Matter-gravity couplings and Lorentz violation

V. Alan Kostelecký and Jay D. Tasson

Phys. Rev. D **83**, 016013 — Published 24 January 2011

DOI: 10.1103/PhysRevD.83.016013

Matter-gravity couplings and Lorentz violation

V. Alan Kostelecký and Jay D. Tasson
Physics Department, Indiana University, Bloomington, IN 47405, U.S.A.

The gravitational couplings of matter are studied in the presence of Lorentz and CPT violation. At leading order in the coefficients for Lorentz violation, the relativistic quantum hamiltonian is derived from the gravitationally coupled minimal Standard-Model Extension. For spin-independent effects, the nonrelativistic quantum hamiltonian and the classical dynamics for test and source bodies are obtained. A systematic perturbative method is developed to treat small metric and coefficient fluctuations about a Lorentz-violating and Minkowski background. The post-newtonian metric and the trajectory of a test body freely falling under gravity in the presence of Lorentz violation are established. An illustrative example is presented for a bumblebee model. The general methodology is used to identify observable signals of Lorentz and CPT violation in a variety of gravitational experiments and observations, including gravimeter measurements, laboratory and satellite tests of the weak equivalence principle, antimatter studies, solar-system observations, and investigations of the gravitational properties of light. Numerous sensitivities to coefficients for Lorentz violation can be achieved in existing or near-future experiments at the level of parts in 10^3 down to parts in 10^{15}. Certain coefficients are uniquely detectable in gravitational searches and remain unmeasured to date.

I. INTRODUCTION

General Relativity (GR) is known to provide an accurate description of classical gravitational phenomena over a wide range of distance scales. A foundational property of the gravitational couplings of matter in GR is local Lorentz invariance in freely falling frames. The realization that a consistent theory of quantum gravity at the Planck scale $m_P \simeq 10^{19}$ GeV could induce tiny manifestations of Lorentz violation at observable scales [1] has revived interest in studies of Lorentz symmetry, with numerous sensitive searches for Lorentz violation being undertaken in recent years [2].

Gravitational signals of Lorentz violation are more challenging to study than ones in Minkowski spacetime for several reasons, including the comparative weakness of gravity at the microscopic level and the impossibility of screening gravitational effects on macroscopic scales. Both for purely gravitational interactions and for matter-gravity couplings, Lorentz violations can be classified and enumerated in effective field theory [3]. Several searches for purely gravitational Lorentz violations in this context have recently been performed [4–6] using a treatment in the post-newtonian regime [7]. These results enlarge and complement the impressive breadth of tests of GR performed in the context of the parametrized post-newtonian (PPN) formalism [8].

Although the coupling between matter and gravity has historically been a primary source of insights into the properties of the gravitational field, a general study of matter-gravity couplings allowing for Lorentz violation in the context of effective field theory has been lacking to date. In this work, we address this gap in the literature and investigate the prospects for searches for Lorentz violation involving matter-gravity couplings. Our goal is to elucidate both theoretical and experimental aspects of the subject. We seek a post-newtonian expansion for the equation for the trajectory of a test body moving under gravity in the presence of Lorentz violation, allowing also for Lorentz-violating effects from the composition of the test and source bodies and for effects from possible additional long-range modes associated with Lorentz violation. We also seek to explore the implications of our analysis in a wide variety of experimental and observational scenarios, identifying prospective measurable signals and thereby enabling more complete searches using matter-gravity couplings.

Despite the current lack of a satisfactory quantum theory of gravity, established gravitational and particle phenomena at accessible energy scales can successfully be analyzed using the field-theoretic combination of GR and the Standard Model (SM). This combination therefore serves as a suitable starting point for a comprehensive effective field theory describing observable signals of Planck-scale Lorentz and CPT violation in gravity and particle physics [9]. The present paper adopts this general framework, known as the gravitational Standard-Model Extension (SME) [3], to analyze Lorentz violation in matter-gravity couplings. Each term violating Lorentz symmetry in the SME Lagrange density is a scalar density under observer general coordinate transformations and consists of a Lorentz-violating operator multiplied by a controlling coefficient for Lorentz violation. Under mild assumptions, CPT violation in effective field theory comes with Lorentz violation [10], so the SME also describes general breaking of CPT symmetry. This feature plays a crucial role for certain signals of Lorentz violation in what follows.

In this work, our focus is on gravitational Lorentz violation in matter-gravity couplings, both with and without CPT violation. These couplings introduce operator structures offering sensitivity to coefficients for Lorentz violation that are intrinsically unobservable in Minkowski spacetime. In fact, comparatively large gravitational Lorentz violation in nature could have remained undetected in searches to date because gravity can provide a countershading effect [11], so this line of investigation has a definite discovery potential. Several searches for gravitational Lorentz violation have led to constraints on SME coefficients for Lorentz violation with sensitivities down to parts in 10^9 [4–6, 12], and additional constraints can be inferred by reanalysis of data from equivalence-principle tests [11].

The nature of the Lorentz violation plays a crucial role in determining the physics of matter-gravity couplings. In Riemann geometry, externally prescribing the coefficients for Lorentz violation as fixed background configurations is generically incompatible with the Bianchi identities and hence problematic [3]. This issue can be avoided via spontaneous Lorentz breaking [1], in which a potential term drives the dynamical development of one or more nonzero vacuum values for a tensor field. This mechanism implies the underlying Lagrange density is Lorentz invariant, so the coefficients for Lorentz violation are expressed in terms of vacuum values and can therefore serve as dynamically consistent backgrounds satisfying the Bianchi identities. The presence of a potential driving spontaneous Lorentz violation implies the emergence of massless Nambu-Goldstone (NG) modes [13] associated with field fluctuations along the broken Lorentz generators [14]. If the potential is smooth, massive modes can also appear [15]. Some features of the NG and massive modes are generic, while others are specific to details of the model being considered. In any case, the nature of these modes plays a key role in determining the physical implications of spontaneous Lorentz violation.

For the purposes of the present work, the presence of NG modes is particularly crucial because they can couple to matter and can transmit a long-range force. A careful treatment of these modes is therefore a prerequisite for studies of Lorentz violation in matter-gravity couplings. In what follows, we develop a methodology to extract the dominant Lorentz-violating effects in matter-gravity couplings irrespective of the details of the underlying model for spontaneous Lorentz violation. In effect, the NG modes are treated via a perturbation scheme that takes advantage of symmetry properties of the underlying Lagrange density to eliminate them in favor of gravitational fluctuations and background coefficients for Lorentz violation. This treatment allows leading Lorentz-violating effects from a large class of plausible models to be handled in a single analysis.

The portion of this paper developing theoretical issues spans Secs. II-V. It begins in Sec. II with a review of the SME framework. We present the field-theoretic action, describe its linearization, and discuss observability issues for the coefficients for Lorentz violation. We also describe the two notions of perturbative order used in the subsequent analysis, one involving Lorentz and gravitational fluctuations and the other based on a post-newtonian expansion. Section III concerns the relativistic and nonrelativistic quantum mechanics arising from the quantum field theory. One technical issue is extracting a meaningful quantum theory in the presence of gravitational fluctuations. We resolve this issue via a judicious field redefinition, which yields a hamiltonian that is hermitian with respect to the usual scalar product for wave functions and that reduces correctly to known limiting cases. We construct the relativistic quantum hamiltonian at leading order in Lorentz violation and gravity fluctuations. For the spin-independent terms, we perform a Foldy-Wouthuysen transformation to obtain the nonrelativistic hamiltonian. Section IV treats the classical dynamics corresponding to the quantum theory. The point particle action is presented and the structure of test and source bodies is discussed. The equations of motion for a test particle and the modified Einstein equations are derived. We describe the methodology for handling coefficient and metric fluctuations. Combining the results determines the Lorentz-violating trajectory of a test body. The results are illustrated in Sec. V in the context of a special class of models of spontaneous Lorentz violation known as bumblebee models.

The remaining research sections of the paper, Secs. VI-XI, concern implications of our theoretical analysis for experiments and observations. Section VI contains basic facts concerning frame choices and outlines the canonical Sun-centered frame used for reporting measurements. We also consider the sensitivities to coefficients for Lorentz violation that can be attained in practical situations. Section VII treats laboratory tests near the surface of the Earth using neutral bulk matter, neutral atoms, or neutrons. The theoretical description of these tests is presented to third post-newtonian order, and some generic features of the test-body motion are discussed. A wide variety of gravimeter and equivalence-principle tests is analyzed for sensitivities to coefficients for Lorentz violation that are presently unconstrained. Satellite-based searches for Lorentz violation using equivalence-principle experiments are studied in Sec. VIII. A generic situation is analyzed, and the results are applied to major proposed satellite tests. Section IX treats gravitational searches using charged particles, antihydrogen, and particles from the second and third generations of the SM. Estimated sensitivities in future tests are provided, and illustrative toy models for antihydrogen studies are discussed. Searches for Lorentz violation using solar-system observations are described in Sec. X. We consider measurements of coefficients for Lorentz violation accessible via lunar and satellite ranging and via studies of perihelion precession. Section XI addresses various tests involving the effects of gravitational Lorentz violation on the properties of light. We analyze the Shapiro time delay, the gravitational Doppler shift, and the gravitational redshift, and we consider the implications for a variety of existing and proposed searches of these types. Finally, in Sec. XII we summarize the paper and tabulate the various estimated actual and attainable sensitivities to coefficients for Lorentz and CPT violation obtained in the body of this work.

Throughout the paper, we follow the conventions of Refs. [3] and [7]. In particular, the Minkowski metric is diagonal with entries $(-1, +1, +1, +1)$. Greek indices are used for spacetime coordinates, while Latin indices are used for local Lorentz coordinates. Appendix A of Ref. [3] provides a summary of most other conventions. Note that parentheses surrounding index pairs in the present work denote symmetrization with a factor of one half.

II. FRAMEWORK

The focus of this work is the study of relativity violations in realistic matter-gravity interactions. The basic field theory of relevance concerns a single fermion field ψ coupled to dynamical gravity and incorporating Lorentz and CPT violation. In this section, we summarize the action for the model, describe the linearization procedure, discuss conditions for the observability of effects, and present the perturbation scheme developed for the analysis to follow.

A. Action

The theory of interest is a special case of the gravitationally coupled SME [3]. The action can be written as

$$S = S_G + S_\psi + S'. \tag{1}$$

The first term in this expression is the action S_G containing the dynamics of the gravitational field, including any coefficients for Lorentz violation in that sector. The geometric framework is a Riemann-Cartan spacetime, which allows both the Riemann curvature tensor $R^\kappa{}_{\lambda\mu\nu}$ and the torsion tensor $T^\lambda{}_{\mu\nu}$. To incorporate fermion-gravity interactions, the vierbein formalism [16] is adopted, with the vierbein $e^\mu{}_a$ and the spin connection $\omega_\mu{}^{ab}$ taken as the fundamental gravitational objects. In the limit of zero torsion and Lorentz invariance, S_G reduces to the Einstein-Hilbert action of General Relativity,

$$S_G \;\to\; \frac{1}{2\kappa}\int d^4x\ (eR - 2e\Lambda), \tag{2}$$

where $\kappa \equiv 8\pi G_N$, e is the vierbein determinant, and Λ is the cosmological constant.

The second term in Eq. (1) is the action S_ψ for the fermion sector of the SME. In this work, we limit attention to terms in this sector with no more than one derivative, which is the gravitationally coupled analogue of the minimal SME in Minkowski spacetime. In this limit, the action S_ψ for a single Dirac fermion ψ of mass m can be written as

$$S_\psi = \int d^4x (\tfrac{1}{2} i e e^\mu{}_a \overline{\psi}\Gamma^a \overleftrightarrow{D_\mu} \psi - e\overline{\psi}M\psi). \tag{3}$$

In the present context, the action of the covariant derivative D_μ on ψ is

$$D_\mu\psi \equiv \partial_\mu\psi + \tfrac{1}{4} i\omega_\mu{}^{ab}\sigma_{ab}\psi. \tag{4}$$

It is convenient to introduce the symbol

$$(\overline{\psi D_\mu}) \equiv \partial_\mu\overline{\psi} - \tfrac{1}{4} i\omega_\mu{}^{ab}\overline{\psi}\sigma_{ab} \tag{5}$$

for the action of the covariant derivative on the Dirac-conjugate field $\overline{\psi}$. The action (3) contains the covariant derivative in a combination defined by

$$\overline{\chi}\Gamma^a \overleftrightarrow{D_\mu} \psi \equiv \overline{\chi}\Gamma^a D_\mu\psi - (\overline{\chi D_\mu})\Gamma^a\psi. \tag{6}$$

The symbols Γ^a and M appearing in the action (3) are defined by

$$\begin{aligned}\Gamma^a \;\equiv\;& \gamma^a - c_{\mu\nu}e^{\nu a}e^\mu{}_b\gamma^b - d_{\mu\nu}e^{\nu a}e^\mu{}_b\gamma_5\gamma^b \\ &- e_\mu e^{\mu a} - if_\mu e^{\mu a}\gamma_5 - \tfrac{1}{2} g_{\lambda\mu\nu}e^{\nu a}e^\lambda{}_b e^\mu{}_c\sigma^{bc}\end{aligned} \tag{7}$$

and

$$M \equiv m + a_\mu e^\mu{}_a\gamma^a + b_\mu e^\mu{}_a\gamma_5\gamma^a + \tfrac{1}{2} H_{\mu\nu}e^\mu{}_a e^\nu{}_b\sigma^{ab}. \tag{8}$$

The first term of Eq. (7) leads to the usual Lorentz-invariant kinetic term for the Dirac field, while the first term of Eq. (8) leads to a Lorentz-invariant mass. A term of the form $im_5\gamma_5$ could also appear in M, but here we suppose it is absorbed into m via a chiral field redefinition. The coefficient fields for Lorentz violation a_μ, b_μ, $c_{\mu\nu}$, $d_{\mu\nu}$, e_μ, f_μ, $g_{\lambda\mu\nu}$, $H_{\mu\nu}$ typically vary with spacetime position. The coefficient field $H_{\mu\nu}$ is antisymmetric, while $g_{\lambda\mu\nu}$ is antisymmetric in $\lambda\mu$. Note the use of an uppercase letter for $H_{\mu\nu}$, which avoids confusion with the metric fluctuation $h_{\mu\nu}$. The CPT-odd operators for Lorentz violation are associated with the coefficient fields a_μ, b_μ, e_μ, f_μ, and $g_{\lambda\mu\nu}$.

The form of the action (3) implies the torsion $T^\lambda{}_{\mu\nu}$ enters the fermion action only via minimal coupling. This coupling has the same form as that of the coefficient field b_μ, so the effects of minimal torsion can be incorporated into a matter-sector analysis by replacing b_μ with the effective coefficient field

$$(b_{\rm eff})_\mu \equiv b_\mu + \tfrac{1}{8} T^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma\mu}. \tag{9}$$

Note that nonminimal torsion couplings can be incorporated into the more general coefficient fields appearing in the full SME. Nonminimal torsion couplings and their experimental constraints are discussed in Ref. [17].

The final term in Eq. (1) is the action S' containing the dynamics associated with the coefficient fields for Lorentz violation. Addressing possible contributions from this sector is the subject of Sec. IV C.

B. Linearization

For the purposes of this work, it suffices to consider weak gravitational fields in a Minkowski-spacetime background. Under these circumstances, the Latin local indices can be replaced with Greek spacetime indices, so the weak-field forms of the metric, vierbein, and spin connection can be written as

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, \\ e_{\mu\nu} &= \eta_{\nu\sigma} e_\mu{}^\sigma \approx \eta_{\mu\nu} + \tfrac{1}{2} h_{\mu\nu} + \chi_{\mu\nu}, \\ \omega_{\lambda\mu\nu} &= \eta_{\mu\rho}\eta_{\nu\sigma}\omega_\lambda{}^{\rho\sigma} \\ &\approx \partial_\lambda \chi_{\mu\nu} - \tfrac{1}{2}\partial_\mu h_{\lambda\nu} + \tfrac{1}{2}\partial_\nu h_{\lambda\mu} \\ &\quad + \tfrac{1}{2}\left(T_{\lambda\mu\nu} + T_{\mu\lambda\nu} - T_{\nu\lambda\mu}\right). \end{aligned} \tag{10}$$

The quantities $\chi_{\mu\nu}$ contain the six Lorentz degrees of freedom in the vierbein.

The coefficient fields for Lorentz violation are expected to acquire vacuum values through spontaneous Lorentz breaking. An arbitrary coefficient field $t_{\lambda\mu\nu\ldots}$ can therefore be expanded about its vacuum value $\overline{t}_{\lambda\mu\nu\ldots}$,

$$t_{\lambda\mu\nu\ldots} = \overline{t}_{\lambda\mu\nu\ldots} + \tilde{t}_{\lambda\mu\nu\ldots}, \tag{11}$$

where the fluctuation $\tilde{t}_{\lambda\mu\nu\ldots}$ includes massless Nambu-Goldstone (NG) modes and massive modes [14, 15]. The vacuum value $\overline{t}_{\lambda\mu\nu\ldots} \equiv \langle t_{\lambda\mu\nu\ldots}\rangle$ is called the coefficient for Lorentz violation. One can instead choose to expand the contravariant coefficient field $t^{\lambda\mu\nu\cdots}$,

$$t^{\lambda\mu\nu\cdots} = \overline{t}^{\lambda\mu\nu\cdots} + \widetilde{t}^{\lambda\mu\nu\cdots}, \tag{12}$$

where $\overline{t}^{\lambda\mu\nu\cdots}$ is related to $\overline{t}_{\lambda\mu\nu\ldots}$ by raising with $\eta^{\mu\nu}$. The reader is cautioned that the relation between $\tilde{t}_{\lambda\mu\nu\ldots}$ and the index-lowered version $\widetilde{t}_{\lambda\mu\nu\ldots}$ of $\widetilde{t}^{\lambda\mu\nu\cdots}$ involves terms containing contractions of $\overline{t}_{\lambda\mu\nu\ldots}$ with $h_{\mu\nu}$. This paper uses the expansion (11) and gives expressions in terms of $\tilde{t}_{\lambda\mu\nu\ldots}$.

To provide a smooth match between our analysis and previous work on the matter sector of the SME in Minkowski spacetime and on the gravitational sector in asymptotically Minkowski spacetime, we make two assumptions about the coefficients for Lorentz violation. First, we assume they are constant in asymptotically inertial cartesian coordinates,

$$\partial_\alpha \overline{t}_{\lambda\mu\nu\ldots} = 0. \tag{13}$$

This preserves translation invariance and hence energy-momentum conservation in the asymptotically Minkowski regime. It also ensures that our barred coefficients correspond to the usual coefficients for Lorentz and CPT violation investigated in the minimal SME in Minkowski spacetime [18]. Second, we assume that the vacuum values $\overline{t}_{\lambda\mu\nu\ldots}$ are sufficiently small to be treated perturbatively. This is standard and plausible, since any Lorentz violation in nature is expected to be small. These two assumptions suffice for most of the analysis that follows. To obtain the leading Lorentz-violating corrections to $h_{\mu\nu}$ without specifying a dynamical model for the coefficient fields for Lorentz violation, one further assumption is required, which is presented in Sec. IV C.

C. Observability

A given coefficient for Lorentz violation can lead to observable effects only if it cannot be eliminated from the Lagrange density via field redefinitions or coordinate choices [3, 18–25]. In this subsection, we outline some implications of this fact relevant to the present work.

1. *Field redefinitions*

One result of key interest here is that matter-gravity couplings can obstruct the removal of some coefficients that are unphysical in the Minkowski-spacetime limit. For example, in the single-fermion theory in Minkowski spacetime, the coefficient $a_\mu \equiv \overline{a}_\mu$ for Lorentz and CPT violation in Eq. (8) is unobservable because it can be eliminated by the spinor redefinition

$$\psi(x) \to \exp[if(x)]\psi(x) \tag{14}$$

with $f(x) = \overline{a}_\mu x^\mu$. However, in Riemann or Riemann-Cartan spacetimes we have $a_\mu \equiv \overline{a}_\mu + \tilde{a}_\mu$, so this redefinition typically leaves the four components of the fluctuation $\tilde{a}_\mu$ in the theory. Instead, the redefinition (14) with an appropriate $f(x)$ can be used to move one component of the coefficient field a_μ into the other three, unless a_μ is constant or the total derivative of a scalar [3]. Note that in the presence of gravity this freedom may be insufficient to eliminate any components of the coefficient $\overline{a}_\mu$ because the components of the fluctuation $\tilde{a}_\mu$ can depend on all four components of $\overline{a}_\mu$ through the equations of motion. This line of reasoning shows that gravitational couplings provide a unique sensitivity to the coefficient $\overline{a}_\mu$, which can be exploited in various experiments [11].

Another type of field redefinition can be written in the generic form [3]

$$\psi(x) \to [1 + v(x)\cdot\Gamma]\psi(x), \tag{15}$$

where Γ represents one of γ^a, $\gamma_5\gamma^a$, σ^{ab} and $v(x)$ is a complex function carrying the appropriate local Lorentz indices. This can be viewed as a position-dependent component mixing in spinor space. Field redefinitions of this type can be used to demonstrate the leading-order equivalence of observable physical effects due to certain coefficients for Lorentz violation. An example relevant in the present context is a redefinition involving a real vector $v_a(x)$. Together with assumption (13), this redefinition can be used to show that at leading order in Lorentz violation the coefficients a_μ and e_μ always appear in the combination

$$(a_{\text{eff}})_\mu \equiv a_\mu - m e_\mu, \tag{16}$$

up to derivatives of fluctuations. Combining this result with the above discussion of the redefinition (14) shows that observables involving gravitational couplings offer the prospect of measuring $(a_{\text{eff}})_\mu$.

Related ideas can be used to simplify the weak-field limit of the theory (3). In particular, the antisymmetric part $\chi_{\mu\nu}$ of the vierbein can be removed everywhere except for possible contributions to fluctuations of the coefficient fields, by applying the field redefinition

$$\begin{aligned}\psi(x) &\rightarrow \exp[-\tfrac{1}{4}i\chi_{\mu\nu}(x)\sigma^{\mu\nu}]\psi(x)\\ &\approx \left(1-\tfrac{1}{4}i\chi_{\mu\nu}\sigma^{\mu\nu}-\tfrac{1}{32}\chi_{\mu\nu}\chi_{\alpha\beta}\sigma^{\mu\nu}\sigma^{\alpha\beta}\right)\psi(x).\end{aligned} \quad (17)$$

Note that this redefinition takes the form of a Lorentz transformation on ψ but that the other fields in the Lagrange density remain unaffected. Note also that in the absence of Lorentz violation $\chi_{\mu\nu}$ can be removed entirely, a fact compatible with the interpretation in Ref. [14] of the role of $\chi_{\mu\nu}$ in Lorentz-violating theories. In the remainder of this work, we assume the redefinition (17) has been performed on the Lagrange density, so that quantities such as Γ^a, M, and $e^\mu{}_a$ are understood to acquire no contributions from $\chi_{\mu\nu}$ except possibly through the fluctuations of the coefficient fields for Lorentz violation.

2. Coordinate choices

The observability of certain combinations of coefficients for Lorentz violation is also affected by the freedom to make coordinate choices. Intuitively, the key point is that any one sector of the SME can be used to define the scales of the four coordinates, to establish the meaning of isotropy, and to set the synchronization scheme. The freedom therefore exists to choose the sector in which the effective background spacetime metric takes the form of the Minkowski metric $\eta_{\mu\nu}$. This implies that in any experimental configuration there are always 10 unobservable combinations of coefficients for Lorentz violation.

As an illustration, consider the SME restricted to the single-fermion and photon sectors [18, 22]. In the fermion sector, the 10 relevant coefficient components are the vacuum values $\overline{c}_{\mu\nu}$ of the coefficient fields $c_{\mu\nu}$ in Eq. (7) because these coefficients enter S_ψ in the same way as the metric. In the photon sector, the SME Lagrange density contains a term

$$\mathcal{L}_{\text{photon}} \supset -\tfrac{1}{4}e(k_F)_{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu}, \quad (18)$$

and the 10 relevant coefficient components can be shown to be the trace $(\overline{k_F})^\alpha{}_{\mu\alpha\nu}$. At leading order, the coordinate transformation

$$x^\mu \rightarrow x^{\mu'} = x^\mu - \tfrac{1}{2}(\overline{k_F})^{\alpha\mu}{}_{\alpha\nu}x^\nu \quad (19)$$

redefines the background metric to take the form $\eta_{\mu\nu}$ in the photon sector. The effective metric in the fermion sector is then also changed, with the observable coefficient combination becoming $\overline{c}_{\mu\nu} + (\overline{k_F})^\alpha{}_{\mu\alpha\nu}/2$. The orthogonal combination $2\overline{c}_{\mu\nu} - (\overline{k_F})^\alpha{}_{\mu\alpha\nu}$ is thus unobservable in any experiments involving only these two sectors. Alternatively, one could perform the coordinate transformation

$$x^\mu \rightarrow x^{\mu'} = x^\mu + \overline{c}^\mu{}_\nu x^\nu, \quad (20)$$

which instead redefines the fermion-sector background metric to be $\eta_{\mu\nu}$ and at leading order produces the effective photon-sector coefficient $2\overline{c}_{\mu\nu} + (\overline{k_F})^\alpha{}_{\mu\alpha\nu}$. This coordinate choice is equally valid for analysis, and as before the orthogonal combination $2\overline{c}_{\mu\nu} - (\overline{k_F})^\alpha{}_{\mu\alpha\nu}$ is unobservable.

Similar results apply for experimental searches for Lorentz violation involving comparisons of different fermion species. Labeling the species by w, each has a coefficient $\overline{c}^w_{\mu\nu}$. Then, for example, the effective metric for any one species X can be reduced to $\eta_{\mu\nu}$ by a coordinate transformation with $\overline{c}^X_{\mu\nu}$ of the form (20). The resulting effective coefficients for the remaining species involve the differences $\overline{c}^w_{\mu\nu} - \overline{c}^X_{\mu\nu}$, which in this coordinate scheme become the relevant observable combinations of coefficients.

In the gravity sector, the situation is more involved because a geometrically consistent treatment of Lorentz violation generically requires the incorporation of effects from the NG modes to ensure the Bianchi identities are satisfied [3]. It turns out that the 10 relevant coefficient components in the gravity sector are the vacuum values $\overline{s}_{\mu\nu}$ of the coefficient fields $s_{\mu\nu}$ in the gravity-sector SME Lagrange density

$$\mathcal{L}_{\text{gravity}} \supset \frac{1}{16\pi G_N} e s^{\mu\nu} R_{\mu\nu}. \quad (21)$$

We find that at leading order the transformation (19) generates an accompanying shift $\overline{s}_{\mu\nu} \rightarrow \overline{s}_{\mu\nu} - (\overline{k_F})^\alpha{}_{\mu\alpha\nu}$, while the transformation (20) produces the shift $\overline{s}_{\mu\nu} \rightarrow \overline{s}_{\mu\nu} + 2\overline{c}_{\mu\nu}$.

One way to obtain these results is to consider the leading-order effect of a metric shift on the equations of motion, and then to match to the known results [7] for observable effects in a post-newtonian expansion. Consider, for example, the restriction of the SME to the Einstein-Hilbert action and the single-fermion action S_ψ with nonzero $\overline{c}_{\mu\nu}$ only. At leading order, the coordinate transformation (20) removes $\overline{c}_{\mu\nu}$ from the fermion action at the cost of introducing a metric shift $g_{\mu\nu} \rightarrow g_{\mu\nu} - 2\overline{c}_{\mu\nu}$ in the Einstein-Hilbert term. The resulting equations of motion involve the Einstein tensor $G_{\mu\nu}(g - 2\overline{c})$ with shifted argument, which can be written in terms of the Einstein tensor $G_{\mu\nu}(g)$ for the original metric and an effective energy-momentum tensor $\phi^{\overline{c}}_{\mu\nu}$. We find

$$\begin{aligned}G_{\mu\nu}(g-2\overline{c}) &= G_{\mu\nu}(g) - \phi^{\overline{c}}_{\mu\nu},\\ \phi^{\overline{c}}_{\mu\nu} &\approx 2(\eta_{\mu\nu}\overline{c}^{\alpha\beta}R_{\alpha\beta} - 2\overline{c}^\alpha{}_{(\mu}R_{\nu)\alpha} + \tfrac{1}{2}\overline{c}_{\mu\nu}R\\ &\quad + \overline{c}^{\alpha\beta}R_{\alpha\mu\nu\beta}).\end{aligned} \quad (22)$$

The trace-reversed form $\Phi^{\overline{c}}_{\mu\nu}$ of $\phi^{\overline{c}}_{\mu\nu}$ matches the post-newtonian term $\Phi^{\overline{s}}_{\mu\nu}$ arising from Eq. (21) and given explicitly as Eq. (24) of Ref. [7] with the combination $\overline{s}_{\mu\nu}$ replaced by $2\overline{c}_{\mu\nu}$. The transformation (20) therefore produces the shift $\overline{s}_{\mu\nu} \to \overline{s}_{\mu\nu} + 2\overline{c}_{\mu\nu}$, as claimed. A similar line of reasoning verifies the claimed shift $\overline{s}_{\mu\nu} \to \overline{s}_{\mu\nu} - (\overline{k_F})^{\alpha}{}_{\mu\alpha\nu}$ for the coordinate transformation (19).

To keep expressions compact throughout this work, we choose to work with coordinates satisfying

$$(\overline{k_F})^{\alpha}{}_{\mu\alpha\nu} = 0. \tag{23}$$

To obtain results valid for arbitrary coordinate choices, the following substitutions can be applied throughout:

$$\begin{aligned} \overline{c}^{w}_{\mu\nu} &\to \overline{c}^{w}_{\mu\nu} + \tfrac{1}{2}(\overline{k_F})^{\alpha}{}_{\mu\alpha\nu}, \\ \overline{s}_{\mu\nu} &\to \overline{s}_{\mu\nu} - (\overline{k_F})^{\alpha}{}_{\mu\alpha\nu}. \end{aligned} \tag{24}$$

We emphasize that all coordinate choices are equivalent for theoretical work or for data analysis, with the choice (23) adopted here being purely one of convenience.

D. Perturbation scheme

In this work, we are interested in experimental searches for Lorentz violation involving gravitational effects on matter. Many of these searches involve test particles moving in background solutions to the equations of motion for gravity and for the coefficient fields. Since the gravitational fields involved are weak and since no compelling evidence for Lorentz violation exists to date, any effects are expected to be small. We can therefore focus attention on perturbative modifications to the behavior of test particles. This subsection describes the scheme we use to track perturbative orders in the construction of the relativistic quantum hamiltonian and in the subsequent developments.

Perturbative effects on physical observables can arise through modifications to the background coefficient fields $t_{\lambda\mu\nu\ldots}$ for Lorentz violation and to the background gravitational field $g_{\mu\nu}$, or directly through modifications to the equation of motion for the test particle. The analysis of these effects is simplified by introducing an appropriate notion of perturbative order. Several ordering schemes are possible. In this work, we adopt a scheme that tracks the orders in the coefficients for Lorentz violation $\overline{t}_{\lambda\mu\nu\ldots}$ and in the metric fluctuation $h_{\mu\nu}$. The overall perturbative order of a given term in an equation is denoted as O(m,n), where m represents the order in $\overline{t}_{\lambda\mu\nu\ldots}$ and n the order in $h_{\mu\nu}$. Within this scheme, the fluctuations $\tilde{t}_{\lambda\mu\nu\ldots}$ of the coefficient fields for Lorentz violation are viewed as secondary quantities that are determined via their equations of motion in terms of the coefficients for Lorentz violation and the gravitational field. There is also a subsidiary notion of order associated with the usual post-newtonian expansion of $h_{\mu\nu}$ itself. We denote a pth-order term in this latter expansion as PNO(p). However, performing an explicit post-newtonian expansion at the initial stage would complicate the ensuing analysis, so in what follows we often write results in terms of $h_{\mu\nu}$ while commenting as needed on the post-newtonian counting.

To preserve a reasonable scope, this work focuses on dominant perturbative effects involving both Lorentz violation and gravity. We next discuss the relevant perturbative orders required to achieve this goal.

Consider first contributions from the fluctuation $\tilde{t}_{\lambda\mu\nu\ldots}$ of the coefficient fields. The detailed structure of $\tilde{t}_{\lambda\mu\nu\ldots}$ depends on the nature of the action S' in Eq. (1). In the scheme adopted here, $\tilde{t}_{\lambda\mu\nu\ldots}$ can be viewed as a series in $\overline{t}_{\lambda\mu\nu\ldots}$ and $h_{\mu\nu}$ of the form

$$\tilde{t}_{\lambda\mu\nu\ldots} = \tilde{t}^{(0,0)}_{\lambda\mu\nu\ldots} + \tilde{t}^{(0,1)}_{\lambda\mu\nu\ldots} + \tilde{t}^{(1,0)}_{\lambda\mu\nu\ldots} + \tilde{t}^{(1,1)}_{\lambda\mu\nu\ldots} + \ldots . \tag{25}$$

For spontaneous Lorentz breaking, $\tilde{t}_{\lambda\mu\nu\ldots}$ includes massive modes and massless NG modes [15, 26]. In this work, we suppose the massive modes either are frozen or have negligible degree of excitation. Incorporating their possible effects into the analysis of matter-gravity phenomena would be of potential interest but lies beyond our present scope. In contrast, the massless NG modes play a key role in what follows. Their fate can include identification with the photon in Einstein-Maxwell theory [14], with the graviton in GR [27, 28], or with a new force [11, 29, 30], or they can be absorbed in the torsion via the Lorentz-Higgs effect [14]. In some models the NG modes can be interpreted as composite photons [31] or gravitons [32]. We consider below the perturbative orders required for the various possibilities.

Suppose first the NG modes correspond to photons, or more generally to a known force field other than gravity. The term $\tilde{t}^{(0,0)}_{\lambda\mu\nu\ldots}$ then contains conventional Lorentz-invariant terms describing this field in Minkowski spacetime, while $\tilde{t}^{(0,1)}_{\lambda\mu\nu\ldots}$ contains conventional leading-order gravitational interactions with the field. Effects from both these terms are therefore part of the conventional description of the force. The term $\tilde{t}^{(1,0)}_{\lambda\mu\nu\ldots}$ describes possible Lorentz violations in Minkowski spacetime involving the known field, many of which are tightly constrained by experiments [2]. For the purposes of this work, which focuses on Lorentz violation involving gravity, we can take this term as experimentally negligible. The dominant term of interest is therefore $\tilde{t}^{(1,1)}_{\lambda\mu\nu\ldots}$, which lies at O(1,1).

If the NG modes correspond to gravitons as, for example, in the cardinal model [27], then the expansion (25) contains no terms at O(m,0). The term $\tilde{t}^{(0,1)}_{\lambda\mu\nu\ldots}$ corresponds to the gravitational fluctuations $h_{\mu\nu}$, and its effects are part of the conventional description of gravity. The dominant term of interest is therefore again $\tilde{t}^{(1,1)}_{\lambda\mu\nu\ldots}$.

If instead the NG modes correspond to a presently unobserved force field, then $\tilde{t}^{(0,0)}_{\lambda\mu\nu\ldots}$ and $\tilde{t}^{(0,1)}_{\lambda\mu\nu\ldots}$ describe unobserved Lorentz-invariant effects in Minkowski spacetime and in leading-order gravitational couplings. These modes must therefore be eliminated from the analysis prior to interpretation of observations, via solving the equations of motion or otherwise. In what follows, we suppose this elimination has been performed where needed. The term $\vec{t}^{(1,0)}_{\lambda\mu\nu\ldots}$ describes possible Lorentz violation in Minkowski-spacetime involving the unknown field. For present purposes, we take this term to be experimentally negligible, although in principle it might offer a novel way to access certain types of presently unobserved interactions at exceptional sensitivities. The remaining term $\vec{t}^{(1,1)}_{\lambda\mu\nu\ldots}$ displayed in Eq. (25) describes the dominant Lorentz-violating gravitational effects involving the unknown interaction. As discussed in Sec. II C 1, certain Lorentz-violating effects are observable only in the presence of gravity, and so under suitable circumstances observable experimental signals from $\vec{t}^{(1,1)}_{\lambda\mu\nu\ldots}$ could arise [11] despite the tight existing experimental constraints [35] on the direct observation of additional interactions due to the lower-order terms $\vec{t}^{(0,0)}_{\lambda\mu\nu\ldots}$ and $\vec{t}^{(0,1)}_{\lambda\mu\nu\ldots}$. In scenarios with an unobserved force, we must therefore also allow for O(1,1) effects involving $\vec{t}^{(1,1)}_{\lambda\mu\nu\ldots}$.

