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Three-Algebras in $\mathcal{N} = 5, 6$ Superconformal Chern-Simons Theories: Representations and Relations

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In this work we present 3-algebraic constructions and representations for three-dimensional $\mathcal{N} = 5$ supersymmetric Chern-Simons theories, and show how they relate to theories with additional supersymmetries. The $\mathcal{N} = 5$ structure constants give theories with $\mathrm{Sp}(2N) \times \mathrm{SO}(M)$ gauge symmetry, as well as more exotic symmetries known from gauged supergravity. We find explicit lifts from $\mathcal{N} = 6$ to 8, and $\mathcal{N} = 5$ to 6 and 8, for appropriate gauge groups.

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I. INTRODUCTION

Over the past few years, there has been an explosion of interest in three-dimensional supersymmetric Chern-Simons gauge theories. Much progress was sparked by the $\mathcal{N} = 8$ theory put forth in [1],[2],[3], and independently in [4], that was proposed to describe the world volume theory of coincident M2-branes [5, 6].

The theory contains 8 scalars, X^I , which take values in the transverse space, and a 16-component real fermion Ψ , which is a two-component real $d = 3$ spinor in one of the 8-dimensional spinor representations of the $SO(8)$ R-symmetry group; the supersymmetry parameter ϵ is in the other. The fields take values in a 3-algebra, defined by a totally antisymmetric triple product, given by

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d. \quad (1)$$

The invariant, symmetric inner-product $(T^a, T^b) = h^{ab}$ raises and lowers indices, so that f^{abcd} is real and totally antisymmetric. The theory is gauged, with gauge field

$$\tilde{A}_\mu{}^a{}_d = f^{abc}{}_d A_{\mu bc}. \quad (2)$$

The gauge field is constrained, so the degrees of freedom balance between bosons and fermions. The 3-algebra satisfies the so-called fundamental identity,

$$\begin{aligned} [T^a, T^b, [T^c, T^d, T^e]] &= [[T^a, T^b, T^c], T^d, T^e] + [T^c, [T^a, T^b, T^d], T^e] \\ &+ [T^c, T^d, [T^a, T^b, T^e]], \end{aligned} \quad (3)$$

which implies that the gauge transformations act as derivations. These constraints define the $\mathcal{N} = 8$ theory, of which there is only one (unitary) example: $f^{abcd} \sim \varepsilon^{abcd}$ and $h^{ab} \sim \delta^{ab}$, for which the gauge group is $SO(4)$.

More theories can be found by reducing the number of supersymmetries. These include the ABJM theories, with $\mathcal{N} = 6$ supersymmetry and $U(N) \times U(N)$ gauge symmetry [7], and the ABJ theories [8], with $\mathcal{N} = 6$ and $U(N) \times U(M)$ gauge symmetry, as well as $\mathcal{N} = 5$ with $Sp(2N) \times O(M)$. Similar theories were constructed in [9]. A classification of the possible $\mathcal{N} = 6$ theories of ABJM type was presented in [10].

None of these constructions made use of a 3-algebra, so it is natural to ask whether they play any role in theories with $\mathcal{N} < 8$. In fact, the most general $\mathcal{N} = 6$ theory was constructed from a 3-algebra in [11]. One realization gives rise to an $\mathcal{N} = 6$ theory with $SU(N) \times SU(N)$ gauge symmetry; another describes the $\mathcal{N} = 6$ $U(N) \times U(M)$ ABJ theories. It has recently been shown that the $SU(N) \times SU(N)$ theory is related to the $U(N) \times U(N)$ ABJM theory [12], so the 3-algebraic approach indeed describes the complete set of $\mathcal{N} = 6$ ABJM and ABJ theories.

Given these results, one would also like to know the role that 3-algebras play in $\mathcal{N} = 5$ theories. The quaternionic unitary 3-algebras were classified in [13], where it was found that they are in one-one correspondence with the $\mathcal{N} = 5$ Chern-Simons theories presented in [9] and [15]. In this paper we take a more direct approach and construct the most general three-dimensional $\mathcal{N} = 5$ superconformal Chern-Simons theories from first principles. We work in components and close the supersymmetry transformations on the fields. We find that the theories depend on real structure constants with four upstairs indices, satisfying $\mathcal{N} = 5$ versions of the fundamental identity. When the structure constants obey $f^{abcd} = -f^{bacd} = f^{cdab}$, they give rise to $\mathcal{N} = 5$ truncations of $\mathcal{N} = 6$ theories, with supersymmetry transformations given in [11]. When they obey $g^{abcd} = g^{bacd} = g^{cdab}$, with $g^{(abc)d} = 0$, the theories are purely $\mathcal{N} = 5$. For this case, our $\mathcal{N} = 5$ transformation laws agree with those presented in [14]. Our results are in accord with the classification derived in [13]. In addition, they clarify the connection between $\mathcal{N} = 5$ and $\mathcal{N} = 6$ theories and show that they both arise as independent solutions to a single set of constraints.

In what follows we also present explicit 3-algebra representations for various $\mathcal{N} = 5$ theories. We recover all the examples discussed in [9, 13, 15]. We find an $Sp(2N) \times SO(M)$ theory of ABJ type, with matter fields transforming in the bifundamental representation of the gauge group, as well as an $SO(4) \times SU(2)$ theory with one free parameter. We also find more exotic theories with gauge groups $G_2 \times SU(2)$, with bifundamental matter, and $SO(7) \times SU(2)$, with matter in the 8-dimensional spinor representation of $SO(7)$. These theories can also be found using the ‘‘embedding tensor’’ approach to $d = 3$, $\mathcal{N} = 8$ gauged supergravity in the conformal limit [17, 18], or using $\mathcal{N} = 1$ superspace, as was done in [16].

Finally, in this paper we also show how to lift certain theories with $\mathcal{N} = 5$ and $\mathcal{N} = 6$ supersymmetry to $\mathcal{N} = 6$ and $\mathcal{N} = 8$. We first lift the $\mathcal{N} = 6$ theory with $SU(2) \times SU(2) \simeq SO(4)$ gauge symmetry to $\mathcal{N} = 8$. We then lift the $Sp(2N) \times SO(2)$ invariant $\mathcal{N} = 5$ theory to $\mathcal{N} = 6$. As a third example, we lift the $\mathcal{N} = 5$ theory with $SO(4) \times SU(2)$ gauge symmetry to $\mathcal{N} = 6$ at one point in its parameter space. At that point, the gauge symmetry is reduced to $SO(4) = SU(2) \times SU(2)$, as required for $\mathcal{N} = 6$ supersymmetry.

The layout of the paper is as follows. In the next section, we review the 3-algebraic construction of the $\mathcal{N} = 6$ theories. We present specific representations of the various gauge groups that arise, and we demonstrate the lift to

$\mathcal{N} = 8$. We then turn our attention to $\mathcal{N} = 5$ and construct the most general theory based on a 3-algebra. We find the fundamental identity, and solve it in terms of structure constants of two different kinds. We discuss explicit representations, and present the lifts from $\mathcal{N} = 5$ to $\mathcal{N} = 6$.

