decouples, when the non-linear effects are taken into account, and, as a result, the theory smoothly passes over to the GR limit.

In this section, we shall show that the same happens here in the SVW setup, too. That is, when we do the linear perturbations of the de Sitter background, we have the strong coupling problem, as shown explicitly in the last section. But, the exactly solutions of the theory have a smoothy GR limit. As a matter of fact, this can already be seen clearly if one simply looks at the corresponding Friedmann equation [19],

$$\left(1 - \frac{3}{2}\xi\right)H^2 = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda. \qquad (4.1)$$

(The other independent equation is the well-known conservation law of energy and momentum, $\dot{\rho} + 3H(\rho + p) = 0$. For detail, see [11, 13, 21].) From the above expression we can see that replacing G in all the solutions obtained in GR by $\tilde{G} \equiv G/(1 - 3\xi/2)$, we shall obtain all the cosmological (flat) solutions in the HL theory.

In the following, we shall go a little bit further, and show that this is true also in the sense of non-linear perturbations of the de Sitter spacetime. Let us first note that, once nonlinear effects are taken into account, the separation of scalar, vector and tensor become impossible. This is well-known in GR when we consider second-order perturbations, where all the sectors of the first-order perturbations become the sources of the second-order ones [50]. Taking these into account, let us consider the non-perturbative solutions of the type,

$$N = a(\eta), \quad N_i = a^2(\eta)n_i(t, \mathbf{x}),$$
$$g_{ij} = a^2(\eta)e^{-2\psi(t,\mathbf{x})}\delta_{ij}. \qquad (4.2)$$

After simple but tedious calculations [cf. Appendix A], we find the following exact solutions of the corresponding HL equations with a non-zero cosmological constant Λ,

$$a(\eta) = -\left(\frac{3(2 - 3\xi)}{2\Lambda}\right)^{1/2}\frac{1}{\eta},$$
$$\psi = \psi_0, \quad n_i = n_0(-y, x, 0), \qquad (4.3)$$

where ψ_0 and n_0 are two integration constants. Without loss of generality, one can set $\psi_0 = 0$ by rescaling of the coordinates x^i (and redefinition of the constant n_0). The constant n_0, on the other hand, represents the rotation of the spacetime, and cannot be gauged away, as can be seen from the following analysis. If one considers the rotation as perturbations, one can see that it corresponds to the sum of infinitely high order perturbation terms, some of which will become singular in the limit $\xi \to 0$, as it is expected from the analysis given in the last section. But, the analytical solutions themselves indeed have a finite and smoothy limit, $\xi \to 0$, as one can see from Eq.(4.3). In particular, when $\xi = 0$, the above solutions reduce to a rotating de Sitter spacetime. In fact, introducing the cylindrical coordinates r and θ via the relations $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we find that the metric can be written in the form,

$$ds^2\big|_{\xi=0} = \frac{1}{(-H\eta)^2}\Big\{ - d\eta^2 + dr^2 + dz^2 + (d\theta + n_0 d\eta)^2 \Big\}, \qquad (4.4)$$

with $H = \sqrt{3/\Lambda}$, and n_0 represents the angular velocity of the rotation.

V. CONCLUSIONS

In this paper, we have considered two different issues raised recently in the studies of the HL theory, the stability of background spacetime and strong coupling, by paying main attention on the SVW setup [21], which represents the most general HL theory with projectability condition. Although the Minkowski spacetime is not stable in such a setup, the de Sitter spacetime is, due to two different kinds of effects: one is from the high-order derivatives of the spacetime curvature, and the other is from the exponential expansion of the de Sitter space. By combining these effects properly, one can make the instability found in the Minkowski background never raise even for small-scale modes. The condition is simply that of Eq.(2.29), from which we can see that if the IR limit is sufficiently closed to the relativistic fixed point ($\xi = 0$ or $c_\psi = 0$), it can be satisfied. In particular, at the fixed point, all the modes become stabilized.

To stabilize the massless spin-0 graviton, another way is to invoke a Higgs-like mechanism to give it a mass term, $m_\psi^2\psi^2$, in the effective action [29]. Massive gravity in 4-dimensional spacetimes have been intensively studied

recently, see, for example, [43] and references therein). Then, it can be shown that the equation for the metric perturbation ψ_k in the Minkowski background reads, $\ddot{\psi}_k + \omega_k^2 \psi_k = 0$, but now with

$$\omega_k^2 = |c_\psi|^2 \left(m_\psi^2 - k^2 + \frac{k^4}{M_A^2} + \frac{k^6}{M_B^4} \right). \tag{5.1}$$

Clearly, if m_ψ is large enough, ω_k^2 is always non-negative for any given k. It is not difficult to show that such a condition is

$$m_\psi \geq \frac{M_B \sqrt{4 + r^2}}{\left(\sqrt{3 + r^4} + r^2 \right)^{1/2} \left(r^4 + r^2 \sqrt{3 + r^4} + 6 \right)^{1/2}}, \tag{5.2}$$

from which we find that $m_\psi(r \sim 1) \simeq M_B$.

The strong coupling problem has been also investigated, and found that it cannot be solved by the Blas-Pujolas-Sibiryakov mechanism, initially designed for the case without projectability condition. Strong coupling itself is not a problem, but an indication that the linear perturbations are broken when energies involved in processes of interest are higher than the strong coupling energy. Then, nonlinear effects are needed to be taken into account. If the relativistic limit (or very closed to it) can be obtained in the IR, after the non-linear effects are taken into account, the theory is still viable. Two typical examples are the massive gravity [39] and the DGP brane model [44], although the physics behind of them is different [See Footnote 1 given in the Introduction]. In this paper, we have constructed a class of non-perturbative cosmological solutions in the SVW setup, and shown explicitly that it reduces smoothly to the rotating de Sitter spacetime. This can be considered as a cosmological generalization of the spherical case studied recently by Mukohyama [40].