The remaining possibility is that NG modes are absorbed into the torsion. To handle this case, note that the matter-sector role of minimal torsion can be treated in parallel with the coefficient field b_μ for Lorentz violation according to Eq. (9). Existing experimental constraints on minimal torsion components are tight, lying below 10^{-27}-10^{-31} GeV [17, 33, 34], so effects involving minimal torsion can be treated in the same way as those corresponding to a presently unobserved force field. We can therefore limit attention to O(1,1) effects as before. In principle, any nonminimal torsion components can be treated in a similar fashion because they play a role analogous to coefficient fields in the nonminimal SME, but effects of this type lie beyond our present scope.

Consider next the metric fluctuation $h_{\mu\nu}$. For applications to gravitational tests with matter, $h_{\mu\nu}$ can be treated as a background field obtained by solving the appropriate equation of motion, which is the modified Einstein equation in the presence of Lorentz violation. It can therefore be viewed as the sum of a Lorentz-invariant piece $h^{(0,1)}_{\mu\nu}$ with a series of corrections of increasing perturbative order in $\bar{t}_{\lambda\mu\nu\ldots}$,

$$h_{\mu\nu} = h^{(0,1)}_{\mu\nu} + h^{(1,1)}_{\mu\nu} + h^{(2,1)}_{\mu\nu} + \ldots. \qquad (26)$$

When we specify the perturbative order of an expression containing $h_{\mu\nu}$ in what follows, it is understood that the correct terms from the above series are included.

For a given expression, establishing the relevant perturbative order for our analysis typically involves a combination of experimental restrictions and theoretical considerations. As an illustration, we outline here the reasoning establishing the appropriate perturbative orders in the construction of the relativistic quantum hamiltonian. First, note that terms quadratic in $h_{\mu\nu}$ involve PNO(4) and higher. Since the sensitivity of current laboratory and solar-system tests lies at the PNO(4) level, we must keep these terms but can discard terms cubic in $h_{\mu\nu}$ and ones involving the product of coefficients for Lorentz violation with terms quadratic in $h_{\mu\nu}$. The Lorentz-invariant part of the hamiltonian can therefore be truncated at O(0,2), while the Lorentz-violating part can be truncated at O(1,1). To maintain consistent post-newtonian counting, the O(0,2) terms must be limited to PNO(5), while the O(1,1) terms are limited to PNO(3). Next, note that for laboratory and solar-system tests the variations in $h_{\mu\nu}$ over the experimental scale L are small compared to $h_{\mu\nu}$, $|\partial_\alpha h_{\mu\nu}| \ll |h_{\mu\nu}/L|$. For example, the typical value of the gravitational acceleration on the surface of the Earth is $g \simeq 10^{-32}$ GeV, which is tiny compared to the ratio of the gravitational potential $|h_{\mu\nu}| \simeq 10^{-9}$ and the size of a typical laboratory experiment $L \simeq 10^{15}$ GeV^{-1}. Terms in the relativistic hamiltonian proportional to derivatives of $h_{\mu\nu}$ can therefore be limited to O(0,1). Finally, note that products of Lorentz-violating terms lead to higher-order effects with operator structures matching ones already appearing in the fermion sector of the Minkowski-spacetime SME. These are already accessible in nongravitational experiments. It therefore suffices for our purpose to restrict attention to terms at leading order in Lorentz violation. To summarize, the construction of the perturbative relativistic quantum-mechanical hamiltonian can be limited to terms at perturbative orders O(0,1), O(1,0), O(1,1), and O(0,2), except for terms involving derivatives of the gravitational fields, which can be limited to O(0,1).

III. QUANTUM THEORY

This section studies the quantum mechanics associated with the fermion action S_ψ in Eq. (3). We begin in Sec. III A by addressing the issue of the unconventional time dependence arising from the Dirac equation derived from S_ψ. The relativistic quantum-mechanical hamiltonian H is then obtained in Sec. III B, and the relevant parts of the nonrelativistic limit $H_{\rm NR}$ are extracted in Sec. III C.

A. Time dependence

In the weak-field limit, the Lagrange density $\mathcal{L}_\psi$ for the action (3) takes the schematic form

$$\mathcal{L} = \tfrac{1}{2} i[\overline{\psi}(\gamma^\mu + C^\mu)\partial_\mu\psi - (\partial_\mu\overline{\psi})(\gamma^\mu + C^\mu)\psi] - \overline{\psi}D\psi, \qquad (27)$$

where C^μ and D represent spacetime-dependent operators without derivatives acting on ψ. These operators satisfy the conditions

$$(\gamma^0 C^\mu)^\dagger = \gamma^0 C^\mu, \qquad (\gamma^0 D)^\dagger = \gamma^0 D, \qquad (28)$$

and C^μ is perturbative.

The Euler-Lagrange equations obtained from Eq. (27) yield a Dirac equation with unconventional time dependence,

$$i(\gamma^0 + C^0)\partial_0\psi = [-i(\gamma^j + C^j)\partial_j - \tfrac{1}{2} i\partial_\mu C^\mu + D]\psi. \qquad (29)$$

This equation differs from the standard Dirac form by the presence of C^0, which impedes the interpretation of the operator acting on ψ on the right-hand side as the hamiltonian. In this subsection, two approaches addressing this issue at first order in C^μ are discussed.

1. *Field-redefinition method*

One method for constructing the hamiltonian has been developed in the context of the SME in Minkowski spacetime [36–38]. It uses an appropriate field redefinition at the level of the action to ensure the Dirac equation emerges with conventional time dependence. In typical applications, the field redefinition is defined perturbatively at the desired order in C^μ.

For present purposes, it suffices to work at first order in C^μ. The appropriate field redefinition is

$$\psi \ = \ A\chi, \qquad A \equiv 1 - \tfrac{1}{2}\gamma^0 C^0. \tag{30}$$

The resulting hamiltonian can be written as

$$H_\chi = H^{(0)} + H_\chi^{(1)}, \tag{31}$$

where $H^{(0)}$ is the hamiltonian in the absence of C^μ, given by

$$H^{(0)} = -i\gamma^0\gamma^j\partial_j + \gamma^0 D. \tag{32}$$

The correction $H_\chi^{(1)}$ is first-order in C^μ and takes the form

$$\begin{aligned} H_\chi^{(1)} \ &= \ -i\gamma^0(C^j - \tfrac{1}{2}C^0\gamma^0\gamma^j + \tfrac{1}{2}\gamma^0\gamma^j C^0)\partial_j \\ &\quad -\tfrac{1}{2}i(\gamma^j\partial_j C^0 + \gamma^0\partial_j C^j) \\ &\quad -\tfrac{1}{2}\gamma^0(C^0\gamma^0 D + D\gamma^0 C^0). \end{aligned} \tag{33}$$

The subscript χ serves as a reminder that the operator acts on the spinor χ. Note that the hamiltonian H_χ is hermitian with respect to the usual scalar product in flat space,

$$\langle\chi_1,\chi_2\rangle_{\rm f} = \int d^3x\ \chi_1^\dagger\chi_2. \tag{34}$$

This implies, for example, that energies can be calculated in the usual way.

Some physical insight into the field-redefinition method is obtained by noting that the combination $\gamma^\mu + C^\mu$ takes the generic form

$$\gamma^\mu + C^\mu = E^\mu{}_a\gamma^a, \tag{35}$$

where $E^\mu{}_a$ can be interpreted as an effective inverse vierbein. It reduces to the conventional inverse vierbein $e^\mu{}_a$ in the purely gravitational case but includes coefficients for Lorentz violation when Lorentz symmetry is broken. This suggests that the field-redefinition method can be interpreted as transforming the problematic situation of a fermion on an effective manifold with vierbein components $E^0{}_a$ into the physically equivalent but tractable theory of a different fermion field on a manifold with vierbein components $E^0{}_a = \delta^0{}_a$, in which the hamiltonian is hermitian with respect to a conventional scalar product.

2. *Parker method*

Another method has been presented by Parker [39] in the context of field theory in curved spacetime. It involves multiplying the Dirac equation by a suitable factor that removes the unconventional time dependence to the desired order in C^μ. The resulting hamiltonian is hermitian with respect to a modified scalar product.

Applying this method at first order in C^μ requires left-multiplying both sides of Eq. (29) with $\gamma^0(1-C^0\gamma^0)$. The ensuing hamiltonian can be written as

$$\hat{H}_\psi = H^{(0)} + H_\psi^{(1)}, \tag{36}$$

where $H^{(0)}$ is given in Eq. (32) and the first-order correction in C^μ is

$$H_\psi^{(1)} = -i\gamma^0 C^j\partial_j + i\gamma^0 C^0\gamma^0\gamma^j\partial_j - \tfrac{1}{2}i\gamma^0\partial_\mu C^\mu - \gamma^0 C^0\gamma^0 D. \tag{37}$$

The subscript ψ indicates an operator acting on the original spinor ψ.

In this method, the modified Dirac equation implies a modified continuity equation and hence requires a modified scalar product. At first order in C^μ, the continuity equation is

$$\partial_0[\psi^\dagger(1+\gamma^0C^0)\psi] + \partial_j[\psi^\dagger\gamma^0(\gamma^j + C^j)\psi] = 0, \tag{38}$$

and the probability density can be identified as the combination $\psi^\dagger(1+\gamma^0C^0)\psi$. The corresponding scalar product is

$$\langle\psi_1,\psi_2\rangle_{\rm P} = \int d^3x\ \psi_1^\dagger(1+\gamma^0C^0)\psi_2. \tag{39}$$

Provided $H_\psi^{(1)}$ is time independent, the hamiltonian $\hat{H}_\psi$ is hermitian with respect to this modified scalar product and so quantum-mechanical calculations can proceed. When C^0 is time dependent, hermiticity with respect to the product (39) can be restored by adding an extra term [40]. We thereby obtain the hermitian hamiltonian

$$H_\psi = \tfrac{1}{2}i\gamma^0\partial_0 C^0 + \hat{H}_\psi \tag{40}$$

at first order in C^0.

3. *Comparison*

The hamiltonians H_χ in Eq. (31) and H_ψ in Eq. (40) typically have different forms. For our present purposes, however, they are physically equivalent because they give rise to the same eigenenergies at first order.

To demonstrate this, first note that the difference ΔH between the two hamiltonians can be written as

$$\Delta H = H_\chi^{(1)} - H_\psi^{(1)} = H^{(0)}A - AH^{(0)}, \tag{41}$$

where A is given by Eq. (30). This can be shown by manipulation of the field redefinition and the Dirac equation or verified by direct calculation. The physical quantities of interest are the eigenenergies. These are obtained as perturbations $E^{(1)}_{\chi,\psi}$ to the unperturbed values $E^{(0)}$, calculable at first order as

$$E^{(1)}_{\psi,\chi} = \langle \psi^{(0)}, H^{(1)}_{\psi,\chi}\psi^{(0)}\rangle_{\rm f}, \tag{42}$$

where $\psi^{(0)}$ are solutions to the unperturbed Schrödinger equation $H^{(0)}\psi^{(0)} = E^{(0)}\psi^{(0)}$. However, the expectation value of ΔH is zero in this scalar product, so the first-order perturbations to the energies are identical for both hamiltonians.

The above first-order result suffices for the present work, although we anticipate equivalence also holds at higher orders in a complete analysis. For our purposes, the field-redefinition method proves technically and conceptually easier because the hamiltonian is hermitian with respect to the usual scalar product. We therefore adopt the field-redefinition method in the remainder of this work, and all references to H implicitly refer to H_χ.

B. Relativistic hamiltonian

The relativistic hamiltonian for the action S_ψ in Eq. (3) can be obtained via the field-redefinition method. At the appropriate perturbative order, we find the operator A of Eq. (30) takes the form

$$\begin{aligned} A \;=\; 1 - \tfrac{1}{2}\gamma^0\Big[& e e^0{}_a\Gamma^a - \gamma^0 - \tfrac{3}{4}e^{(0,1)}e^{(0,1)}\gamma^0 \\ & - \tfrac{3}{4}e^{(0,1)}h_{0\mu}\gamma^\mu - \tfrac{3}{16}h_{0\mu}h_{0\nu}\gamma^\mu\gamma^0\gamma^\nu \\ & - \tfrac{3}{2}e^{(0,1)}\Gamma^0 - \tfrac{3}{8}h_{0\mu}\left(\Gamma^0\gamma^0\gamma^\mu + \gamma^\mu\gamma^0\Gamma^0\right)\Big]. \end{aligned} \tag{43}$$

Implementing the field redefinition and varying the transformed Lagrange density results in a Dirac equation from which the hamiltonian H can be identified.

The hamiltonian H can be split into pieces according to perturbative order,

$$H = H^{(0,0)} + H^{(0,1)} + H^{(1,0)} + H^{(1,1)} + H^{(0,2)}. \tag{44}$$

The component $H^{(0,0)}$ is the conventional hamiltonian for the Lorentz-invariant Minkowski-spacetime limit of the theory. The first-order Lorentz-invariant piece is

$$\begin{aligned} H^{(0,1)} \;=\; & \tfrac{1}{2}i(h_{jk} + h_{00}\eta_{jk})\gamma^0\gamma^k\partial^j + ih_{j0}\partial^j - \tfrac{1}{2}mh_{00}\gamma^0 \\ & + \tfrac{1}{4}\partial^j h^{0k}\epsilon_{jkl}\gamma_5\gamma^0\gamma^l + \tfrac{1}{2}i\partial^j h_{j0} \\ & + \tfrac{1}{4}i(\partial_j h_{00} + \partial^k h_{jk})\gamma^0\gamma^j. \end{aligned} \tag{45}$$

It represents the first-order correction to the conventional hamiltonian $H^{(0,0)}$ arising from gravitational and inertial effects.

The first-order correction to the conventional hamiltonian $H^{(0,0)}$ arising from Lorentz violation can be written

$$\begin{aligned} H^{(1,0)} \;=\; & \overline{a}_0 - m\overline{e}_0 + 2i\overline{c}_{(j0)}\partial^j - (m\overline{c}_{00} - i\overline{e}_j\partial^j)\gamma^0 \\ & - \overline{f}_j\partial^j\gamma^0\gamma_5 + [\overline{a}_j + i(\overline{c}_{00}\eta_{jk} + \overline{c}_{jk})\partial^k]\gamma^0\gamma^j \\ & + (-\overline{b}_0 - 2i\overline{d}_{(j0)}\partial^j)\gamma_5 + (i\overline{H}_{0j} + 2\overline{g}_{j(k0)}\partial^k)\gamma^j \\ & - [\overline{b}_j + i(\overline{d}_{jk}\partial^k + \overline{d}_{00}\partial_j) - \tfrac{1}{2}m\overline{g}^{kl0}\epsilon_{jkl}]\gamma_5\gamma^0\gamma^j \\ & - (\tfrac{1}{2}\overline{H}^{kl}\epsilon_{jkl} + m\overline{d}_{j0})\gamma_5\gamma^j \\ & - i\epsilon_{jlm}(\overline{g}^{l00}\eta^{km} + \tfrac{1}{2}\overline{g}^{lmk})\partial_k\gamma_5\gamma^j. \end{aligned} \tag{46}$$

This result matches the one previously obtained for the SME in Minkowski spacetime [37] when the change in metric signature is incorporated. When minimal torsion is included in the analysis, its background value enters Eq. (46) through the replacement $b_\mu \to (b_{\rm eff})_\mu$ specified by Eq. (9). It can be constrained through a reinterpretation of experiments searching for nonzero $\overline{b}_\mu$ [17, 33, 34].

An interesting issue is the extent to which the gravitational and inertial effects in Eq. (45) mimic the Lorentz-violating effects in Eq. (46). For example, in a rotating frame of reference the term $\partial^j h^{0k}\epsilon_{jkl}\gamma_5\gamma^0\gamma^l/4$ in Eq. (45) contains a coupling of the rotation to the spin of the particle with the same operator structure as the term $-\overline{b}_j\gamma_5\gamma^0\gamma^j$ in Eq. (46). At this order, a frame rotation can therefore mimic potential signals arising from a nonzero coefficient $\overline{b}_j$ for Lorentz and CPT violation. This effect has been observed in tests with a spin-polarized torsion pendulum [41]. The same term in Eq. (45) also contains gravitomagnetic effects that are in principle observable in tests searching for $\overline{b}_\mu$ if sufficient sensitivity is reached. Certain Lorentz-violating effects can be separated from gravitational and inertial effects because the former generate time-varying signals due to the motion of the Earth and can have flavor dependence, but a complete separation may be problematic.

The O(1,1) contribution to H can be separated as

$$\begin{aligned} H^{(1,1)} \;=\; & H^{(1,1)}_h + H^{(1,1)}_a + H^{(1,1)}_b + H^{(1,1)}_c + H^{(1,1)}_d \\ & + H^{(1,1)}_e + H^{(1,1)}_f + H^{(1,1)}_g + H^{(1,1)}_H. \end{aligned} \tag{47}$$

Here, the term $H^{(1,1)}_h$ arises from Lorentz-violating corrections to the metric fluctuation $h_{\mu\nu}$. It is given by

$$\begin{aligned} H^{(1,1)}_h \;=\; & ih^{(1,1)}_{j0}\partial^j - \tfrac{1}{2}mh^{(1,1)}_{00}\gamma^0 \\ & + \tfrac{1}{2}i(h^{(1,1)}_{jk} + h^{(1,1)}_{00}\eta_{jk})\gamma^0\gamma^k\partial^j. \end{aligned} \tag{48}$$

The other terms in $H^{(1,1)}$ are labeled according to the type of coefficient for Lorentz violation involved. The contributions involving the four-component coefficients a_μ and b_μ of mass dimension one are

$$H^{(1,1)}_a \;=\; \overset{\leftharpoonup}{a}_0 - \overline{a}^j h_{j0} + (\overset{\leftharpoonup}{a}_j - \tfrac{1}{2}\overline{a}_j h_{00} - \tfrac{1}{2}\overline{a}^k h_{jk})\gamma^0\gamma^j \tag{49}$$

and

$$\begin{aligned} H^{(1,1)}_b \;=\; & \big(-\overset{\leftharpoonup}{b}_0 + \overline{b}^j h_{j0}\big)\gamma_5 \\ & + (\overset{\leftharpoonup}{b}_j - \tfrac{1}{2}\overline{b}_j h_{00} - \tfrac{1}{2}\overline{b}^k h_{jk})\gamma^0\gamma_5\gamma^j, \end{aligned} \tag{50}$$

respectively. In the latter equation, effects from minimal torsion are included via the replacement $b_\mu \to (b_{\rm eff})_\mu$ given in Eq. (9).

The contributions involving the dimensionless coefficients $c_{\mu\nu}$ and $d_{\mu\nu}$ are

$$\begin{aligned} H_c^{(1,1)} &= i\big[\tilde{c}_{00}\eta_{jk} + \tilde{c}_{kj} + \tfrac{1}{2}\overline{c}_{00}\left(h_{00}\eta_{jk} - h_{jk}\right) \\ &\quad + 2\overline{c}_{(l0)}h^{l0}\eta_{jk} + \tfrac{1}{4}\overline{c}_{k0}h_{j0} - \tfrac{1}{4}\overline{c}_{j0}h_{k0} \\ &\quad - \tfrac{1}{2}\overline{c}_{lj}\left(h_{00}\eta^l{}_k + h^l{}_k\right) - \overline{c}_{kl}h^l{}_j\big]\gamma^0\gamma^k\partial^j \\ &\quad -2i\big(\tilde{c}^{(j0)} + \overline{c}_{(k0)}h^{jk} + \overline{c}^{jk}h_{k0}\big)\partial_j \\ &\quad -m\big(\tilde{c}_{00} + \tfrac{1}{2}\overline{c}_{00}h_{00} + 2\overline{c}_{(j0)}h^{j0}\big)\gamma^0 \qquad (51) \end{aligned}$$

and

$$\begin{aligned} H_d^{(1,1)} &= 2i\big(\tilde{d}^{(j0)} + \overline{d}_{(k0)}h^{jk} + \overline{d}^{(jk)}h_{k0}\big)\gamma_5\partial_j \\ &\quad +i\big(\tilde{d}_{00} + \tfrac{1}{2}\overline{d}_{00}h_{00} + 2\overline{d}_{(k0)}h^{k0}\big)\gamma^0\gamma_5\gamma^j\partial_j \\ &\quad +i\big[\tilde{d}_{kj} + \tfrac{1}{4}\overline{d}_{k0}h_{j0} - \tfrac{1}{4}\overline{d}_{j0}h_{k0} - \tfrac{1}{2}\overline{d}_{00}h_{jk} \\ &\quad - \tfrac{1}{2}\overline{d}_{lj}\left(h_{00}\eta^l{}_k + h^l{}_k\right) - \overline{d}_{kl}h^l{}_j\big]\gamma^0\gamma_5\gamma^k\partial^j \\ &\quad +m\big(\tilde{d}^{j0}\eta_{jk} - \tfrac{1}{2}\overline{d}^{j0}h_{jk} - \overline{d}_{kj}h^{j0}\big)\gamma_5\gamma^k \\ &\quad -\tfrac{1}{4}im\overline{d}^{j0}h^{k0}\epsilon_{jkl}\gamma^l, \qquad (52) \end{aligned}$$

respectively. The dimensionless four-component coefficients e_μ and f_μ generate the expressions

$$\begin{aligned} H_e^{(1,1)} &= i\big(\tilde{e}_j + \tfrac{1}{4}\overline{e}_0 h_{j0} - \tfrac{1}{2}\overline{e}_j h_{00} - \overline{e}^k h_{jk}\big)\gamma^0\partial^j \\ &\quad -m\tilde{e}_0 + m\overline{e}^j h_{j0} + \tfrac{1}{4}m\overline{e}_0 h_{j0}\gamma^0\gamma^j \qquad (53) \end{aligned}$$

and

$$H_f^{(1,1)} = -\big(\tilde{f}_j + \tfrac{1}{4}\overline{f}_0 h_{j0} - \tfrac{1}{2}\overline{f}_j h_{00} - \overline{f}^k h_{jk}\big)\gamma^0\gamma_5\partial^j, \qquad (54)$$

while the dimensionless coefficient $g_{\lambda\mu\nu}$ leads to

$$\begin{aligned} H_g^{(1,1)} &= \big(2\tilde{g}_{k(j0)} - \tfrac{1}{4}\overline{g}_{l00}h^{l0}\eta_{jk} - \overline{g}_{l(j0)}h^l{}_k - 2\overline{g}_{k(l0)}h^l{}_j \\ &\quad + 2\overline{g}_{k(jl)}h^{l0}\big)\gamma^k\partial^j \\ &\quad +i\big[\tilde{g}^{l00}\eta^{jk} - \tfrac{1}{2}\tilde{g}^{klj} - \tfrac{1}{2}\overline{g}^{k00}\left(h_{00}\eta^{jl} - h^{jl}\right) \\ &\quad - \tfrac{1}{2}\overline{g}_{n00}h^{ln}\eta^{jk} - \tfrac{1}{4}\overline{g}^{jk0}h^{l0} \\ &\quad + 2\overline{g}^{l(n0)}h_{n0}\eta^{jk} - \tfrac{1}{8}\overline{g}^{kl0}h^{j0} + \tfrac{1}{4}\overline{g}^{klj}h_{00} \\ &\quad + \tfrac{1}{2}\overline{g}^{kln}h^j{}_n - \tfrac{1}{2}\overline{g}^{knj}h^l{}_n\big]\epsilon_{klm}\gamma_5\gamma^m\partial_j \\ &\quad -\tfrac{1}{2}m\big(\tilde{g}^{jk0} + \overline{g}^{km0}h^j{}_m + \overline{g}^{jkm}h_{m0}\big)\epsilon_{jkl}\gamma^0\gamma_5\gamma^l \\ &\quad +\tfrac{1}{2}mi\overline{g}_{jk0}h^{k0}\gamma^0\gamma^j + \tfrac{1}{8}m\overline{g}^{jk0}h^{l0}\epsilon_{jkl}\gamma_5. \qquad (55) \end{aligned}$$

Finally, the antisymmetric coefficient $H_{\mu\nu}$ of mass dimension one contributes

$$\begin{aligned} H_H^{(1,1)} &= -\tfrac{1}{2}\big(\tilde{H}^{jk} - \overline{H}^{jm}h^k{}_m - \tfrac{1}{2}\overline{H}^{jk}h_{00}\big)\epsilon_{jkl}\gamma_5\gamma^l \\ &\quad -i\big(\tilde{H}_{j0} + \tfrac{1}{2}\overline{H}^{k0}h_{jk} + \overline{H}_{jk}h^{k0}\big)\gamma^j. \qquad (56) \end{aligned}$$

In the above expressions, the fluctuations of the various coefficient fields appear in H only at perturbative order O(1,1). For most purposes, it is necessary to find expressions for these fluctuations prior to using the hamiltonian H in a given analysis. This issue is addressed further in Sec. IV C, where the spin-independent coefficient fluctuations $(\tilde{a}_{\rm eff})_\mu$ and $\tilde{c}_{\mu\nu}$ are considered in more detail.

The remaining piece of the hamiltonian H lies at perturbative order O(0,2) and represents the second-order Lorentz-invariant contribution from gravitational and inertial effects. It takes the form

$$\begin{aligned} H^{(0,2)} &= i\big(\tfrac{1}{8}h_{00}h_{00}\eta_{jk} + \tfrac{1}{2}h_{l0}h^{l0}\eta_{jk} - \tfrac{1}{4}h_{00}h_{jk} \\ &\quad - \tfrac{3}{8}h_{jl}h^l{}_k\big)\gamma^0\gamma^k\partial^j - ih_{k0}h^{jk}\partial_j \\ &\quad -m\big(\tfrac{1}{8}h_{00}h_{00} + \tfrac{1}{2}h_{j0}h^{j0}\big)\gamma^0. \qquad (57) \end{aligned}$$

C. Nonrelativistic hamiltonian

Most experimental tests of interest in this work are nonrelativistic. In this section, we use a Foldy-Wouthuysen transformation [42] to extract from the relativistic hamiltonian H the parts of the nonrelativistic hamiltonian $H_{\rm NR}$ relevant for the subsequent analyses.

The Foldy-Wouthuysen transformation is a systematic procedure for determining the nonrelativistic content of certain relativistic quantum-mechanical hamiltonians. For a massive four-component Dirac fermion, the transformation generates a series expansion in powers of the fermion momentum. In the present case, the transformation can be implemented as usual, but care must be taken to keep track of both the order in momentum and the perturbative order O(m,n) in coefficients for Lorentz violation and in gravitational fluctuations.

Performing the Foldy-Wouthuysen transformation for the complete hamiltonian H of Eq. (44) is cumbersome and also unnecessary for our scope because most attained sensitivities to spin couplings are unlikely to be improved by studying the suppressed effects from gravitational couplings. However, only limited sensitivity currently exists to spin-independent effects controlled by the coefficients $\overline{a}_\mu$ and $\overline{e}_\mu$ because these are unobservable for baryons and charged leptons in Minkowski spacetime. In the remainder of this work we focus on general spin-independent effects, which are associated with the coefficients $\overline{a}_\mu$, $\overline{c}_{\mu\nu}$, and $\overline{e}_\mu$. Since the minimal torsion coupling also involves spin, this focus implies also disregarding nonrelativistic effects due to torsion, effectively restricting attention to the limiting Riemann geometry. Although beyond our current scope, a Foldy-Wouthuysen analysis incorporating spin-dependent effects could lead to additional torsion sensitivities beyond those obtained via searches for $\overline{b}_\mu$ [3, 17, 33, 43].

In the relativistic quantum theory, the upper two components of the four-component wave function describe the particle while the lower two describe the antiparticle. The hamiltonian H can be separated into an odd part $\mathcal{O}$ containing terms that mix the upper and lower components and an even part $\mathcal{E}$ that involves no mixing. The idea of the Foldy-Wouthuysen method is to find a momentum-dependent unitary transformation S in the Hilbert space such that the 4×4 hamiltonian $\widetilde{H} = e^{iS}He^{-iS}$ is 2×2 block diagonal. The leading 2×2 block of $\widetilde{H}$ then represents the desired nonrelativistic hamiltonian $H_{\rm NR}$. The full transformation S is obtained at the desired level of accuracy via an iterated series of incremental transformations reducing the off-diagonal content to the appropriate order.

We proceed by separating the hamiltonian H into an odd part $\mathcal{O}_0^{(m,n)}$ and an even part $\mathcal{E}_0^{(m,n)}$ at each perturbative order O(m,n). A subscript is used to specify the iteration number of the transformation, with 0 corresponding to the zeroth iteration. The relativistic hamiltonian H can therefore be written as

$$\begin{aligned} H_0 \equiv H &= m\gamma^0 + \mathcal{O}_0^{(0,0)} + \mathcal{O}_0^{(0,1)} + \mathcal{O}_0^{(1,0)} + \mathcal{O}_0^{(1,1)} \\ &\quad + \mathcal{E}_0^{(0,0)} + \mathcal{E}_0^{(0,1)} + \mathcal{E}_0^{(1,0)} + \mathcal{E}_0^{(1,1)}. \end{aligned} \tag{58}$$

The Foldy-Wouthuysen sequence is then defined iteratively as

$$\begin{aligned} H_{n+1} &= e^{iS}H_n e^{-iS} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{[iS_n,[iS_n,\cdots[iS_n,}_{k \text{ commutations with } iS_n} H_0]\cdots]], \end{aligned} \tag{59}$$

where

$$S_n = \frac{-i\gamma^0\mathcal{O}_n}{2m}. \tag{60}$$

At each stage, the sum on k is truncated once the appropriate order in momentum and small quantities is reached. The iteration continues until the hamiltonian is even at the desired order. Here, we proceed to O(1,1) in the small quantities and to second order in the momentum, which requires three iterations and yields a hamiltonian H_3.

The desired spin-independent contributions to the nonrelativistic hamiltonian $H_{\rm NR}$ can be separated according to perturbative order and origin as

$$\begin{aligned} H_{\rm NR} &\equiv H_3 \\ &= H_{\rm NR}^{(0,0)} + H_{\rm NR}^{(0,1)} + H_{\rm NR}^{(1,0)} \\ &\quad + H_{{\rm NR},a_{\rm eff}}^{(1,1)} + H_{{\rm NR},c}^{(1,1)} + H_{{\rm NR},h}^{(1,1)} + H_{\rm NR}^{(0,2)}. \end{aligned} \tag{61}$$

Here, $H_{\rm NR}^{(0,0)}$ is the conventional Minkowski-spacetime hamiltonian. The conventional Lorentz-invariant contributions $H_{\rm NR}^{(0,1)}$ due to the metric fluctuation can be written as

$$\begin{aligned} H_{\rm NR}^{(0,1)} &= -\tfrac{1}{2}mh_{00}^{(0,1)} - h_{0k}^{(0,1)}p^k - \frac{1}{4m}h_{00}^{(0,1)}p^2 \\ &\quad - \frac{1}{2m}h_{jk}^{(0,1)}p^jp^k. \end{aligned} \tag{62}$$

The leading-order perturbation $H_{\rm NR}^{(1,0)}$ due to Lorentz violation and independent of $h_{\mu\nu}$ is identical to the Minkowski-spacetime result given as Eq. (4) of Ref. [37].

The corrections of primary interest for our purposes lie at perturbative order O(1,1). The contribution from a_μ and e_μ can be written in terms of the effective coefficient $(a_{\rm eff})_\mu$ introduced in Eq. (16), and it takes the form

$$\begin{aligned} H_{{\rm NR},a_{\rm eff}}^{(1,1)} &= (\stackrel{\ast}{a}_{\rm eff})_0 + (\overline{a}_{\rm eff})_k h^{0k} - \frac{1}{m}(\overline{a}_{\rm eff})^j h_{jk}p^k \\ &\quad + \frac{1}{m}\left((\stackrel{\ast}{a}_{\rm eff})_j - \tfrac{1}{2}(\overline{a}_{\rm eff})_j h_{00}\right)p^j. \end{aligned} \tag{63}$$

The O(1,1) contribution from $c_{\mu\nu}$ can be written

$$\begin{aligned} H_{{\rm NR},c}^{(1,1)} &= -m\left(\stackrel{\ast}{c}_{00} + \tfrac{1}{2}\overline{c}_{00}h_{00} + 2\overline{c}_{(k0)}h^{0k}\right) \\ &\quad -2\left(\stackrel{\ast}{c}_{(j0)} + \overline{c}_{(jk)}h^{0k} - \overline{c}_{(0k)}h_j{}^k\right)p^j \\ &\quad -\frac{1}{m}\big(\tfrac{1}{2}\stackrel{\ast}{c}_{00}\eta_{jk} + \stackrel{\ast}{c}_{jk} + \tfrac{1}{4}\overline{c}_{00}h_{00}\eta_{jk} + \overline{c}_{(l0)}h^{0l}\eta_{jk} \\ &\qquad - \tfrac{1}{2}\overline{c}_{jk}h_{00} - \tfrac{1}{2}\overline{c}_{00}h_{jk} - 2\overline{c}_{(jl)}h^l{}_k\big)p^jp^k, \end{aligned} \tag{64}$$

while the O(1,1) contribution from Lorentz-violating effects on the metric fluctuation is

$$\begin{aligned} H_{{\rm NR},h}^{(1,1)} &= -\tfrac{1}{2}mh_{00}^{(1,1)} - h_{0k}^{(1,1)}p^k - \frac{1}{4m}h_{00}^{(1,1)}p^2 \\ &\quad - \frac{1}{2m}h_{jk}^{(1,1)}p^jp^k. \end{aligned} \tag{65}$$

The remaining contribution to $H_{\rm NR}$ is the O(0,2) contribution involving quadratic products of $h_{\mu\nu}$. This can be written as

$$\begin{aligned} H_{\rm NR}^{(0,2)} &= -\tfrac{1}{2}m\left(h_{0j}h^{0j} + \tfrac{1}{4}h_{00}h_{00}\right) + h_{0j}h^{jk}p_k \\ &\quad -\frac{1}{m}\big(\tfrac{1}{16}h_{00}h_{00}\eta_{jk} + \tfrac{1}{4}h_{0l}h^{0l}\eta_{jk} \\ &\qquad - \tfrac{1}{4}h_{00}h_{jk} - \tfrac{1}{2}h_{jl}h^l{}_k\big)p^jp^k. \end{aligned} \tag{66}$$

IV. CLASSICAL THEORY

For many analyses of Lorentz violation in matter-gravity couplings, a classical description suffices. This section considers the classical limit of the quantum theory discussed above, focusing on the limit involving the coefficient fields $(a_{\rm eff})_\mu$ and $c_{\mu\nu}$. A suitable classical relativistic action for a point particle is presented, and its application to modeling test and source bodies is described. The modified Einstein equation and the equation for the trajectory of a test body are obtained. We also discuss the treatment of the coefficient fluctuations $(\stackrel{\ast}{a}_{\rm eff})_\mu$, $\stackrel{\ast}{c}_{\mu\nu}$ and the procedure for determining the background gravitational field in the presence of Lorentz violation.