II. REVIEW OF THE $\mathcal{N} = 6$ CONSTRUCTION

In this section, we review the relevant features of the construction in [11]. We start by decomposing the $\text{SO}(8)$ global symmetry into $\text{SO}(6) \times \text{SO}(2) = \text{SU}(4) \times \text{U}(1)$. The matter fields are a scalar Z_a^A and a spinor Ψ_{Aa} , both with $\text{U}(1)$ charges $+1$, together with their conjugates \bar{Z}_A^a and $\bar{\Psi}^{Aa}$, where $A = 1, \dots, 4$ is the $\text{SU}(4)$ index and a spans a representation of some gauge group. The 3-algebra structure constants $f^{ab}{}_{cd}$ are no longer necessarily real or totally antisymmetric, but satisfy $f^{ab}{}_{cd} = -f^{ba}{}_{cd} = f^{ba}{}_{dc} = f^{*ab}$. The six supersymmetry parameters ε^{AB} are antisymmetric in A and B , and obey the reality condition

$$\varepsilon^{AB} = \frac{1}{2} \varepsilon^{ABCD} \varepsilon_{CD}. \quad (4)$$

The $\mathcal{N} = 6$ supersymmetry transformations on the scalar and the fermion are

$$\begin{aligned} \delta Z_d^A &= i \bar{\varepsilon}^{AD} \Psi_{Dd}, \\ \delta \Psi_{Dd} &= \gamma^\mu \varepsilon_{AD} D_\mu Z_d^A \\ &\quad + f^{ab}{}_{cd} Z_a^A Z_b^B \bar{Z}_A^c \varepsilon_{BD} + f^{ab}{}_{cd} Z_a^A Z_b^B \bar{Z}_D^c \varepsilon_{AB}, \end{aligned} \quad (5)$$

where the gauge-covariant derivative on the scalar is defined by

$$D_\mu Z_d^A = \partial_\mu Z_d^A - \tilde{A}_\mu{}^a{}_d Z_a^A. \quad (6)$$

The transformations on the scalar close according to the supersymmetry algebra,

$$[\delta_1, \delta_2] Z_d^A = v^\mu D_\mu Z_d^A + \tilde{\Lambda}^a{}_d Z_a^A, \quad (7)$$

where

$$v^\mu = \frac{i}{2} \bar{\varepsilon}_2^{CD} \gamma^\mu \varepsilon_{1CD} \quad (8)$$

and

$$\tilde{\Lambda}^a{}_d = i \bar{\varepsilon}_{[2}^{CE} \varepsilon_{1]BE} f^{ab}{}_{cd} Z_b^B \bar{Z}_C^c, \quad (9)$$

where the antisymmetrization is done without a factor of $\frac{1}{2}$.

The transformations on the fermions close similarly,

$$[\delta_1, \delta_2] \Psi_{Dd} = v^\mu D_\mu \Psi_{Dd} + \tilde{\Lambda}^a{}_d \Psi_{Da}, \quad (10)$$

provided the equations of motion are satisfied:

$$\begin{aligned} E_{Dd} &= \gamma^\mu D_\mu \Psi_{Dd} - 2 f^{ab}{}_{cd} \Psi_{Ba} Z_b^B \bar{Z}_D^c \\ &\quad + f^{ab}{}_{cd} \Psi_{Da} Z_b^B \bar{Z}_B^c + \varepsilon_{ABCD} f^{ab}{}_{cd} \Psi^{Cc} Z_a^A Z_b^B = 0. \end{aligned} \quad (11)$$

Finally, the gauge field transformations

$$\delta \tilde{A}_\mu{}^a{}_d = -i f^{ab}{}_{cd} (\bar{\varepsilon}^{BC} \gamma_\mu \Psi_{Bb} \bar{Z}_C^c + \bar{\varepsilon}_{BC} \gamma_\mu \Psi^{Cc} Z_b^B) \quad (12)$$

close as follows,

$$[\delta_1, \delta_2] \tilde{A}_\mu{}^a{}_d = D_\mu (\tilde{\Lambda}^a{}_d) + v^\nu \tilde{F}_{\mu\nu}{}^a{}_d + \mathcal{O}(Z^4), \quad (13)$$

provided the field strength obeys the following condition:

$$\begin{aligned} \tilde{F}_{\mu\nu}{}^a{}_d &= -\partial_\mu \tilde{A}_\nu{}^a{}_d + \partial_\nu \tilde{A}_\mu{}^a{}_d + \tilde{A}_\nu{}^a{}_b \tilde{A}_\mu{}^b{}_d - \tilde{A}_\mu{}^a{}_b \tilde{A}_\nu{}^b{}_d \\ &= -\varepsilon_{\mu\nu\lambda} (D^\lambda Z_b^B \bar{Z}_B^c - Z_b^B D^\lambda \bar{Z}_B^c - i \bar{\Psi}^{Bc} \gamma^\lambda \Psi_{Bb}) f^{ab}{}_{cd}. \end{aligned} \quad (14)$$

Canceling the $\mathcal{O}(Z^4)$ -terms leads to the $\mathcal{N} = 6$ fundamental identity,

$$f^{ef}{}_{gb} f^{cb}{}_{ad} + f^{fe}{}_{ab} f^{cb}{}_{gd} + f_{ga}^*{}^{fb} f^{ce}{}_{bd} + f_{ag}^*{}^{eb} f^{cf}{}_{bd} = 0. \quad (15)$$

The fundamental identity ensures that the gauge transformation acts as a derivation. With these ingredients, it is not hard to construct the $\mathcal{N} = 6$ Lagrangian, written in terms of the 3-algebra. In the next section, we discuss representations of the $\mathcal{N} = 6$ gauge groups.

III. $\mathcal{N} = 6$ REPRESENTATIONS

A representation of the 3-algebra can be constructed from rectangular $M \times N$ matrices, X, Y, Z , as follows:

$$[X, Y; Z] = XZ^\dagger Y - YZ^\dagger X, \quad (16)$$

where Z^\dagger is the conjugate transpose of Z . This can be interpreted as a gauge transformation on X_{dl} , acting via left and right multiplication, with X carrying bifundamental indices d and l ,

$$\begin{aligned} \delta X_{dl} &= [X, Y; Z]_{dl} \\ &= X_{dk} Z^{\dagger kb} Y_{bl} - Y_{dk} Z^{\dagger kb} X_{bl}. \end{aligned} \quad (17)$$

In this case, the 3-algebra structure constants are given by

$$f^{abj}{}_{ckdl} = \delta_d^a \delta_c^b \delta_k^i \delta_l^j - \delta_c^a \delta_d^b \delta_l^i \delta_k^j. \quad (18)$$

The structure constants have the correct symmetries and satisfy the $\mathcal{N} = 6$ fundamental identity.

Using (9), it is a simple matter to determine the gauge theories that are constructed in this way. For this particular 3-algebra, we find

$$\delta Z_{dl}^A = \tilde{\Lambda}^{ai}{}_{dl} Z_{ai}^A = i\tilde{\varepsilon}_{[2}^{CE} \varepsilon_{1]BE} Z_{bl}^B \bar{Z}_C^{bk} Z_{dk}^A - i\tilde{\varepsilon}_{[2}^{CE} \varepsilon_{1]BE} Z_{dj}^B \bar{Z}_C^{cj} Z_{cl}^A, \quad (19)$$

The matrix $\tilde{\Lambda}^{ai}{}_{dl}$ is anti-Hermitian, with a non-vanishing trace for $M \neq N$ and a vanishing trace for $M = N$. As expected, these $\mathcal{N} = 6$ theories have $U(N) \times U(M)$ and $SU(N) \times SU(N)$ gauge symmetry. The original $U(N) \times U(N)$ ABJM model can be recovered by gauging the global $U(1)$ symmetry, as was done in [12].