VI.   APPENDIX A: NON-LINEAR COSMOLOGICAL PERTURBATIONS

In this Appendix, we present some basic expressions that are useful for the studies of the non-linear cosmological perturbations, given by

$$\begin{aligned}
N(\eta) &= a(\eta), \quad N_i = a^2(\eta) n_i(\eta, \mathbf{x}), \\
g_{ij} &= a^2(\eta) e^{-2\psi(\eta,\mathbf{x})} \delta_{ij}.
\end{aligned} \tag{A.1}$$

For the sake of simplicity, we set $c = 1$. Then, we find that

$$\begin{aligned}
K_{ij} &= a\Big(e^{-2\psi}(\psi' - \mathcal{H})\delta_{ij} + n_{(i,j)} - n_k\psi^{,k}\delta_{ij} \\
&\quad + n_i\psi_{,j} + n_j\psi_{,i} \Big), \\
K &= \frac{e^{2\psi}}{a} \left(3e^{-2\psi}(\psi' - \mathcal{H}) - n_k\psi^{,k} + \partial^k n_k \right), \\
R_{ij} &= \psi_{,ij} + \psi_{,i}\psi_{,j} + \left(\partial^2\psi - (\partial\psi)^2 \right)\delta_{ij}, \\
R &= \frac{2e^{2\psi}}{a^2} \left(2\partial^2\psi - (\partial\psi)^2 \right), \tag{A.2}
\end{aligned}$$

where $n_{(i,j)} \equiv (n_{i,j} + n_{j,i})/2$. Then, we obtain that

$$\begin{aligned}
\mathcal{L}_K &= K_{ij}K^{ij} - \lambda K^2 \\
&= \frac{e^{4\psi}}{a^2} \Big\{ 3(1 - 3\lambda)e^{-4\psi}(\psi' - \mathcal{H})^2 \\
&\quad + (1 - \lambda)n_k\psi^{,k}\left(n_k\psi^{,k} - 2\partial^k n_k \right) \\
&\quad - 2(1 - 3\lambda)e^{-2\psi}(\psi' - \mathcal{H})\left(n_k\psi^{,k} - \partial^k n_k \right) \\
&\quad + n_{(k,l)}\left(n^{(k,l)} + 2n^k\psi^{,l} + 2n^l\psi^{,k} \right) \\
&\quad + 2n_k n^k\psi_{,l}\psi^{,l} - \lambda\left(\partial^k n_k \right)^2 \Big\}, \tag{A.3}
\end{aligned}$$

$$\begin{aligned}
R_{ij}R^{ij} &= \frac{e^{4\psi}}{a^4} \Big\{ 5(\partial^2\psi)^2 + 6(\partial^2\psi)(\partial\psi)^2 + 2(\partial\psi)^4 \\
&\quad + \psi_{kl}\left(\psi^{,kl} + 2\psi^{,k}\psi^{,l} \right) \Big\}, \tag{A.4}
\end{aligned}$$

$$\begin{aligned}
R_j^i R_k^j R_i^k &= \frac{e^{4\psi}}{a^4} \Big\{ \psi^{,i}_{,j}\psi^{,j}_{,k}\psi^{,k}_{,i} + 3\psi^{,ij}\psi_{,jk}\psi^{,k}\psi_{,i} \\
&\quad + 3\psi_{,ij}\psi^{,ij}\left(\partial^2\psi - (\partial\psi)^2 \right) \\
&\quad + 3\psi_{,ij}\psi^{,i}\psi^j\left(2\partial^2\psi - (\partial\psi)^2 \right) \\
&\quad + 6(\partial^2\psi)^2\left(\partial^2\psi - 2(\partial\psi)^2 \right) \\
&\quad + (\partial\psi)^4\left(9\partial^2\psi - 2(\partial\psi)^2 \right), \tag{A.5}
\end{aligned}$$

and

$$\begin{aligned}
(\nabla_i R_{jk})(\nabla^i R^{jk}) &= \frac{e^{6\psi}}{a^6} \Big\{ F_{ijk}F^{ijk} + 3G_k G^k \\
&\quad + 4F^i\left(F^k_{\ ki} + 2F_i \right) + 2G^i\left(F_{ik}^{\ \ k} + 2F_i \right) \Big\}, \tag{A.6}
\end{aligned}$$

where

$$\begin{aligned}
F_{ijk} &= \psi_{,ijk} + 2\psi_{,ij}\psi_{,k} + 2\psi_{,ik}\psi_{,j} + 2\psi_{,jk}\psi_{,i} \\
&\quad + 4\psi_{,i}\psi_{,j}\psi_{,k}, \\
G_i &= \left(\partial^2\psi - (\partial\psi)^2 \right)_{,i} + 2\left(\partial^2\psi - (\partial\psi)^2 \right)\psi_{,i}, \\
F_i &= -\left(\psi^{,k}_{,i} + \delta_i^k(\partial\psi)^2 \right)\psi_{,k}, \tag{A.7}
\end{aligned}$$

and $F^i \equiv \delta^{ik}F_k$, etc.

With these expressions, one can write down the Hamiltonian and momentum constraints, and the dynamical equations given in [11, 13], which are too complicated to provide here.

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