A. Particle action

The classical action $S_{\rm c}$ corresponding to the action S of Eq. (1) can be written as

$$S_{\rm c} = S_G + S_u + S'. \tag{67}$$

As before, S_G describes the gravitational dynamics, while S' contains the dynamics associated with the coefficient fields for Lorentz violation. The partial action S_u is the classical relativistic point-particle limit of the action S_ψ for the fermion sector. In this subsection, we discuss S_u and extend it to describe test and source bodies.

1. Point particle

At leading order in Lorentz violation, we find

$$S_u = \int d\lambda \left(-m\sqrt{-(g_{\mu\nu} + 2c_{\mu\nu})u^\mu u^\nu} - (a_{\rm eff})_\mu u^\mu \right). \tag{68}$$

In this expression, the particle path $x^\mu = x^\mu(\lambda)$ is parametrized by λ, and $u^\mu = dx^\mu/d\lambda$ is the four-velocity of the particle. As usual, a gauge choice for λ is required to fix the path-reparametrization invariance and to define the proper time of the particle on shell. We adopt here the conventional proper-time interval

$$d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}. \tag{69}$$

The leading-order form (68) of the classical action can be deduced in several ways. At the intuitive level, the term involving $(a_{\rm eff})_\mu$ has the same structure as the usual coupling of a classical relativistic particle to an electromagnetic 4-potential, and this is consistent with the coupling of a_μ in the field-theory action (3). Similarly, the coefficient $c_{\mu\nu}$ enters Eq. (68) as a shift in the metric, which is compatible with the way it appears in the field theory (3). In a different vein, the contributions from $c_{\mu\nu}$ to the relativistic particle action have previously been discussed in the context of the photon sector [22], where the appearance of $c_{\mu\nu}$ as a metric shift is related to the coordinate choices discussed in Sec. II C 2. The validity of the action (68) can also be verified by extracting the leading-order terms from the all-orders expression obtained by construction of the exact relativistic dispersion relation [44]. In the present context, we can demonstrate explicitly that the action (68) reproduces the corresponding terms in the nonrelativistic hamiltonian $H_{\rm NR}$ generated from the Foldy-Wouthuysen transformation as Eq. (61). This involves expanding the action (68) to the appropriate orders in velocities and Lorentz violation, extracting the conjugate 3-momentum, constructing the corresponding hamiltonian, and matching it to $H_{\rm NR}$ in Eq. (61). These methods all confirm that Eq. (68) is the correct leading-order form of the relativistic classical action.

The energy-momentum tensor $T_u^{\mu\nu}$ for the point particle can be derived from the action (68) by variation with respect to the metric, as usual. We obtain

$$T_u^{\mu\nu} = -\int d\tau \frac{m u^\mu u^\nu \delta^4(x - x'(\tau))}{\sqrt{g}\sqrt{1 - 2c_{\alpha\beta}u^\alpha u^\beta}}, \tag{70}$$

where the proper-time interval is given by Eq. (69). Note that no contributions from $(a_{\rm eff})_\mu$ appear in this expression. This follows from the adoption of $(a_{\rm eff})_\mu$ with lower index as the coefficient field, which implies that the contraction $(a_{\rm eff})_\mu u^\mu$ in Eq. (68) contains no metric. Working with $(a_{\rm eff})^\mu$ instead is possible but less convenient. It would produce a contribution to $T_u^{\mu\nu}$ along with corresponding changes in the contributions to the energy-momentum tensor $T'^{\mu\nu}$ associated with S', leading to the same physical results.

2. Test and source bodies

The experiments and observations considered in this work involve bodies B acting as test bodies T or as sources S. Many of these bodies consist of atoms or macroscopic matter rather than individual particles. It is therefore useful to extend the point-particle action (68) to an action $S_u^{\rm B}$ for a body B. This requires consideration of several issues.

One issue arises because the interactions involved in binding electrons, protons, and neutrons into atoms and macroscopic matter contribute additional Lorentz-violating effects. This issue appears also in the study of fermion-sector SME coefficients in the Minkowski-spacetime limit [45]. However, for the gravitational tests of interest here, it is reasonable to assume that these interaction effects are small compared to the propagation effects.

Another issue arises from the spacetime dependence of the coefficient fields $(a_{\rm eff})_\mu$ and $c_{\mu\nu}$, which implies Lorentz-violating effects may vary over the region filled by the body. Most of the test bodies we consider are small, so it is reasonable to approximate the coefficient fields as constant across the extent of the body. This corresponds to the usual approximation of constant metric fluctuation $h_{\mu\nu}$ across a test body. However, some of the source bodies we consider are comparatively large, so some variation of the coefficient fields over the source is plausible. This could produce Lorentz-violating effects of various types, including possible dependence on the mass moments of the body. In what follows, we suppose that the variation of the coefficient fields is sufficiently mild and smooth that these effects can be neglected for the bodies we consider. A more comprehensive treatment of this issue would be of potential interest but lies outside our present scope.

With the above assumptions and for most purposes in this work, a given body B can be modeled as a composite particle with constituents located at a single spacetime point and having the same 4-velocity, held together by binding energy. The body B can then be assigned an effective mass $m^{\rm B}$, expressed in terms of its constituent particles as

$$m^{\rm B} = \sum_w N^w m^w + m'^{\rm B}. \tag{71}$$

Here, w ranges over the particle species forming the body B. For example, w can be taken to include the electron e, the proton p, and the neutron n whenever B is an atom or made of ordinary macroscopic matter. The symbol N^w denotes the number of particles of type w in the body, and $m'^{\rm B}$ represents the contribution to the mass from the binding energy. In practice, the exact values of N^w are readily obtained for test bodies on the atomic or molecular scale, while estimating N^w for macroscopic test and source bodies in the laboratory is straightforward. When considering the Earth as the source body, we adopt the estimates $N^e_\oplus = N^p_\oplus \approx N^n_\oplus = 1.8\times 10^{51}$ based on recent studies of the bulk Earth composition [46]. The difference $N^n_\oplus - N^p_\oplus \simeq 10^{49}$ is primarily due to the iron core. The radial variation in neutron content is neglected in what follows, although it might be of interest in more detailed studies.

Similarly, the Lorentz-violating properties of B can be represented via effective coefficient fields $(a^{\rm B}_{\rm eff})_\mu$ and $(c^{\rm B})_{\mu\nu}$ for the body. These can be viewed as the sum of vacuum values and coefficient fluctuations,

$$(a^{\rm B}_{\rm eff})_\mu = (\overline{a}^{\rm B}_{\rm eff})_\mu + (\tilde{a}^{\rm B}_{\rm eff})_\mu, \quad (c^{\rm B})_{\mu\nu} = (\overline{c}^{\rm B})_{\mu\nu} + (\tilde{c}^{\rm B})_{\mu\nu}, \tag{72}$$

in parallel with the point-particle case. The form of the action (68) implies that the coefficient field $(a^{\rm B}_{\rm eff})_\mu$ takes the form

$$(a^{\rm B}_{\rm eff})_\mu = \sum_w N^w (a^w_{\rm eff})_\mu + (a'^{\rm B}_{\rm eff})_\mu, \tag{73}$$

where $(a'^{\rm B}_{\rm eff})_\mu$ is a possible coefficient field associated with the binding energy that contributes to $(a^{\rm B}_{\rm eff})_\mu$. Also, expanding the action S_u for small Lorentz violation shows that at leading order the body coefficient field $(c^{\rm B})_{\mu\nu}$ can be taken as

$$(c^{\rm B})_{\mu\nu} = \frac{1}{m^{\rm B}}\left(\sum_w N^w m^w (c^w)_{\mu\nu} + m'^{\rm B}(c'^{\rm B})_{\mu\nu}\right), \tag{74}$$

where $(c'^{\rm B})_{\mu\nu}$ is associated with the binding energy.

The two contributions $(a'^{\rm B}_{\rm eff})_\mu$ and $(c'^{\rm B})_{\mu\nu}$ describe Lorentz violation arising from the particles associated with the forces binding together the body B. These particles are primarily gravitons, gluons, or photons and are associated with boson fields, for which the CPT-violating terms are expected to be small or zero. In the minimal SME, no such terms exist for gravitons, while for photons and gluons they can reasonably be assumed to vanish [3]. Also, the relevant photon coefficient $(k_{AF})^\mu$ is constrained well below levels relevant for this work [2]. Possible CPT-violating contributions from other sea particles largely cancel due to particle-antiparticle pairings or are suppressed in loops involving weak interactions. We therefore approximate the contributions from $(a'^{\rm B}_{\rm eff})_\mu$ as negligible,

$$(a'^{\rm B}_{\rm eff})_\mu \simeq 0, \tag{75}$$

in this work. In contrast, all the force fields have CPT-even terms that can be expected to contribute to $(c'^{\rm B})_{\mu\nu}$, so the resulting size of $(c'^{\rm B})_{\mu\nu}$ may well be of the same order as $(c^w)_{\mu\nu}$ and cannot be neglected.

Given the above discussion, we conclude that the leading-order approximation to the classical action $S^{\rm B}_u$ for a body B can be written in the simple form

$$\begin{aligned} S^{\rm B}_u \;\approx\; \int d\lambda \Big(&- m^{\rm B}\sqrt{-(g_{\mu\nu} + 2(c^{\rm B})_{\mu\nu})u^\mu u^\nu} \\ &-(a^{\rm B}_{\rm eff})_\mu u^\mu\Big), \end{aligned} \tag{76}$$

where $m^{\rm B}$, $(a^{\rm B}_{\rm eff})_\mu$, and $(c^{\rm B})_{\mu\nu}$ are given by Eqs. (71), (73), and (74), respectively. In this expression, u^μ is the 4-velocity of the body B, which follows a world line parametrized by λ. This form is convenient for calculational purposes. Note, however, that the derivation establishes validity of this form of the action only at leading order in $(c^{\rm B})_{\mu\nu}$.

The model action (76) for a body B suffices for most situations of interest in this work. In a few cases where the body acts as a gravitational source S, it is also useful to incorporate dominant effects arising from its rotation. For this purpose, we treat S as rigid at leading order and assume that the distribution of electrons, protons, and neutrons is approximately uniform throughout it. For the bodies we consider, this assumption is good to within an order of magnitude. The density ρ of S can be taken as the mass per unit volume and approximated as uniform. For large source bodies such as the Earth, some results could in principle also depend on spherical moments of inertia [7], but these effects are neglected here. The angular velocity $\vec{\omega}$ of rotation is defined in the frame at rest relative to S with origin at the center of mass, which can be identified with the location of S.

B. Equations of motion

The primary experimental observables arising from the classical theory involve the relative motion of particles. To investigate the motion of a test particle in the presence of gravitational sources, the modified Einstein equation must be solved for the background metric and the equation for the particle trajectory must be found. In this subsection, we derive the equations of motion from the action (67) in terms of the metric fluctuation $h_{\mu\nu}$ and the coefficient fluctuations $(\tilde{a}^{\rm B}_{\rm eff})_\mu$, $(\tilde{c}^{\rm B})_{\mu\nu}$. The issue of expressing these fluctuations in terms of the vacuum values $(\overline{a}^{\rm B}_{\rm eff})_\mu$, $(\overline{c}^{\rm B})_{\mu\nu}$ for a given distribution of matter is addressed in the following subsection, Sec. IV C. We conclude the present subsection with comments about the implications of Lorentz violation for the equivalence principle.

1. Modified Einstein equation

Varying the action (67) in Riemann spacetime with respect to the metric yields the modified Einstein equation

$$G^{\mu\nu} = T_{\rm G}^{\mu\nu} + \kappa T_u^{\mu\nu} + \kappa T'^{\mu\nu}, \tag{77}$$

where $G^{\mu\nu}$ is the Einstein tensor and the terms on the right-hand side form the energy-momentum tensor. The contribution $T_{\rm G}^{\mu\nu}$ arises from Lorentz violation in the pure-gravity sector. The energy-momentum tensor $T_u^{\mu\nu}$ for the matter is given in Eq. (70). The remaining energy-momentum contribution $T'^{\mu\nu}$ arises from the dynamics of the coefficient fields for Lorentz violation and is determined by S'.

Taking the covariant divergence of Eq. (77) and using the Bianchi identities shows that the geometry requires the total energy-momentum tensor to be locally conserved. The theory can be consistent only if this result is compatible with the explicit form of the energy-momentum tensor. This requires careful accounting of contributions from the massless NG modes arising from the spontaneous Lorentz breaking [3]. In the general case, these modes are contained in the fluctuations $\tilde{t}_{\lambda\mu\nu\ldots}$.

For the pure-gravity sector, the relevant analysis is given in Refs. [3, 7] and can be subsumed as needed in the present context. For the matter sector, the NG modes produce no relevant contribution to the energy-momentum tensor $T_u^{\mu\nu}$ at the post-newtonian order appropriate for the tests considered here. The key point is that the leading Lorentz-violating effects of coefficient fluctuations $\tilde{t}_{\lambda\mu\nu\ldots}$ arise at PNO(2) or beyond, as shown in Sec. II D. Since these fluctuations are accompanied by an additional factor of G_N in the modified Einstein equation (77), they affect the metric only at PNO(4) or beyond. However, for the tests considered below it suffices to work at PNO(3) for Lorentz-violating terms, so the coefficient fluctuations $(\tilde{a}^{\rm B}_{\rm eff})_\mu$ and $(\tilde{c}^{\rm B})_{\mu\nu}$ appearing in $T_u^{\mu\nu}$ can be neglected in Eq. (77).

In contrast, the contributions to $T'^{\mu\nu}$ arising from the dynamics of the coefficient fluctuations $\tilde{t}_{\lambda\mu\nu\ldots}$ are of potential relevance in solving the modified Einstein equation for the metric. The specific effects associated with the coefficients $(\tilde{a}^{\rm B}_{\rm eff})_\mu$ and $(\tilde{c}^{\rm B})_{\mu\nu}$ are derived in Sec. IV C.

2. Particle trajectory

The equation of motion for a classical test particle T is obtained by varying the action (67) with respect to the particle position 4-vector x^μ. In the absence of Lorentz violation, this is the geodesic equation. However, in the presence of Lorentz violation, the trajectories of test particles T no longer match the geodesics of the spacetime.

Expanding to O(1,1), the equation of motion can be written as

$$\begin{aligned}
\ddot{x}^\mu &= -{\Gamma_{(0,1)}}^{\mu}{}_{\alpha\beta} u^\alpha u^\beta \\
&\quad -{\Gamma_{(1,1)}}^{\mu}{}_{\alpha\beta} u^\alpha u^\beta + 2\eta^{\mu\gamma}(\overline{c}^{\rm T})_{(\gamma\delta)} {\Gamma_{(0,1)}}^{\delta}{}_{\alpha\beta} u^\alpha u^\beta \\
&\quad +2(\overline{c}^{\rm T})_{(\alpha\beta)} {\Gamma_{(0,1)}}^{\alpha}{}_{\gamma\delta} u^\beta u^\gamma u^\delta u^\mu + \partial^\mu (\tilde{c}^{\rm T})_{\alpha\beta} u^\alpha u^\beta \\
&\quad -2\eta^{\mu\gamma}\partial_\alpha (\tilde{c}^{\rm T})_{(\gamma\beta)} u^\alpha u^\beta - \partial_\gamma (\tilde{c}^{\rm T})_{(\alpha\beta)} u^\alpha u^\beta u^\gamma u^\mu \\
&\quad -\frac{1}{m^{\rm T}}[\partial^\mu (\tilde{a}^{\rm T}_{\rm eff})_\alpha - \eta^{\mu\beta}\partial_\alpha (\tilde{a}^{\rm T}_{\rm eff})_\beta] u^\alpha,
\end{aligned} \tag{78}$$

where each dot on x^μ represents a derivative using the proper-time interval (69). The superscript T denotes quantities associated with the test particle. The first term on the right-hand side is the usual geodesic contribution, where ${\Gamma_{(0,1)}}^{\mu}{}_{\alpha\beta}$ is the linearized Christoffel symbol. A Christoffel symbol with subscript (1,1) also appears in Eq. (78). It is defined as the linearized Christoffel symbol with $h_{\mu\nu}$ replaced by $h_{\mu\nu}^{(1,1)}$. This introduces matter-sector coefficients associated with the gravitational source, along with any gravity-sector coefficients that may be included in the analysis.

Once the forms of $h_{\mu\nu}$, $(\tilde{a}_{\rm eff})_\mu$, and $\tilde{c}_{\mu\nu}$ are established, Eq. (78) can be used to determine to O(1,1) the motion of a classical test particle in a curved but asymptotically flat spacetime with nonzero coefficients for Lorentz violation a_μ, e_μ, and $c_{\mu\nu}$. Obtaining expressions for $h_{\mu\nu}$, $(\tilde{a}_{\rm eff})_\mu$, and $\tilde{c}_{\mu\nu}$ is the subject of Sec. IV C.

Although unnecessary for the present work, we can comment in passing about the effects of nongravitational interactions on the particle trajectory. Any such interactions can be viewed as introducing an additional contribution α^μ to the right-hand side of Eq. (78). Using the perturbation scheme of Sec. II D, this additional acceleration α^μ can be expanded as a sum over terms $\alpha^\mu_{(m,n)}$, one at each perturbative order O(m,n). Notice that, although the interaction itself is nongravitational, contributions to $\alpha^\mu_{(m,n)}$ with $n \neq 0$ can be induced from gravitational couplings in the interaction sector. Similarly, Lorentz-violating contributions to $\alpha^\mu_{(m,n)}$ can originate from coefficients for Lorentz violation in the interaction sector. If we also expand $\ddot{x}^\mu$ as

$$\ddot{x}^\mu = \ddot{x}^\mu_{(0,0)} + \ddot{x}^\mu_{(0,1)} + \ddot{x}^\mu_{(1,0)} + \ddot{x}^\mu_{(1,1)} + \ldots, \tag{79}$$

then we obtain the following additional terms for the particle 4-acceleration $\ddot{x}^\mu$:

$$\begin{aligned}
\ddot{x}^\mu_{(0,0)} &\supset \alpha^\mu_{(0,0)}, \\
\ddot{x}^\mu_{(0,1)} &\supset \alpha^\mu_{(0,1)}, \\
\ddot{x}^\mu_{(1,0)} &\supset \alpha^\mu_{(1,0)} - 2\eta^{\mu\alpha}(\overline{c}^{\rm T})_{(\alpha\beta)}\alpha^\beta_{(0,0)} \\
&\quad -2(\overline{c}^{\rm T})_{(\alpha\beta)}\alpha^\alpha_{(0,0)} u^\beta u^\mu, \\
\ddot{x}^\mu_{(1,1)} &\supset \alpha^\mu_{(1,1)} - 2\eta^{\mu\alpha}(\overline{c}^{\rm T})_{(\alpha\beta)}\alpha^\beta_{(0,1)} \\
&\quad -2(\overline{c}^{\rm T})_{(\alpha\beta)}\alpha^\alpha_{(0,1)} u^\beta u^\mu + 2h^{\mu\alpha}(\overline{c}^{\rm T})_{(\alpha\beta)}\alpha^\beta_{(0,0)} \\
&\quad -2\eta^{\mu\alpha}(\tilde{c}^{\rm T})_{(\alpha\beta)}\alpha^\beta_{(0,0)}.
\end{aligned} \tag{80}$$

The trajectory at O(1,n) is affected both directly by $\alpha^\mu_{(1,n)}$ and indirectly by combinations of $\alpha^\mu_{(0,n)}$ with the coefficients for Lorentz violation. The origin of the indirect terms can be traced to the additional factor of $2c_{\mu\nu}$ in the action (68) relative to the conventional proper-time interval (69).

3. *Implications for the equivalence principle*

The deviations from geodesic motion implied by Eq. (78) can be species dependent because the couplings to the coefficient fields $(a_{\rm eff})_\mu$ and $c_{\mu\nu}$ can vary with particle flavor. This leads to apparent violations of the weak equivalence principle (WEP), which stipulates that the motion of uncharged test particles is independent of internal structure or composition [8].

One implication of this observation is that experiments designed to test the WEP are also sensitive to $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$. Since all the WEP violations implied by Eq. (78) are accompanied by effects associated with the breaking of rotation and boost symmetries, the experimental signatures associated with $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ typically differ from those in other scenarios for violations of the WEP. The latter portion of this work discusses in some detail the role that experiments testing the WEP can play in searches for Lorentz violation.

The flavor dependence of the coefficient couplings leads to the philosophical question of whether spontaneous Lorentz violation in the matter sector violates the WEP or merely mimics violations of the WEP. The issue hinges on the interpretation of the term 'uncharged test particle.' In models with spontaneous Lorentz violation in the matter sector, the NG modes couple to test particles and so mediate an interaction. This interaction can be identified with Einstein-Maxwell electrodynamics [14], GR gravity [27], an effect on torsion [14], or a new force [11, 29, 30]. If the term 'uncharged' is taken in the restrictive sense to mean that the test particle is unaffected by standard forces such as electrodynamics, then the trajectory deviations of 'uncharged' test particles caused by nonzero $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ coefficients represent violations of the WEP. If instead the term 'uncharged' indicates the test particle cannot have nongravitational couplings of any kind, then no violations of the WEP occur. However, in this latter case 'uncharged' test particles may be nonexistent in the matter sector of the SME, where generically all particles experience nonzero $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ coefficients. We emphasize that the above discussion is a matter of philosophical classification only, without impact on the practical issue of using tests of the WEP to search for Lorentz violation via the deviations from geodesic motion described by Eq. (78).

The WEP is subsumed in certain other equivalence principles, such as the Einstein equivalence principle or the strong equivalence principle. These incorporate also aspects of local Lorentz invariance and local position invariance. Since nonzero coefficients $(\overline{a}_{\rm eff})_\mu$, $\overline{c}_{\mu\nu}$ correspond directly to local Lorentz violation, and since Lorentz violation can be position dependent, the deviations from geodesic motion described by Eq. (78) can represent violations of these broader equivalence principles arising in more than one way. Several related philosophical issues remain open, including classifying violations of various equivalence principles according to properties of the coefficients for Lorentz violation and identifying implications for relations such as the Schiff conjecture [8, 47]. We note also in passing that comments analogous to those above bear on the philosophical issue of whether theories with matter-sector couplings to spontaneous Lorentz violation constitute metric theories of gravity.

C. Coefficient and metric fluctuations

To solve the equation of motion (78) for the trajectory of a test particle, explicit expressions for the metric fluctuation $h_{\mu\nu}$ and the coefficient fluctuations $(\tilde{a}_{\rm eff})_\mu$, $\tilde{c}_{\mu\nu}$ are required. Within a specific model with known action S' for the coefficient fields, these expressions can be obtained by direct calculation. An illustration of this is provided in Sec. V. However, in the interest of generality, it is useful to establish results valid for a large class of models. In this subsection, we outline a procedure to obtain expressions for $h_{\mu\nu}$ and for the generic coefficient fluctuations $\tilde{t}_{\lambda\mu\nu\ldots}$ when S' is largely unknown, and we obtain explicit results for $h_{\mu\nu}$, $(\tilde{a}_{\rm eff})_\mu$, and $\tilde{c}_{\mu\nu}$ applicable to the equation of motion (78). These results are used in later sections of this work in establishing experimental signatures for Lorentz violation.

1. *Methodology*

Consider first the metric fluctuation $h_{\mu\nu}$. In the perturbation scheme of Sec. II D, the expansion of $h_{\mu\nu}$ takes the form (26). To determine the test-particle trajectory at order O(1,1) via Eq. (78), it is necessary to obtain explicit expressions for $h^{(0,1)}_{\mu\nu}$ and $h^{(1,1)}_{\mu\nu}$.

The Lorentz-invariant contribution $h^{(0,1)}_{\mu\nu}$ can be obtained in the usual way as the leading-order solution of the Einstein equation, taking the Lorentz-invariant part of the energy-momentum tensor as the source. To PNO(3), the standard solution can be written in harmonic coordinates as

$$h^{(0,1)}_{00} \ = \ 2U, \quad h^{(0,1)}_{0j} = -4V^j, \quad h^{(0,1)}_{jk} = 2U\delta^{jk}, \tag{81}$$

where U and V^j are the usual post-newtonian potentials defined as

$$\begin{aligned} U \ &= \ G_N \int d^3x' \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|}, \\ V^j \ &= \ G_N \int d^3x' \frac{\rho(\vec{x}', t) v^j(\vec{x}', t)}{|\vec{x} - \vec{x}'|}. \end{aligned} \tag{82}$$

In these expressions, the density $\rho(\vec{x}',t)$ and the 3-velocity $v^j(\vec{x}',t)$ are properties of the source in the chosen asymptotic inertial frame. As described in Sec. IV A 2, it suffices in this work to use the approximation (76) for the source-body action S_u^B, so the energy-momentum tensor takes the generic form (70).

The Lorentz-violating component $h_{\mu\nu}^{(1,1)}$ can conveniently be viewed as a sum over individual contributions arising from each coefficient field for Lorentz violation,

$$h_{\mu\nu}^{(1,1)} = (h_a^{(1,1)})_{\mu\nu} + (h_b^{(1,1)})_{\mu\nu} + ... + (h_H^{(1,1)})_{\mu\nu}. \quad (83)$$

This means each coefficient can be treated in turn. However, the procedure for determining a particular contribution can take different paths depending on the type of gravitational coupling of the coefficient field.

A simple case arises for any coefficient field $t_{\lambda\mu\nu...}$ that is minimally coupled to gravity. Then, only the vacuum values $\overline{t}_{\lambda\mu\nu...}$ in the expansion (11) contribute to the energy-momentum tensor $T_u^{\mu\nu}$ of the source at the relevant order. The key point is that the solution for $h_{\mu\nu}^{(1,1)}$ at PNO(3) arises from the combination $\kappa T_u^{\mu\nu}$, which itself already lies at PNO(2). However, as discussed in Sec. II D, the coefficient fluctuations of interest here are $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$ and also lie at PNO(2). These fluctuations therefore cannot contribute to $h_{\mu\nu}^{(1,1)}$ below PNO(4). As a result, $h_{\mu\nu}^{(1,1)}$ can be found directly by solving the modified Einstein equation with attention limited to the vacuum values $\overline{t}_{\lambda\mu\nu...}$.

We note in passing that this procedure is consistent with the no-go result for explicit Lorentz violation in gravity [3] even though $\overline{t}_{\lambda\mu\nu...}$ can be interpreted as a coefficient for explicit breaking. This is because we are working to O(1,1) and PNO(3), for which $D_\alpha \overline{t}_{\lambda\mu\nu...} \sim \mathrm{O}(h\overline{t}) \sim \mathrm{O}(1,1)$. As a result, the covariant derivative of $\kappa T_u^{\mu\nu}$ is compatible with the Bianchi identities $D_\mu G^{\mu\nu} = 0$ at this perturbative order. In effect, the comparatively low perturbative order implies that a constant vacuum value remains consistent with the geometry of spontaneous Lorentz breaking. It is also noteworthy that an independent contribution from $T'^{\mu\nu}$ may exist that satisfies local conservation and hence is compatible with the Bianchi identities to the relevant perturbative order. This would also represent a consistent theory, albeit a different one. The two theories involving the coefficient field $(a_{\mathrm{eff}})_\mu$ with minimal and with nonminimal gravitational couplings provide an illustration of this, as is discussed in the next subsection.

In the simple case with minimal coupling, once $h_{\mu\nu}^{(1,1)}$ has been found, it remains only to determine the direct contributions to the equation of motion (78) arising from the fluctuations $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$. For this purpose, we can apply the requirement that the system of the source S and the test body T conserves the total 4-momentum P_μ. For a two-body system, this implies the force law must be antisymmetric upon exchange of S and T. Otherwise, the forces on each body due to the other would violate Newton's third law, and the system would self-accelerate.

At PNO(2) and in the absence of Lorentz violation, the relevant force between S and T can be directly identified as $m\ddot{x}^j$. At higher order and in the presence of Lorentz violation, it is simpler to impose conservation of the total 4-momentum, $dP_\mu/dt = 0$. In principle, P_μ can be found by adding the conjugate momenta of S and T obtained from the two-body action. In practice, for the perturbative order to which we work, it suffices to obtain the conjugate momentum for T alone and require antisymmetry of its time derivative under the exchange of S and T. The constraints fixing the fluctuations $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$ arise from the $\mu = j$ components of $dP_\mu/dt = 0$, all at PNO(3) except for terms at PNO(4) involving the velocities of both S and T. We remark in passing that extending this treatment to higher perturbative orders requires also incorporating back-reaction effects on the metric, including gravitational radiation.

The above procedure holds for coefficient fields that are minimally coupled to gravity. If nonminimal curvature couplings also occur in S', then additional terms involving the coefficient fields $t_{\lambda\mu\nu...}$ can appear in the energy-momentum tensor and hence can affect the modified Einstein equation. The curvature couplings intertwine the kinetic contributions from $h_{\mu\nu}$ and $\tilde{t}_{\lambda\mu\nu...}$, so $\tilde{t}_{\lambda\mu\nu...}$ can contribute to the solution for $h_{\mu\nu}^{(1,1)}$ at PNO(3). To proceed without specifying S', we therefore need additional information about $\tilde{t}_{\lambda\mu\nu...}$.

In the present work, the necessary information can be extracted from the general structure of the equation of motion for $t_{\lambda\mu\nu...}$ and the symmetries of the theory. When linearized, this equation of motion can be written as the sum of a differential operator acting on $\tilde{t}_{\lambda\mu\nu...}$ and a source term at most linear in $h_{\mu\nu}$. The differential operator can involve arbitrary powers of $\overline{t}_{\lambda\mu\nu...}$ but is independent of $h_{\mu\nu}$. The contributions $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$ of interest are at O(1,1) and hence are linear in both $\overline{t}_{\lambda\mu\nu...}$ and $h_{\mu\nu}$. At this order, the solution for $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$ can therefore be written as a sum of terms, each containing up to one power of $h_{\mu\nu}$ along with some number n of powers of $\overline{t}_{\lambda\mu\nu...}$ in the numerator and $n-1$ powers of $\overline{t}_{\lambda\mu\nu...}$ in the denominator. This expansion of $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$ in terms of $\overline{t}_{\lambda\mu\nu...}$ and $h_{\mu\nu}$ is constrained by two requirements. One arises from the restriction of $\tilde{t}_{\lambda\mu\nu...}$ to NG modes, which must maintain the extremum of the action. The solution for $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$ must therefore obey the NG conditions at O(1,1). The second is the requirement that $\tilde{t}_{\lambda\mu\nu...}$ must transform as expected under diffeomorphisms, as a consequence of the spontaneous nature of the symmetry breaking and the requirement of observer general coordinate invariance [48]. It turns out that these two restrictions suffice to express $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$ in terms of $\overline{t}_{\lambda\mu\nu...}$ and $h_{\mu\nu}$ in the cases of interest here.

Once the expression for $\tilde{t}_{\lambda\mu\nu...}^{(1,1)}$ has been found, $h_{\mu\nu}$ can be obtained by combining information from the modified Einstein equation and the trajectory equation. The modified Einstein equation yields directly the piece of $h_{\mu\nu}$ arising from $\overline{t}_{\lambda\mu\nu\ldots}$ in the energy-momentum tensor. Inserting this result and the expression for $\tilde{t}^{(1,1)}_{\lambda\mu\nu\ldots}$ in the trajectory equation and imposing conservation of the total 4-momentum of the source and test body as before determines the missing piece of $h_{\mu\nu}$ arising from $\tilde{t}^{(1,1)}_{\lambda\mu\nu\ldots}$.

In both minimal and nonminimal cases, the net result of the above procedure is a form of the trajectory equation in which $h_{\mu\nu}$ and $\tilde{t}_{\lambda\mu\nu\ldots}$ can be replaced with specified gravitational potentials and the vacuum values $\overline{t}_{\lambda\mu\nu\ldots}$. The solution of the trajectory equation can then proceed. In the next two subsections, we apply these methods to obtain the relevant contributions to the fluctuations from the coefficient fields $(a_{\rm eff})_\mu$ and $c_{\mu\nu}$.

2. Fluctuations and $c_{\mu\nu}$

The treatment of the coefficient field $c_{\mu\nu}$ provides an example involving the comparatively simple case of minimal coupling to gravity. Although nonminimal curvature couplings to $c_{\mu\nu}$ could be considered, these are of lesser interest in the context of searches for Lorentz violation because direct signals from $\overline{c}_{\mu\nu}$ already appear for minimal coupling. We therefore neglect nonminimal couplings to $c_{\mu\nu}$ here. In this subsection, the relevant contributions to the metric fluctuation $(h_c^{(1,1)})_{\mu\nu}$ are obtained to third post-newtonian order, and effects from $\tilde{c}_{\mu\nu}$ at O(1,1) are considered.

Following the procedure outlined in the previous subsection, we begin with the modified Einstein equation (77). The relevant energy-momentum tensor for the source S is given by the expression (70) expanded to leading order in Lorentz violation and with $c_{\mu\nu}$ replaced by $(c^{\rm S})_{\mu\nu}$. Solving for the metric fluctuation to PNO(3), we obtain in harmonic gauge

$$\begin{aligned}
(h_c^{(1,1)})_{00} &= 2(c^{\rm S})_{00}U + 4(c^{\rm S})_{(j0)}V^j, \\
(h_c^{(1,1)})_{0j} &= -4(c^{\rm S})_{00}V^j, \\
(h_c^{(1,1)})_{jk} &= 2(c^{\rm S})_{00}U\delta^{jk}.
\end{aligned} \tag{84}$$

A consistent expansion to PNO(3) requires only PNO(1) terms in $(h_c^{(1,1)})_{0j}$ and none in $(h_c^{(1,1)})_{jk}$, but we display PNO(3) terms in $(h_c^{(1,1)})_{0j}$ and PNO(2) terms in $(h_c^{(1,1)})_{jk}$ because they are useful in part of the analysis to follow.

The next step is to examine the contributions to the equation of motion (78) from $\tilde{c}_{\mu\nu}$ at O(1,1). The conjugate momentum can be extracted from the action (68) with $(a_{\rm eff})_\mu$ set to zero. Conservation of the total 4-momentum P_μ of the system is ensured by the requirement that its time derivative be antisymmetric under the exchange of the source S and the test body T. We find that P_μ is conserved to PNO(3) without contributions from $\tilde{c}_{\mu\nu}$. This establishes the PNO(3) result

$$\partial_\lambda \tilde{c}^{(1,1)}_{\mu\nu} = 0, \tag{85}$$

showing that the NG modes associated with $\tilde{c}_{\mu\nu}$ play no role at this perturbative order.

The results (84) and (85) complete the determination of the trajectory equation for the coefficient field $c_{\mu\nu}$. For a given source S, the potentials U and V^j can be calculated explicitly. The effects of $(c^{\rm S})_{\mu\nu}$ and $(c^{\rm T})_{\mu\nu}$ on the trajectory of the test body T can therefore be investigated in various regimes of experimental interest. This line of reasoning is pursued beginning in Sec. VI.

3. Fluctuations and $(a_{\rm eff})_\mu$

For the coefficient field $(a_{\rm eff})_\mu$, the case of minimal coupling to gravity is of lesser interest. The modified Einstein equation is unaffected because $(a_{\rm eff})_\mu$ is absent from the energy-momentum tensor (70). Also, only $(\tilde{a}_{\rm eff})_\mu$ enters the trajectory equation. Since it is indistinguishable from an electromagnetic field, it provides no relevant contributions at O(1,1). We therefore expand the treatment to the case of nonminimal curvature couplings, for which $(\overline{a}_{\rm eff})_\mu$ becomes measurable [11]. In effect, the fluctuations $(\tilde{a}_{\rm eff})_\mu$ become observable by virtue of their nonminimal gravitational couplings.