A second choice of structure constants is given by

$$f^{ab}{}_{cd} = J^{ab} J_{cd} + (\delta_c^a \delta_d^b - \delta_d^a \delta_c^b), \quad (20)$$

where $J^{ab} = i(\sigma^2 \otimes \mathbf{I}_{N \times N})^{ab}$ is the antisymmetric invariant tensor of $\text{Sp}(2N)$. The $f^{ab}{}_{cd}$ also obey the fundamental identity and have the correct symmetries. As before, we close the algebra to find the gauge transformation on Z_d^A ,

$$\begin{aligned} \delta Z_d^A &= \tilde{\Lambda}^a{}_d Z_a^A = i\tilde{\varepsilon}_{[2}^{CE} \varepsilon_{1]BE} (Z_d^B \bar{Z}_C^a + J^{ab} J_{cd} Z_b^B \bar{Z}_C^c) Z_a^A \\ &\quad - i\tilde{\varepsilon}_{[2}^{CE} \varepsilon_{1]BE} Z_b^B \bar{Z}_C^b Z_d^A. \end{aligned} \quad (21)$$

This transformation is a sum of two parts. The first is of the form $\delta' Z_d^A = \tilde{\Lambda}'^a{}_d Z_a^A$; the second is a phase. It is easy to see that $J_{ab} \Lambda^{bc} J^{cd} = \Lambda'^d{}_a$, so the gauge group is simply $\text{Sp}(2N) \times U(1)$.

IV. LIFT: $\mathcal{N} = 6 \rightarrow \mathcal{N} = 8$

From the above construction, it is possible to find an explicit lift from the $\mathcal{N} = 6$ theory with $SU(2) \times SU(2)$ gauge symmetry to the unique $\mathcal{N} = 8$ theory. We begin by writing the matter fields $Z_{\alpha\dot{\alpha}}^A$ in $\text{SO}(4)$ notation,

$$Z_d^A = Z_{\alpha\dot{\alpha}}^A \bar{\sigma}_d^{\dot{\alpha}\alpha}, \quad (22)$$

using the ordinary Pauli matrices of [19] (except taking $\sigma^0 \rightarrow i\sigma^0 = i\bar{\sigma}^0$ to make the gauge-space Euclidean). Because of the well-known identity

$$(\bar{\sigma}^a \sigma^b \bar{\sigma}^c - \bar{\sigma}^c \sigma^b \bar{\sigma}^a)^{\dot{\alpha}\alpha} = -2\varepsilon^{abcd} \bar{\sigma}_d^{\dot{\alpha}\alpha},$$

the representation of the $SU(2) \times SU(2)$ transformation given in (16) exactly reproduces the 3-algebra of the $\mathcal{N} = 8$ theory, with $f^{abcd} = \varepsilon^{abcd}$ (we absorb the constant of proportionality into ε^{abcd}).

In this notation, we start with the original $\mathcal{N} = 6$ supersymmetry transformations presented above, parameterized by ε^{AB} , and construct two additional supersymmetries, parameterized by a *complex* spinor η of global $U(1)$ charge +2. The most general supersymmetry transformations consistent with these assignments are

$$\begin{aligned} \delta Z_d^A &= i\tilde{\varepsilon}^{AD} \Psi_{Dd} + i\Theta_1 \eta \Psi_d^A \\ \delta \Psi_D^d &= \gamma^\mu \varepsilon_{AD} D_\mu Z^{Ad} + \Theta_2 \gamma^\mu \eta D_\mu \bar{Z}_D^d \\ &\quad + \varepsilon^{abcd} Z_a^A Z_b^B \bar{Z}_{Dc} \varepsilon_{AB} - \varepsilon^{abcd} Z_a^A Z_b^B \bar{Z}_{Bc} \varepsilon_{AD} \\ &\quad - \Theta_3 \varepsilon^{abcd} Z_a^A \bar{Z}_{Ab} \bar{Z}_{Dc} \eta + \Theta_4 \varepsilon_{ABCD} \varepsilon^{abcd} \eta^* Z_a^A Z_b^B Z_c^C, \end{aligned} \quad (23)$$

for some complex numbers $\Theta_1, \Theta_2, \Theta_3, \Theta_4$. Note that since the gauge group is $\text{SO}(4)$, the gauge indices can be raised and lowered at will.

Imposing the supersymmetry algebra on the scalar transformation leads to $\Theta_1 = \Theta_3$ and $\Theta_1 = \Theta_2$. In particular, we find

$$[\delta_1, \delta_2]Z_d^A = v^\mu D_\mu Z_d^A + \tilde{\Lambda}^a{}_d Z_a^A, \quad (24)$$

where

$$v^\mu = \frac{i}{2}\bar{\varepsilon}_2^{BC}\gamma^\mu\varepsilon_{1BC} + i|\Theta_1|^2\bar{\eta}_{[2}\gamma^\mu\eta_{1]}^* \quad (25)$$

and

$$\begin{aligned} \tilde{\Lambda}^{ad} &= i\bar{\varepsilon}_{[2}^{CE}\varepsilon_{1]BE}\varepsilon^{abcd}Z_b^B\bar{Z}_{Cc} + 3i\Theta_4\bar{\varepsilon}_{[2BC}\eta_{1]}^*\varepsilon^{abcd}Z_b^B Z_c^C \\ &\quad + i\Theta_1\bar{\eta}_{[2}\varepsilon_{1]}^{BC}\varepsilon^{abcd}\bar{Z}_{Bb}\bar{Z}_{Cc} + i|\Theta_1|^2\bar{\eta}_{[2}\eta_{1]}^*\varepsilon^{abcd}Z_b^B\bar{Z}_{Bc}. \end{aligned} \quad (26)$$

Anti-Hermiticity of the generator $\tilde{\Lambda}^{ad}$ requires $\Theta_1 = -3\Theta_4^*$. This leaves only Θ_1 independent; it can be absorbed into the parameter η .

With these results, the supersymmetry transformations are

$$\begin{aligned} \delta Z_d^A &= i\bar{\varepsilon}^{AD}\Psi_{Dd} + i\bar{\eta}\Psi_d^A \\ \delta\Psi_D^d &= \gamma^\mu\varepsilon_{AD}D_\mu Z^{Ad} + \gamma^\mu\eta D_\mu\bar{Z}_D^d \\ &\quad + \varepsilon^{abcd}Z_a^A Z_b^B\bar{Z}_{Dc}\varepsilon_{AB} - \varepsilon^{abcd}Z_a^A Z_b^B\bar{Z}_{Bc}\varepsilon_{AD} \\ &\quad - \varepsilon^{abcd}Z_a^A\bar{Z}_{Ab}\bar{Z}_{Dc}\eta - \frac{1}{3}\varepsilon_{ABCD}\varepsilon^{abcd}\eta^*Z_a^A Z_b^B Z_c^C. \end{aligned} \quad (27)$$

Closing on the fermion gives

$$\begin{aligned} [\delta_1, \delta_2]\Psi_{Dd} &= v^\mu D_\mu\Psi_{Dd} + \tilde{\Lambda}^a{}_d\Psi_{Da} \\ &\quad + \frac{i}{2}\bar{\varepsilon}_{[2}^{CB}\varepsilon_{1]CD}E_{Bd} - \frac{i}{4}\bar{\varepsilon}_2^{BE}\gamma^\mu\varepsilon_{1BE}\gamma_\mu E_{Dd} \\ &\quad + i\bar{\eta}_{[2}\varepsilon_{1]CD}E_d^C - \frac{i}{2}(\bar{\eta}_{[2}\eta_{1]}^* + \bar{\eta}_{[2}^*\gamma^\mu\eta_{1]}\gamma_\mu)E_{Dd}, \end{aligned} \quad (28)$$

as required, where E_{Dd} denotes the fermion equation of motion (11). The same calculation also fixes the transformation of the gauge field,

$$\begin{aligned} \delta\tilde{A}_\mu^{ad} &= -i\varepsilon^{abcd}\bar{\varepsilon}_{BC}\gamma_\mu\Psi_b^B Z_c^C - i\varepsilon^{abcd}\bar{\varepsilon}^{BC}\gamma_\mu\Psi_{Bb}\bar{Z}_{Cc} \\ &\quad + i\varepsilon^{abcd}\bar{\eta}^*\gamma_\mu\Psi_{Bb}Z_c^B + i\varepsilon^{abcd}\bar{\eta}\gamma_\mu\Psi_b^B\bar{Z}_{Bc}. \end{aligned} \quad (29)$$

Closing on \tilde{A}_μ^{ad} imposes the constraint (14).