Following the procedure of Sec. IV C 1, the first step is to obtain an expression for the fluctuations $(\tilde{a}_{\rm eff})^{(1,1)}_\mu$ originating from the source S using the NG condition and the requirement of diffeomorphism covariance. At O(1,1), the NG condition can be written as

$$(\tilde{a}_{\rm eff})^{(1,1)}_\mu(\overline{a}^{\rm B}_{\rm eff})^\mu = \tfrac{1}{2}(\overline{a}^{\rm B}_{\rm eff})_\mu h^{\mu\nu}(\overline{a}^{\rm B}_{\rm eff})_\nu. \tag{86}$$

We find that the contributions to $(\tilde{a}_{\rm eff})^{(1,1)}_\mu$ consistent with this equation and with diffeomorphism covariance take the form

$$(\tilde{a}_{\rm eff})^{(1,1)}_\mu = \tfrac{1}{2}\alpha h_{\mu\nu}(\overline{a}^{\rm B}_{\rm eff})^\nu - \tfrac{1}{4}\alpha(\overline{a}^{\rm B}_{\rm eff})_\mu h^\nu{}_\nu + \partial_\mu\Psi \tag{87}$$

in harmonic coordinates. Here, the constant α is calculable but varies with the specifics of the theory, typically being determined in terms of the coupling constants that control the nonminimal couplings. The function Ψ contains effects proportional to $h_{\mu\nu}$ and $(\overline{a}^{\rm S}_{\rm eff})_\mu$ that are unphysical by virtue of the discussion in Sec. II C 1, so it is disregarded in what follows.

At this stage, the result (87) can be combined with the modified Einstein equation and the trajectory equation to determine the contributions to $h^{(1,1)}_{\mu\nu}$ proportional to $(\overline{a}^{\rm S}_{\rm eff})_\mu$. Working at PNO(3), we find these contributions can be written in harmonic gauge as

$$\begin{aligned}
(h_a^{(1,1)})_{00} &= \frac{2}{m}[2\alpha(\overline{a}^{\rm S}_{\rm eff})_0U + \alpha(\overline{a}^{\rm S}_{\rm eff})_jV^j - \alpha(\overline{a}^{\rm S}_{\rm eff})_jW^j], \\
(h_a^{(1,1)})_{0j} &= \frac{1}{m}[\alpha(\overline{a}^{\rm S}_{\rm eff})_jU + \alpha(\overline{a}^{\rm S}_{\rm eff})_kU^{jk} \\
&\qquad -\alpha(\overline{a}^{\rm S}_{\rm eff})_0V^j - \alpha(\overline{a}^{\rm S}_{\rm eff})_0W^j], \\
(h_a^{(1,1)})_{jk} &= \frac{2}{m}[-\alpha(\overline{a}^{\rm S}_{\rm eff})_0U\delta^{jk} + \alpha(\overline{a}^{\rm S}_{\rm eff})_0U^{jk}].
\end{aligned} \tag{88}$$

Paralleling the case of the coefficient field $c_{\mu\nu}$, we have kept here PNO(3) terms in $(h_a^{(1,1)})_{0j}$ and PNO(2) terms in $(h_a^{(1,1)})_{jk}$ as a convenience for the analysis to follow. In Eq. (88), U and V^j are the post-newtonian potentials defined in Eq. (82). Additional potentials U^{jk} and W^j also appear, defined by

$$\begin{aligned} U^{jk} &= G_N \int d^3x' \frac{\rho(\vec{x}',t)(\vec{x}-\vec{x}')^j(\vec{x}-\vec{x}')^k}{|\vec{x}-\vec{x}'|^3}, \\ W^j &= G_N \int d^3x' \frac{\rho(\vec{x}',t)v_k(\vec{x}',t)(\vec{x}-\vec{x}')^j(\vec{x}-\vec{x}')^k}{|\vec{x}-\vec{x}'|^3}. \end{aligned} \tag{89}$$

The results (87) and (88) fix the form of the contributions involving the coefficient field $(a_{\text{eff}})_\mu$ to the equation of motion (78). Modifications of the trajectory of a test body T arising from nonzero values of the coefficients $(\overline{a}^{\text{S}}_{\text{eff}})_\mu$ and $(\overline{a}^{\text{T}}_{\text{eff}})_\mu$ can therefore be studied at third post-newtonian order. The resulting experimental signals are discussed starting in Sec. VI.

V. EXAMPLE: BUMBLEBEE MODEL

In this section, we examine a specific model and demonstrate how it fits into the general theory developed above. This discussion is included solely for illustrative purposes and is inessential to the development of the paper. In particular, the analyses of experimental signals in subsequent sections are independent of this specific model, so the reader can proceed directly to Sec. VI if desired.

A. Bumblebee model

Bumblebee models are theories in which spontaneous Lorentz violation is induced by a potential $V(B^\mu)$ for a vector field B^μ [49]. As an illustration of the general theoretical treatment of Sec. IV, we consider here a specific and comparatively simple bumblebee model and study its matter-gravity couplings. A discussion of generic models of vacuum-valued vectors coupled to gravity including references to the substantial early literature can be found in Sec. III A of Ref. [15], while some more recent papers are listed in Ref. [50]. Discussions of various stability issues with these models are given in Ref. [51].

The action S_B for the specific bumblebee model of interest here can be written as

$$\begin{aligned} S_B &= S_G + S_{Bu} + S'_B \\ &= \int d^4x\ e\mathcal{L}_G + \int d\tau\ \mathcal{L}_{Bu} + \int d^4x\ e\mathcal{L}'_B. \end{aligned} \tag{90}$$

The form of this action corresponds to that of the general action (67). The term S_G is the usual Einstein-Hilbert action (2), with cosmological constant chosen as $\Lambda = 0$ for this illustrative case. The term S_{Bu} represents the matter-bumblebee coupling, while S'_B contains the bumblebee dynamics, including the potential V triggering spontaneous Lorentz violation.

For the classical lagrangian $\mathcal{L}_{Bu}$ describing the matter-bumblebee coupling, we choose the expression

$$\mathcal{L}_{Bu} = -m\sqrt{-(g_{\mu\nu} + 2\zeta_2 B_\mu B_\nu)u^\mu u^\nu} + \zeta_1 B_\mu u^\mu. \tag{91}$$

Here, ζ_1 and ζ_2 are coupling constants that can vary with the particle species. Where needed in what follows, we distinguish the coupling constants for a source body S and a test particle T by superscripts: ζ_1^{S}, ζ_1^{T}, ζ_2^{S}, ζ_2^{T}. Note that the lagrangian (91) could be viewed as the point-particle limit of a quantum field theory, in parallel with the derivation for the general theory (68).

For the Lagrange density $e\mathcal{L}'_B$ determining the dynamics of the bumblebee field, we take

$$e\mathcal{L}'_B = -\tfrac{1}{4}eB^{\mu\nu}B_{\mu\nu} - eV + \sigma_1 eB^\mu B^\nu R_{\mu\nu}, \tag{92}$$

where the field strength is $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The coupling constant σ_1 is sometimes written $\sigma_1 = \xi/2\kappa$ in the literature [3, 15]. The potential V has the form

$$V = V(B^\mu B_\mu \pm b^2), \tag{93}$$

where b^2 is a real number. Where a definite form is needed in the calculations to follow, we adopt for simplicity the smooth quadratic potential

$$V = \lambda(B^\mu B_\mu \pm b^2)^2/2. \tag{94}$$

In any event, the potential is assumed to induce a nonzero vacuum expectation value for the bumblebee field, which we denote by $b_\mu \equiv \langle B_\mu \rangle$ following standard usage, where $b^\mu b_\mu = \mp b^2$. Denoting the bumblebee fluctuation about the vacuum value by $\tilde{B}_\mu$, we can expand

$$B_\mu = b_\mu + \tilde{B}_\mu \tag{95}$$

in parallel with Eq. (11).

A match can be made between the bumblebee action (90) and the general action (67) by identifying the various coefficient fields for Lorentz violation with specific combinations of the bumblebee field. The term $\mathcal{L}_{Bu}$ corresponds to nonzero coefficient fields $(a_{\text{eff}})_\mu$ and $c_{\mu\nu}$, given by

$$\begin{aligned} (a_{\text{eff}})_\mu &= \zeta_1 B_\mu \\ c_{\mu\nu} &= \zeta_2(B_\mu B_\nu - \tfrac{1}{4}g_{\mu\nu}B_\alpha B^\alpha). \end{aligned} \tag{96}$$

It is also necessary to introduce an additional scalar field k, defined as

$$k = \tfrac{1}{2}\zeta_2 B_\alpha B^\alpha, \tag{97}$$

which normally can be disregarded in the SME context because it is Lorentz invariant. In the presence of k, the general action (68) is slightly modified, with lagrangian now given by the expression

$$\mathcal{L}_u \;=\; -m\sqrt{-(g_{\mu\nu} + kg_{\mu\nu} + 2c_{\mu\nu})u^\mu u^\nu} + (a_{\mathrm{eff}})_\mu u^\mu. \tag{98}$$

The term $\mathcal{L}'_B$ yields nonzero coefficients fields $s^{\mu\nu}$ and u in the pure-gravity sector, given by [7]

$$\begin{aligned} s^{\mu\nu} &= \xi B^\mu B^\nu - \tfrac{1}{4}\xi B_\alpha B^\alpha g^{\mu\nu}, \\ u &= \tfrac{1}{4}\xi B_\alpha B^\alpha. \end{aligned} \tag{99}$$

Note that in this model only a single field B_μ underlies all the coefficient fields $(a_{\mathrm{eff}})_\mu$, $c_{\mu\nu}$, k, $s_{\mu\nu}$, and u. It follows that searches for Lorentz violation that are sensitive to any one of these coefficient fields could provide information constraining the others, at least in part. This special feature of the bumblebee model may not extend to models with more complicated field structure. Note also that the coefficient field k in the matter sector is a species-dependent analogue of the coefficient field u in the gravity sector. A nonzero value of k can introduce apparent WEP violations. Since these are Lorentz invariant, the resulting phenomenology lacks the various time dependences that characterize WEP violations resulting from Lorentz breaking.

B. Solving the Model

Given the action (90), we can illustrate by direct calculation the correspondence between results from the bumblebee model and ones from the general SME-based approach developed in Sec. IV. For this purpose, it suffices to work at lowest nontrivial order in the couplings ζ_1, ζ_2, and σ_1. We focus here on observable effects arising from the identifications of $(a_{\mathrm{eff}})_\mu$, $c_{\mu\nu}$, and k in Eqs. (96) and (97). Observable effects involving $s^{\mu\nu}$ and u as defined in Eq. (99) are studied in Ref. [7].

The basic goal is to predict effects such as trajectory deviations for given values of $(\overline{a}_{\mathrm{eff}})_\mu$ and $\overline{c}_{\mu\nu}$. As discussed in Sec. IV C 3, observability of $(\overline{a}_{\mathrm{eff}})_\mu$ involves nonminimal couplings, so in the present context we can expect dominant effects from $(\overline{a}_{\mathrm{eff}})_\mu$ to be proportional to the product $\zeta_1\sigma_1$. In constrast, dominant observable effects from $\overline{c}_{\mu\nu}$ are generated directly from ζ_2.

We remark in passing that the special bumblebee model considered here is experimentally viable provided the sizes of ζ_1 and ζ_2 are compatible with existing constraints on long-range spin-independent forces [35]. The proportionality of $(\overline{a}_{\mathrm{eff}})_\mu$ to $\zeta_1\sigma_1$ implies the model can yield Lorentz-violating effects involving large b_μ that are detectable only in gravitational experiments [11].

In this subsection, working at the appropriate perturbative order and taking the newtonian limit where useful, we obtain and solve the bumblebee equation of motion and the modified Einstein equation. These results suffice to determine the trajectory equation for a test particle in terms of the vacuum value b_μ of the bumblebee model. Comparison to the general SME-based approach developed in Sec. IV yields an explicit match for $(\overline{a}_{\mathrm{eff}})_\mu$, $\overline{c}_{\mu\nu}$, $\overline{k}$ in terms of the couplings ζ_1, ζ_2, σ_1 and the vacuum value b_μ.

1. Bumblebee equation

Consider first the equation of motion for the bumblebee field, which follows from varying the action S_B. At the perturbative order of interest, this equation takes the form

$$\partial^\mu B_{\mu\nu} \;=\; 2V'b_\nu - 2\sigma_1 b^\mu R_{\mu\nu} + \zeta_1^{\mathrm{S}} j_\nu + \ldots, \tag{100}$$

where

$$j_\nu = \int d\tau\; u_\nu \delta^4(x-x') \tag{101}$$

is the source 4-current at the relevant order. In Eq. (100), the prime on V denotes a derivative with respect to the argument, while the ellipsis indicates that source terms proportional to ζ_2^{S} exist but provide no observable contributions to the order at which we work. Adopting the smooth quadratic potential (94) and the expansion (95), the bumblebee equation can be written

$$\begin{aligned} (\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu - 4\lambda b_\mu b_\nu)\stackrel{+}{B}{}^\mu &= \\ -2\lambda b_\nu b^\alpha b^\beta h_{\alpha\beta} &- 2\sigma_1 b^\alpha R_{\alpha\nu} + \zeta_1^{\mathrm{S}} j_\nu. \end{aligned} \tag{102}$$

The idea is to solve this expression for the fluctuations $\stackrel{+}{B}{}^\mu$ so they can be eliminated from the analysis as needed.

The solution can be obtained in momentum space with the propagator chosen as a suitable Green function [7]. The appropriate bumblebee propagator is

$$K^{\mu\nu}(p) \;=\; -\frac{\eta^{\mu\nu}}{p^2} + \frac{(b^\mu p^\nu + b^\nu p^\mu)}{p^2 b_\alpha p^\alpha} - \frac{(4\lambda b^\alpha b_\alpha + p^2)p^\nu p^\mu}{4\lambda p^2 (b_\alpha p^\alpha)^2} \tag{103}$$

in momentum space. Note that the additional poles in this and following expressions can be understood as a consequence of residual gauge freedom to the order at which we work [52]. Using this propagator, we find the solution

$$\stackrel{+}{B}_\mu(p) = \frac{p_\mu b^\alpha b^\beta h_{\alpha\beta}}{2b^\alpha p_\alpha} + (\stackrel{+}{B}_{\sigma_1})_\mu(p) + (\stackrel{+}{B}_{\zeta_1})_\mu(p) + (\stackrel{+}{B}_{\zeta_2})_\mu(p), \tag{104}$$

where $(\stackrel{+}{B}_{\sigma_1})_\mu(p)$, $(\stackrel{+}{B}_{\zeta_1})_\mu(p)$, and $(\stackrel{+}{B}_{\zeta_2})_\mu(p)$ are contributions to $\stackrel{+}{B}_\mu(p)$ proportional to σ_1, ζ_1^{S}, and ζ_2^{S} respectively. In the limit of vanishing ζ_1^{S} and ζ_2^{S}, the solution (104) reduces to the known result [7] once the conversion from $\stackrel{+}{B}_\mu$ to $\widetilde{B}^\mu$ described in Sec. II B is implemented.

Explicitly, the quantity $(\overset{\leftrightarrow}{B}_{\sigma_1})_\mu$ is given in momentum space by [7]

$$(\overset{\leftrightarrow}{B}_{\sigma_1})_\mu(p) = -\frac{\sigma_1 b_\mu R}{p^2} + \frac{\sigma_1 p_\mu R}{4\lambda b^\alpha p_\alpha} + \frac{\sigma_1 p_\mu b^\alpha b_\alpha R}{p^2 b^\alpha p_\alpha} + \frac{2\sigma_1 b^\alpha R_{\alpha\mu}}{p^2} - \frac{2\sigma_1 p_\mu b^\alpha b^\beta R_{\alpha\beta}}{p^2 b^\alpha p_\alpha}. \quad (105)$$

For the piece proportional to ζ_1, we obtain

$$(\overset{\leftrightarrow}{B}_{\zeta_1})_\mu(p) = -\frac{\zeta_1^{\rm S} j_\mu}{p^2} + \frac{\zeta_1^{\rm S} b_\mu p^\nu j_\nu}{p^2 p_\alpha b^\alpha} + \frac{\zeta_1^{\rm S} p_\mu b^\nu j_\nu}{p^2 p_\alpha b^\alpha} - \frac{\zeta_1^{\rm S} b_\alpha b^\alpha p_\mu p^\nu j_\nu}{p^2 (p_\beta b^\beta)^2} - \frac{\zeta_1^{\rm S} p_\mu p^\nu j_\nu}{4\lambda (p_\alpha b^\alpha)^2}. \quad (106)$$

The remaining term in Eq. (104), which contains contributions proportional to $\zeta_2^{\rm S}$, is irrelevant to the order at which we work.

We emphasize that the explicit solution (104) for the bumblebee fluctuation $\overset{\leftrightarrow}{B}_\mu$ is obtained by direct calculation from the action S_B. This calculation depends on knowledge of the bumblebee dynamics as described by the Lagrange density (92). In contrast, the general SME-based method presented in Sec. IV to obtain an arbitrary coefficient fluctuation $\overset{\leftrightarrow}{t}_{\lambda\mu\nu\ldots}$ replaces the need for complete knowledge of S_B with the judicious use of perturbation theory, the NG constraint, diffeomorphism invariance, and Newton's third law.

2. Modified Einstein equation

Varying the action with respect to the metric yields the modified Einstein equation. At leading order in $\overset{\leftrightarrow}{B}_\mu$ and lowest order in $h_{\mu\nu}$, this equation takes the form

$$G_{\mu\nu} = 2\sigma_1\kappa[b^\alpha\partial_\alpha\partial_{(\mu}B_{\nu)} + b_{(\mu}\partial_\alpha\partial_{\nu)}B^\alpha - b_{(\mu}\Box B_{\nu)} - \eta_{\mu\nu}b^\alpha\partial_\alpha\partial_\beta B^\beta] + 2\kappa V' b_\mu b_\nu + \kappa (T_{Bu})_{\mu\nu}, \quad (107)$$

where contributions from B_μ are understood to be limited to the appropriate perturbative order. Note that V' contributes at most through massive modes at this order, so it plays no role in the present context.

The matter-sector contribution $(T_{Bu})_{\mu\nu}$ to the energy-momentum tensor can be written

$$(T_{Bu})_{\mu\nu} = -\int d\tau \frac{m u_\mu u_\nu \delta^4(x - x'(\tau))}{\sqrt{1 - 2\zeta_2^S b_\alpha b_\beta u^\alpha u^\beta}}. \quad (108)$$

This explicit expression is the bumblebee analogue of the general form (70). As expected, terms proportional to $\zeta_1^{\rm S}$ are absent from Eq. (108), confirming that minimal couplings cannot generate Lorentz violation of the $(\overline{a}_{\rm eff})_\mu$ type in the modified Einstein equation. In this model, the only nonzero observables proportional to $\zeta_1^{\rm S}$ arise through the bumblebee fluctuations (106).

Inserting the solution for B_μ at the appropriate order, the modified Einstein equation (107) can be solved. To match the analysis in the general SME-based method of Sec. IV the harmonic gauge must be used. At zeroth order in Lorentz violation, the conventional metric is reproduced. For simplicity in this illustrative model, we limit consideration at the next order to the newtonian limit for $\overset{\leftrightarrow}{B}_\mu$. This avoids possible complications from the residual gauge invariance, while permitting a complete match to the results of the SME method.

The solution for the metric can be constructed directly from the trace-reversed form of the modified Einstein equation (107). Using the NG condition

$$b_\mu \overset{\leftrightarrow}{B}{}^\mu = \tfrac{1}{2} b^\mu b^\nu h_{\mu\nu} \quad (109)$$

and the bumblebee equation (102), we can write

$$R_{\mu\nu} = 2\sigma_1\kappa[b^\alpha\partial_\alpha\partial_{(\mu}B_{\nu)} - \zeta_1^{\rm S} b_{(\mu}j_{\nu)}] + \kappa (S_{Bu})_{\mu\nu}, \quad (110)$$

where $(S_{Bu})_{\mu\nu}$ is the trace-reversed version of the energy-momentum tensor (108). In the newtonian limit, the first term on the right-hand side is higher-order in time derivatives and so is negligible.

Expanding $(S_{Bu})_{\mu\nu}$ to the appropriate order, we find that the O(1,1) modifications of the metric fluctuation h_{00} are given in terms of the bumblebee vacuum value b_μ by

$$h_{00}^{(1,1)} = (4\sigma_1\zeta_1^{\rm S} + \zeta_2^{\rm S} m b_0)\frac{2G_N b_0}{r}. \quad (111)$$

The first term arises from the bumblebee fluctuations via the nonminimal couplings and is the bumblebee analogue of Eq. (88). The second term arises directly from the energy-momentum tensor $(T_{Bu})_{\mu\nu}$ of the source S and corresponds to Eq. (84). To complete the match to the general SME analysis of Sec. IV, it remains to apply this result to determine the deviations from geodesic motion of a test particle T.

3. Particle trajectory

The equation of motion for a test point particle T in the presence of the bumblebee field B_μ and the metric $h_{\mu\nu}$ can be obtained by varying the action S_{Bu} with respect to x^μ. At leading order in the fluctuations, this yields

$$\ddot{x}^\mu = -\Gamma_{(0,1)}{}^\mu{}_{\alpha\beta} u^\alpha u^\beta + \ddot{x}^\mu_{\zeta_1} + \ddot{x}^\mu_{\zeta_2}, \quad (112)$$

where $\Gamma_{(0,1)}{}^\mu{}_{\alpha\beta}$ is the linearized Christoffel symbol. The terms $\ddot{x}^\mu_{\zeta_1}$ and $\ddot{x}^\mu_{\zeta_2}$ represent contributions to $\ddot{x}^\mu$ proportional to $\zeta_1^{\rm T}$ and $\zeta_2^{\rm T}$, respectively. The above equation is the bumblebee analogue of the equation of motion (78) obtained for the general SME analysis in Sec. IV.

The explicit form of the quantity $\ddot{x}^\mu_{\zeta_1}$ takes the form

$$\ddot{x}^\mu_{\zeta_1} = -(\Gamma_{\zeta_1})^\mu{}_{\alpha\beta} u^\alpha u^\beta - \frac{\zeta_1^{\rm T}}{m^{\rm T}} g^{\mu\nu}(\partial_\nu \overset{\leftrightarrow}{B}_\alpha - \partial_\alpha \overset{\leftrightarrow}{B}_\nu) u^\alpha, \quad (113)$$

where $(\Gamma_{\zeta_1})^\mu{}_{\alpha\beta}$ contains terms proportional to ζ_1^{S} that enter via Lorentz-violating corrections to the metric. The piece of $\tilde{B}_\mu$ contributing to this equation at the relevant order can be written as

$$(\tilde{B}_{\sigma_1})_\mu = \sigma_1 h_{\mu\nu} b^\nu - \tfrac{1}{2}\sigma_1 b_\mu h_\alpha{}^\alpha + \dots \tag{114}$$

in harmonic coordinates, where the ellipsis represents terms that play no role within our approximations. The latter equation is the bumblebee analogue of the SME result (87) for the coefficient fluctuation $(\tilde{a}_{\mathrm{eff}})_\mu$, and as expected it yields contributions to the trajectory equation proportional to the product $\sigma_1\zeta_1^{\mathrm{T}}$.

The quantity $\ddot{x}^\mu_{\zeta_2}$ can be written

$$\begin{aligned}
\ddot{x}^\mu_{\zeta_2} &= -(\Gamma_{\zeta_2})^\mu{}_{\alpha\beta} u^\alpha u^\beta + 2\zeta_2^{\mathrm{T}} \eta^{\mu\nu} b_\nu b_\lambda \Gamma_{(0,n)}{}^\lambda{}_{\alpha\beta} u^\alpha u^\beta \\
&\quad + 2\zeta_2^{\mathrm{T}} b_\alpha b_\beta \Gamma_{(0,1)}{}^\alpha{}_{\nu\lambda} u^\beta u^\mu u^\nu u^\lambda + 2\zeta_2^{\mathrm{T}} b_\alpha \partial^\mu \tilde{B}_\beta u^\alpha u^\beta \\
&\quad - 2\zeta_2^{\mathrm{T}} \eta^{\mu\nu} (b_\nu \partial_\alpha \tilde{B}_\beta + b_\beta \partial_\alpha \tilde{B}_\nu) u^\alpha u^\beta \\
&\quad - 2\zeta_2^{\mathrm{T}} b_\alpha \partial_\nu \tilde{B}_\beta u^\alpha u^\beta u^\mu u^\nu,
\end{aligned} \tag{115}$$

where $(\Gamma_{\zeta_2})^\mu{}_{\alpha\beta}$ contains terms proportional to ζ_2^{S} that enter via Lorentz-violating corrections to the metric. In this equation, only the first term in Eq. (104) produces a relevant contribution to the fluctuation $\tilde{B}_\mu$ in the present context, which matches the SME result (85) for the coefficient fluctuation $\tilde{c}_{\mu\nu}$.

At this stage, we can verify the conservation of total 4-momentum of the system of the source S and test body T, as described in Sec. IV C 1. Substituting for $\tilde{B}_\mu$ and $h_{00}^{(1,1)}$ in the trajectory equation reveals the antisymmetry under interchange of S and T required to satisfy Newton's third law. We can also complete the correspondence between the bumblebee model and the general SME-based analysis of Sec. IV by making the identifications

$$\begin{aligned}
\alpha &= 2\sigma_1, \qquad (\overline{a}_{\mathrm{eff}})_\mu = \zeta_1 b_\mu, \\
\overline{c}_{\mu\nu} &= \zeta_2 b_\mu b_\nu + \tfrac{1}{4}\zeta_2 \eta_{\mu\nu} b^2, \qquad \overline{k} = \tfrac{1}{2}\zeta_2 b^2,
\end{aligned} \tag{116}$$

which can be obtained by matching Eqs. (111) and (114) to the SME results (84), (88), and (87).

VI. EXPERIMENTAL BASICS

In the remainder of this paper, we apply the theoretical framework developed above to explore some experimental prospects for detecting Lorentz violation through matter-gravity couplings. As before, we adopt coordinates satisfying the condition (23), which produces simplified expressions without the photon-sector coefficients $(\overline{k_F})^\alpha{}_{\mu\alpha\nu}$. The primary focus is on signals involving the coefficients $(\overline{a}^w_{\mathrm{eff}})_\mu$ and $(\overline{c}^w)_{\mu\nu}$. Certain effects associated with the coefficient $\overline{s}_{\mu\nu}$ in the pure-gravity sector are also considered.

In the present section, we provide some basic information broadly applicable to searches for Lorentz violation, including an outline of frame conventions and a discussion of sensitivities to coefficient combinations. Each subsequent section addresses a particular class of experimental searches. Section VII examines tests with ordinary neutral matter in Earth-based laboratories, while Sec. VIII studies satellite-based searches with ordinary matter. Section IX considers more exotic laboratory and satellite-based tests, including ones using charged particles, antimatter, and particles beyond the first generation. Section X addresses solar-system observations, including lunar and satellite ranging and measurements of perihelion precession. Finally, Sec. XI considers signals from photon-gravity couplings.

A. Frames

A substantial advantage of the SME framework is the ability to compare signals for Lorentz violation across a wide variety of experiments and observations. To facilitate these comparisons, it is useful to report search results in a canonical inertial frame.

In Minkowski spacetime, the canonical frame is a Sun-centered celestial-equatorial frame [21], which is approximately inertial over the time scales of most searches. In this frame, the Z axis is aligned with the rotation axis of the Earth, while the X axis points from the Earth to the Sun at the vernal equinox. The origin of the time coordinate T is the time when the Earth crosses the Sun-centered X axis at the vernal equinox.

For post-newtonian investigations involving gravitational effects in the solar system, the canonical frame is identified with an asymptotically Minkowski frame that is comoving with the rest frame of the solar system and that coincides with the canonical Sun-centered frame [7]. In this Sun-centered frame, cartesian coordinates are denoted by

$$x^\Xi = (T, X^J) = (T, X, Y, Z) \tag{117}$$

and are labeled with capital Greek indices. Also, we write

$$\mathbf{e}_\Xi = (\mathbf{e}_T, \mathbf{e}_J) \tag{118}$$

for the corresponding coordinate basis vectors.

Various types of observers appear in the analyses below, including ones at rest in an Earth-centered frame, in a laboratory frame, in a satellite frame, and others. The corresponding frames are specified as needed in the sections that follow. Among the sets of basis vectors having generic applicability are one for an observer at rest in the Sun-centered frame and a related one for an observer in uniform motion relative to the Sun-centered frame. We summarize these two sets briefly here.

For an observer at rest at the point $(T, \vec{X})$ in the Sun-centered frame, $dX^J/dT = 0$. Suitable basis vectors are denoted as $\mathbf{e}_\mu$ with $\mu = (t, j)$, and they can be written as [7]

$$\begin{aligned}
\mathbf{e}_t &= \delta^T{}_t [1 + \tfrac{1}{2} h_{TT}(T, \vec{X}) + \mathrm{PNO}(4)] \mathbf{e}_T, \\
\mathbf{e}_j &= \delta^J{}_j [\mathbf{e}_J - \tfrac{1}{2} h_J{}^K(T, \vec{X}) \mathbf{e}_K] + \delta^J{}_j h_{TJ}(T, \vec{X}) \mathbf{e}_T.
\end{aligned} \tag{119}$$

This basis is orthonormal.

If the observer is in motion with four-velocity u^Ξ in the Sun-centered frame, then an appropriate set of basis vectors can be taken as $\mathbf{e}_{\hat{\mu}}$ with $\hat{\mu} = (\hat{t}, \hat{j})$, where the components $(\mathbf{e}_{\hat{t}})^\Xi$ are identified with the four-velocity, $(\mathbf{e}_{\hat{t}})^\Xi = u^\Xi$. This basis is given by [7]

$$\begin{aligned} \mathbf{e}_{\hat{t}} &= \delta^t{}_{\hat{t}}(1 + \tfrac{1}{2}v^2)\mathbf{e}_t + v^j \mathbf{e}_j, \\ \mathbf{e}_{\hat{j}} &= \delta^j{}_{\hat{j}} v^k R_{kj} \mathbf{e}_t + \delta^j{}_{\hat{j}}(\delta^{kl} + \tfrac{1}{2}v^k v^l) R_{lj} \mathbf{e}_k, \end{aligned} \quad (120)$$

where v^j is the coordinate velocity of the observer in the frame (119), and R_{jk} implements the appropriate rotation. This basis is also orthonormal.

In applying the above equations, the relevant contributions to the metric fluctuation and the observer velocity can be obtained from the modified Einstein equations and from the equation of motion of the observer. Note that the results typically depend on coefficients for Lorentz violation. Also, some simplifying assumptions can usually be adopted without loss of generality. For example, in certain laboratory experiments the contributions to the metric fluctuation $h_{\Xi\Lambda}$ sourced by the energy-momentum tensor of the Sun can safely be neglected.

We remark in passing that the above definition of the Sun-centered frame could be sharpened in various ways, such as allowing for the precession and nutation of the Earth, establishing the vernal equinox via the centroids of bodies, and incorporating the motion of the Sun with respect to the center of the solar system. Some of these effects may allow additional sensitivities to Lorentz violation via the resulting time dependence of the standard frame. Note also that the notion of parallelism used in the Minkowski-spacetime definition of the Sun-centered frame is inapplicable in the context of curved spacetime. One way to address this latter issue is to define the Z axis so that $\mathbf{e}_{\hat{z}}$ aligns with the spin axis of the Earth after Eqs. (119) and (120) with $R_{jk} = 0$ and the appropriate velocity are applied. For the various searches considered in this work, the standard definition of the Sun-centered frame suffices. A more complete investigation of these issues is of potential interest but lies beyond our present scope.

B. Sensitivities

In the following sections, we consider the observational effects of the coefficients $(\overline{a}^w_{\rm eff})_\mu$, $(\overline{c}^w)_{\mu\nu}$ in the matter sector and $\overline{s}_{\mu\nu}$ in the pure-gravity sector. This subsection offers some comments about attainable sensitivities to these coefficients.

Measurement of the coefficients $(\overline{a}^w_{\rm eff})_\mu$ is of particular interest because they are virtually unexplored to date. The existence of the field redefinitions described in Sec. II C 1 means that observation of effects from $(\overline{a}^w_{\rm eff})_\mu$ requires either flavor-changing physics or gravitational couplings. At the level of quarks, the flavor-changing weak interactions have been used to provide access to observables involving differences of two coefficients $(\overline{a}^w)_\mu$ with w including second- and third-generation quarks [53, 54]. Flavor oscillations can also be used to constrain the coefficients $\overline{a}_\mu$ in the neutrino sector, where they form 3×3 matrices in flavor space [55]. However, to date gravitational couplings have been used to obtain sensitivity only to limited combinations of the 12 independent components of the SME coefficients $(\overline{a}^e)_\mu$, $(\overline{a}^p)_\mu$, $(\overline{a}^n)_\mu$ for electrons, protons, and neutrons [11, 12]. These coefficients are otherwise unconstrained and could be comparatively large, so they offer interesting prospects for further investigation in gravitational tests.

In the present context, we can extend the single bound on $(\overline{a}^w)_\mu$ given in Ref. [11] by taking advantage of the result of Sec. II C 1 that $(\overline{a}^w)_\mu$ always appears at leading order with $(\overline{e}^w)_\mu$ in the combination $(\overline{a}^w_{\rm eff})_\mu$ given by Eq. (16). Using this result immediately yields a constraint on three of the independent components of $(\overline{a}^w_{\rm eff})_\mu$ for electrons, protons, and neutrons, given as

$$|\alpha(\overline{a}^e_{\rm eff})_T + \alpha(\overline{a}^p_{\rm eff})_T - 0.8\alpha(\overline{a}^n_{\rm eff})_T| < 1\times 10^{-11}\ \text{GeV} \quad (121)$$

at the 90% confidence level.

In contrast, many of the coefficients $(\overline{c}^w)_{\mu\nu}$ are readily observable in nongravitational experiments. Nonetheless, gravitational tests offer additional opportunities to achieve sensitivities to $(\overline{c}^w)_{\mu\nu}$, including some components that are unmeasured to date. For electrons, protons, and neutrons, there are 27 independent observable symmetric coefficients $(\overline{c}^w)_{\mu\nu}$. A compilation of existing limits on $(\overline{c}^w)_{\mu\nu}$ for different flavors w is given in Ref. [2].

The coefficients $\overline{s}_{\mu\nu}$ lie in the pure-gravity sector of the minimal SME and therefore can be measured only in the gravitational context. The corresponding post-newtonian corrections to the gravitational field are known [7, 56]. Constraints on most of the nine independent components of $\overline{s}_{\mu\nu}$ have been obtained using a variety of techniques, including among others perihelion-precession studies, lunar laser ranging, atom-interferometer gravimetry, and laboratory and space-based experiments [4–7]. All these analyses disregard matter effects. In this work, we show that Lorentz violation in the matter sector can contribute in different ways to signals involving the coefficients $\overline{s}_{\mu\nu}$.

For all the coefficients $(\overline{a}^w_{\rm eff})_\mu$, $(\overline{c}^w)_{\mu\nu}$, $\overline{s}_{\mu\nu}$, the effects of interest here involve gravitational couplings to matter. It is therefore reasonable to expect that the best sensitivities to Lorentz violation are associated with couplings to dominant gravitational effects. This suggests that tests with high sensitivity to Newton gravity are of particular interest. As described in Sec. IV B 3, the flavor dependence of the coefficients for Lorentz violation implies that WEP tests also lie in this category.

Many of the signals sought in gravity tests require ancillary measurements of time and distance. These typically involve matter in some form, and they may introduce additional Lorentz-violating effects beyond those comprising the direct signal of interest. However, most of these additional effects are negligible in the present context because the corresponding coefficients are tightly constrained via tests in Minkowski spacetime [2], whereas sensitivities in gravitational tests are typically substantially reduced by the weak gravitational field. Among the coefficients of interest in the present work, this issue is relevant only to $(\overline{c}^w)_{\mu\nu}$ because $(\overline{a}^w_{\rm eff})_\mu$ and $\overline{s}_{\mu\nu}$ are unobservable in Minkowski-spacetime tests and because we adopt the coordinate choice (23) making unobservable the photon-sector coefficients $(\overline{k_F})^\alpha{}_{\mu\alpha\nu}$. Among the coefficients $(\overline{c}^w)_{\mu\nu}$ for ordinary matter, the neutron-sector coefficients $(\overline{c}^n)_{\mu\nu}$ are the least well constrained at present. Their effects may therefore be important for certain tests, in which case a detailed analysis of the measurement method may be necessary.