The above transformations are manifestly $\text{SU}(4) \times \text{U}(1)$ covariant. However, they must also be covariant under $\text{SO}(8)$, the $\mathcal{N} = 8$ R-symmetry group. As a check, therefore, we compute their transformations under the twelve remaining generators of $\text{SO}(8)/(\text{SU}(4) \times \text{U}(1))$, which we denote g^{AB} , with $\text{U}(1)$ charge 2. The transformations are

$$\begin{aligned} \delta Z_a^A &= g^{AB}\bar{Z}_{Ba} \\ \delta\Psi_{Ba} &= -\frac{1}{2}\varepsilon_{BCDE}g^{DE}\Psi_a^C \\ \delta\varepsilon^{AB} &= g^{AB}\eta^* + \frac{1}{2}\varepsilon^{ABCD}g_{CD}^*\eta \\ \delta\eta &= -\frac{1}{2}g^{AB}\varepsilon_{AB}, \end{aligned} \quad (30)$$

consistent with the fact that Z_a^A , Ψ_{Bb} and ε^{AB} live in different $\text{SO}(8)$ representations. The transformations (30) close into $\text{SU}(4) \times \text{U}(1)$, as required by the $\text{SO}(8)$ algebra. Moreover, it is not hard to show that the supersymmetry transformations (27) and (29) are covariant under (30), as they must be. Thus, for the case of $\text{SO}(4)$ gauge symmetry, the supersymmetry transformations (27) and (29) do indeed lift the $\mathcal{N} = 6$ theory to $\mathcal{N} = 8$.

V. $\mathcal{N} = 5$ CONSTRUCTION

In this section, we proceed along similar lines to construct the most general $\mathcal{N} = 5$ theories that make use of a 3-algebra. We start by decomposing the $\text{SO}(8)$ global symmetry into $\text{SO}(5) \times \text{SO}(3) = \text{Sp}(4) \times \text{SU}(2)$. We take the eight scalar fields to have the index structure X_{id}^A , where $A = 1, \dots, 4$ and $i = 1, 2$ are indices that refer to the $\text{Sp}(4)$ R-symmetry and the global $\text{SU}(2)$, respectively; the index d spans a representation of the gauge group. The $\text{Sp}(4)$ indices are raised or lowered with the $\text{Sp}(4)$ -invariant tensor,

$$\omega^{AB} = i(\sigma^2 \otimes \mathbf{I}_{2 \times 2})^{AB},$$

for which $\omega^{AB}\omega_{BC} = -\delta_C^A$. Here and elsewhere we adopt the convention $X^A = \omega^{AB}X_B$, $X_A = -\omega_{AB}X^B$ for any symplectic structure. The supersymmetry parameters are real spinors ξ^{AB} , antisymmetric in A and B and traceless,

$$\omega_{AB}\xi^{AB} = 0, \quad (31)$$

so the ξ^{AB} are in the $\mathbf{5}$ of $\text{Sp}(4)$. The superpartner fermions are real spinors as well, with index structure Ψ_{Aid} .

The most general supersymmetry transformations are of the following form,

$$\begin{aligned} \delta X_{id}^A &= i\bar{\xi}^{AD}\Psi_{Did} \\ \delta\Psi_{Dld} &= \gamma^\mu\xi_{AD}D_\mu X_{ld}^A \\ &\quad + \omega_{BD}\xi_{AC}\epsilon^{jk}(f^{abc}{}_d X_{la}^A X_{jb}^B X_{kc}^C + g^{abc}{}_d X_{ka}^A X_{lb}^B X_{jc}^C) \\ &\quad + \omega_{AC}\xi_{BD}\epsilon^{jk}(h^{abc}{}_d X_{la}^A X_{jb}^B X_{kc}^C + j^{abc}{}_d X_{ka}^A X_{lb}^B X_{jc}^C), \end{aligned} \quad (32)$$

where the Levi-Civita tensor ϵ^{ij} raises and lowers the $\text{SU}(2)$ indices. Without loss of generality, we may take $g^{abc}{}_d$ and $j^{abc}{}_d$ to be symmetric in a and c .

The tensors $g^{abc}{}_d$, $h^{abc}{}_d$, and $j^{abc}{}_d$ are fixed by closing the supersymmetry algebra on the scalar,

$$[\delta_1, \delta_2]X_{id}^A = v^\mu D_\mu X_{id}^A + \tilde{\Lambda}^a{}_d X_{ia}^A, \quad (33)$$

with $v^\mu = \frac{i}{2}\bar{\xi}_2^{BC}\gamma^\mu\xi_1_{BC}$. We find

$$f^{abc}{}_d = 2g^{cab}{}_d = h^{acb}{}_d = 2j^{cab}{}_d, \quad (34)$$

which implies

$$\tilde{\Lambda}^a{}_d X_{ia}^A = \frac{i}{2}\epsilon^{jk}\bar{\xi}_{[2}^{EF}\xi_{1]CF}\omega_{EB}f^{abc}{}_d X_{jb}^B X_{kc}^C X_{ia}^A. \quad (35)$$

Because of conflicting symmetries, $\tilde{\Lambda}^a{}_d$ vanishes, so no gauge transformation appears in the closure of the algebra.

With these conditions, the fermion supersymmetry transformation becomes

$$\begin{aligned} \delta\Psi_{Dld} &= \gamma^\mu\xi_{AD}D_\mu X_{ld}^A \\ &\quad + \epsilon^{jk}(\omega_{BD}\xi_{AC} + \omega_{AC}\xi_{BD}) \\ &\quad \times (f^{abc}{}_d X_{jb}^B X_{kc}^C X_{la}^A - \frac{1}{2}f^{abc}{}_d X_{la}^B X_{kc}^C X_{jb}^A). \end{aligned} \quad (36)$$

Closing this transformation leads to a trivial theory. All interaction terms cancel in the equation of motion. Indeed, upon closer inspection, it is possible to show that the interaction terms in the fermion transformation (36) also vanish, as indeed they must.

To find a nontrivial $\mathcal{N} = 5$ theory, we need to impose a less restrictive global symmetry group. Therefore, in what follows, we will take the global symmetry group to be the R-symmetry group $\text{SO}(5) = \text{Sp}(4)$. Since $\text{Sp}(4) \subset \text{SU}(4)$, we can carry over many results from $\mathcal{N} = 6$.

We start by examining the supersymmetry parameters. We write the $\mathcal{N} = 6$ parameters ε^{AB} in terms of the $\mathcal{N} = 5$ parameters ξ^{AB} , together with a *real* R-symmetry singlet spinor η , as follows:

$$\begin{aligned} \varepsilon^{AB} &= \xi^{AB} + i\omega^{AB}\eta \\ \varepsilon_{AB} &= \xi_{AB} - i\omega_{AB}\eta. \end{aligned} \quad (37)$$

In an $\mathcal{N} = 5$ theory, the $\text{Sp}(4)$ indices are raised and lowered using the antisymmetric tensors ω^{AB} and ω_{AB} , respectively. For the $\mathcal{N} = 5$ parameters ξ_{AB} , this convention is consistent with the $\text{SU}(4)$ R-symmetry of the $\mathcal{N} = 6$ theory:

$$\begin{aligned}\xi^{AB} &\equiv \omega^{AC}\omega^{BD}\xi_{CD} \\ &= \frac{1}{2}(\omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC} - \omega^{AB}\omega^{CD})\xi_{CD} \\ &= \frac{1}{2}\varepsilon^{ABCD}\xi_{CD}.\end{aligned}\tag{38}$$

The sign change in the singlet follows from the group theory,

$$\begin{aligned}\omega^{AB}\eta &\equiv \omega^{AC}\omega^{BD}\omega_{CD}\eta \\ &= -\frac{1}{2}(\omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC} - \omega^{AB}\omega^{CD})\omega_{CD}\eta \\ &= -\frac{1}{2}\varepsilon^{ABCD}\omega_{CD}\eta.\end{aligned}\tag{39}$$

It is also necessary for the closure of the supersymmetry transformations, as can be checked for the free case.