Another consideration relevant for identifying sensitivities in tests with atoms or bulk matter is the role of the contributions from binding energy. In some cases, accounting for these contributions can disentangle effects from different coefficients, thereby producing additional independent sensitivities. This can occur when coefficients from two or more sectors are involved, either directly within a WEP test or indirectly via comparison of results obtained for different bodies. In the remainder of this subsection, we discuss this possibility for the coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ in turn.

Consider first combinations of the coefficients $(\overline{a}^w_{\rm eff})_\mu$. Following the discussion in Sec. IV A 2, a body B has an effective coefficient $(a^{\rm B}_{\rm eff})_\mu$ given by Eqs. (73) and (75). The dimensionless quantity relevant for a test is $(\overline{a}^{\rm B}_{\rm eff})_\mu/m^B$, and comparisons involving two bodies therefore appear as the difference of two quantities of this form. For two neutral bodies involving bound electrons, protons, and neutrons, this difference can be expanded as follows:

$$\begin{aligned}\sum_w \left(\frac{N^w_1}{m_1}-\frac{N^w_2}{m_2}\right)(\overline{a}^w_{\rm eff})_\mu \ = \ & \\ &\frac{N^p_1N^n_2-N^n_1N^p_2}{m_1m_2}m^n(\overline{a}^{e+p-n}_{\rm eff})_\mu \\ &+\frac{N^p_1m'_2-N^p_2m'_1}{m_1m_2}(\overline{a}^{e+p}_{\rm eff})_\mu \\ &+\frac{N^n_1m'_2-N^n_2m'_1}{m_1m_2}(\overline{a}^{n}_{\rm eff})_\mu. \qquad (122)\end{aligned}$$

Here, the numbers of particles of species w for the two bodies are N^w_1, N^w_2, and m'_1, m'_2 are the binding-energy contributions to the masses m_1, m_2 of the two bodies, as defined in Eq. (71). Also, we define

$$\begin{aligned}(\overline{a}^{e+p}_{\rm eff})_\mu &= (\overline{a}^{e}_{\rm eff})_\mu+(\overline{a}^{p}_{\rm eff})_\mu, \\ (\overline{a}^{e+p-n}_{\rm eff})_\mu &= (\overline{a}^{e+p}_{\rm eff})_\mu-\frac{m^e+m^p}{m^n}(\overline{a}^{n}_{\rm eff})_\mu. \qquad (123)\end{aligned}$$

When the contributions from binding energy are neglected in Eq. (122), the linear combination $(\overline{a}^{e+p-n}_{\rm eff})_\mu$ of coefficients becomes the sole observable involving $(\overline{a}^w_{\rm eff})_\mu$ in a comparison of two bodies, with the effect scaled by their difference in species content. However, incorporating the binding energy in the analysis introduces the last two terms in Eq. (122), revealing that the effects of $(\overline{a}^{e}_{\rm eff})_\mu+(\overline{a}^{p}_{\rm eff})_\mu$ and $(\overline{a}^{n}_{\rm eff})_\mu$ vary differently with the content of the bodies. This allows the possibility of independent measurements of $(\overline{a}^{e}_{\rm eff})_\mu+(\overline{a}^{p}_{\rm eff})_\mu$ and $(\overline{a}^{n}_{\rm eff})_\mu$. Note that the sensitivity of such measurements is typically an order of magnitude less than that of measurements of $(\overline{a}^{e+p-n}_{\rm eff})_\mu$ due the appearance of ratios of the form m'/m.

Next, consider combinations of the coefficients $(\overline{c}^w)_{\mu\nu}$. For a body B, the effective coefficient $(c^{\rm B})_{\mu\nu}$ is a dimensionless quantity given by Eq. (74). As discussed in Sec. IV A 2, nonzero Lorentz-violating contributions from the binding energy given by the coefficients $(c'^{\rm B})_{\mu\nu}$ are expected to exist, along with the usual binding-energy contributions m' to the body mass. It turns out that these Lorentz-violating contributions impede the use of binding energy to extract additional independent sensitivities to combinations of the coefficients $(\overline{c}^w)_{\mu\nu}$. To see this, consider two neutral bodies as before, and expand the analogue of Eq. (122) to get

$$\begin{aligned}\sum_w \left(\frac{N^w_1}{m_1}-\frac{N^w_2}{m_2}\right)m^w(\overline{c}^w)_{\mu\nu} \ = \ & \\ &\frac{N^p_1N^n_2-N^n_1N^p_2}{m_1m_2}m^nm^p(\overline{c}^{e+p-n})_{\mu\nu} \\ &+\frac{N^p_1m'_2-N^p_2m'_1}{m_1m_2}m^p(\overline{c}^{e+p})_{\mu\nu} \\ &+\frac{N^n_1m'_2-N^n_2m'_1}{m_1m_2}m^n(\overline{c}^{n})_{\mu\nu} \\ &+(m^e+m^p)\frac{N^p_2m'_1(\overline{c}'^1)_{\mu\nu}-N^p_1m'_2(\overline{c}'^2)_{\mu\nu}}{m_1m_2} \\ &+m^n\frac{N^n_2m'_1(\overline{c}'^1)_{\mu\nu}-N^n_1m'_2(\overline{c}'^2)_{\mu\nu}}{m_1m_2}, \qquad (124)\end{aligned}$$

where we introduce

$$\begin{aligned}(\overline{c}^{e+p})_{\mu\nu} &= \frac{m^e}{m^p}(\overline{c}^{e})_{\mu\nu}+(\overline{c}^{p})_{\mu\nu}, \\ (\overline{c}^{e+p-n})_{\mu\nu} &= (\overline{c}^{e+p})_{\mu\nu}-\frac{m^e+m^p}{m^p}(\overline{c}^{n})_{\mu\nu}. \qquad (125)\end{aligned}$$

When binding-energy effects are neglected, $(\overline{c}^{e+p-n})_{\mu\nu}$ becomes the only observable combination of the coefficients $(\overline{c}^w)_{\mu\nu}$ in gravitational tests comparing two bodies. Including the binding-energy terms as in Eq. (124) shows that bodies with different species content can exhibit distinct effects. Although it seems unlikely that nonzero effects at order m'/m in a variety of bodies would cancel sufficiently well to evade detection altogether, the appearance of the unknown coefficients $(c'^{\rm B})_{\mu\nu}$ makes it infeasible at present to extract unambiguous independent measurements on combinations of the coefficients $(\overline{c}^w)_{\mu\nu}$ using binding-energy effects.

VII. LABORATORY TESTS

This section considers some sensitive laboratory tests with ordinary neutral bulk matter, neutral atoms, and neutrons performed on or near the surface of the Earth. The basic theory for these tests is developed in Sec. VII A, while Secs. VII B, VII C, VII D, and VII E consider signals and sensitivities attainable in a variety of terrestrial searches. More exotic laboratory tests with charged particles, antimatter, and particles beyond the first generation are considered in Sec. IX.

Terrestrial experiments seeking gravitational Lorentz violation using ordinary matter can be classified either as gravimeter tests or as WEP tests. In gravimeter tests, the basic idea is to seek variations either in the gravitational force on a test body or in its gravitational acceleration. The corresponding signals originate in the time dependence of laboratory coefficients for Lorentz violation induced by the rotation of the apparatus and the rotation and revolution of the Earth. These signals can be interpreted as an effective time variation of the Newton gravitational constant G_N. In WEP tests, the idea is to compare either the gravitational force between two different bodies or their relative gravitational acceleration. The corresponding signals, which can be instantaneous or time-varying, are sensitive to differences between the coefficients associated with different species of matter.

Lorentz violation can introduce deviations from Newton's second law, so the distinction between force and acceleration can be important. This distinction implies the two classes of gravimeter and WEP tests can each be further subdivided into two categories, force-comparison tests and free-fall tests. The basic idea of a free-fall test is to search for a time or composition dependence in the gravitational acceleration of a freely falling test body by monitoring its motion. The idea of a force-comparison test is to balance the gravitational force experienced by a test body with a second force, investigating changes in the equilibrium arising from the time or species dependence of the laboratory coefficients for Lorentz violation. The force comparison can be achieved either by using a seesaw arrangement to balance the gravitational forces on test bodies of different composition, which constitutes a force-comparison WEP test, or by using a nongravitational force to counter the gravitational force on the test body, which represents a force-comparison gravimeter test.

We thus have four categories of possible laboratory tests with ordinary matter. In what follows, each is considered in a separate subsection. Free-fall gravimeter tests, including searches with freely falling corner cubes and with atom interferometers, are considered in Sec. VII B. Force-comparison gravimeter tests using mechanical and superconducting gravimeters are studied in VII C. Free-fall WEP tests, which come in a wide variety of forms, are considered in Sec. VII D. Finally, force-comparison WEP tests are discussed in Sec. VII E, with focus on a torsion-balance configuration.

Table I provides a list of some conventions adopted in this section for the analyses of laboratory tests. Many of the quantities are self explanatory. The laboratory speed V_L is due to the rotation of the Earth and depends on the laboratory colatitude χ. The relative time $T_\oplus$ involves a convenient choice of origin, measured from any instant when the $\hat{y}$ axis in the laboratory frame and the Y axis of the Sun-centered frame coincide. Using $T_\oplus$ instead of the canonical time T in the Sun-centered frame introduces a phase ψ in the analysis. The angle ζ is defined in terms of the component accelerations $\mathrm{a}_{\hat{x}}$, $\mathrm{a}_{\hat{z}}$ of a test body along $\hat{x}$, $\hat{z}$ in the laboratory frame,

$$\zeta = \tan^{-1}(\mathrm{a}_{\hat{x}}/\mathrm{a}_{\hat{z}}). \tag{126}$$

At leading order, ζ is approximated by the ratio of the usual Newton centripetal and gravitational accelerations, $\zeta \simeq 10^{-3}$. It represents the angular deviation from the vertical at the location of the laboratory of a plumb line or of a test body in free fall.

Table I. Notation for laboratory tests.

Quantity	Definition
R	mean Earth-Sun distance
$R_\oplus$	mean Earth radius
Ω	mean Earth orbital frequency
ω	mean sidereal frequency
ω_e	apparatus rotation frequency
$V_\oplus = \Omega R$	mean Earth orbital speed
V_L	laboratory rotational speed
$T_\oplus$	relative time
η	inclination of Earth orbit
χ	laboratory colatitude
$\psi = \omega(T_\oplus - T)$	phase induced by $T_\oplus$
$\zeta \approx \omega^2 R_\oplus / \sin(2\chi) g$	deviation angle

A. Theory

The relevant observables for laboratory tests of Lorentz symmetry in gravity are the motions of test bodies relative to the Earth and relative to each other. These observables can be obtained from the action for a test body, evaluated at the appropriate post-newtonian order and expressed in laboratory coordinates.

Consider the action S_u^{B} for a test body given in Eq. (76), with the gravitational field of the Earth acting as the source S. The corresponding lagrangian $L_{a,c}^{(3)}$ describing the motion of the test body T at PNO(3) can be constructed by expanding S_u^{B} with B≡T. The solution for the metric fluctuation $h_{\mu\nu}$ at this order is obtained from the general expressions (81), (84), and (88), with the Earth treated as a rigid rotating source S as described in Sec. IV A 2. In what follows, we neglect the gravitational fields of other bodies such as the Sun, although in a more detailed treatment these could be incorporated using similar methods.

For laboratory searches, it is convenient to begin calculations in an Earth-centered frame with coordinates denoted by $x^{\tilde{\mu}} = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$. At leading order, the spatial components of the Earth-centered basis are taken to coincide with those of the Sun-centered frame, and $\tilde{t} = T$. In the Earth-centered frame, we find

$$\begin{aligned} L^{(3)}_{a,c} = & \tfrac{1}{2} m^{\mathrm{T}} \left(1 + (\overline{c}^{\mathrm{T}})_{\tilde{t}\tilde{t}} + 2(\overline{c}^{\mathrm{T}})_{\tilde{t}\tilde{j}} v_{\tilde{j}}\right) v_{\tilde{k}} v_{\tilde{k}} \\ & + m^{\mathrm{T}} (\overline{c}^{\mathrm{T}})_{\tilde{j}\tilde{k}} v_{\tilde{j}} v_{\tilde{k}} \\ & + \frac{G_N m^{\mathrm{S}} m^{\mathrm{T}}}{r} \Big[1 + \frac{2\alpha}{m^{\mathrm{T}}} (\overline{a}^{\mathrm{T}}_{\mathrm{eff}})_{\tilde{t}} + \frac{2\alpha}{m^{\mathrm{S}}} (\overline{a}^{\mathrm{S}}_{\mathrm{eff}})_{\tilde{t}} \\ & \qquad + \frac{\alpha}{m^{\mathrm{S}}} (\overline{a}^{\mathrm{S}}_{\mathrm{eff}})_{\tilde{j}} v_{\tilde{j}} \\ & \qquad + (\overline{c}^{\mathrm{T}})_{\tilde{t}\tilde{t}} + (\overline{c}^{\mathrm{S}})_{\tilde{t}\tilde{t}} + 2(\overline{c}^{\mathrm{T}})_{(\tilde{t}\tilde{j})} v_{\tilde{j}} \Big] \\ & + \frac{G_N m^{\mathrm{T}}}{r^3} \alpha (\overline{a}^{\mathrm{S}}_{\mathrm{eff}})_{\tilde{j}} x_{\tilde{j}} x_{\tilde{k}} v_{\tilde{k}} \\ & + \frac{G_N m^{\mathrm{S}} m^{\mathrm{T}}}{5r^3} R_\oplus^2 \epsilon_{\tilde{j}\tilde{k}\tilde{l}} \omega_{\tilde{k}} x_{\tilde{l}} \Big[\frac{2\alpha}{m^{\mathrm{T}}} (\overline{a}^{\mathrm{T}}_{\mathrm{eff}})_{\tilde{j}} + \frac{\alpha}{m^{\mathrm{S}}} (\overline{a}^{\mathrm{S}}_{\mathrm{eff}})_{\tilde{j}} \\ & \qquad + 2(\overline{c}^{\mathrm{S}})_{(\tilde{t}\tilde{j})} \Big], \end{aligned} \tag{127}$$

where $r = \sqrt{x_{\tilde{j}} x_{\tilde{j}}}$. This expression contains the conventional Newton kinetic and potential terms for a test body T moving in the gravitational field of S, along with a series of corrections that depend on the coefficients $(\overline{a}^{\mathrm{B}}_{\mathrm{eff}})_{\tilde{\mu}}$ and $(\overline{c}^{\mathrm{B}})_{\tilde{\mu}\tilde{\nu}}$. Some of these additional terms are motional, analogous to centrifugal effects, and some are gravitational, including ones analogous to gravitomagnetic effects. Effects from the Earth's motion about the Sun are implicitly included via the dependence of $(\overline{a}^{\mathrm{B}}_{\mathrm{eff}})_{\tilde{\mu}}$ and $(\overline{c}^{\mathrm{B}})_{\tilde{\mu}\tilde{\nu}}$ on the orbital speed $V_\oplus$. This dependence can be made explicit by expressing the coefficients in Sun-centered coordinates instead of Earth-centered ones.

To obtain results applicable to laboratory tests, the result (127) must be transformed from the Earth-centered frame to the laboratory frame. We denote the laboratory coordinates by $x^{\hat{\mu}}$, where the spatial coordinates $x^{\hat{j}}$ are taken to coincide with the standard SME conventions for a laboratory on the surface of the Earth [21]. In the laboratory, the $\hat{x}$ axis points South, the $\hat{y}$ axis points East, and the $\hat{z}$ axis points towards the local zenith. To the required post-newtonian order, $\tilde{t} = \hat{t}$ and the coordinate location of the laboratory in the Earth-centered frame can be written [7]

$$\vec{\xi} = R_\oplus (\sin\chi \cos(\omega_\oplus T + \phi), \sin\chi \sin(\omega_\oplus T + \phi), \cos\chi). \tag{128}$$

The transformation between the two sets of spatial coordinates can therefore be written

$$x_{\tilde{j}} = \xi_{\tilde{j}} + R_{\tilde{j}\hat{j}} x_{\hat{j}}, \tag{129}$$

where $R_{\tilde{j}\hat{j}}$ is the relevant rotation between the bases of the laboratory and the Earth-centered frames. Note that Eq. (129) implies the coefficients $(\overline{a}^{\mathrm{B}}_{\mathrm{eff}})_{\hat{\mu}}$ and $(\overline{c}^{\mathrm{B}})_{\hat{\mu}\hat{\nu}}$ in the laboratory frame acquire implicit dependences on the laboratory speed V_L and on the sidereal frequency ω, which arise from the rotation of the Earth.

The inclusion of the Earth's rotation in the analysis implies the laboratory frame is noninertial. The structure of the first few terms in Eq. (127) reveals that inertial forces in the laboratory couple to $(\overline{c}^{\mathrm{T}})_{\hat{\mu}\hat{\nu}}$, which can result in nongravitational Lorentz-violating effects comparable in size to the gravitational ones of interest. We therefore incorporate these nongravitational effects in our subsequent analyses. In practice, this means effects proportional to the centrifugal acceleration $\omega^2 R_\oplus \approx 10^{-3} g$ must be considered.

In what follows, we consider effects up to and including PNO(3). The leading PNO(3) effects are proportional to the speed $V_\oplus$ of the Earth as it revolves about the Sun and are of order $gV_\oplus \approx 10^{-4} g$, where $g = G_N m^{\mathrm{S}}/R_\oplus^2$ for a laboratory on the surface of the Earth. This yields sensitivity to various components of the coefficients $(\overline{a}^w_{\mathrm{eff}})_\mu$ and $(\overline{c}^w)_{\mu\nu}$. For some laboratory tests, it is advantageous to consider also PNO(3) effects proportional to the smaller speed V_L of the laboratory due to the rotation of the Earth, which are of order $gV_L \approx 10^{-6} g$. The benefit arises in two ways. First, inclusion of the boost V_L introduces effects proportional to $(\overline{a}^w_{\mathrm{eff}})_\mu$ that vary sidereally instead of annually. This offers access to $(\overline{a}^w_{\mathrm{eff}})_\mu$ for measurements conducted on comparatively short time scales, albeit at a sensitivity reduced by about two orders of magnitude. Second, certain laboratory tests have greater sensitivity to forces in the $\hat{x}$ and $\hat{y}$ directions than to ones in the $\hat{z}$ direction. The inclusion of effects from $(\overline{a}^w_{\mathrm{eff}})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ that are proportional to V_L can then introduce new sensitivities or improve existing ones.

So far, modifications to the trajectory of the test body arising from the coefficients $\overline{s}_{\mu\nu}$ have been disregarded. However, it is straightforward to incorporate these in the lagrangian at PNO(2) because the coordinate choices made here are consistent with those of Ref. [7] at this perturbative order. In the laboratory frame, we find the PNO(2) contribution from $\overline{s}_{\mu\nu}$ to the lagrangian of the test body can be written

$$L^{(2)}_s = m^{\mathrm{T}} g (\overline{s}_{\hat{z}\hat{x}} \hat{x} + \overline{s}_{\hat{z}\hat{y}} \hat{y} - \tfrac{1}{2} \overline{s}_{\hat{z}\hat{z}} \hat{z} - \tfrac{3}{2} \overline{s}_{\hat{t}\hat{t}} \hat{z}). \tag{130}$$

It turns out that $L^{(2)}_s$ suffices to achieve sensitivity to $\overline{s}_{\mu\nu}$ at PNO(3). The point is that the leading PNO(3) effects are proportional to $V_\oplus$, while inclusion of effects proportional to V_L offers no additional benefit in this case for the tests we consider. The coefficients $\overline{s}_{\mu\nu}$ are species independent, so they are unobservable in WEP tests. Moreover, inspection of $L^{(2)}_s$ reveals that the coefficients $\overline{s}_{\hat{\mu}\hat{\nu}}$ already vary at the sidereal frequency through the transformation to the Sun frame.

In the laboratory frame, the PNO(3) lagrangian $L^{(3)}_{a,c,s}$ obtained from Eq. (127) and incorporating effects from $\overline{s}_{\hat{\mu}\hat{\nu}}$ via Eq. (130) is somewhat lengthy in form. As an illustration of its structure and implications, we can restrict attention to its PNO(2) limit $L^{(2)}_{a,c,s}$. We find

$$\begin{aligned} L^{(2)}_{a,c,s} &= \tfrac{1}{2}m^{\rm T}(1+(\overline{c}^{\rm T})_{\hat t\hat t})\dot x_{\hat j}\dot x_{\hat j} + m^{\rm T}(\overline{c}^{\rm T})_{\hat j\hat k}\dot x_{\hat j}\dot x_{\hat k} \\ &\quad -m^{\rm T}g\Big[1+\frac{2\alpha}{m^{\rm T}}(\overline{a}^{\rm T}_{\rm eff})_{\hat t}+\frac{2\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_{\hat t} \\ &\qquad + (\overline{c}^{\rm T})_{\hat t\hat t}+(\overline{c}^{\rm S})_{\hat t\hat t}+\tfrac{3}{2}\overline{s}_{\hat t\hat t}+\tfrac{1}{2}\overline{s}_{\hat z\hat z}\Big]z \\ &\quad +m^{\rm T}g(\overline{s}_{\hat z\hat x}x+\overline{s}_{\hat z\hat y}y). \end{aligned} \tag{131}$$

Varying this result yields the Euler-Lagrange equations of motion, which we can express in the form of the modified force law

$$F_{\hat j} = m_{\hat j\hat k}\ddot x_{\hat k}. \tag{132}$$

At this perturbative order, the inertial and gravitational forces acting on the test particle are given by

$$\begin{aligned} F_{\hat x} &= m^{\rm T}g\overline{s}_{\hat z\hat x}, \\ F_{\hat y} &= m^{\rm T}g\overline{s}_{\hat z\hat y}, \\ F_{\hat z} &= -m^{\rm T}g\Big[1+\frac{2\alpha}{m^{\rm T}}(\overline{a}^{\rm T}_{\rm eff})_{\hat t}+\frac{2\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_{\hat t} \\ &\qquad + (\overline{c}^{\rm T})_{\hat t\hat t}+(\overline{c}^{\rm S})_{\hat t\hat t}+\tfrac{3}{2}\overline{s}_{\hat t\hat t}+\tfrac{1}{2}\overline{s}_{\hat z\hat z}\Big], \end{aligned} \tag{133}$$

while

$$m_{\hat j\hat k} = m^{\rm T}\left(1+(\overline{c}^{\rm T})_{\hat t\hat t}\right)\delta_{\hat j\hat k} + 2m^{\rm T}(\overline{c}^{\rm T})_{(\hat j\hat k)}. \tag{134}$$

is the effective inertial mass.

These results reveal the generic feature that the gravitational force $F_{\hat j}$ acquires tiny corrections both along the $\hat z$ direction and perpendicular to it. Also, the response of the test body deviates slightly from the direction of the applied force because the effective inertial mass $m_{\hat j\hat k}$ depends on the coefficients $(\overline{c}^{\rm T})_{\hat\mu\hat\nu}$. In principle, some of these effects are detectable in sensitive laboratory tests, and the corresponding signals are discussed using PNO(3) results in the following subsections.

Some coefficients appear in combinations that are challenging to separate in laboratory tests. This is true, for example, of the coefficients $\alpha(\overline{a}^{\rm T}_{\rm eff})_T$ and $(\overline{c}^{\rm T})_{TT}$. Consider for simplicity the scenario with only isotropic Lorentz violation in the Sun-centered frame, where the nonzero coefficients are $\alpha(\overline{a}^{\rm T}_{\rm eff})_T$ and $(\overline{c}^{\rm T})_{TT} = 3(\overline{c}^{\rm T})_{XX} = 3(\overline{c}^{\rm T})_{YY} = 3(\overline{c}^{\rm T})_{ZZ}$. In the laboratory frame, $(\overline{a}^{\rm T}_{\rm eff})_{\hat t} \approx \alpha(\overline{a}^{\rm T}_{\rm eff})_T$ and $(\overline{c}^{\rm T})_{\hat t\hat t} \approx (\overline{c}^{\rm T})_{TT}$ up to boost factors. These coefficients therefore cannot be readily separated in gravimeter tests, which depend on time variations from anisotropic effects. Moreover, inspection of the PNO(2) lagrangian (131) reveals that if $3\alpha(\overline{a}^{\rm T}_{\rm eff})_{\hat t} = m^{\rm T}(\overline{c}^{\rm T})_{\hat t\hat t}$ then the contributions of $\alpha(\overline{a}^{\rm T}_{\rm eff})_{\hat t}$ and $(\overline{c}^{\rm T})_{\hat t\hat t}$ to the effective inertial and gravitational masses are identical. The combination $\alpha(\overline{a}^{\rm T}_{\rm eff})_T - m^{\rm T}(\overline{c}^{\rm T})_{TT}/3$ therefore cannot be readily separated in conventional WEP tests either. Note that WEP tests comparing a particle and its antiparticle can in principle evade this difficulty because the sign of $\alpha(\overline{a}^{\rm T}_{\rm eff})_T$ differs between the two. Another possibility would be to compare matter with light, an option considered further in Sec. XI.

B. Free-fall gravimeter tests

In this subsection, we consider laboratory tests that monitor the motion of a test body in free fall near the surface of the Earth. The equation of motion for the test body can be obtained from the PNO(3) lagrangian $L^{(3)}_{a,c,s}$ described in Sec. VII A. Its explicit form is lengthy. However, all information relevant for present purposes is contained in its solution expressed to the desired perturbative order. This solution can be written in the form

$$x_{\hat j} = (x_o)_{\hat j} + (v_o)_{\hat j}t + \tfrac{1}{2}{\rm a}_{\hat j}t^2, \tag{135}$$

where the test body has initial position $\vec{x}_o$ and initial velocity $\vec{v}_o$. The quantities of interest in searches for gravitational Lorentz violation are the components ${\rm a}_{\hat j}$ of the acceleration of the test body in laboratory coordinates.

For purposes of data analysis and reporting sensitivities to coefficients for Lorentz violation, it is useful to express the acceleration components ${\rm a}_{\hat j}$ in a form that displays explicitly the time variation and the dependence on particle species. In free-fall gravimeter tests, the time variation appears at frequencies 0, ω, 2ω, $\omega\pm\Omega$, $2\omega\pm\Omega$, and Ω, which are collectively labeled as n in what follows. The dependence on particle species arises from the composition of the test and source bodies. It is characterized by the label w, which ranges over e, p, n for ordinary matter.

Table II. Amplitudes for the acceleration ${\rm a}_{\hat x}$.

Amplitude	Phase
$A^w_0 = m^w\sin\chi\cos\chi[(\overline{c}^w)_{XX}+(\overline{c}^w)_{YY}-2(\overline{c}^w)_{ZZ}]$	0
$A^w_\omega = 2m^w(\overline{c}^w)_{(XZ)}\cos2\chi+\frac{2}{5}V_L\alpha(\overline{a}^w_{\rm eff})_Y\cos\chi$	ψ
$A'^w_\omega = \frac{1}{5}V_L\left[\alpha(\overline{a}^w_{\rm eff})_Y+2m^w(\overline{c}^w)_{(TY)}\right]\cos\chi$	ψ
$B^w_\omega = 2m^w(\overline{c}^w)_{(YZ)}\cos2\chi-\frac{2}{5}V_L\alpha(\overline{a}^w_{\rm eff})_X\cos\chi$	ψ
$B'^w_\omega = -\frac{1}{5}V_L\left[\alpha(\overline{a}^w_{\rm eff})_X+2m^w(\overline{c}^w)_{(TX)}\right]\cos\chi$	ψ
$A^w_{2\omega} = \frac{1}{2}m^w((\overline{c}^w)_{XX}-(\overline{c}^w)_{YY})\sin2\chi$	2ψ
$B^w_{2\omega} = m^w(\overline{c}^w)_{(XY)}\sin2\chi$	2ψ
$A^w_{\omega+\Omega} = -m^wV_\oplus(\overline{c}^w)_{(TX)}\sin\eta\cos2\chi$	ψ
$B^w_{\omega+\Omega} = -m^wV_\oplus\big[(\overline{c}^w)_{(TY)}\sin\eta - (\overline{c}^w)_{(TZ)}(1-\cos\eta)\big]\cos2\chi$	ψ
$A^w_{\omega-\Omega} = -m^wV_\oplus(\overline{c}^w)_{(TX)}\sin\eta\cos2\chi$	ψ
$B^w_{\omega-\Omega} = -m^wV_\oplus\big[(\overline{c}^w)_{(TY)}\sin\eta + (\overline{c}^w)_{(TZ)}(1+\cos\eta)\big]\cos2\chi$	ψ
$A^w_{2\omega+\Omega} = -\frac{1}{2}m^wV_\oplus(\overline{c}^w)_{(TY)}(1-\cos\eta)\sin2\chi$	2ψ
$B^w_{2\omega+\Omega} = \frac{1}{2}m^wV_\oplus(\overline{c}^w)_{(TX)}(1-\cos\eta)\sin2\chi$	2ψ
$A^w_{2\omega-\Omega} = \frac{1}{2}m^wV_\oplus(\overline{c}^w)_{(TY)}(1+\cos\eta)\sin2\chi$	2ψ
$B^w_{2\omega-\Omega} = -\frac{1}{2}m^wV_\oplus(\overline{c}^w)_{(TX)}(1+\cos\eta)\sin2\chi$	2ψ
$A^w_\Omega = -m^wV_\oplus\big[(\overline{c}^w)_{(TY)}\cos\eta -2(\overline{c}^w)_{(TZ)}\sin\eta\big]\sin2\chi$	0
$B^w_\Omega = m^wV_\oplus(\overline{c}^w)_{(TX)}\sin2\chi$	0

Table III. Amplitudes for the acceleration $\mathrm{a}_{\hat{y}}$.

Amplitude	Phase
$C^w_0 = m^w V_L (\overline{c}^w)_{(TZ)} \sin 2\chi$	0
$C^w_\omega = 2m^w (\overline{c}^w)_{(YZ)} \cos\chi$ $- \frac{2}{5} V_L \alpha (\overline{a}^w_{\rm eff})_X + 2m^w V_L (\overline{c}^w)_{(TX)} \sin^2\chi$	ψ
$C'^w_\omega = -\frac{1}{5} V_L \big[\alpha (\overline{a}^w_{\rm eff})_X + 2(\overline{c}^w)_{(TX)}\big]$	ψ
$D^w_\omega = -2m^w (\overline{c}^w)_{(XZ)} \cos\chi$ $+ 2m^w V_L (\overline{c}^w)_{(TY)} \sin^2\chi - \frac{2}{5} V_L \alpha (\overline{a}^w_{\rm eff})_Y$	ψ
$D'^w_\omega = -\frac{1}{5} V_L \big[2m^w (\overline{c}^w)_{(TY)} + \alpha (\overline{a}^w_{\rm eff})_Y\big]$	ψ
$C^w_{2\omega} = 2m^w (\overline{c}^w)_{(XY)} \sin\chi$	2ψ
$D^w_{2\omega} = -m^w ((\overline{c}^w)_{XX} - (\overline{c}^w)_{YY}) \sin\chi$	2ψ
$C^w_{\omega+\Omega} = m^w V_\oplus \big[(\overline{c}^w)_{(TZ)} (1-\cos\eta)$ $- (\overline{c}^w)_{(TY)} \sin\eta\big] \cos\chi$	ψ
$D^w_{\omega+\Omega} = m^w V_\oplus (\overline{c}^w)_{(TX)} \sin\eta \cos\chi$	ψ
$C^w_{\omega-\Omega} = -m^w V_\oplus \big[(\overline{c}^w)_{(TZ)} (1+\cos\eta)$ $+ (\overline{c}^w)_{(TY)} \sin\eta\big] \cos\chi$	ψ
$D^w_{\omega-\Omega} = m^w V_\oplus (\overline{c}^w)_{(TX)} \sin\eta \cos\chi$	ψ
$C^w_{2\omega+\Omega} = m^w V_\oplus (\overline{c}^w)_{(TX)} (1-\cos\eta) \sin\chi$	2ψ
$D^w_{2\omega+\Omega} = m^w V_\oplus (\overline{c}^w)_{(TY)} (1-\cos\eta) \sin\chi$	2ψ
$C^w_{2\omega-\Omega} = -m^w V_\oplus (\overline{c}^w)_{(TX)} (1+\cos\eta) \sin\chi$	2ψ
$D^w_{2\omega-\Omega} = -m^w V_\oplus (\overline{c}^w)_{(TY)} (1+\cos\eta) \sin\chi$	2ψ

For the $\hat{x}$ component of the acceleration, some calculation yields an expression of the form

$$\begin{aligned} \mathrm{a}_{\hat{x}} \;=\; & \omega^2 R_\oplus \sin\chi\cos\chi \\ & + g \sum_{n,w} \Big[\left(\frac{N^w}{m^{\rm T}} A^w_n + \frac{N^w_\oplus}{m^{\rm S}} A'^w_n + \tfrac{1}{3} A_n \right) \cos(\omega_n T + \psi_n) \\ & + \left(\frac{N^w}{m^{\rm T}} B^w_n + \frac{N^w_\oplus}{m^{\rm S}} B'^w_n + \tfrac{1}{3} B_n \right) \sin(\omega_n T + \psi_n) \Big]. \end{aligned} \tag{136}$$

In this equation, the amplitudes A^w_n, A'^w_n, B^w_n, B'^w_n contain the coefficients for Lorentz violation $(\overline{a}^w_{\rm eff})_\mu$, $(\overline{c}^w)_{\mu\nu}$ and hence depend on particle species. These amplitudes and their associated phases are listed in Table II. The remaining amplitudes A_n, B_n contain the coefficients $\overline{s}_{\mu\nu}$ from the gravitational sector, which are independent of the composition of the test body. These amplitudes can be obtained from the amplitudes A^w_n, B^w_n by the substitutions $m^w \to 1$, $(\overline{c}^w)_{\Sigma\Xi} \to \frac{1}{2}\overline{s}_{\Sigma\Xi}$, and $(\overline{a}^w_{\rm eff})_\Xi \to 0$, disregarding contributions proportional to V_L.

The $\hat{y}$ component of the acceleration can be decomposed similarly. We find

$$\begin{aligned} \mathrm{a}_{\hat{y}} \;=\; & \sum_{n,w} g \Big[\left(\frac{N^w}{m^{\rm T}} C^w_n + \frac{N^w_\oplus}{m^{\rm S}} C'^w_n + \tfrac{1}{3} C_n \right) \cos(\omega_n T + \psi_n) \\ & + \left(\frac{N^w}{m^{\rm T}} D^w_n + \frac{N^w_\oplus}{m^{\rm S}} D'^w_n + \tfrac{1}{3} D_n \right) \sin(\omega_n T + \psi_n) \Big]. \end{aligned} \tag{137}$$

The amplitudes C^w_n, C'^w_n, D^w_n, D'^w_n depend on particle species through the coefficients $(\overline{a}^w_{\rm eff})_\mu$, $(\overline{c}^w)_{\mu\nu}$ and are listed in Table III, along with the corresponding phases. The remaining amplitudes C_n, D_n are obtained from C^w_n, D^w_n using the substitutions $m^w \to 1$, $(\overline{c}^w)_{\Sigma\Xi} \to \frac{1}{2}\overline{s}_{\Sigma\Xi}$, and $(\overline{a}^w_{\rm eff})_\Xi \to 0$, disregarding contributions proportional to V_L as before.