We next consider the fields. The $\mathbf{4}$ of $\text{Sp}(4)$ is obtained from the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $\text{SU}(4)$ by imposing a reality condition. For the case at hand, we impose the following constraints on the fields of the $\mathcal{N} = 6$ theory:[22]

$$\begin{aligned}\bar{Z}_A^a &= -J^{ab}\omega_{AB}Z_b^B \\ \Psi^{Aa} &= -J^{ab}\omega^{AB}\Psi_{Bb}.\end{aligned}\tag{40}$$

Here ω_{AB} is the antisymmetric $\text{Sp}(4)$ invariant tensor, while J_{ab} is an invariant (antisymmetric) tensor of the gauge group, with $J_{ab}J^{bc} = -\delta_a^c$. The minus sign in the second term renders the constraint consistent with the $\mathcal{N} = 5$ supersymmetry transformations. The constraint is inconsistent with the transformation parametrized by η , so it explicitly breaks $\mathcal{N} = 6$ supersymmetry to $\mathcal{N} = 5$.

With this constraint, we can write the $\mathcal{N} = 5$ supersymmetry transformations entirely in terms of the fields Z_a^A and Ψ_{Dd} . The most general transformations take the following form,

$$\begin{aligned}\delta Z_d^A &= i\bar{\xi}^{AD}\Psi_{Dd} \\ \delta\Psi_{Dd} &= \gamma^\mu\xi_{AD}D_\mu Z_d^A + f_1^{abc}{}_d Z_a^A Z_b^B Z_c^C \xi_{DC}\omega_{AB} \\ &\quad + f_2^{abc}{}_d Z_a^A Z_b^B Z_c^C \xi_{AB}\omega_{DC},\end{aligned}\tag{41}$$

where, without loss of generality, we take $f_1^{abc}{}_d$ and $f_2^{abc}{}_d$ to be antisymmetric in their first two indices. Closing on the scalar gives

$$[\delta_1, \delta_2]Z_d^A = v^\mu D_\mu Z_d^A + \tilde{\Lambda}^a{}_d Z_a^A,\tag{42}$$

with

$$\tilde{\Lambda}^a{}_d = i f_2^{abc}{}_d Z_b^B Z_c^C \omega_{DC} \bar{\xi}_{[2}^{DF} \xi_{1]BF},\tag{43}$$

where

$$f_1^{abc}{}_d = \frac{1}{2}(f_2^{bca}{}_d - f_2^{acb}{}_d).\tag{44}$$

This implies

$$\begin{aligned}\delta\Psi_{Dd} &= \gamma^\mu\xi_{AD}D_\mu Z_d^A - f_2^{acb}{}_d Z_a^A Z_b^B Z_c^C \xi_{DC}\omega_{AB} \\ &\quad + f_2^{abc}{}_d Z_a^A Z_b^B Z_c^C \xi_{AB}\omega_{DC}.\end{aligned}\tag{45}$$

Closing on the fermion gives

$$\begin{aligned}[\delta_1, \delta_2]\Psi_{Dd} &= v^\mu D_\mu \Psi_{Dd} + \tilde{\Lambda}^a{}_d \Psi_{Da} \\ &\quad - \frac{i}{2}\bar{\xi}_{[1}^{AC} \xi_{2]AD} E_{Cd} + \frac{i}{4}(\bar{\xi}_1^{AB} \gamma_\nu \xi_{2AB}) \gamma^\nu E_{Dd},\end{aligned}$$

with the following fermion equation of motion:

$$\begin{aligned}
E_{Dd} &= \gamma^\mu D_\mu \Psi_{Dd} \\
&\quad - f_2^{abc}{}_d (\Psi_{Dc} Z_a^A Z_b^B + \Psi_{Db} Z_a^A Z_c^B) \omega_{AB} \\
&\quad + 2f_2^{abc}{}_d (\Psi_{Ab} Z_a^A Z_c^C + \Psi_{Ac} Z_a^A Z_b^C) \omega_{DC} = 0.
\end{aligned} \tag{46}$$

With these assignments, the gauge field transforms as

$$\delta \tilde{A}_\mu{}^a{}_d = -i(f_2^{acb}{}_d + f_2^{abc}{}_d) \omega^{BE} \bar{\xi}_{EC} \gamma_\mu \Psi_{Bb} Z_c^C. \tag{47}$$

Closing on the gauge field imposes additional constraints:

$$\begin{aligned}
&f_2^{abc}{}_g (f_2^{edg}{}_f + f_2^{egd}{}_f) Z_a^A Z_b^B Z_c^C Z_d^D \omega_{AD} \omega_{BC} = 0 \\
&f_2^{abc}{}_g (f_2^{edg}{}_f + f_2^{egd}{}_f) Z_a^A Z_b^B Z_c^C Z_d^D \bar{\xi}_{AB} [1\gamma^\mu \xi_2]_{CD} = 0.
\end{aligned} \tag{48}$$

These two constraints must be satisfied by the $\mathcal{N} = 5$ fundamental identity.

Up to now, we have worked in complete generality. To proceed further, we impose symmetries on the structure constants $f_2^{abc}{}_d$. The most obvious choice is

$$f_2^{abcd} = f^{abcd} = -f^{bacd} = f^{cdab}, \tag{49}$$

as in $\mathcal{N} = 6$. With this choice, the calculations work out just as before. In particular, the conditions (48) are satisfied by the $\mathcal{N} = 5$ restriction of the $\mathcal{N} = 6$ fundamental identity:

$$J_{gj} (f^{abfg} f^{jhcd} + f^{agfd} f^{hbjc} + f^{ahfg} f^{jdbc} + f^{agfc} f^{bhjd}) = 0. \tag{50}$$

In this case, the supersymmetry transformations are those of ref. [11].

A second and more interesting choice is to take

$$f_2^{abcd} = g^{acbd} - g^{bcad}, \tag{51}$$

where

$$g^{acbd} = g^{cabd} = g^{bdac} \tag{52}$$

so f_2^{abcd} has all the right symmetries. As we shall see, this choice generates $\mathcal{N} = 5$ theories that are not restrictions of $\mathcal{N} = 6$. The conditions (48) are satisfied if[23]

$$g^{(acb)d} = 0 \tag{53}$$

and

$$J_{gj} (g^{afbg} g^{jchd} + g^{afgd} g^{hjbc} + g^{afhgc} g^{jdbc} + g^{afgc} g^{bjhd}) = 0. \tag{54}$$

This is the $\mathcal{N} = 5$ fundamental identity, which was also found in [15] by taking the conformal limit of three-dimensional $\mathcal{N} = 8$ gauged supergravity.