For the $\hat{z}$ component of the acceleration, we obtain

$$\begin{aligned} \mathrm{a}_{\hat{z}} \;=\; & -g + \omega^2 R_\oplus \sin^2\chi \\ & + \sum_{n,w} g \Big[\left(\frac{N^w}{m^{\rm T}} E^w_n + \frac{N^w_\oplus}{m^{\rm S}} E'^w_n + \tfrac{1}{3} E_n \right) \cos(\omega_n T + \psi_n) \\ & + \left(\frac{N^w}{m^{\rm T}} F^w_n + \frac{N^w_\oplus}{m^{\rm S}} F'^w_n + \tfrac{1}{3} F_n \right) \sin(\omega_n T + \psi_n) \Big]. \end{aligned} \tag{138}$$

The amplitudes E^w_n, E'^w_n, F^w_n, F'^w_n depend on particle species via the coefficients $(\overline{a}^w_{\rm eff})_\mu$, $(\overline{c}^w)_{\mu\nu}$. The amplitudes E_n, F_n are independent of species and given in terms of the coefficients $\overline{s}_{\mu\nu}$. All these amplitudes and their phases are provided in Table IV.

In principle, the results of a free-fall laboratory test using any gravimeter can be analyzed with the above equations. The dominant effects appear at different frequencies for different coefficients, so the time scale of data taking in a given experiment affects the breadth of its reach in coefficient space. Also, each signal frequency can be expected to have distinct systematics. For example, dominant effects from the coefficients $(\overline{a}_{\rm eff})_J$ occur at the annual frequency Ω, for which seasonal systematics are relevant. Note that all the Lorentz-violating effects can be accessed at or near the sidereal frequency ω, although in some cases at reduced sensitivity.

At least two kinds of devices can be classified as free-fall gravimeters: falling corner cubes, and matter interferometers. Falling corner cubes, which typically are sensitive only to the direction of the free-fall motion, are used to monitor time variations of the gravitational field for geodesy and other geophysical purposes [57]. In principle, they are of interest for free-fall gravimeter tests of Lorentz violation. However, matter interferometers presently carry several advantages over falling corner cubes in this context. They are slightly more sensitive, some types can sense accelerations in more than one direction, and the composition of the test body can be determined more readily. We therefore focus on matter interferometers in this subsection, revisiting the use of both falling corner cubes and interferometers in the context of free-fall WEP tests in Sec. VII D.

Table IV. Amplitudes for the acceleration $\mathrm{a}_{\hat{z}}$.

Amplitude	Phase
$E^w_0 = -2\alpha(\overline{a}^w_{\text{eff}})_T + 2m^w(\overline{c}^w)_{ZZ}\cos^2\chi + m^w\left((\overline{c}^w)_{XX} + (\overline{c}^w)_{YY}\right)\sin^2\chi$	0
$E'^w_0 = -2\alpha(\overline{a}^w_{\text{eff}})_T - m^w(\overline{c}^w)_{TT}$	0
$E^w_\omega = 2m^w(\overline{c}^w)_{(XZ)}\sin 2\chi - \frac{4}{5}V_L\alpha(\overline{a}^w_{\text{eff}})_Y\sin\chi$	ψ
$E'^w_\omega = -\frac{4}{5}V_L\left(3\alpha(\overline{a}^w_{\text{eff}})_Y + m^w(\overline{c}^w)_{(TY)}\right)\sin\chi$	ψ
$F^w_\omega = 2m^w(\overline{c}^w)_{(YZ)}\sin 2\chi + \frac{4}{5}V_L\alpha(\overline{a}^w_{\text{eff}})_X\sin\chi$	ψ
$F'^w_\omega = \frac{4}{5}V_L\left(3\alpha(\overline{a}^w_{\text{eff}})_X + m^w(\overline{c}^w)_{(TX)}\right)\sin\chi$	ψ
$E^w_{2\omega} = m^w((\overline{c}^w)_{XX} - (\overline{c}^w)_{YY})\sin^2\chi$	2ψ
$F^w_{2\omega} = 2m^w(\overline{c}^w)_{(XY)}\sin^2\chi$	2ψ
$E^w_{\omega+\Omega} = -m^wV_\oplus(\overline{c}^w)_{(TX)}\sin\eta\sin 2\chi$	ψ
$F^w_{\omega+\Omega} = -m^wV_\oplus\left[(\overline{c}^w)_{(TY)}\sin\eta - (\overline{c}^w)_{(TZ)}(1-\cos\eta)\right]\sin 2\chi$	ψ
$E^w_{\omega-\Omega} = -m^wV_\oplus(\overline{c}^w)_{(TX)}\sin\eta\sin 2\chi$	ψ
$F^w_{\omega-\Omega} = -m^wV_\oplus\left[(\overline{c}^w)_{(TY)}\sin\eta + (\overline{c}^w)_{(TZ)}(1+\cos\eta)\right]\sin 2\chi$	ψ
$E^w_{2\omega+\Omega} = -m^wV_\oplus(\overline{c}^w)_{(TY)}(1-\cos\eta)\sin^2\chi$	2ψ
$F^w_{2\omega+\Omega} = m^wV_\oplus(\overline{c}^w)_{(TX)}(1-\cos\eta)\sin^2\chi$	2ψ
$E^w_{2\omega-\Omega} = m^wV_\oplus(\overline{c}^w)_{(TY)}(1+\cos\eta)\sin^2\chi$	2ψ
$F^w_{2\omega-\Omega} = -m^wV_\oplus(\overline{c}^w)_{(TX)}(1+\cos\eta)\sin^2\chi$	2ψ
$E^w_\Omega = 2V_\oplus\alpha((\overline{a}^w_{\text{eff}})_Y\cos\eta + (\overline{a}^w_{\text{eff}})_Z\sin\eta) - 2m^wV_\oplus\left[(\overline{c}^w)_{(TY)}\cos\eta\sin^2\chi + 2(\overline{c}^w)_{(TZ)}\sin\eta\cos^2\chi\right]$	0
$E'^w_\Omega = 2V_\oplus\alpha((\overline{a}^w_{\text{eff}})_Y\cos\eta + (\overline{a}^w_{\text{eff}})_Z\sin\eta) + 2m^wV_\oplus((\overline{c}^w)_{(TY)}\cos\eta + (\overline{c}^w)_{(TZ)}\sin\eta)$	0
$F^w_\Omega = -2V_\oplus\alpha(\overline{a}^w_{\text{eff}})_X + 2m^wV_\oplus(\overline{c}^w)_{(TX)}\sin^2\chi$	0
$F'^w_\Omega = -2V_\oplus\alpha(\overline{a}^w_{\text{eff}})_X - 2m^wV_\oplus(\overline{c}^w)_{(TX)}$	0
$E_0 = -\frac{1}{2}\overline{s}_{ZZ}\cos^2\chi - \frac{1}{4}\left(\overline{s}_{XX} + \overline{s}_{YY}\right)\sin^2\chi - \frac{3}{2}\overline{s}_{TT}$	0
$E_\omega = -\frac{1}{2}\overline{s}_{(XZ)}\sin 2\chi$	ψ
$F_\omega = -\frac{1}{2}\overline{s}_{(YZ)}\sin 2\chi$	ψ
$E_{2\omega} = -\frac{1}{4}(\overline{s}_{XX} - \overline{s}_{YY})\sin^2\chi$	2ψ
$F_{2\omega} = -\frac{1}{4}\overline{s}_{(XY)}\sin^2\chi$	2ψ
$E_{\omega+\Omega} = \frac{1}{4}V_\oplus\overline{s}_{(TX)}\sin\eta\sin 2\chi$	ψ
$F_{\omega+\Omega} = \frac{1}{4}V_\oplus\left[\overline{s}_{(TY)}\sin\eta - \overline{s}_{(TZ)}(1-\cos\eta)\right]\sin 2\chi$	ψ
$E_{\omega-\Omega} = \frac{1}{4}V_\oplus\overline{s}_{(TX)}\sin\eta\sin 2\chi$	ψ
$F_{\omega-\Omega} = \frac{1}{4}V_\oplus\left[\overline{s}_{(TY)}\sin\eta + \overline{s}_{(TZ)}(1+\cos\eta)\right]\sin 2\chi$	ψ
$E_{2\omega+\Omega} = \frac{1}{4}V_\oplus\overline{s}_{(TY)}(1-\cos\eta)\sin^2\chi$	2ψ
$F_{2\omega+\Omega} = -\frac{1}{4}V_\oplus\overline{s}_{(TX)}(1-\cos\eta)\sin^2\chi$	2ψ
$E_{2\omega-\Omega} = -\frac{1}{4}V_\oplus\overline{s}_{(TY)}(1+\cos\eta)\sin^2\chi$	2ψ
$F_{2\omega-\Omega} = \frac{1}{4}V_\oplus\overline{s}_{(TX)}(1+\cos\eta)\sin^2\chi$	2ψ
$E_\Omega = V_\oplus\left[\overline{s}_{(TY)}\cos\eta(\frac{1}{2}\sin^2\chi + 3) + \overline{s}_{(TZ)}\sin\eta(\cos^2\chi + 3)\right]$	0
$F_\Omega = -V_\oplus\overline{s}_{(TX)}(\frac{1}{2}\sin^2\chi + 3)$	0

Matter interferometers, which permit quantum-mechanical laboratory measurements of the motion of falling matter, have attained impressive sensitivities to gravitational acceleration [58] and to rotational accelerations via the Sagnac effect [59]. In the context of gravitational Lorentz violation, matter interferometry has been used to measure combinations of the coefficients $\overline{s}_{\mu\nu}$ and $(\overline{k_F})^\alpha{}_{\mu\alpha\nu}$ [5] based on the gravimeter analysis of effects from the pure-gravity sector of the SME [7]. Here, we extend the latter analysis to include effects from the coefficients $(\overline{a}^w_{\text{eff}})_\mu$, $(\overline{c}^w)_{\mu\nu}$ and generalize it to other interferometer configurations.

The basic idea of a matter interferometer is to place the matter in a superposition of spatially separated quantum states, which may acquire a measurable relative phase. In the gravitational tests considered here, the behavior of the interferometer is close to the classical limit, and a convenient way to perform the analysis is to proceed semiclassically via path integration along the classical motion [58, 60]. The phase difference between the final states can then be viewed as a sum of three contributions: the phase difference acquired from the momentum transfers used to control the beams, the phase difference accumulated from the classical action along the different paths, and in some configurations a phase difference coming from a final separation of the states. It turns out that the dominant effect for acceleration sensing is the phase difference acquired through momentum transfers. Since leading-order Lorentz-violating motional effects appear as modified accelerations, the phase difference from the momentum transfers is the relevant contribution in the present context.

For definiteness, suppose the interferometer paths trace a parallelogram. This includes the limiting case of temporal path separation, where the parallelogram has zero area. Other shapes could also be analysed using the equation of motion (135). In the Lorentz-invariant case, the standard result for the phase shift due to the Earth's gravitational field is $\Delta\phi = k_{\hat{z}}g\tau^2$, where $\vec{k}$ is the magnitude of the momentum transfer in the beam splitter and τ is the time of flight between impulses. With Lorentz violation present, we find the phase shift $\Delta\phi$ takes a similar form but with the Newton gravitational acceleration replaced by the accelerations in Eq. (135), giving

$$\Delta\phi = k_{\hat{j}}\mathrm{a}_{\hat{j}}\tau^2. \tag{139}$$

The signal frequencies associated with Lorentz violation can be identified by substitution of the expressions (136)-(138) for the acceleration components $\mathrm{a}_{\hat{j}}$. Note that Lorentz-violating effects on the atomic energy levels could generate additional contributions to the phase difference but are already tightly constrained in other experiments and so can typically be neglected. Note also that possible Lorentz-violating effects varying with the particle spins, which are described explicitly by the relativistic hamiltonian of Sec. III B, are disregarded here as outside our present scope. A comprehensive investigation of their implications for matter interferometry may be of interest [61].

Several atom interferometers currently or recently operating are relevant to free-fall gravimeter searches for Lorentz violation. An impressive sensitivity of about $1 \times 10^{-10} g$ to the vertical acceleration was achieved by Peters, Chung, and Chu [58]. In another apparatus, a differential-acceleration sensitivity of $3 \times 10^{-9} g/\sqrt{\mathrm{Hz}}$ has been demonstrated [62]. An interferometer designed for experiments in space is expected to achieve sensitivity of about $3 \times 10^{-9} g$ in ground operations [63]. Initial sensitivities to accelerations in each direction of about $6 \times 10^{-7} g$ after 10 minutes of averaging have been attained in a device using highly parabolic trajectories [64]. Recent estimates suggest that future measurements of vertical acceleration could achieve sensitivities at the level of about $10^{-15} g$ [65].

Given this information and the phase shift (139), we can use Tables II, III, and IV to obtain crude estimates for attainable sensitivities to coefficients for Lorentz violation in existing or near-future atom interferometers. With present capabilities, sensitivities at the level of parts in 10^5 could in principle be obtained to combinations of the coefficients $\alpha(\overline{a}^w_{\mathrm{eff}})_J$, $J = X, Y, Z$ and of several currently unconstrained components of the coefficients $(\overline{c}^w)_{\Sigma\Xi}$, including $(\overline{c}^n)_{(TJ)}$ for the neutron. The relevant signals are associated with the Earth's boost as it revolves about the Sun, so they exhibit an annual periodicity. The next generation of atom interferometers could in principle improve this sensitivity to parts in 10^{10}. The boost of the laboratory due to the Earth's rotation provides sensitivities that have sidereal periodicities instead but that are weaker by a factor of about 100. Note that the boost suppressions could in principle be avoided for certain coefficients, including presently unbounded combinations involving $(\overline{c}^n)_{ZZ}$, by the use of an interferometer sensitive to the accelerations $a_{\hat{x}}$, $a_{\hat{y}}$ that is placed on a rotating turntable. Note also that individual sensitivities to neutron coefficients can in principle be extracted by performing atom interferometry with different neutral atoms having distinct proton-to-neutron ratios. Another possibility with weaker existing sensitivity includes neutron interferometry [66], which could provide independent and clean bounds on neutron coefficients.

C. Force-comparison gravimeter tests

Another class of gravimeter tests is based on the idea of countering the gravitational force with an appropriate electromagnetic force. Force-comparison gravimeter tests can be performed with gravimeters based on systems of springs and masses [57] and with superconducting gravimeters [57, 67, 68]. At present, the latter devices have sensitivities competitive with those of existing atom interferometers. Certain experiments studying short-range gravity may also offer relevant sensitivities [69].

The signals for gravitational Lorentz violation in a given force-comparison gravimeter can be extracted from the PNO(3) lagrangian $L^{(3)}_{a,c,s}$ discussed in Sec. VII A. Since macroscopic bodies are involved, the analysis must include an assessment of their composition. Note also that conventional intuition from Newton's second law can be misleading because the effective inertial masses depend on the coefficients $(\overline{c}^w)_{\mu\nu}$, as discussed following Eq. (134).

Superconducting gravimeters have already been proposed as suitable devices for measuring the gravity-sector coefficients $\overline{s}_{\mu\nu}$ for Lorentz violation [7]. Here, we extend this discussion to include effects from the coefficients $(\overline{a}^w_{\mathrm{eff}})_\mu$ and $(\overline{c}^w)_{\mu\nu}$. The analysis proceeds directly from the PNO(3) lagrangian $L^{(3)}_{a,c,s}$ by noting that the device is designed to maintain $\ddot{x}_{\hat{j}} = 0$. The applied force required to hold this constraint can be taken as the relevant observable and can be written

$$F_{\hat{z}'} = F_{\hat{z}} \cos\zeta + F_{\hat{x}} \sin\zeta, \tag{140}$$

where ζ is the deviation angle defined in Eq. (126). To maintain consistent counting of small effects, we restrict terms independent of velocity to first order in ζ and terms containing a power of velocity to zeroth order in ζ.

We find that the relevant contributions to the force $F_{\hat{z}'}$ can be decomposed by frequency as

$$\begin{aligned} F_{\hat{z}'} &= m^{\mathrm{T}} g(1 - \zeta\tan\chi - \tfrac{3}{2}\zeta^2) - m^{\mathrm{T}} g \\ &\quad\times \sum_{n\neq 0,w} \left[\left(\frac{N^w}{m^{\mathrm{T}}} G^w_n + \frac{N^w_\oplus}{m^{\mathrm{S}}} E'^w_n + \tfrac{1}{3} G_n \right) \cos(\omega_n T + \psi_n) \right. \\ &\qquad \left. + \left(\frac{N^w}{m^{\mathrm{T}}} H^w_n + \frac{N^w_\oplus}{m^{\mathrm{S}}} F'^w_n + \tfrac{1}{3} H_n \right) \sin(\omega_n T + \psi_n) \right], \end{aligned} \tag{141}$$

where constant effects that are unobservable in superconducting gravimeters are neglected. In this expression, the amplitudes G^w_n, H^w_n and their phases are given in Table V, while E'^w_n and F'^w_n are listed in Table IV. The remaining amplitudes G_n and H_n can be expressed in terms of amplitudes given in Tables II and IV as

$$G_n = A_n\zeta + E_n, \qquad H_n = B_n\zeta + F_n. \tag{142}$$

The frequency decomposition (141) can be examined to extract crude estimates of attainable sensitivities to Lorentz violation. In this way, we estimate that the presently unbounded coefficients $\alpha(\overline{a}^w_{\mathrm{eff}})_J$ and $(\overline{c}^w)_{(TJ)}$ could be measured at the level of parts in 10^7 using existing data from superconducting gravimeters [68]. Improved sensitivities are likely to be attainable in a dedicated experiment of this type.

Table V. Amplitudes for the force $F_{\hat{z}'}$.

Amplitude	Phase
$G^w_\omega = 2m^w\zeta(\overline{c}^w)_{(XZ)}$	
$\quad - \frac{4}{5}V_L\alpha(\overline{a}^w_{\rm eff})_Y \sin\chi - 2m^w V_L(\overline{c}^w)_{(TY)}\sin\chi$	ψ
$H^w_\omega = 2m^w\zeta(\overline{c}^w)_{(YZ)}$	
$\quad + \frac{4}{5}V_L\alpha(\overline{a}^w_{\rm eff})_X \sin\chi + 2m^w V_L(\overline{c}^w)_{(TX)}\sin\chi$	ψ
$G^w_{2\omega} = m^w\zeta((\overline{c}^w)_{XX} - (\overline{c}^w)_{YY})$	2ψ
$H^w_{2\omega} = 2m^w\zeta(\overline{c}^w)_{(XY)}$	2ψ
$G^w_\Omega = 2V_\oplus\alpha((\overline{a}^w_{\rm eff})_Y\cos\eta + (\overline{a}^w_{\rm eff})_Z\sin\eta)$	
$\quad + 2m^w V_\oplus\big[(\overline{c}^w)_{(TY)}\cos\eta\sin^2\chi$	
$\quad\quad + 2(\overline{c}^w)_{(TZ)}\sin\eta\cos^2\chi\big]$	0
$H^w_\Omega = 2V_\oplus\alpha(\overline{a}^w_{\rm eff})_X + 2m^w V_\oplus(\overline{c}^w)_{(TX)}\sin^2\chi$	0

For tests of short-range gravity and certain other applications, it is useful to consider the standard case of two point masses m_1 and m_2 at coordinate locations $\vec{x}_1$ and $\vec{x}_2$. With this setup, the modified Newton potential V at PNO(2) in the laboratory frame can be obtained from $L^{(3)}_{a,c,s}$. We find

$$V = -\frac{G_N m_1 m_2}{|\vec{x}_1 - \vec{x}_2|}\big[1 + \frac{2\alpha}{m_1}(\overline{a}^1_{\rm eff})_{\hat{t}} + \frac{2\alpha}{m_2}(\overline{a}^2_{\rm eff})_{\hat{t}} + (\overline{c}^1)_{\hat{t}\hat{t}} + (\overline{c}^2)_{\hat{t}\hat{t}} + \tfrac{1}{2}\hat{x}^{\hat{j}}\hat{x}^{\hat{k}}\overline{s}^{\hat{j}\hat{k}}\big], \quad (143)$$

where $\hat{x} = (\vec{x}_1 - \vec{x}_2)/|\vec{x}_1 - \vec{x}_2|$. This modified potential exhibits the usual inverse-distance dependence, and it generalizes Eq. (137) of Ref. [7]. The corresponding modified Newton force typically has a component perpendicular to the unit vector $\hat{x}$, while obtaining the accelerations requires determining also the effective inertial masses. As usual, any motion of the masses relative to the Sun-centered frame implies time dependence of the laboratory-frame coefficients. In principle, the above modified Newton potential could be used in conjunction with integration or finite-element methods to determine the effects of the coefficients $(\overline{a}^w_{\rm eff})_\mu$, $(\overline{c}^w)_{\mu\nu}$, $\overline{s}_{\mu\nu}$ on the behavior of two interacting bodies.

D. Free-fall WEP tests

In this subsection, we consider WEP tests in which signals for Lorentz violation can be sought by monitoring the relative motion of two freely falling bodies of different composition. Typical free-fall WEP tests are sensitive to motion along the direction of the net acceleration $a_{\hat{z}'}$. This acceleration is the combination

$$a_{\hat{z}'} = a_{\hat{z}}\cos\zeta + a_{\hat{x}}\sin\zeta \quad (144)$$

of the component accelerations (136) and (138), weighted by the deviation angle ζ given in Eq. (126). In what follows, terms containing both a boost factor and a factor of ζ are treated as higher order and negligible, as in the previous subsection.

The relevant observable for free-fall WEP tests is the relative position $\Delta\hat{z}'$ of two test bodies 1 and 2 in a given drop. It can be written as

$$\Delta\hat{z}' = \big[(v_o)^1_{\hat{z}'} - (v_o)^2_{\hat{z}'}\big]\,\hat{t} + \tfrac{1}{2}\left(a^1_{\hat{z}'} - a^2_{\hat{z}'}\right)\hat{t}^2. \quad (145)$$

This relative position varies with the canonical time T. Decomposing by frequency yields the expression

$$\Delta\hat{z}' = \sum_{n,w}\left(\frac{N^w_1}{m_1} - \frac{N^w_2}{m_2}\right)T^2 \times[I^w_n\cos(\omega_n T + \psi_n) + J^w_n\sin(\omega_n T + \psi_n)]. \quad (146)$$

The amplitudes I^w_n and J^w_n can be expressed as

$$I^w_n = A^w_n\zeta + E^w_n, \qquad J^w_n = B^w_n\zeta + F^w_n, \quad (147)$$

where A^w_n, B^w_n are listed in Table II and E^w_n, F^w_n are given in Table IV, along with the associated phases ψ_n. In Eq. (146), the quantities N^w_1 and N^w_2 are the numbers of particles of type w appearing in the test bodies 1 and 2, respectively, while m_1 and m_2 are the corresponding conventional masses.

The frequency decomposition (146) of the signal (145) can be used to provide rough estimates of attainable sensitivities to Lorentz violation in existing or near-future free-fall WEP tests. We combine values for the fractional acceleration sensitivity $\Delta a/a$ discussed in Refs. [65, 70–75] with the result (146) to compile some estimates in Table VI. In this table, the first row lists the fractional acceleration sensitivity, while each of the other rows concerns a particular combination of coefficients. For brevity, in the first column we adopt the notations

$$(\overline{a}_{\rm eff})_{Y+Z} = (\overline{a}_{\rm eff})_Y\cos\eta + (\overline{a}_{\rm eff})_Z\sin\eta,$$
$$(\overline{c}^w)_Q = (\overline{c}^w)_{XX} + (\overline{c}^w)_{YY} - 2(\overline{c}^w)_{ZZ}, \quad (148)$$

along with those introduced in Eqs. (123) and (125). We follow common procedure in the literature [2] by taking $(\overline{c}^w)_{TT}$, $(\overline{c}^w)_{XX} - (\overline{c}^w)_{YY}$, and $(\overline{c}^w)_Q$ as the relevant independent combinations of the traceless coefficients $(\overline{c}^w)_{\Sigma\Xi}$.

Each column in Table VI lists estimated attainable sensitivities on the moduli of various quantities in specified types of free-fall WEP test, expressed to the nearest order of magnitude. Values listed with neither brackets nor braces are limits based on published data that are implied by our present analysis. Values shown in brackets are our estimate of sensitivities that could in principle be obtained from a suitable reanalysis of existing data. Values shown in braces represent our estimate of sensitivities attainable using data from future tests.

The second column of the table concerns free-fall WEP tests using falling corner cubes [70, 71]. In the second entry of this column, we present a single bound on the time-independent portion of the signal implied by existing data. The remainder of this column lists crude estimates of sensitivities that could be attained through sidereal and annual analysis of the same data.

The third and fourth columns of the table list sensitivities from free-fall WEP tests using atom interferometry. In the second entry of the third column, we present a single bound extracted from existing data [72]. The fourth column concerns proposals for future tests with atom interferometers [65], based on the idea that the relative vertical acceleration of two different atoms may be measured using a simultaneous dual-species fountain [76].

The remaining columns of the table concern other proposed free-fall WEP tests. Crude estimates are provided of the sensitivities that might be achieved in the Principle of Equivalence Measurement (POEM) [73], via balloon drops in the General Relativity Accuracy Test (GReAT) [74], and using the Bremen drop tower [75].

In the table, the estimates for the coefficients listed in the second and third rows and for $(\overline{c}^n)_Q$ in the penultimate row all arise from the time-independent component of the data. A nonzero signal for any of these measurements would therefore be challenging to distinguish from other potential sources of WEP violation. Note that obtaining the independent sensitivities in the third and penultimate rows requires combining data taken in free-fall WEP tests performed at different colatitudes χ.

Independent sensitivities can also be achieved via other techniques. One possibility is positronium interferometry [77], which via comparison with ordinary matter could yield a bound on different linear combinations of $(\overline{a}^w_{\rm eff})_T$ and $(\overline{c}^w)_{TT}$. Also, some independent measurements can be extracted by combining results from free-fall WEP tests with those from the force-comparison WEP tests discussed below.

The next generation of the POEM experiment [73] is the proposed Sounding Rocket POEM (SR-POEM) [78], which is a WEP test designed to measure to 10^{-16} the relative acceleration of freely falling test bodies on a sounding rocket during certain phases of its flight. Although not terrestrial, this experiment can also be analyzed using the methods presented here. A competitive sensitivity is anticipated for measurements of the combination $\alpha(\overline{a}^{e+p-n}_{\rm eff})_T - m^p(\overline{c}^{e+p-n})_{TT}/3$ of isotropic coefficients. Obtaining sufficient data to resolve the periodic changes necessary for sensitivity to other coefficient combinations would be challenging.

Table VI. Sensitivities for free-fall WEP tests.

Coefficient combination	Falling corner-cube [70, 71]	Atom interferometry [72]	Atom interferometry [65]	Tossed masses [73]	Balloon drop [74]	Drop tower [75]
$\frac{\Delta \mathrm{a}}{\mathrm{a}}$	10^{-10}	10^{-7}	$\{10^{-17}\}$	$\{10^{-14}\}$	$\{10^{-15}\}$	$\{10^{-12}\}$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_T - \frac{1}{3}m^p(\overline{c}^{e+p-n})_{TT} + (\frac{1}{2}\cos^2\chi - \frac{1}{6})m^n(\overline{c}^n)_Q$	10^{-8} GeV	10^{-5} GeV	$\{10^{-15}$ GeV$\}$	$\{10^{-12}$ GeV$\}$	$\{10^{-13}$ GeV$\}$	$\{10^{-10}$ GeV$\}$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_T - \frac{1}{3}m^p(\overline{c}^{e+p-n})_{TT}$	...	...	$\{10^{-15}$ GeV$\}$	$\{10^{-12}$ GeV$\}$	$\{10^{-13}$ GeV$\}$	$\{10^{-10}$ GeV$\}$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_X$	$[10^{-4}$ GeV$]$	...	$\{10^{-11}$ GeV$\}$	$\{10^{-8}$ GeV$\}$	$\{10^{-9}$ GeV$\}$	$\{10^{-6}$ GeV$\}$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_{Y+Z}$	$[10^{-4}$ GeV$]$	...	$\{10^{-11}$ GeV$\}$	$\{10^{-8}$ GeV$\}$	$\{10^{-9}$ GeV$\}$	$\{10^{-6}$ GeV$\}$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_Y$	$[10^{-2}$ GeV$]$	...	$\{10^{-9}$ GeV$\}$	$\{10^{-6}$ GeV$\}$	$\{10^{-7}$ GeV$\}$	$\{10^{-4}$ GeV$\}$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_Z$	$[10^{-2}$ GeV$]$	...	$\{10^{-9}$ GeV$\}$	$\{10^{-6}$ GeV$\}$	$\{10^{-7}$ GeV$\}$	$\{10^{-4}$ GeV$\}$
$(\overline{c}^n)_Q$	...	...	$\{10^{-15}\}$	$\{10^{-12}\}$	$\{10^{-13}\}$	$\{10^{-10}\}$
$(\overline{c}^n)_{(TJ)}$	$[10^{-4}]$	...	$\{10^{-11}\}$	$\{10^{-8}\}$	$\{10^{-9}\}$	$\{10^{-6}\}$

E. Force-comparison WEP tests

Typical force-comparison WEP tests can be viewed as comparing the motion of two or more bodies joined through electromagnetic forces with that predicted by an equation of the modified form (132). The predicted motion depends on the details of the configuration, so a unified analysis for all force-comparison WEP tests is impractical. Here, we consider as an illustration a sensitive existing force-comparison WEP test based on a torsion pendulum [35, 81]. Exceptional sensitivity to Lorentz violation can be achieved using a torsion pendulum with a spin-weighted bob [34, 41, 82], but here we treat instead a bob with a dipolar composition. We remark in passing that another interesting option for force-comparison WEP tests is the use of superconducting gravimeters to compare gravitational forces on parts of the Earth having different compositions [83], although present sensitivities to coefficients for Lorentz violation are likely to be somewhat weaker.

For the torsion pendulum, a simple model of the bob is a dumbbell viewed as a rod with test bodies 1 and 2 placed on each end. The two test bodies are composed of different materials, and the bob is suspended by a torsion fiber attached at the midpoint between them. The resulting pendulum is typically rotated in the laboratory to improve the modulation of the signal. The relevant observable in such tests is the twist angle $\theta(T)$ of the torsion fiber. This angle can be calculated from the Newton second law as modified by the presence of Lorentz violation.

At zeroth order in Lorentz violation, the pendulum hangs at an angle ζ from the local vertical in the laboratory given by Eq. (126). Lorentz-violating corrections to ζ exist, but these make no contribution to the signal to the order at which we work. This angle represents the equilibrium position for the swing mode of the pendulum. Lorentz-violating modifications to this position could drive small excitations of the swing mode, but experiments tuned to the torsion mode are typically comparatively insensitive to other modes [81].

The orientation of the bob about the axis perpendicular to both the torsion fiber and to the dumbbell dipole moment can also be considered. This is the equilibrium position for the wobble mode of the pendulum. For simplicity, we assume here that the bob is suspended at a point P equidistant between the centers of mass of the test bodies and that the test bodies are constructed to ensure the dumbbell is perpendicular to the torsion fiber. In the absence of Lorentz violation, this implies equality of the two masses m_1 and m_2. However, in the presence of Lorentz violation, m_1 and m_2 differ at leading order in the coefficients for Lorentz violation. As the pendulum rotates, this difference could shift the dumbbell orientation away from its equilibrium point and generate small excitations of the wobble mode about P. Again, experiments are comparatively insensitive to this mode.

Table VII. Amplitudes for torsion-pendulum tests.

Amplitude	Phase α_n
$K_{\omega_e} = -\frac{1}{2}\left(m^w(\overline{c}^w)_{XX} + m^w(\overline{c}^w)_{YY} - 2\alpha(\overline{a}^w_{\text{eff}})_T\right)\left(1+\frac{\omega^2 R_\oplus}{g}\sin^2\chi\right)\sin 2\chi$	ϕ
$K_{\omega_e+\Omega} = -\frac{1}{2}V_\oplus\alpha((\overline{a}^w_{\text{eff}})_Y\cos\eta + (\overline{a}^w_{\text{eff}})_Z\sin\eta)\sin 2\chi + \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TY)}\cos\eta\sin 2\chi$	ϕ
$L_{\omega_e+\Omega} = -\frac{1}{2}V_\oplus\alpha(\overline{a}^w_{\text{eff}})_X\sin 2\chi + \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}\sin 2\chi$	ϕ
$K_{\omega_e-\Omega} = -\frac{1}{2}V_\oplus\alpha((\overline{a}^w_{\text{eff}})_Y\cos\eta + (\overline{a}^w_{\text{eff}})_Z\sin\eta)\sin 2\chi + \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TY)}\cos\eta\sin 2\chi$	ϕ
$L_{\omega_e-\Omega} = \frac{1}{2}V_\oplus\alpha(\overline{a}^w_{\text{eff}})_X\sin 2\chi - \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}\sin 2\chi$	ϕ
$K_{\omega_e+\omega} = m^w(\overline{c}^w)_{(XZ)}\left(1-\frac{\omega^2 R_\oplus}{g}\cos^2\chi\right)\sin^2\chi - \frac{gV_L\alpha}{5\omega^2 R_\oplus}(\overline{a}^w_{\text{eff}})_Y(1+\cos\chi)$	$\phi+\psi$
$L_{\omega_e+\omega} = -m^w(\overline{c}^w)_{(YZ)}\left(1-\frac{\omega^2 R_\oplus}{g}\cos^2\chi\right)\sin^2\chi - \frac{gV_L\alpha}{5\omega^2 R_\oplus}(\overline{a}^w_{\text{eff}})_X(1+\cos\chi)$	$\phi+\psi$
$K_{\omega_e-\omega} = m^w(\overline{c}^w)_{(XZ)}\left(1-\frac{\omega^2 R_\oplus}{g}\cos^2\chi\right)\sin^2\chi + \frac{gV_L\alpha}{5\omega^2 R_\oplus}(\overline{a}^w_{\text{eff}})_Y(1-\cos\chi)$	$\phi-\psi$
$L_{\omega_e-\omega} = m^w(\overline{c}^w)_{(YZ)}\left(1-\frac{\omega^2 R_\oplus}{g}\cos^2\chi\right)\sin^2\chi + \frac{gV_L\alpha}{5\omega^2 R_\oplus}(\overline{a}^w_{\text{eff}})_X(\cos\chi-1)$	$\phi-\psi$
$K_{\omega_e+2\omega} = -\frac{1}{2}m^w((\overline{c}^w)_{XX} - (\overline{c}^w)_{YY})\left(\sin\chi + \sin\chi\cos\chi + \frac{\omega^2 R_\oplus}{g}\sin^3\chi\cos\chi\right)$	$\phi+2\psi$
$L_{\omega_e+2\omega} = m^w(\overline{c}^w)_{(XY)}\left(\sin\chi + \sin\chi\cos\chi + \frac{\omega^2 R_\oplus}{g}\sin^3\chi\cos\chi\right)$	$\phi+2\psi$
$K_{\omega_e-2\omega} = \frac{1}{2}m^w((\overline{c}^w)_{XX} - (\overline{c}^w)_{YY})\left(\sin\chi - \sin\chi\cos\chi - \frac{\omega^2 R_\oplus}{g}\sin^3\chi\cos\chi\right)$	$\phi-2\psi$
$L_{\omega_e-2\omega} = m^w(\overline{c}^w)_{(XY)}\left(\sin\chi - \sin\chi\cos\chi - \frac{\omega^2 R_\oplus}{g}\sin^3\chi\cos\chi\right)$	$\phi-2\psi$
$K_{\omega_e+\omega+\Omega} = -\frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}\sin\eta\sin^2\chi$	$\phi+\psi$
$L_{\omega_e+\omega+\Omega} = \frac{1}{2}m^w V_\oplus\left[(\overline{c}^w)_{(TY)}\sin\eta - (\overline{c}^w)_{(TZ)}(1-\cos\eta)\right]\sin^2\chi$	$\phi+\psi$
$K_{\omega_e+\omega-\Omega} = -\frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}\sin\eta\sin^2\chi$	$\phi+\psi$
$L_{\omega_e+\omega-\Omega} = \frac{1}{2}m^w V_\oplus\left[(\overline{c}^w)_{(TY)}\sin\eta + (\overline{c}^w)_{(TZ)}(1+\cos\eta)\right]\sin^2\chi$	$\phi+\psi$
$K_{\omega_e-\omega+\Omega} = -\frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}\sin\eta\sin^2\chi$	$\phi-\psi$
$L_{\omega_e-\omega+\Omega} = -\frac{1}{2}m^w V_\oplus\left[(\overline{c}^w)_{(TY)}\sin\eta + (\overline{c}^w)_{(TZ)}(1+\cos\eta)\right]\sin^2\chi$	$\phi-\psi$
$K_{\omega_e-\omega-\Omega} = -\frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}\sin\eta\sin^2\chi$	$\phi-\psi$
$L_{\omega_e-\omega-\Omega} = -\frac{1}{2}m^w V_\oplus\left[(\overline{c}^w)_{(TY)}\sin\eta - (\overline{c}^w)_{(TZ)}(1-\cos\eta)\right]\sin^2\chi$	$\phi-\psi$
$K_{\omega_e+2\omega+\Omega} = \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TY)}(1-\cos\eta)\sin\chi(1+\cos\chi)$	$\phi+2\psi$
$L_{\omega_e+2\omega+\Omega} = \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}(1-\cos\eta)\sin\chi(1+\cos\chi)$	$\phi+2\psi$
$K_{\omega_e+2\omega-\Omega} = -\frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TY)}(1+\cos\eta)\sin\chi(1+\cos\chi)$	$\phi+2\psi$
$L_{\omega_e+2\omega-\Omega} = -\frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}(1+\cos\eta)\sin\chi(1+\cos\chi)$	$\phi+2\psi$
$K_{\omega_e-2\omega+\Omega} = -\frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TY)}(1+\cos\eta)\sin\chi(\cos\chi-1)$	$\phi-2\psi$
$L_{\omega_e-2\omega+\Omega} = \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}(1+\cos\eta)\sin\chi(\cos\chi-1)$	$\phi-2\psi$
$K_{\omega_e-2\omega-\Omega} = \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TY)}(1-\cos\eta)\sin\chi(\cos\chi-1)$	$\phi-2\psi$
$L_{\omega_e-2\omega-\Omega} = \frac{1}{2}m^w V_\oplus(\overline{c}^w)_{(TX)}(1-\cos\eta)\sin\chi(1-\cos\chi)$	$\phi-2\psi$

To analyze the torsion mode, it is convenient to express the relevant contributions to the difference $m_1 - m_2$ of the test-body masses in a form displaying the dependence on particle species. This form can be obtained from Eq. (133), giving

$$m_1 - m_2 = -\sum_w \left(N_1^w - N_2^w\right)\left(2\alpha(\overline{a}^w_{\text{eff}})_T + m^w(\overline{c}^w)_{TT}\right). \tag{149}$$

At leading order in Lorentz violation, the oscillations of the system are determined by the second-order differential equation

$$I\frac{d^2\theta}{dT^2}+2\gamma I\frac{d\theta}{dT}+\kappa\theta=\tau, \tag{150}$$

where γ is the torsional damping constant and κ is the torsional spring constant. The moment of inertia I can be taken as

$$I=(m_1+m_2)r_0^2+I', \tag{151}$$

where r_0 is the distance from P to the test bodies and I' is the moment of inertia of the remaining matter comprising the dumbbell. The torque τ includes Lorentz-violating effects and is determined by the forces on the test bodies calculated from the PNO(3) lagrangian $L^{(3)}_{a,c,s}$.