Substituting $g^{abc}{}_d$ for $f_2^{abc}{}_d$ in (41), (45) and (47), we find the $\mathcal{N} = 5$ supersymmetry transformations

$$\begin{aligned}
\delta Z_d^A &= i\bar{\xi}^{AD} \Psi_{Dd} \\
\delta \Psi_{Dd} &= \gamma^\mu \xi_{AD} D_\mu Z_d^A - g^{abc}{}_d Z_a^A Z_b^B Z_c^C \xi_{DB} \omega_{AC} \\
&\quad + 2g^{abc}{}_d Z_a^A Z_b^B Z_c^C \xi_{AC} \omega_{DB} \\
\delta \tilde{A}_\mu{}^a{}_d &= 3ig^{bca}{}_d \omega^{BE} \bar{\xi}_{EC} \gamma_\mu \Psi_{Bb} Z_c^C.
\end{aligned} \tag{55}$$

These transformations close into a translation and a gauge variation, with parameter

$$\tilde{\Lambda}^a{}_d = -\frac{3i}{2} g^{bca}{}_d Z_b^B Z_c^C \omega_{DC} \bar{\xi}_{[2}^{DF} \xi_1]_{BF}. \tag{56}$$

These are the same transformations that were found, starting from different assumptions, in ref. [14].

VI. $\mathcal{N} = 5$ REPRESENTATIONS

In this section we construct $\mathcal{N} = 5$ gauge theories, built from symmetric structure constants g^{abcd} , with gauge transformations

$$\delta Z_d^A = \tilde{\Lambda}^a{}_d Z_a^A = -\frac{3i}{2} g^{bca}{}_d Z_b^B Z_c^C \omega_{DC} \bar{\xi}_{[2}^{DF} \xi_{1]BF} Z_a^A. \quad (57)$$

We will see that there are a host of such theories, including some with free parameters or exceptional gauge groups, in stark contrast to $\mathcal{N} = 6$ or 8.

We start by constructing a set of g^{abcd} that lead to an $\text{Sp}(2N) \times \text{SO}(M)$ gauge group. There are four combinations of the invariant tensors of $\text{Sp}(2N)$ and $\text{SO}(M)$ that have the symmetries (52):

$$\begin{aligned} g_1^{ajibckdl} &= (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}) J^{ij} J^{kl} \\ g_2^{ajibckdl} &= (J^{ik} J^{jl} + J^{jk} J^{il}) \delta^{ab} \delta^{cd} \\ g_3^{(\pm)ajibckdl} &= (\delta^{ac} \delta^{bd} \pm \delta^{ad} \delta^{bc}) (J^{ik} J^{jl} \pm J^{jk} J^{il}), \end{aligned} \quad (58)$$

where $i, j, \dots = 1, \dots, 2N$ are $\text{Sp}(2N)$ indices, and $a, b, \dots = 1, \dots, M$ are $\text{SO}(M)$. From them, we must select linear combinations that satisfy (53) and the fundamental identity (54).

In fact, there are just two linear combinations that do the job:

$$\begin{aligned} g^{ajibckdl} &= g_1^{ajibckdl} - g_2^{ajibckdl} \\ g^{ajibckdl} &= g_3^{(+)}{}^{ajibckdl} + g_3^{(-)}{}^{ajibckdl}. \end{aligned} \quad (59)$$

Let us focus in detail on the first case. The structure constants are

$$g^{ajibckdl} = (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}) J^{ij} J^{kl} - \delta^{ab} \delta^{cd} (J^{ik} J^{jl} + J^{jk} J^{il}). \quad (60)$$

They give rise to the following gauge transformation:

$$\begin{aligned} \delta Z^{Adl} &= -\frac{3i}{2} \bar{\xi}_{[2}^{DF} \xi_{1]BF} \omega_{DC} Z_{bk}^B Z_b^{Cl} Z^{Adk} \\ &\quad -\frac{3i}{2} \bar{\xi}_{[2}^{DF} \xi_{1]BF} \omega_{DC} Z_b^{Bk} Z_k^{Cd} Z_b^{Al}. \end{aligned} \quad (61)$$

The two terms are $\text{Sp}(2N)$ and $\text{SO}(M)$ transformations, respectively, with matter fields in the fundamental representations of each [8, 15, 16].

For the second case, the structure constants are simply

$$g^{ajibckdl} = J^{ik} J^{jl} \delta^{ac} \delta^{bd} + J^{il} J^{jk} \delta^{ad} \delta^{bc}. \quad (62)$$

The indices are in standard direct product form, so the theory has gauge group $\text{Sp}(2MN)$, with matter fields in the $2MN$ dimensional fundamental representation.

For the special case of $\text{SO}(4) \times \text{Sp}(2) \simeq \text{SO}(4) \times \text{SU}(2)$, it is possible to add another term to the structure constants [15, 16]:

$$g^{ajibckdl} = g_1^{ajibckdl} - g_2^{ajibckdl} + \alpha \varepsilon^{abcd} J^{ij} J^{kl}, \quad (63)$$

where ε^{abcd} is the totally antisymmetric $\text{SO}(4)$ -invariant tensor. The resulting $g^{ajibckdl}$ satisfy (53) and the fundamental identity, for any choice of the free parameter α . The gauge group closes into $\text{SO}(4) \times \text{SU}(2)$ for $\alpha \neq \infty$. In the next section, we will see that this example, in the limit $\alpha \rightarrow \infty$, has gauge group $\text{SO}(4)$. In this limit, it lifts to $\mathcal{N} = 6$ and 8.

There are also two ‘‘exceptional’’ theories with $\mathcal{N} = 5$. The first arises from the tensor

$$g^{ajibckdl} = g_1^{ajibckdl} - g_2^{ajibckdl} + \beta C^{abcd} J^{ij} J^{kl}, \quad (64)$$

where $a, b, \dots = 1, \dots, 7$ and $i, j, \dots = 1, 2$ are $\text{SO}(7)$ and $\text{SU}(2)$ indices, respectively. Here C^{abcd} is the totally antisymmetric tensor that is dual to the octonionic structure constants C_{efg} ,

$$C^{abcd} = \frac{1}{3!} \varepsilon^{abcdefg} C_{efg}. \quad (65)$$

[For a concise introduction to G_2 , $SO(7)$ and the octonians, as well as a host of useful identities, see Section 2 and Appendix A of [20].] The tensor (64) satisfies (53) and the fundamental identity for $\beta = 0$ or $\beta = \frac{1}{2}$. When $\beta = 0$, the $g^{aibjckdl}$ are just the $Sp(2) \times SO(7)$ structure constants discussed above.

When $\beta = \frac{1}{2}$, the gauge group is $G_2 \times SU(2)$. In this case, the structure constants take the form

$$g^{aibjckdl} = (\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} + \frac{1}{2}C^{abcd})J^{ij}J^{kl} - \delta^{ab}\delta^{cd}(J^{ik}J^{jl} + J^{jk}J^{il}), \quad (66)$$

where $i, j, \dots = 1, 2$. The gauge transformation is then

$$\delta Z^{Adl} = \tilde{\Lambda}^{aidl} Z_{ai}^A,$$

with

$$\begin{aligned} \tilde{\Lambda}^{aidl} = & \frac{3i}{2}\bar{\xi}_{[2}^{DF}\xi_{1]BF}\omega_{DC}\delta^{ad}Z_b^{Bi}Z_b^{Cl} \\ & - \frac{3i}{4}\bar{\xi}_{[2}^{DF}\xi_{1]BF}\omega_{DC}(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \frac{1}{2}C^{abcd})J^{jk}J^{il}Z_{bj}^B Z_{ck}^C. \end{aligned} \quad (67)$$

The first term is clearly an $SU(2)$ transformation. The second is a $G_2 \subset SO(7)$ transformation, as can be seen by recognizing that the operator

$$\mathcal{P}_{14}^{abcd} = \frac{1}{3}\left(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \frac{1}{2}C^{abcd}\right) \quad (68)$$

is a projector from the adjoint **21** of $SO(7)$ to the adjoint **14** of G_2 ,

$$\mathcal{P}_{14}^{abcd}C_{bce} = 0. \quad (69)$$

In this way we construct the $\mathcal{N} = 5$, $G_2 \times SU(2)$ gauge theory from a 3-algebra, recovering the result found in [15, 16].

The second exceptional theory has $SO(7) \times SU(2)$ gauge symmetry with matter transforming in the *spinor* **8** of $SO(7)$ [15, 16]. To find the structure constants, we start with the tensor

$$g^{aibjckdl} = \delta^{ab}\delta^{cd}(J^{ik}J^{jl} + J^{jk}J^{il}) + \gamma\Gamma_{mn}^{ab}\Gamma_{mn}^{cd}J^{ij}J^{kl}. \quad (70)$$

where $a, b, \dots = 1, \dots, 8$ and $i, j, \dots = 1, 2$, and $\Gamma_{mn}^{ab} = \frac{1}{2}(\Gamma_m\Gamma_n - \Gamma_n\Gamma_m)^{ab}$ is built from the $SO(7)$ gamma matrices. The $g^{aibjckdl}$ have the correct symmetries and satisfy the fundamental identity for $\gamma = -\frac{1}{6}$, in which case the structure constants become

$$g^{aibjckdl} = \delta^{ab}\delta^{cd}(J^{ik}J^{jl} + J^{jk}J^{il}) - \frac{1}{6}\Gamma_{mn}^{ab}\Gamma_{mn}^{cd}J^{ij}J^{kl}. \quad (71)$$

The gauge transformations reduce to

$$\delta Z^{Adl} = \tilde{\Lambda}^{aidl} Z_{ai}^A, \quad (72)$$

where

$$\begin{aligned} \tilde{\Lambda}^{aidl} = & -\frac{3i}{2}\bar{\xi}_{[2}^{DF}\xi_{1]BF}\omega_{DC}\delta^{ad}Z_b^{Bi}Z_b^{Cl} \\ & + \frac{i}{8}\bar{\xi}_{[2}^{DF}\xi_{1]BF}\omega_{DC}\Gamma_{mn}^{ad}\Gamma_{mn}^{bc}J^{jk}J^{il}Z_{bj}^B Z_{ck}^C. \end{aligned} \quad (73)$$

We see that the gauge group is $SO(7) \times SU(2)$, with the matter fields in the spinor representation of each.

VII. LIFTS: $\mathcal{N} = 5 \rightarrow \mathcal{N} = 6$

In this section, we lift two theories with $\mathcal{N} = 5$ supersymmetry to $\mathcal{N} = 6$, along the lines of the lift from $\mathcal{N} = 6$ to $\mathcal{N} = 8$. In particular, we lift the $\mathcal{N} = 5$ theories with $Sp(2N) \times SO(2)$ and $SO(4) \times SU(2)$ gauge symmetry to $\mathcal{N} = 6$ theories with $Sp(2N) \times U(1)$ and $SO(4)$ gauge symmetry, respectively. As we showed previously, the latter theory can then be lifted to $\mathcal{N} = 8$.

To carry out the lifts, we first define unconstrained complex-conjugate scalars \mathcal{Z}_a^A and $\bar{\mathcal{Z}}_A^a$, consistent with the constraint (40):

$$\begin{aligned}\mathcal{Z}_a^A &= Z_{a1}^A + iZ_{a2}^A \\ \bar{\mathcal{Z}}_A^a &= \bar{Z}_A^{a1} - i\bar{Z}_A^{a2}.\end{aligned}\tag{74}$$

Supersymmetry then requires that the superpartner Ξ_{Aa} be defined as follows:

$$\begin{aligned}\Xi_{Aa} &= \Psi_{Aa1} + i\Psi_{Aa2} \\ \Xi^{*Aa} &= \Psi^{Aa1} - i\Psi^{Aa2}.\end{aligned}\tag{75}$$

The indices 1 and 2 refer to either SU(2) or SO(2), while a refers to SO(4) or Sp(2N), respectively. The constraint (40) allows us to write the complex-conjugate expressions in terms of the original fields. Note that this procedure only works when one of the $\mathcal{N} = 5$ gauge groups is SU(2) or SO(2).

We first consider the theory with Sp(2N) \times SO(2) gauge symmetry, where $a, b, \dots = 1, \dots, 2N$ are Sp(2N) indices, and $i, j, \dots = 1, 2$ are SO(2). The conjugate scalar $\bar{\mathcal{Z}}_A^a$ takes the form

$$\bar{\mathcal{Z}}_A^a = -\omega_{AB} J^{ab} (Z_{b1}^B - iZ_{b2}^B),\tag{76}$$

and likewise for the conjugate spinor Ξ^{*Aa} . With these definitions, it is straightforward to check that the $\mathcal{N} = 5$ transformations, with

$$g^{ajibckdl} = -\frac{2}{3} ((\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})J^{ab}J^{cd} - \delta^{ij}\delta^{kl}(J^{ac}J^{bd} + J^{bc}J^{ad})),\tag{77}$$

coincide with the $\mathcal{N} = 6$ transformations, with

$$f^{ab}{}_{cd} = J^{ab}J_{cd} + (\delta_c^a\delta_d^b - \delta_d^a\delta_c^b),\tag{78}$$

for five of the six supersymmetries.

To find the sixth, we plug $\varepsilon_{AB} \rightarrow -i\omega_{AB}\eta$ into the transformations (5) and collect terms. After some calculation, we find:

$$\begin{aligned}\delta Z_{dl}^A &= -\omega^{AD}\bar{\eta}\Psi_{Ddl} \\ \delta\Psi_{Ddl} &= -i\gamma^\mu\omega_{AD}\eta D_\mu Z_{dl}^A \\ &\quad + if^{ab}{}_{cd}(\omega_{AB}\omega_{CD} - \omega_{AC}\omega_{BD}) \\ &\quad \times (\epsilon_{ik}\epsilon_{jl} + \epsilon_{jk}\epsilon_{il} + i\delta_{ij}\epsilon_{kl})Z_{ai}^AZ_{bj}^BZ_{ck}^C\eta \\ \delta\tilde{A}_\mu{}^{iadt} &= if^{abcd}(\bar{\eta}\gamma_\mu\Psi_{Bbj}Z_{ck}^B - \bar{\eta}\gamma_\mu\Psi_{Bck}Z_{bj}^B)(\delta^{jk}\epsilon^{il} + \epsilon^{jk}\delta^{il}),\end{aligned}\tag{79}$$

where ϵ^{ij} is the antisymmetric, invariant tensor of SO(2). This is the extra supersymmetry transformation that lifts the $\mathcal{N} = 5$ theory with Sp(2N) \times SO(2) gauge symmetry to the $\mathcal{N} = 6$ theory with Sp(2N) \times U(1).

Finally, we consider the $\mathcal{N} = 5$ theory with SO(4) \times SU(2) gauge symmetry, with $g^{aibkckdl}$ given in (63), in the limit $\alpha \rightarrow \infty$. In this limit, the structure constants reduce to

$$g^{aibkckdl} \rightarrow \alpha\varepsilon^{abcd}\epsilon^{ij}\epsilon^{kl},\tag{80}$$

where $a, b, \dots = 1, \dots, 4$ are SO(4) indices, and $i, j, \dots = 1, 2$ are SU(2), and ϵ^{ij} is the antisymmetric, invariant tensor of SU(2). We first compute the gauge transformation. Using (43), we find

$$\delta Z_{dl}^D \propto \bar{\xi}_{[2}^{EF}\xi_{1]BF}\omega_{EC}\varepsilon^{abcd}\epsilon^{jk}Z_{bj}^BZ_{ck}^CZ_{al}^D.\tag{81}$$

This is a pure SO(4) gauge transformation; it suggests that the SO(4) \times SU(2) invariant $\mathcal{N} = 5$ theory, in the $\alpha \rightarrow \infty$ limit, can be lifted to the SO(4) theory with $\mathcal{N} = 6$ and 8.