The damping term in Eq. (150) ensures that free oscillations vanish in the steady state. The time dependence of the steady-state solution is therefore determined by the rotations of the pendulum relative to the Sun-centered frame. Neglecting possible torques other than those implied by $L^{(3)}_{a,c,s}$, the steady-state solution can be written in frequency-decomposed form as

$$\begin{aligned}\theta(T) \;=\; \sum_{n,w}&\frac{(N_1^w-N_2^w)\,\omega^2R_\oplus r_0}{I\sqrt{(\omega_0^2-\omega_n^2)^2+4\gamma^2\omega_n^2}}\\ &\times\big[K_n\sin(\omega_nT+\beta_n+\alpha_n)\\ &\quad+L_n\cos(\omega_nT+\beta_n+\alpha_n)\big],\end{aligned} \tag{152}$$

where $\beta_n=2\gamma\omega_n/(\omega_0^2-\omega_n^2)$ and where α_n is a phase fixing the relationship between the time coordinate in the turntable frame and the Sun-centered time T. The amplitudes K_n, L_n and the phase α_n are given in Table VII. With the exception of the first row in the table, these signals for Lorentz violation are distinguished from other potential sources of WEP violation by their characteristic time dependence.

In the above analysis, the assumption of a steady-state solution implies the pendulum motion is governed by leading-order Lorentz violation, while the torque τ is taken as the only relevant source of Lorentz violation. Note that Lorentz-violating contributions to the moment of inertia I can be neglected here because they enter only at higher order. These are a manifestation of the angular-momentum nonconservation that accompanies Lorentz violation, and they are analogous to the Lorentz-violating contributions to the effective inertial mass in the modified Newton second law (132). In principle, the rotation of the apparatus in the laboratory introduces similar effects proportional to $\overline{c}_{JK}$ and $\omega_e^2r_0$. These may be comparable in magnitude to effects listed in Table VII that are suppressed by $\omega^2R_\oplus V_\oplus$, but they offer no additional advantage in terms of sensitivity and so are disregarded here.

The above analysis can be used to extract constraints on Lorentz violation from the results of the torsion-pendulum WEP tests reported in Refs. [35, 81]. The attained sensitivity to the differential acceleration of Be and Ti test bodies at the level of 10^{-15} ms^{-2} [35] is the experimental basis for our limit (121), which extends an earlier bound [11] to include the coefficients $\overline{e}_\mu$. In the remainder of this subsection, we revisit this issue to incorporate the slightly weaker constraints from torsion pendulum experiments using Al, Be, Cu, and Si test bodies [81], and we consider implications of nonzero $(\overline{c}^w)_{\mu\nu}$.

First, suppose $(\overline{c}^w)_{\mu\nu}=0$. Inclusion of data from tests with different materials permits the extraction of some independent sensitivities to neutron coefficients and to combinations of electron and proton coefficients. This treatment relies on differences in binding energy between the materials involved, so the signal sensitivity of 10^{-15} ms^{-2} relative to $\omega^2R_\oplus\simeq3\times10^{-3}$ ms^{-2} is suppressed both by the typical material-dependence factor of $(N_1^w-N_2^w)\simeq10^{-2}$ appearing in Eq. (152) and by another order of magnitude from the binding-energy difference. By combining available Be-Ti and Al-Be data [81] we obtain the estimated bounds

$$\begin{aligned}|\alpha(\overline{a}^{e+p}_{\rm eff})_T| &\lesssim 10^{-10}\ \text{GeV},\\ |\alpha(\overline{a}^{n}_{\rm eff})_T| &\lesssim 10^{-10}\ \text{GeV},\end{aligned} \tag{153}$$

valid for $(\overline{c}^w)_{\mu\nu}=0$.

If instead nonzero coefficients $(\overline{c}^w)_{\mu\nu}$ are present, then we obtain the estimated bound

$$|\alpha(\overline{a}^{e+p-n}_{\rm eff})_T-\tfrac13m^p(\overline{c}^{e+p-n})_{TT}-\tfrac16m^n(\overline{c}^n)_Q|\lesssim10^{-11}\ \text{GeV}. \tag{154}$$

The contributions due to the spatial neutron coefficient $(\overline{c}^n)_Q$ cannot be disentangled from those due to the temporal components at this order in the analysis. However, this separation becomes feasible when the result (154) is combined with the limit achieved via free-fall WEP tests given in row 2 of Table VI. We thereby obtain the constraints

$$\begin{aligned}|\alpha(\overline{a}^{e+p-n}_{\rm eff})_T-\tfrac13m^p(\overline{c}^{e+p-n})_{TT}| &\lesssim 10^{-8}\ \text{GeV},\\ |(\overline{c}^n)_Q| &\lesssim 10^{-8}.\end{aligned} \tag{155}$$

As discussed following Eq. (125), the possibility of $(\overline{c}^w)_{\mu\nu}$-type Lorentz-violating effects in the binding energy impedes its direct use in extracting independent sensitivities to $|\alpha(\overline{a}^{e+p}_{\rm eff})_T-\frac13m^p(\overline{c}^{e+p})_{TT}|$ and $|\alpha(\overline{a}^{n}_{\rm eff})_T-\frac13m^p(\overline{c}^{n})_{TT}|$.

In addition to the constraints (153), (154), and (155), other new bounds could be placed on the moduli of certain coefficients for Lorentz violation by reanalysing the time dependence of the data obtained in the experiments of Refs. [35, 81] using the result (152). Crude estimates of these sensitivities are given in Table VIII. These are obtained disregarding binding-energy considerations but making the strong assumption that all relevant frequencies in Table VII can be studied in the data. Allowing for binding-energy effects could yield independent sensitivities to the neutron coefficients and to a combination of proton and electron coefficients, both reduced by roughly a factor of 10.

Table VIII. Sensitivities for torsion-pendulum tests.

Coefficient	Sensitivity
$\overline{c}^n_{(TJ)}$	$[10^{-7}]$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_X$	$[10^{-8}$ GeV$]$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_{Y+Z}$	$[10^{-7}$ GeV$]$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_Y$	$[10^{-8}$ GeV$]$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_Z$	$[10^{-7}$ GeV$]$

VIII. SATELLITE-BASED WEP TESTS

Space-based platforms offer certain advantages in tests of gravity [84] and searches for Lorentz violation [85]. The long free-fall times that may be attainable on a drag-free spacecraft make satellite-based WEP tests particularly attractive. Several proposals are in an advanced stage of development, including the Micro-Satellite à traînée Compensée pour l'Observation du Principe d'Equivalence (MicroSCOPE) [86], the Satellite Test of the Equivalence Principle (STEP) [87], and the Galileo Galilei (GG) mission [88]. A WEP reach similar to that of STEP has also been suggested for the Grand Unification and Gravity Explorer (GaUGE) mission [89].

The basic idea underlying these missions is to monitor the relative motion of test bodies made of different materials as they orbit the Earth in a satellite. In the presence of nonzero coefficients for Lorentz violation $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$, the orbits of the test bodies become material dependent. In this section, we determine the resulting apparent WEP violations and then obtain crude estimates of the sensitivities to $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ attainable in MicroSCOPE, STEP, and GG.

A. Theory

The basic observable for a satellite-based WEP test is the differential local acceleration between the test bodies. The typical design goal is to achieve excellent sensitivity to one or two components of this acceleration. For present purposes, we can idealize the situation as a pair of test bodies aboard a satellite traveling in a circular orbit. In what follows, we allow for the possibility that the test bodies are also rotating about an axis perpendicular both to the direction of motion of the satellite and to the direction of acceleration sensitivity.

Some notation relevant for our analysis of satellite-based WEP tests is summarized in Table IX. Paralleling the analysis of terrestrial experiments in Sec. VII A, it is convenient to introduce an Earth-centered frame with coordinates $x^{\tilde\mu} = (\tilde t, \tilde x, \tilde y, \tilde z)$, chosen so that $\tilde t = T$ and so that the spatial components match those of the Sun-centered frame at leading post-newtonian order. The Earth-centered coordinates can be related to the Sun-centered ones as discussed in Sec. VI A. The angles ξ_1, ξ_2 in the table are defined relative to the basis vectors of the Earth-centered frame. The notation for properties of the test masses 1 and 2 follows that of Sec. VI B.

Table IX. Notation for satellite-based WEP tests.

Quantity	Definition
$R_\oplus$	mean Earth radius
$V_\oplus$	mean Earth orbital speed
r^J	Earth-satellite separation
ω_s	satellite orbital frequency
ω_r	satellite rotational frequency
ξ_1	inclination of satellite orbit
ξ_2	longitude of satellite-orbit node
θ_1	phase fixing satellite location at $T=0$
θ_2	phase fixing satellite orientation at $T=0$

Establishing the signal arising from nonzero coefficients for Lorentz violation requires the transformation from the Sun-centered frame to a frame comoving with the satellite. The satellite frame serves as the equivalent of the laboratory frame for terrestrial searches. We denote coordinates in the satellite frame by $x^{\hat\mu}$.

Since the satellite orbit is inclined relative to the Earth-centered frame, it is also useful to introduce an intermediate frame aligned with the satellite orbit and hence rotated with respect to the Earth-centered frame. The intermediate coordinates are denoted by $x^{\mu'}$. The rotation transformation from $x^{j'}$ to $x^{\tilde j}$ can be written as the matrix

$$R_1^{\tilde j k'} = \begin{pmatrix} \cos\xi_2 & -\cos\xi_1\sin\xi_2 & \sin\xi_1\sin\xi_2 \\ \sin\xi_2 & \cos\xi_1\cos\xi_2 & -\sin\xi_1\cos\xi_2 \\ 0 & \sin\xi_1 & \cos\xi_1 \end{pmatrix} \tag{156}$$

using the angles ξ_1 and ξ_2 defined in Table IX.

The connection between the satellite coordinates and the Earth-centered coordinates can be written

$$x^{\tilde j} = R_1^{\tilde j k'}(R_2^{k'\hat l}x^{\hat l} + x_s^{k'}). \tag{157}$$

Here, $x_s^{k'}$ is the world line of the satellite in the intermediate coordinate system. This world line can be parametrized as

$$x_s^{k'} = (r\cos(\omega_s T + \theta_1), r\sin(\omega_s T+\theta_1), 0)\,, \tag{158}$$

where r is the magnitude of the Earth-satellite separation. The satellite therefore orbits in the x'-y' plane. Also, in Eq. (157) the rotation $R_2^{k'\hat l}$ of the satellite is given by the matrix

$$R_2^{k'\hat l} = \begin{pmatrix} \cos(\omega_r T+\theta_2) & -\sin(\omega_r T+\theta_2) & 0 \\ \sin(\omega_r T+\theta_2) & \cos(\omega_r T+\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{159}$$

The axis of the satellite rotation is therefore along $\hat z$.

Table X. Amplitudes for satellite-based WEP tests.

Amplitude	phase
$P_{\omega_r} = m^w r\omega_s\left[(\overline{c}^w)_{(TY)}\sin\xi_1 + (\overline{c}^w)_{(TX)}\cos\xi_1\right] + \frac{\omega R_\oplus^2\alpha\cos\xi_2}{5r}\left[(\overline{a}^w_{\text{eff}})_X\cos\xi_1 + (\overline{a}^w_{\text{eff}})_Y\sin\xi_1\right]$	θ_2
$Q_{\omega_r} = m^w r\omega_s\left[(\overline{c}^w)_{(TX)}\sin\xi_1\cos\xi_2 - (\overline{c}^w)_{(TY)}\cos\xi_1\cos\xi_2 - (\overline{c}^w)_{(TZ)}\sin\xi_2\right] + \frac{\omega R_\oplus^2\alpha}{5r}\left[(\overline{a}^w_{\text{eff}})_X\sin\xi_1 - (\overline{a}^w_{\text{eff}})_Y\cos\xi_1\right]$	θ_2
$P_{\omega_r+\omega_s} = 2m^w\Big[\cos\xi_2\cos 2\xi_1(\overline{c}^w)_{(XY)} + \sin\xi_2\sin\xi_1(\overline{c}^w)_{(YZ)} + \frac{1}{2}\sin 2\xi_1\cos\xi_2((\overline{c}^w)_{YY} - (\overline{c}^w)_{XX}) + \sin\xi_2\cos\xi_1(\overline{c}^w)_{(XZ)}\Big]$	$\theta_1+\theta_2$
$Q_{\omega_s+\omega_r} = m^w\Big[(\cos^2\xi_2\cos^2\xi_1 - \sin^2\xi_1 + \frac{1}{2}\sin^2\xi_2)((\overline{c}^w)_{XX} - (\overline{c}^w)_{YY}) + \frac{1}{2}\sin^2\xi_2((\overline{c}^w)_{XX} + (\overline{c}^w)_{YY} - 2(\overline{c}^w)_{ZZ}) - \cos\xi_1\sin 2\xi_2(\overline{c}^w)_{(YZ)} + \sin\xi_1\sin 2\xi_2(\overline{c}^w)_{(XZ)} + \sin 2\xi_1(1+\cos^2\xi_2)(\overline{c}^w)_{(XY)}\Big]$	$\theta_1+\theta_2$
$Q_{\omega_s-\omega_r} = m^w\Big[\left(\cos^2\xi_1\sin^2\xi_2 + \frac{1}{2}\cos^2\xi_2 + \frac{1}{2}\right)((\overline{c}^w)_{XX} - (\overline{c}^w)_{YY}) - \frac{1}{2}\sin^2\xi_2\,((\overline{c}^w)_{XX} + (\overline{c}^w)_{YY} - 2(\overline{c}^w)_{ZZ}) + 2(\overline{c}^w)_{YY} + \sin 2\xi_1\left(1-\cos^2\xi_2\right)(\overline{c}^w)_{(XY)} - \sin\xi_1\sin 2\xi_2(\overline{c}^w)_{(XZ)} + \cos\xi_1\sin 2\xi_2(\overline{c}^w)_{(YZ)}\Big] - 2\alpha(\overline{a}^w_{\text{eff}})_T$	$\theta_1-\theta_2$
$P_{2\omega_s-\omega_r} = -m^w r\omega_s\left[(\overline{c}^w)_{(TX)}\cos\xi_1 + (\overline{c}^w)_{(TY)}\sin\xi_1\right] - \frac{3\omega R_\oplus^2\alpha\cos\xi_2}{5r}\left[(\overline{a}^w_{\text{eff}})_X\cos\xi_1 + (\overline{a}^w_{\text{eff}})_Y\sin\xi_1\right]$	$2\theta_1-\theta_2$
$Q_{2\omega_s-\omega_r} = m^w r\omega_s\left[(\overline{c}^w)_{(TY)}\cos\xi_1\cos\xi_2 - (\overline{c}^w)_{(TX)}\sin\xi_1\cos\xi_2 + (\overline{c}^w)_{(TZ)}\sin\xi_2\right] - \frac{3\omega R_\oplus^2\alpha}{5r}\left[(\overline{a}^w_{\text{eff}})_X\sin\xi_1 - (\overline{a}^w_{\text{eff}})_Y\cos\xi_1\right]$	$2\theta_1-\theta_2$
$P_{\Omega+\omega_s+\omega_r} = m^w V_\oplus\Big[\left(\cos^2\xi_1 - \sin^2\xi_1\cos^2\xi_2 - \cos\eta\cos\xi_2\cos 2\xi_1 - \sin\eta\sin\xi_2\cos\xi_1\right)(\overline{c}^w)_{(TX)} + \sin\xi_1\sin\xi_2\left(\cos\xi_2 - \cos\eta\right)(\overline{c}^w)_{(TZ)} + \left(\cos\xi_1 + \cos\xi_1\cos^2\xi_2 - \sin\eta\sin\xi_2 - 2\cos\eta\cos\xi_1\cos\xi_2\right)\sin\xi_1(\overline{c}^w)_{(TY)}\Big]$	$\theta_1+\theta_2$
$Q_{\Omega+\omega_s+\omega_r} = m^w V_\oplus\Big[\left(2\cos\xi_1\cos\xi_2 - \sin\eta\sin\xi_2\cos\xi_2 - \cos\eta\cos\xi_1(1+\cos^2\xi_2)\right)\sin\xi_1(\overline{c}^w)_{(TX)} - \left(\cos 2\xi_1\cos\xi_2 - \sin\eta\cos\xi_1\sin\xi_2\cos\xi_2 + \cos\eta(1-\cos^2\xi_1\sin^2\xi_2)\right)(\overline{c}^w)_{(TY)} - \left(\cos\xi_1 - \sin\eta\sin\xi_2 - \cos\eta\cos\xi_1\right)\sin\xi_2(\overline{c}^w)_{(TZ)}\Big]$	$\theta_1+\theta_2$
$P_{\Omega+\omega_s-\omega_r} = m^w V_\oplus\Big[\left(1-\sin^2\xi_1\sin^2\xi_2\right)(\overline{c}^w)_{(TX)} + \frac{1}{2}\sin 2\xi_1\sin^2\xi_2(\overline{c}^w)_{(TY)} - \frac{1}{2}\sin\xi_1\sin 2\xi_2(\overline{c}^w)_{(TZ)}\Big] - \alpha V_\oplus(\overline{a}^w_{\text{eff}})_X$	$\theta_1-\theta_2$
$Q_{\Omega+\omega_s-\omega_r} = -m^w V_\oplus\Big[\frac{1}{2}\left(\cos\eta\sin 2\xi_1\sin^2\xi_2 - \sin\eta\sin\xi_1\sin 2\xi_2\right)(\overline{c}^w)_{(TX)} + \left(\frac{1}{2}\sin\eta\cos\xi_1\sin 2\xi_2 + (1-\sin^2\xi_2\cos^2\xi_1)\cos\eta\right)(\overline{c}^w)_{(TY)} + \left(\sin\eta\sin^2\xi_2 + \frac{1}{2}\cos\eta\cos\xi_1\sin 2\xi_2\right)(\overline{c}^w)_{(TZ)}\Big] + \alpha V_\oplus\left[(\overline{a}^w_{\text{eff}})_Z\sin\eta + (\overline{a}^w_{\text{eff}})_Y\cos\eta\right]$	$\theta_1-\theta_2$
$P_{\Omega-\omega_s+\omega_r} = m^w V_\oplus\Big[(1-\sin^2\xi_1\sin^2\xi_2)(\overline{c}^w)_{(TX)} + \frac{1}{2}\sin 2\xi_1\sin^2\xi_2(\overline{c}^w)_{(TY)} - \frac{1}{2}\sin\xi_1\sin 2\xi_2(\overline{c}^w)_{(TZ)}\Big] - \alpha V_\oplus(\overline{a}^w_{\text{eff}})_X$	$-\theta_1+\theta_2$
$Q_{\Omega-\omega_s+\omega_r} = m^w V_\oplus\Big[\frac{1}{2}\left(\sin\eta\sin\xi_1\sin 2\xi_2 - \cos\eta\sin 2\xi_1\sin^2\xi_2\right)(\overline{c}^w)_{(TX)} - \left(\frac{1}{2}\sin\eta\cos\xi_1\sin 2\xi_2 + \cos\eta(1-\cos^2\xi_1\sin^2\xi_2)\right)(\overline{c}^w)_{(TY)} - \left(\sin\eta\sin^2\xi_2 + \frac{1}{2}\cos\eta\cos\xi_1\sin 2\xi_2\right)(\overline{c}^w)_{(TZ)}\Big] + \alpha V_\oplus\left[(\overline{a}^w_{\text{eff}})_Z\sin\eta + (\overline{a}^w_{\text{eff}})_Y\cos\eta\right]$	$-\theta_1+\theta_2$
$P_{\Omega-\omega_s-\omega_r} = m^w V_\oplus\Big[\left(\cos^2\xi_1 - \sin^2\xi_1\cos^2\xi_2 + \sin\eta\cos\xi_1\sin\xi_2 + \cos\eta\cos 2\xi_1\cos\xi_2\right)(\overline{c}^w)_{(TX)} + \left(\frac{1}{2}\sin 2\xi_1(1+\cos^2\xi_2) + \sin\eta\sin\xi_1\sin\xi_2 + \cos\eta\sin 2\xi_1\cos\xi_2\right)(\overline{c}^w)_{(TY)} + \left(\frac{1}{2}\sin 2\xi_2 + \cos\eta\sin\xi_2\right)\sin\xi_1(\overline{c}^w)_{(TZ)}\Big]$	$-\theta_1-\theta_2$
$Q_{\Omega-\omega_s-\omega_r} = m^w V_\oplus\Big[-\left(\sin 2\xi_1\cos\xi_2 + \frac{1}{2}\sin\eta\sin\xi_1\sin 2\xi_2 + \frac{1}{2}\cos\eta\sin\xi_1(1+\cos^2\xi_2)\right)(\overline{c}^w)_{(TX)} + \left(\cos 2\xi_1\cos\xi_2 + \frac{1}{2}\sin\eta\cos\xi_1\sin\xi_2 - \cos\eta(\sin^2\xi_1 - \cos^2\xi_1\cos^2\xi_2)\right)(\overline{c}^w)_{(TY)} + \left(\cos\xi_1\sin\xi_2 + \sin\eta\sin^2\xi_2 + \frac{1}{2}\cos\eta\cos\xi_1\sin 2\xi_2\right)(\overline{c}^w)_{(TZ)}\Big]$	$-\theta_1-\theta_2$

For our purposes, it suffices to obtain explicitly the local differential acceleration $\Delta a^{\hat{x}}$ of the test bodies in the $\hat{x}$ direction. We have

$$\Delta a^{\hat{x}} \equiv \frac{d^2\Delta\hat{x}}{d\hat{t}^2} = \Delta a^{\hat{x}}_{\rm tidal} + \Delta a^{\hat{x}}_{\rm LV} + \ldots . \tag{160}$$

The first term on the right-hand side of this expression is the conventional Newton tidal term. It takes the form

$$\begin{aligned}\Delta a^{\hat{x}}_{\rm tidal} \;=\; &-\Big(\tfrac{3}{2}\omega_s^2 \cos(2\omega_r T - 2\omega_s T + \theta_2 - \theta_1) \\ &+\omega_r^2 + \tfrac{1}{2}\omega_s^2\Big)\Delta\hat{x}.\end{aligned} \tag{161}$$

The second term in Eq. (160) contains Lorentz-violating contributions to the differential acceleration. It can be written

$$\begin{aligned}\Delta a^{\hat{x}}_{\rm LV} \;=\; &r\omega_s^2 \sum_{w,n} \left(\frac{N_1^w}{m_1} - \frac{N_2^w}{m_2}\right) \\ &\times\Big(P_n \sin(\omega_n T + \alpha_n) + Q_n \cos(\omega_n T + \alpha_n)\Big).\end{aligned} \tag{162}$$

The amplitudes P_m, Q_m and the corresponding phases are provided in Table X. Finally, the ellipsis in Eq. (160) represents higher-order general-relativistic corrections and Lorentz-violating effects at the same post-newtonian order as $\Delta a^{\hat{x}}_{\rm tidal}$. The latter are typically of lesser interest. If desired, the differential acceleration $\Delta a^{\hat{y}}$ along $\hat{y}$ can be obtained by performing the transformation $\omega_r T \rightarrow \omega_r T - \pi/2$ on Eq. (160).

B. MicroSCOPE and STEP

Within our idealized scenario, MicroSCOPE [86] and STEP [87] can be analysed in parallel. Each apparatus consists of a pair of cylindrical test bodies made of different material but having a common symmetry axis. The test bodies are free to move along this axis. In satellite coordinates, this direction lies along $\hat{x}$ and is perpendicular both to the direction of motion of the satellite and to the axis of the satellite rotation.

One prosaic origin of relative motion of the test bodies along the $\hat{x}$ direction could be the influence of tidal forces on a misalignment of the two centers of mass, which would lead to the acceleration $\Delta a^{\hat{x}}_{\rm tidal}$ in Eq. (160). This can be separated from the acceleration due to WEP violations stemming from Lorentz-invariant sources, which enters with the characteristic frequency $\omega_s - \omega_r$. Here, we are interested in a WEP-violating acceleration $\Delta a^{\hat{x}}_{\rm LV}$ arising from the coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ for Lorentz violation. This can be distinguished from both the above effects through careful separation of the frequencies associated with the amplitudes in Table X, except for the amplitude $Q_{\omega_s-\omega_r}$.

The sensitivity goals of MicroSCOPE and STEP are $\Delta a/r\omega_s^2 < 10^{-15}$ and $\Delta a/r\omega_s^2 < 10^{-18}$, respectively. These sensitivities and the results in Table X can be used to obtain rough estimates of the reach of these experiments for studies of Lorentz violation. For this purpose, we take the quantity $N_1^w/m_1 - N_2^w/m_2$ appearing in Eq. (162) to be of order 10^{-2} GeV^{-1}, which is the best available value with the Pt-Ir, Be, and Nb test bodies presently proposed for STEP. Note that the bounds scale linearly with this difference, so a careful choice of test-body material can maximize sensitivity to Lorentz violation. Moreover, combining results for different test materials can yield additional independent sensitivities. Note also that the experimental reach may vary with the choice of orbit. For definiteness, we suppose the sines and cosines of ξ_1 and ξ_2 are of order one.

Our crude estimates for attainable sensitivities to the moduli of $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ for MicroSCOPE and STEP are presented in the first three columns of Table XI. In each row, the listed sensitivities are obtained under the assumption that all coefficients vanish except those appearing in the first entry. The key factor underlying the difference in reach for the various coefficient combinations is the boost entering the relevant amplitude in Table X. Amplitudes containing $V_\oplus$ are suppressed by roughly 10^{-4}, while those containing $r\omega_s$ are suppressed by about 10^{-5}. As before, the braces indicate the estimated sensitivities involve data from future tests.

C. Galileo Galilei

Certain design features of GG [88] differ from those of MicroSCOPE and STEP in ways that are significant for studies of Lorentz violation. Although GG also uses coaxial cylindrical test bodies, it is sensitive to accelerations in the plane perpendicular to the axis of the cylinders. Also, the cylinders are rotated about their axis at a comparatively high frequency of about 2 Hz.

In applying the generic analysis of Sec. VIII A to GG, it is convenient to take the cylinder axes to lie along $\hat{z}$. The experiment is then sensitive to accelerations in the $\hat{x}$-$\hat{y}$ plane. The differential acceleration $\Delta a^{\hat{x}}$ along $\hat{x}$ is given in Eq. (160), while $\Delta a^{\hat{y}}$ can be obtained by adjusting the phase θ_2.

The sensitivity goal of GG is $\Delta a/r\omega_s^2 < 10^{-17}$. In Table XI, we present rough estimates of the corresponding reach for measurements of the coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ for Lorentz violation, obtained using the result (162). The values for GG in the table are based on the same assumptions as those discussed above for MicroSCOPE and STEP. This includes the material-dependent factor, with the proposed materials for the GG test bodies being Be and Cu. The boost factors leading to the varying sensitivities for GG listed in the table are also of the same order of magnitude as for the other satellite experiments.

We remark in passing that the comparatively high rotation rate for the GG cylinders could introduce additional Lorentz-violating effects. Typically, the presence of nonzero $(\overline{c}^w)_{\mu\nu}$ introduces modifications to the effective moment of inertia of a body. This can affect the dynamical balance of the system, which can lead to observable signals. For example, potential effects of this type on the timing of pulsar signals have been used to constrain some combinations of $(\overline{c}^n)_{\mu\nu}$ [90]. In the present context, the observable signals could include a material-dependent Lorentz-violating wobble varying at the satellite frequency and at the Earth's orbital frequency. It is conceivable that these Lorentz-violating effects could be detected by the GG apparatus that senses the test-body location. Notice that the signals would be independent of gravity. They may be detectable using sophisticated terrestrial dynamical-balancing equipment, perhaps including that used in the Galileo Galilei on the Ground (GGG) experiment [91]. The investigation of these effects represents an interesting open question for future work.

Table XI. Sensitivities for satellite-based WEP tests.

Coefficient	MicroSCOPE	GG	STEP
$\alpha(\overline{a}_{\rm eff}^{e+p-n})_T - \frac{1}{3}m^p(\overline{c}^{e+p-n})_{TT}$	$\{10^{-13}$ GeV$\}$	$\{10^{-15}$ GeV$\}$	$\{10^{-16}$ GeV$\}$
$\alpha(\overline{a}_{\rm eff}^{e+p-n})_X$	$\{10^{-9}$ GeV$\}$	$\{10^{-11}$ GeV$\}$	$\{10^{-12}$ GeV$\}$
$\alpha(\overline{a}_{\rm eff}^{e+p-n})_{Y+Z}$	$\{10^{-9}$ GeV$\}$	$\{10^{-11}$ GeV$\}$	$\{10^{-12}$ GeV$\}$
$\alpha(\overline{a}_{\rm eff}^{e+p-n})_Y$	$\{10^{-7}$ GeV$\}$	$\{10^{-9}$ GeV$\}$	$\{10^{-10}$ GeV$\}$
$\alpha(\overline{a}_{\rm eff}^{e+p-n})_Z$	$\{10^{-7}$ GeV$\}$	$\{10^{-9}$ GeV$\}$	$\{10^{-10}$ GeV$\}$
$(\overline{c}^n)_Q$	$\{10^{-13}\}$	$\{10^{-15}\}$	$\{10^{-16}\}$
$(\overline{c}^n)_{(TJ)}$	$\{10^{-9}\}$	$\{10^{-11}\}$	$\{10^{-12}\}$

IX. EXOTIC GRAVITATIONAL TESTS

In this section, we offer a few remarks about some gravitational searches for Lorentz violation using material test bodies other than neutral bulk matter, neutral atoms, or neutrons. These more exotic searches typically present unique experimental challenges, but they could provide access to combinations of coefficients for Lorentz violation that are awkward or impossible to isolate and measure in other searches discussed in this paper. Here, we briefly consider tests with electrons and ions, studies with antihydrogen, and experiments using particles from the second and third generation of the SM.

A. Tests with electrons and ions

Measurements of the gravitational acceleration of charged matter remain of definite theoretical interest because the WEP and other foundational aspects of gravity are comparatively poorly tested in this regime. In this subsection, we consider possible signals from studies of charged electrons or ions. Given the experimental challenges of these tests and their limited attainable sensitivities, we restrict attention here to effects from $(\overline{a}_{\rm eff}^w)_\mu$, setting other coefficients to zero for simplicity.

In the context of searches for Lorentz violation, gravitational tests with charged matter offer unique access to the coefficients $(\overline{a}_{\rm eff}^w)_\mu$. For example, measurements of this kind can disentangle coefficients for Lorentz violation in the proton and electron sectors. They can also detect certain countershaded effects that are otherwise invisible. In particular, some models have coefficients $(\overline{a}_{\rm eff}^w)_\mu$ proportional to electric charge, which would evade detection in searches with neutral test bodies [11]. This possibility is a natural consequence for theories in which the photon modes are interpreted as Nambu-Goldstone bosons from spontaneous Lorentz breaking and in which $(\overline{a}_{\rm eff}^w)_\mu$ remains physically observable, such as nonminimally coupled bumblebee electrodynamics [14].

One candidate technique to measure gravitational effects from the coefficients $(\overline{a}_{\rm eff}^w)_\mu$ is charged-particle interferometry. Electron interferometry has been used to measure the Sagnac effect at the 30% level [92], while ion interferometry is under investigation as a practical tool for sensitive tests of Coulomb's law [93]. In the present context, electron or ion interferometry offers an interesting alternative prospect to the free-fall tests with neutral matter discussed in Sec. VII. For a given geometry, the observed phase shift can be determined using the methods of Sec. VII B. In the limit of interest here, the vertical acceleration $\mathrm{a}_{\hat z}$ of the electron or ion T in the gravitational field of the Earth S is given at PNO(2) by

$$\mathrm{a}_{\hat z} = -g - \frac{2g\alpha}{m^{\rm T}}(\overline{a}_{\rm eff}^{\rm T})_{\hat t} - \frac{2g\alpha}{m^{\rm S}}(\overline{a}_{\rm eff}^{\rm S})_{\hat t}. \tag{163}$$

As before, the PNO(3) version of this acceleration can be frequency decomposed relative to the Sun-centered frame, with the corresponding amplitudes depending on the coefficients $(\overline{a}_{\rm eff}^w)_\mu$ as given in Table IV.