We now construct the lift. We start by defining the complex-conjugate scalars \mathcal{Z}_a^A and $\bar{\mathcal{Z}}_A^a$. For the case at hand, we find

$$\bar{\mathcal{Z}}_{Aa} = -i\omega_{AB}(Z_{a1}^B - iZ_{a2}^B),\tag{82}$$

and likewise for the spinor Ξ^{*Aa} . As above, it possible to show that the $\mathcal{N} = 5$ transformations with

$$g^{ajibckdl} = -\frac{2}{3}\epsilon^{abcd}\epsilon^{ij}\epsilon^{kl}, \quad (83)$$

and the $\mathcal{N} = 6$ transformations with

$$f^{abcd} = \epsilon^{abcd}, \quad (84)$$

coincide for five of the six supersymmetries.

The sixth supersymmetry is derived in the same way as before. Plugging $\epsilon_{AB} \rightarrow -i\omega_{AB}\eta$ into (5) and collecting terms, we find:

$$\begin{aligned} \delta Z_{dl}^A &= -\omega^{AD}\bar{\eta}\Psi_{Ddl} \\ \delta\Psi_{Ddl} &= -i\gamma^\mu\omega_{AD}\eta D_\mu Z_{dl}^A \\ &\quad + 2\epsilon^{abcd}\omega_{AB}\omega_{CD}\delta_{ik}\delta_{jl}Z_{ai}^AZ_{bj}^BZ_{ck}^C\eta \\ \delta\tilde{A}_\mu{}^{aidl} &= -2i\epsilon^{abcd}\epsilon^{il}\bar{\eta}\gamma_\mu\Psi_{Bbj}Z_{cj}^B. \end{aligned} \quad (85)$$

Note that the interaction term explicitly breaks the $SU(2)$ symmetry. The transformation is just what we need to lift the $\mathcal{N} = 5$ theory with $SU(2) \times SO(4)$ gauge symmetry to the $\mathcal{N} = 6$ theory with $SO(4)$ gauge symmetry. In Section IV, we proved that this theory can again be lifted to $\mathcal{N} = 8$.

It is worth emphasizing that these lifts arise from $\mathcal{N} = 5$ theories that are not simply $\mathcal{N} = 6$ theories with a reality constraint. Instead they arise from purely $\mathcal{N} = 5$ theories, using very special properties of the gauge groups in question.

VIII. CONCLUSIONS

In this paper, we constructed the most general three-dimensional $\mathcal{N} = 5$ superconformal Chern-Simons gauge theory from first principles. We identified the 3-algebra, found the fundamental identity, and constructed various representations of it. We used 3-algebras to demonstrate how certain theories can be lifted to $\mathcal{N} = 6$ or 8 for an appropriate choice of gauge group.

Our results confirm that 3-algebras provide a powerful approach to superconformal Chern-Simons theories in three dimensions [16]. They unify and simplify the construction of theories with $\mathcal{N} \geq 5$. The number of supersymmetries is determined by the structure of the underlying 3-algebra. Antisymmetric structure constants, with $f^{abcd} = -f^{bacd} = f^{cdab}$, give rise to $\mathcal{N} = 6$ theories, corresponding to $U(M|N)$ and $OSp(2|N)$ in the Kac classification [21]. Symmetric structure constants, with $g^{abcd} = g^{bacd} = g^{cdab}$, give $\mathcal{N} = 5$ theories, corresponding to $OSp(M|N)$, $D(2|1;\alpha)$ and the exotic pair $F(4)$ and $G(3)$.

Perhaps our most surprising result is that theories with different gauge groups can be continuously connected through their 3-algebras. How does this occur in an M2 brane construction? We have seen that the $\mathcal{N} = 5$ supersymmetric $SO(4) \times SU(2)$ theory can be continuously deformed to the $\mathcal{N} = 6$ $SO(4)$ theory, changing both gauge group and the number of supersymmetries along the way. It is surely of interest to find the M theory realization of this phenomenon.

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APPENDIX

The theories we consider are constructed in three dimensions, with $\gamma^\mu = \{i\sigma^2, \sigma^1, \sigma^3\}$, with Minkowski metric $\eta^{\mu\nu} = (-, +, +)$. Therefore, $\{\gamma^\mu, \gamma^\nu\} = +2\eta^{\mu\nu}$. In three dimensions, the Fierz transformation is

$$(\bar{\lambda}\chi)\psi = -\frac{1}{2}(\bar{\lambda}\psi)\chi - \frac{1}{2}(\bar{\lambda}\gamma_\nu\psi)\gamma^\nu\chi. \quad (86)$$

We use the symmetrization conventions $F^{A[BG^C]D} = F^{AB}G^{CD} - F^{AC}G^{BD}$ and $F^{A(BG^C)D} = F^{AB}G^{CD} + F^{AC}G^{BD}$, for any parameters F, G , and indices A, B, C, D . We adopt the convention $X^A = \omega^{AB}X_B$, $X_A = -\omega_{AB}X^B$ for any symplectic structure.

Throughout the paper, we denote spinors that are R-symmetry singlets by η . Those in the **6** of $SU(4)$ are denoted by ε^{AB} ; those in the **5** of $Sp(4)$ are denoted by ξ^{AB} . We note the following useful identities, which hold for both ε^{AB} and ξ^{AB} , although they are presented with the latter, with the appropriate definition of ε^{ABCD} :

$$\frac{1}{2}\bar{\xi}_1^{CD}\gamma_\nu\xi_2{}_{CD}\delta_B^A = \bar{\xi}_{[1}^{AC}\gamma_\nu\xi_2]_{BC} \quad (87)$$

$$\begin{aligned} 2\bar{\xi}_{[1}^{AC}\xi_2]_{BD} &= \bar{\xi}_{[1}^{CE}\xi_2]_{DE}\delta_B^A - \bar{\xi}_{[1}^{AE}\xi_2]_{DE}\delta_B^C \\ &\quad + \bar{\xi}_{[1}^{AE}\xi_2]_{BE}\delta_D^C - \bar{\xi}_{[1}^{CE}\xi_2]_{BE}\delta_D^A \end{aligned} \quad (88)$$

$$\begin{aligned} \frac{1}{2}\varepsilon_{ABCD}\bar{\xi}_1^{EF}\gamma_\mu\xi_2{}_{EF} &= \bar{\xi}_{AB[1}\gamma_\mu\xi_2]_{CD} + \bar{\xi}_{AD[1}\gamma_\mu\xi_2]_{BC} \\ &\quad - \bar{\xi}_{BD[1}\gamma_\mu\xi_2]_{AC} \end{aligned} \quad (89)$$

$$\varepsilon^{ABCD} = \omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC} - \omega^{AB}\omega^{CD}. \quad (90)$$

In our calculations concerning G_2 and the spinor representation of $SO(7)$, we made considerable use of the representations and identities listed in [20]. The $SO(7)$ gamma matrices are

$$\Gamma^{mab} = i(C^{mab} + \delta^{ma}\delta^{b8} - \delta^{mb}\delta^{a8}). \quad (91)$$

They lead to the $SO(7)$ generators

$$\Gamma^{mnab} = C^{mnab} + C^{mna}\delta^{b8} - C^{mnb}\delta^{a8} + \delta^{ma}\delta^{nb} - \delta^{mb}\delta^{na}, \quad (92)$$

which require the following $SO(7)$ identities:

$$C^{abe}C^{cde} = -C^{abcd} + \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} \quad (93)$$

$$C^{acd}C^{bcd} = 6\delta^{ab} \quad (94)$$

$$C^{abpq}C^{pqc} = -4C^{abc}. \quad (95)$$

The C^{abc} are the structure constants for the octonian algebra, and

$$C^{abcd} = \frac{1}{3!}\varepsilon^{abcdefg}C_{efg}. \quad (96)$$

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