In principle, a charged-particle interferometer can be used for free-fall gravimeter tests of the type discussed in Sec. VII B or for free-fall WEP tests as in Sec. VII D. A free-fall gravimeter test is insensitive to $(\overline{a}_{\rm eff}^w)_T$ and has only boost-suppressed signals from $(\overline{a}_{\rm eff}^w)_J$, so a substantial improvement over the existing reach of charged-matter interferometers would be required to achieve a sensitivity compatible with perturbative consistency. In contrast, a free-fall WEP test is directly sensitive to $(\overline{a}_{\rm eff}^w)_T$ but requires a simultaneous measurement with two test bodies. One option along these lines could be a direct comparison with neutral matter via a falling corner cube or an atom interferometer.

Another approach to gravitational tests with charged matter is to study the motion of charged particles in a vertical metallic drift tube. This setup is accompanied by gravitationally induced electric forces caused by the sagging of the tube [94], along with a variety of challenging systematics. An experiment of this type with electrons [95] confirmed that the gravitational forces on the electrons in the tube and on the electrons within the metal are comparable to about 10%. An analogous experiment involving cold antiprotons [96] was designed to achieve a sensitivity of 0.1% to the gravitational acceleration [97]. These measurements are all experimentally challenging, and their interpretation is theoretically subtle [98].

In the present context of gravitational Lorentz violation involving the coefficients $(\overline{a}^w_{\rm eff})_\mu$, intuition for the theoretical implications of ballistic tests of this type can be gained by considering the idealized case and working at PNO(2). We suppose a test particle T of charge $q^{\rm T}$ moves along the symmetry axis of a vertical cylindrical metallic drift tube with body comprised of a lattice of ions of type I and conduction electrons e. Disregarding applied fields, stray fields, and various systematics, the overall conventional force on the particle T is the sum of the direct gravitational force on T from the Earth S and the net force on T from the electromagnetic field arising from the gravitationally induced sagging of the tube. The presence of nonzero coefficients $(\overline{a}^w_{\rm eff})_\mu$ introduces corrections to both these forces. At PNO(2), the gravitational force on T is given by

$$(F_{\rm grav})_{\hat z} \;=\; -m^{\rm T} g - 2g\alpha(\overline{a}^{\rm T}_{\rm eff})_{\hat t} - 2g\alpha\frac{m^{\rm T}}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_{\hat t}, \tag{164}$$

while the vertical component of the force on T from the gravitationally induced electric field is

$$\begin{aligned}(F_{\rm em})_{\hat z} \;=\;& \frac{q^{\rm T}}{e}\left(m^e g + 2g\alpha(\overline{a}^e_{\rm eff})_{\hat t} + 2g\alpha\frac{m^e}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_{\hat t}\right)\\ &+\gamma\frac{q^{\rm T}}{e}\left(m^I g + 2g\alpha(\overline{a}^I_{\rm eff})_{\hat t} + 2g\alpha\frac{m^I}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_{\hat t}\right).\end{aligned} \tag{165}$$

In these expressions, m^e and m^I are the masses of an electron and an ion in the tube lattice, respectively, while e and q^I are the corresponding charges. The factor γ is a constant, set by the properties of the metal lattice. In Eq. (165), the first three terms arise from the sagging of the electrons in the tube walls, while the last three are proportional to the dilation derivative of the work function for the metal and originate in the longitudinal compression of the lattice. These expressions reduce to standard ones when the coefficients $(\overline{a}^w_{\rm eff})_\mu$ vanish.

Although the expressions (164) and (165) hold in an idealized situation, they suffice to demonstrate in principle that experiments of this type are sensitive to nonzero coefficients $(\overline{a}^w_{\rm eff})_\mu$, even when these coefficients are undetectable with neutral matter. This is also true if the particle T is an electron, when the sum of the forces $(F_{\rm grav})_{\hat z}$ and $(F_{\rm em})_{\hat z}$ leaves only the last three terms in Eq. (165). In practice, however, the reported reach of drift-tube experiments to date is insufficient to achieve useful sensitivity in gravimeter tests. A WEP test relating a drift-tube setup to an independent gravimeter may be of more interest. For a given experiment, specific sensitivities can be estimated using the analyses presented in Sec. VII.

A third methodology for investigating gravitational Lorentz violation from nonzero coefficients $(\overline{a}^w_{\rm eff})_\mu$ could conceivably be to adopt as the gravimeter a device that wholly confines charged particles. For example, a single charged particle can be trapped for long periods using a Penning trap [99]. Measuring gravitational effects in this way is ambitious, as can be appreciated from the size of the quantity $|m^e g/e| \simeq 6\times 10^{-12}$ V/m. Nonetheless, the feasibility of gravitational measurements with trapped antiprotons at a sensitivity of about 1% has been suggested, using a gravity-induced shift of radial orbits [100]. This would also lead to sensitivity to the coefficients $(\overline{a}^w_{\rm eff})_\mu$ via an analysis similar to those discussed above.

B. Tests with antimatter

The study of antimatter offers another realm in which to search for Lorentz and CPT violation. Antihydrogen has been detected [101, 102] and produced in copious amounts [103], while prospects for studies of trapped cold antihydrogen are excellent [104]. Antihydrogen spectroscopy could yield special sensitivity to nongravitational SME coefficients for Lorentz and CPT violation [105], and the experiment for Atomic Spectroscopy And Collisions Using Slow Antiprotons (ASACUSA) expects to achieve sensitivities of parts in 10^{-7} to the predicted shifts in hyperfine transitions [106].

To study the interaction of gravity and antimatter, various ideas for measuring the gravitational acceleration of antihydrogen have been advanced. Among them are methods involving trapped antihydrogen [107], antihydrogen interferometry [108], antihydrogen free fall from an antiion trap [109], and tests in space [110]. One approved project, the Antimatter Experiment: Gravity, Interferometry, Spectroscopy (AEGIS) [111], has an interoferometric design with an initial sensitivity goal of 1% to the gravitational acceleration of antihydrogen.

In the context of gravitational Lorentz and CPT violation, these experiments offer the prospect of special sensitivities to the coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$. The key point is that a CPT transformation has the net effect of reversing the sign of $(\overline{a}^w_{\rm eff})_\mu$ while leaving $(\overline{c}^w)_{\mu\nu}$ unchanged. As a result, experiments with antihydrogen could in principle observe distinctive and novel behaviors. Moreover, when compared with similar measurements on hydrogen, the results would offer the opportunity for clean separation of effects. For instance, free-fall

WEP tests comparing hydrogen and antihydrogen could yield independent sensitivity to $(\overline{c}^{e+p})_{TT}$. In general, the theoretical treatment of prospective free-fall gravimeter or WEP tests with antihydrogen follows the same path as described in Sec. VII, except with the sign of $(\overline{a}^w_{\rm eff})_\mu$ reversed throughout.

The literature contains numerous attempts to place indirect limits on the possibility of unconventional antimatter-gravity interactions, many of which are reviewed and critiqued in Ref. [112]. In the present context, the SME offers a general field-theoretic approach that can elucidate aspects of this issue and provide new insights about possible limitations on effects. We next present an explicit toy model that evades some previous indirect limits on large unconventional effects in antihydrogen.

For simplicity, we choose to work within the isotropic limit of the SME. In any specified inertial frame O, a subset of Lorentz-violating operators in the SME Lagrange density preserves rotational symmetry. Setting the coefficients of all other operators to zero produces an interesting limiting case. The frame O then becomes a preferred frame, since the rotation invariance is broken in any frame O' boosted with respect to O. Physical effects of Lorentz violation are then isotropic in O but not in O'. The frame O could in principle be identified as the rest frame U of the cosmic microwave background (CMB), the Sun-centered frame S, or any other desired choice. Isotropic models of this type are sometimes called 'fried-chicken' models because of their popularity and simplicity.

In Minkowski spacetime, toy isotropic models can be used to show that Lorentz- and CPT-violating effects could in principle be substantially larger in antihydrogen than hydrogen. One example is the isotropic 'invisible' model (IIM) [113], which is defined in the CMB frame U and yields effects challenging to see in searches with ordinary matter. Denoting coordinates in U by (T', X', Y', Z'), the IIM assumes the only nonzero coefficients for Lorentz violation are $(b^p)_{T'}$ and isotropic $(d^p)_{\Xi'\Xi'}$ obeying the simple condition

$$(b^p)_{T'} = k m^p (d^p)_{T'T'} \tag{166}$$

for a suitable choice of constant k. In the Sun-centered frame S, this one-parameter model generates nonzero coefficients $(b^p)_J$ and $(d^p)_{JT}$. The dominant signals in terrestrial experiments with hydrogen appear in the hyperfine structure and involve the combination $(b^p)_J - m^p(d^p)_{JT}$, which vanishes for suitable k. These experiments can therefore detect only effects suppressed by at least one power of the boost of the Earth around the Sun, which is about 10^{-4} and requires an experiment sensitive to annual modulations. In contrast, the dominant effects in experiments with antihydrogen involve the combination $(b^p)_J + m^p(d^p)_{JT}$, which produces unsuppressed signals in the hyperfine structure. The IIM thus provides a toy field-theoretic scenario in which observable effects in antihydrogen are at least 10,000 times greater than those in hydrogen or other nonrelativistic neutral matter.

The IIM involves spin-dependent operators for Lorentz and CPT violation in Minkowski spacetime. In this work, the focus is on the gravitational couplings of spin-independent operators with coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$. At the end of Sec. VIII A, we remark on the difficulty of observing with matter any signals depending on the combination $\alpha(\overline{a}^{\rm T}_{\rm eff})_T - m^{\rm T}(\overline{c}^{\rm T})_{TT}/3$ of isotropic coefficients. Here, we consider some implications for antimatter gravity of a specific toy model, the isotropic 'parachute' model (IPM), in which unobserved combinations of this type provide the dominant source of Lorentz-violating effects and could yield significant *a priori* differences in the gravitational accelerations of hydrogen and antihydrogen.

To construct the IPM, consider the Lagrange density of the SME in the Sun-centered frame S, with nonzero coefficients restricted to $(\overline{a}^w_{\rm eff})_T$ and isotropic $(\overline{c}^w)_{\Sigma\Xi}$. Following the derivation in the early sections of this work, we can extract the PNO(3) effective classical lagrangian for a test particle T moving in the gravitational field of a source S. This can be written in the suggestive form

$$L_{\rm IPM} = \tfrac{1}{2} m_i^{\rm T} v^2 + \frac{G_N m_g^{\rm T} m_g^{\rm S}}{r}, \tag{167}$$

where $m_i^{\rm T}$ is the effective inertial mass of T, while $m_g^{\rm T}$ and $m_g^{\rm S}$ are the effective gravitational masses of T and S, respectively. All these effective masses are defined in terms of the coefficients $(\overline{a}^w_{\rm eff})_T$, $(\overline{c}^w)_{TT}$ for Lorentz violation and the body masses $m^{\rm B}$ of Eq. (71). We find

$$\begin{aligned} m_i^{\rm B} &= m^{\rm B} + \sum_w \tfrac{5}{3}(N^w + N^{\bar w}) m^w (\overline{c}^w)_{TT} \\ m_g^{\rm B} &= m^{\rm B} + \sum_w \Big((N^w + N^{\bar w}) m^w (\overline{c}^w)_{TT} \\ &\qquad + 2\alpha (N^w - N^{\bar w}) (\overline{a}^w_{\rm eff})_T \Big), \end{aligned} \tag{168}$$

where B is either T or S. These expressions adopt the notation N^w and $N^{\bar w}$ for the number of particles and antiparticles of type w, respectively, while as before m^w is the mass of a particle of type w. Note that for a given body the passive and active gravitational masses are identical, reflecting the preservation of Newton's third law in the model.

For electrons, protons, and neutrons, the IPM is defined by the three conditions

$$\alpha(\overline{a}^w_{\rm eff})_T = \tfrac{1}{3} m^w (\overline{c}^w)_{TT}, \tag{169}$$

where w ranges over e, p, n. Since there are three independent conditions on six real parameters, this produces a three-parameter IPM. The condition (169) ensures that for a matter body B the effective inertial and gravitational masses are equal,

$$m_i^{\rm B} = m_g^{\rm B} \qquad \text{(matter)}, \tag{170}$$

and hence no Lorentz-violating effects appear in gravitational tests to PNO(3) using ordinary matter. However, for an antimatter test body T this condition fails,

$$m_i^{\mathrm{T}} \neq m_g^{\mathrm{T}} \qquad \text{(antimatter)}, \tag{171}$$

so observable signals arise in comparisons between the gravitational responses of matter and antimatter or between different types of antimatter. Ensuring the validity of perturbation theory requires that the coefficients $\alpha(\overline{a}^w_{\mathrm{eff}})_T = m^w(\overline{c}^w)_{TT}/3$ are perturbatively small relative to m^w. With theoretically conceivable values perhaps even as large as $0.5m^w$, the gravitational accelerations of hydrogen and antihydrogen might differ at the 50% level.

Rather than a serious effort at a realistic theory, the IPM is constructed as a simplistic playground within which to explore field-theoretic limitations on unconventional properties of antimatter and antihydrogen. In the next few paragraphs we treat it as such, briefly addressing some concerns about unconventional signals in this context.

One issue is whether energy remains conserved when matter and antimatter have different gravitational interactions [114]. For the analysis of the SME in the present work, this issue is moot because an explicit conserved energy-momentum tensor exists. As an illustration, consider the gedanken experiment in which a particle-antiparticle pair is lowered in a gravitational field, converted to a photon pair, raised to the original location, and finally reconverted to the particle-antiparticle pair. In generic scenarios the particle, antiparticle, and photons each provide different contributions to the energy and so problems can arise. However, in the IPM these complications are avoided. The photons make no contribution because they are conventional, partly via the coordinate choice (23). The particle and antiparticle do contribute to the energy via the coefficient $(\overline{a}^w_{\mathrm{eff}})_T$, but the two contributions cancel. Contributions involving the coefficient $(\overline{c}^w)_{TT}$ exist and combine during the lowering procedure, but the definition (70) of the conserved energy also contains $(\overline{c}^w)_{TT}$ and so the net change remains zero at the end of the experiment. The resolution of this and other illustrative scenarios is less transparent when more nonzero coefficients for Lorentz violation are present, but the existence of a conserved energy-momentum tensor ensures that no contradictions arise.

Another attempt to argue against the possibility of an anomalous antimatter response to gravity is based on the large binding energy content of baryons, atoms, and bulk matter [115]. For hydrogen and antihydrogen, a modern version of the argument could proceed by first noting that the quarks in hydrogen contain only about 10% of the mass with most of the remainder contained in the gluon and sea binding, and then concluding that since the binding forces are comparable for hydrogen and antihydrogen their gravitational response cannot differ by more than about 10%. This type of reasoning implicitly assumes that the gravitational response of a body is determined by its mass and hence also by its binding energy. However, as shown generically in Sec. VIII A, the coefficient $(\overline{a}^w_{\mathrm{eff}})_T$ in the IPM leads to a correction to the gravitational force that is independent of mass and can vary with flavor. Indeed, the binding forces are largely conventional in the IPM, and the gravitational responses of hydrogen and antihydrogen are primarily determined by the flavor content of the valence particles. It is even conceivable in principle that a large gravity effect could be associated purely with the positron, as occurs in the IPM when only $(\overline{a}^e_{\mathrm{eff}})_T$ is nonzero and satisfies the condition (169). A careful treatment of this issue in the IPM would require consideration of radiative effects involving $(\overline{a}^w_{\mathrm{eff}})_T$, $(\overline{c}^w)_{TT}$, and other SME coefficients for Lorentz violation [18, 116], perhaps imposing the condition (169) only after renormalization. In any case, the essential points illustrated with the IPM remain valid: the gravitational response of a body can be independent of mass, can vary with flavor, and can differ between particles and antiparticles.

The gravitational response of antimatter could in principle also be restricted by the results of experiments studying kaons [117] and other neutral-meson systems, which are natural interferometers mixing strong-interaction particle and antiparticle eigenstates via weak-interaction effects. When analyzed in the context of the SME in Minkowski spacetime, neutral-meson mixing places tight constraints on certain differences of the coefficients $(\overline{a}^w_{\mathrm{eff}})_\mu$ for w ranging over several quark flavors [53, 54]. However, these constraints have no dominant implications for leptons or for baryons, which involve three valence quarks rather than a quark and an antiquark as in mesons. Moreover, the neutral-meson constraints necessarily involve valence s, c, and b quarks, which are largely irrelevant for protons and neutrons. In the presence of gravitational interactions, the same line of reasoning holds, with the flavor dependence of Lorentz and CPT violation leading to the conclusion that the IPM evades restrictions from meson oscillations.

We can also use the IPM to illustrate a type of constraint on more realistic model building arising from the extensive searches for Lorentz and CPT violation in Minkowski spacetime. The key point is that the mixing of Lorentz-violating operators under rotations and boosts can imply indirect limits on some coefficients. In the IPM, for example, the coefficient $(\overline{a}^w_{\mathrm{eff}})_T$ is unobservable in Minkowski spacetime, as discussed in Sec. II C, but certain nongravitational experiments could in principle obtain boost-suppressed sensitivity to $(\overline{c}^w)_{TT}$ for some w via measurements of the coefficients $(\overline{c}^w)_{JK}$. As one illustration, a measurement with a Cs-Rb double fountain clock over a total of five weeks in the spring and fall of 2005 achieved a sensitivity of parts in 10^{25} on some combinations of the coefficients $(\overline{c}^p)_{JK}$ [118]. This suggests that continuing an experiment of this type over a longer period could attain parts in 10^{17} on the coefficient $(\overline{c}^p)_{TT}$ by analysing the data allowing for the Earth's orbital boost $V_\oplus \simeq 10^{-4}$. Similarly, a careful analysis of multiple searches for Lorentz violation involving the electron sector could be used to measure $(\overline{c}^e)_{TT}$ at the level of parts in 10^{15} [119]. Although these types of nongravitational studies remain to be performed, they could in principle place experimental limits on the magnitude of the anomalous gravitational response of antihydrogen in the IPM and possibly also in more realistic models. We remark in passing that these kinds of constraints nonetheless leave considerable room for realistic model building, in particular when operators of arbitrary dimension are incorporated in the framework [120].

C. Tests with matter beyond the first generation

Most studies of fermion-gravity couplings to date have involved particles from the first generation of the SM. However, the SME coefficients for Lorentz and CPT violation can differ between sectors, so investigations of higher-generation matter-gravity couplings are of independent interest. Since fermion masses and hence fermion-gravity couplings typically increase with the generation, it is conceivable that an unconventional gravitational coupling may be more readily identified in gravitational tests with higher-generation matter. Comparatively few results exist for the coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ for particles w beyond the first generation [2], so there is considerable room for measurements of effects involving gravity couplings.

The comparatively long lifetime of the muon makes it an interesting candidate for gravitational tests of Lorentz violation with second-generation particles. Several muon coefficients for Lorentz and CPT violation have already been measured [121], but the sensitivities are largely limited to spin-dependent effects. Measurements of Lorentz-violating gravitational couplings of the muon could be achieved via muonium interferometry, with an estimated initial reach of 10% [122]. Interferometry with muonic hydrogen may also be possible [123]. In principle, these experiments could yield first measurements of some components of the coefficients $(\overline{a}^\mu_{\rm eff})_\mu$ and $(\overline{c}^\mu)_{\mu\nu}$ in the muon sector. In particular, free-fall WEP tests using muonium interferometry to search for Lorentz and CPT violation offer the prospect of direct sensitivity to the coefficients $(\overline{a}^\mu_{\rm eff})_T$ and $(\overline{c}^\mu)_{TT}$. In contrast, performing free-fall gravimetric tests with muonium interferometry is unlikely to be useful in the near future because the dominant signals appear at annual frequencies and are suppressed by the boost $V_\oplus$.

Consider for definiteness a free-fall WEP experiment comparing the gravitational acceleration of muonium with that of neutral matter N. Muonium is a bound system containing an antimuon and an electron, so its spin-independent Lorentz-violating gravitational properties are determined by the coefficients $-(\overline{a}^\mu_{\rm eff})_\mu$, $(\overline{c}^\mu)_{\mu\nu}$, $(\overline{a}^e_{\rm eff})_\mu$ and $(\overline{c}^e)_{\mu\nu}$. Following the line of reasoning in Sec. VI B, we find that the dominant observable combination of coefficients for CPT-odd effects in a free-fall WEP experiment is

$$(\overline{a}^{\mu+e-N}_{\rm eff})_\mu = -(\overline{a}^\mu_{\rm eff})_\mu + (\overline{a}^e_{\rm eff})_\mu - \frac{m^\mu + m^e}{m^N}(\overline{a}^N_{\rm eff})_\mu, \quad (172)$$

where m^N is the mass of N and $(\overline{a}^N_{\rm eff})_\mu$ is its effective coefficient for Lorentz and CPT violation. Assuming N is composed of first-generation particles, the existing constraints on coefficients [2] imply that for most models it is a good approximation to neglect all but the first term on the right-hand side of this equation. For CPT-even effects, the relevant observable combination of coefficients is

$$(\overline{c}^{\mu+e-N})_{\mu\nu} = (\overline{c}^\mu)_{\mu\nu} + \frac{m^e}{m^\mu}(\overline{c}^e)_{\mu\nu} - \frac{m^\mu + m^e}{m^\mu}(\overline{c}^N)_{\mu\nu}. \quad (173)$$

Again, only the first term is likely to be significant in practice. Similar expressions hold for muonic hydrogen, with the replacements $e \to p$ for the superscripts and $(\overline{a}^\mu_{\rm eff})_\mu \to -(\overline{a}^\mu_{\rm eff})_\mu$ for the muon coefficient for Lorentz violation.

Searches for Lorentz-violating gravitational couplings of other second- and third-generation particles could also be countenanced. The typically short lifetimes of these particles can in principle be overcome by boosting, so accelerator experiments are likely to provide the best laboratory prospects. Studying the gravitational infall of particles of extraterrestrial origin in the context of free-fall WEP searches might be a source of additional constraints.

The physical mixing of uncharged particles of different flavors w offers an interesting alternative method to achieve sensitivity to the coefficients $(\overline{a}^w_{\rm eff})_\mu$. Examples already yielding SME constraints on Lorentz and CPT violation include the interferometric oscillations of neutral mesons [53, 54] and of neutrinos [55]. Particle mixing implies nondiagonal terms in the propagator matrix, so field redefinitions of the type (14) cannot be used to remove the coefficients $(\overline{a}^w_{\rm eff})_\mu$ from the theory. Differences between the coefficients $(\overline{a}^w_{\rm eff})_\mu$ then become observable even in Minkowski spacetime, offering sensitivity to effects that would otherwise be undetectable. For instance, the recent observation of anomalous CP-violating effects in B-meson oscillations [124] could originate in one or more nonzero coefficients $(\overline{a}^w_{\rm eff})_T$ for Lorentz and CPT violation in the quark sector, since these control CP-odd but T-even operators that contribute to the effective hamiltonian for the mixing [54]. Spin-independent CPT-odd Lorentz violation involving coefficients such as $(\overline{a}^w)_\mu$ could also underlie the observed baryon asymmetry in the Universe [125].

In terms of the perturbative counting scheme of Sec. II D, the existing SME studies using neutral-meson and neutrino oscillations lie at O(1,0). Incorporating leading-order gravitational couplings along the lines in this paper would introduce O(1,1) oscillation effects, including species-dependent modifications of the meson or neutrino trajectories with characteristic time dependences similar to the WEP-violating effects discussed in Sec. VII D. Possible O(1,0) contributions to the oscillations can be distinguished from O(1,1) ones via the dependences on energy, baseline, flavor, and time. The advent of neutrino-oscillation experiments with long and very long baselines of order 100-1000 km and corresponding changes in gravitational potential along the beams may offer particularly interesting options for free-fall WEP tests of Lorentz and CPT violation of this type. A detailed consideration of these possibilities would be a worthwhile subject for future investigation.

X. SOLAR-SYSTEM TESTS

Studies of the motion of bodies within the solar system provide an important source of information about gravitational couplings to matter. In this section, we investigate the effects of nonzero coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ for Lorentz violation in two solar-system contexts: lunar and satellite laser ranging, and perihelion precession. The analysis here neglects effects that act merely to scale the mass of the gravitational source. These are unobservable using solar-system observations alone, but they may be detectable in combined measurements using photon tests. This latter issue is revisited in Sec. XI.

A. Lunar and satellite laser ranging

Lunar and satellite laser ranging provides a sensitive test of gravitational physics. The relevant orbital perturbations to the motion of a satellite orbiting the Earth that arise from nonzero Lorentz violation in the pure-gravity sector of the minimal SME have been established [7] and used to constrain some of the coefficients $\overline{s}_{\mu\nu}$ [4]. Here, we seek to extend these results to include dominant effects from nonzero coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$.

Where possible in this subsection, we follow the conventions of Ref. [7]. A summary of our notation is given in Table XII. The flavor dependence of the matter-gravity couplings leads to composition-dependent factors in some of the equations to follow. To simplify these expressions, it is useful to define the eight combinations

$$\begin{aligned}
n_1^w &= N_1^w + N_2^w, \\
n_2^w &= N_1^w - N_2^w, \\
n_3^w &= M\left(\frac{N_1^w}{m_1} + \frac{N_2^w}{m_2}\right), \\
n_4^w &= M\left(\frac{N_2^w}{m_2} - \frac{N_1^w}{m_1}\right), \\
n_5^w &= \frac{1}{M}(m_1 N_2^w + m_2 N_1^w), \\
n_6^w &= \frac{1}{M}\left(m_1 N_2^w - m_2 N_1^w\right), \\
n_7^w &= \frac{m_2}{m_1}N_1^w + \frac{m_1}{m_2}N_2^w, \\
n_8^w &= \frac{1}{M}\left(\frac{m_2^2}{m_1}N_1^w - \frac{m_1^2}{m_2}N_2^w\right).
\end{aligned} \tag{174}$$

Table XII. Notation for laser-ranging tests.

Quantity	Definition
m_1	satellite mass
N_1^w	number of particles of species w in the satellite
m_2	Earth mass
N_2^w	number of particles of species w in the Earth
$M = m_1 + m_2$	total Earth-satellite mass
$\delta m = m_2 - m_1$	Earth-satellite mass difference
m_n	mass of the nth perturbing body
$M_\odot$	Sun mass
$N_\odot^w$	number of particles of species w in the Sun
r_1^J	satellite position
r_2^J	Earth position
$r^J = r_1^J - r_2^J = (x,y,z)$	Earth-satellite separation, of magnitude $r = \lvert\vec{r}_1 - \vec{r}_2\rvert$
$R^J = (m_1 r_1^J + m_2 r_2^J)/M$	position of Newton center of mass for Earth-satellite system
$\Omega_\oplus = \sqrt{G_N M_\odot/R^3}$	mean Earth orbital frequency
$v^J = v_1^J - v_2^J = dr^J/dT$	relative Earth-satellite velocity
$V^J = (m_1 v_1^J + m_2 v_2^J)/M$	velocity of Newton center of mass for Earth-satellite system

The primary observable in laser-ranging tests is the coordinate acceleration $a_{\rm ES}^J$ of the relative Earth-satellite separation. Working in the Sun-centered frame, we can obtain this acceleration from the equation of motion (78).

The relevant contributions to the coefficient and metric fluctuations from $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ can be found in Sec. IV C, while those from $\overline{s}_{\mu\nu}$ are given in Ref. [7].

Incorporating perturbative effects of other bodies including the Sun, the coordinate acceleration can be written

$$\mathrm{a}^J_{\rm ES} \equiv \frac{d^2 r^J}{dT^2} = \mathrm{a}^J_{\rm N} + \mathrm{a}^J_{\rm T} + \mathrm{a}^J_{\rm Q} + \mathrm{a}^J_{\rm LV} + \dots . \qquad (175)$$

The first three terms in this expression involve effects independent of Lorentz violation. They represent the acceleration due to the Newton gravitational field of the Earth-satellite system, the Newton tidal quadrupole term, and the quadrupole moment of the Earth, respectively. Their explicit form is given in Ref. [7]. The leading Lorentz-violating contributions to the acceleration are represented by the fourth term $\mathrm{a}^J_{\rm LV}$. This term can itself be split into four pieces,

$$\mathrm{a}^J_{\rm LV} = \mathrm{a}^J_{\overline{a}_{\rm eff},\overline{c},{\rm ES}} + \mathrm{a}^J_{\overline{a}_{\rm eff},\overline{c},{\rm tidal}} + \mathrm{a}^J_{\overline{s},{\rm ES}} + \mathrm{a}^J_{\overline{s},{\rm tidal}}. \qquad (176)$$

The first two terms are the ones of interest in the present work and are discussed below. The last two depend on the coefficient $\overline{s}_{\mu\nu}$, with $\mathrm{a}^J_{\overline{s},{\rm ES}}$ arising from the Earth-satellite system and $\mathrm{a}^J_{\overline{s},{\rm tidal}}$ involving perturbations due to other bodies. The explicit form of these two quantities is provided in Ref. [7].

The term $\mathrm{a}^J_{\overline{a}_{\rm eff},\overline{c},{\rm ES}}$ in Eq. (176) provides the Lorentz-violating acceleration of the Earth-satellite system from the matter-gravity couplings $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$. It takes the form

$$\begin{aligned}
\mathrm{a}^J_{\overline{a}_{\rm eff},\overline{c},{\rm ES}} = \frac{G_N}{r^3}\sum_w \Big[&-2n_3^w\alpha(\overline{a}^w_{\rm eff})_T r^J - n_1^w m^w(\overline{c}^w)_{TT} r^J + 2n_7^w m^w \eta^{JK}(\overline{c}^w)_{(KL)} r^L \\
&- 2n_3^w\alpha(\overline{a}^w_{\rm eff})_K V^K r^J - 2n_2^w\alpha(\overline{a}^w_{\rm eff})_K \eta^{JK} v_L r^L + 2n_2^w\alpha(\overline{a}^w_{\rm eff})_K v^K r^J + 2n_7^w m^w(\overline{c}^w)_{(TK)} V^J r^K \\
&- 2n_1^w m^w(\overline{c}^w)_{(TK)} V^K r^J + 2n_6^w m^w(\overline{c}^w)_{(TK)} v^K r^J + 2n_7^w m^w \eta^{JK}(\overline{c}^w)_{(TK)} V_L r^L \\
&- 2(n_6^w - 2n_8^w) m^w \eta^{JK}(\overline{c}^w)_{(TK)} v_L r^L + 2n_8^w m^w(\overline{c}^w)_{(TK)} v^J r^K \Big].
\end{aligned} \qquad (177)$$

In principle, $\mathrm{a}^J_{\overline{a}_{\rm eff},\overline{c},{\rm ES}}$ also acquires contributions proportional to $R_\oplus \omega$, but these are neglected here because they are typically suppressed compared to effects proportional to V^J and v^j.

In Eq. (176), the term $\mathrm{a}^J_{\overline{a}_{\rm eff},\overline{c},{\rm tidal}}$ contains the Lorentz-violating tidal acceleration involving $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$, which arises from perturbing bodies. When the satellite is taken as the Moon, the dominant tidal contributions are due to the Sun and can be written

$$\begin{aligned}
\mathrm{a}^J_{\overline{a}_{\rm eff},\overline{c},{\rm tidal}} = \Omega_\oplus^2 \sum_w \Big\{ &\Big[\frac{N^w_\odot}{m_\odot}\left(2\alpha(\overline{a}^w_{\rm eff})_T + m^w(\overline{c}^w)_{TT}\right) - \frac{2}{M} n_1^w \alpha(\overline{a}^w_{\rm eff})_T\Big]\left(3r^L\hat{R}_L\hat{R}^J - r^J\right) \\
&+ \frac{2}{M} n_4^w \alpha(\overline{a}^w_{\rm eff})_T R^J - 2\frac{m^w}{M} n_4^w \eta^{JK}(\overline{c}^w)_{KL} R^L - 2\frac{m^w}{M} n_7^w \eta^{JK}(\overline{c}^w)_{(KL)}\left(3r^M\hat{R}_M\hat{R}^L - r^L\right) \\
&- 2\frac{m^w}{M}\Big[2n_4^w\eta^{JK}V_L(\overline{c}^w)_{(TK)} - 2n_7^w\eta^{JK}v_L(\overline{c}^w)_{(TK)} + n_4^w V^J(\overline{c}^w)_{(TL)} - n_7^w v^J(\overline{c}^w)_{(TL)}\Big] R^L \\
&+ 4\frac{N^w_\odot}{m_\odot}\alpha(\overline{a}^w_{\rm eff})_K \eta^{J[K}v^{L]} R_L - 4\frac{N^w_\odot}{m_\odot}\alpha(\overline{a}^w_{\rm eff})_K \left(\frac{\delta m}{M}\eta^{J[K}v^{L]} + \eta^{J[K}V^{L]}\right)\left(3r^M\hat{R}_M\hat{R}_L - r_L\right) \\
&- 2\frac{m^w}{M}\Big[2n_7^w\eta^{JK}V_L(\overline{c}^w)_{(TK)} + 2n_8^w\eta^{JK}v_L(\overline{c}^w)_{(TK)} \\
&\qquad + n_7^w V^J(\overline{c}^w)_{(TL)} + n_8^w v^J(\overline{c}^w)_{(TL)}\Big]\left(3r^M\hat{R}_M\hat{R}^L - r^L\right)\Big\}.
\end{aligned} \qquad (178)$$

If instead the satellite is artificial, then there are tidal effects from both the Sun and the Moon. However, these are suppressed relative to the Earth-satellite acceleration (177).

The Lorentz-violating coordinate accelerations given by Eqs. (177) and (178) exhibit some interesting features. The first two terms in Eq. (177) and the first term in Eq. (178) are composition-dependent scalings of the corresponding Newton accelerations. These terms are therefore detectable only by comparison to results obtained using satellites of different compositions. Also, unlike the contributions $\mathrm{a}^J_{\overline{s},\mathrm{tidal}}$ obtained in Ref. [7], here the tidal acceleration (178) from the Sun on the Moon-Earth system involves nontrivial WEP violations because the Moon and the Earth fall differently towards the Sun when the coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ are nonzero. It is also interesting to note that the tidal acceleration (178) contains contributions at PNO(2) that are independent of r^J and hence are enhanced at this order relative to other contributions by a factor of R/r. This too is a consequence of the WEP violations arising from $(\overline{a}_{\rm eff})_\mu$. Similar terms appear at PNO(3) as well.

A typical experiment measures the time of flight for laser photons to travel from the Earth to a reflector on the satellite and back. To analyze the results, the laser-ranging data can be fitted by incorporating Eq. (175) and other conventional perturbing effects into an appropriate modeling code. An alternative approach is to perform an analytical perturbative expansion along the lines of the one performed for the $\overline{s}_{\mu\nu}$ contributions in Ref. [7] and then match to the data. This latter method is adopted in Ref. [4] to constrain combinations of the coefficients $\overline{s}_{\mu\nu}$.

In the present context, we can obtain crude estimates of sensitivites to $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ attainable in lunar laser ranging via either of these procedures, by using term-by-term comparison of the accelerations Eq. (177) and (178) to the accelerations $\mathrm{a}^J_{\overline{s},\mathrm{ES}}$ and $\mathrm{a}^J_{\overline{s},\mathrm{tidal}}$ obtained for the coefficient $\overline{s}_{\mu\nu}$ in Ref. [7]. With the precision already achieved in lunar laser ranging [126], we thereby find estimated sensitivities at parts in 10^{10} to combinations of $(\overline{c}^w)_{(JK)}$ and $\overline{s}_{JK}$, and parts in 10^6 to various combinations of $(\overline{a}^w_{\rm eff})_J$, $(\overline{c}^w)_{(TJ)}$ and $\overline{s}_{TJ}$. Actual measurements at roughly these levels can be expected to result from a reanalysis of existing data. A significant further improvement is likely to be possible using data from the Apache Point Observatory Lunar Laser-Ranging Operation (APOLLO) [127]. Assuming that millimeter ranging is achieved as expected and disregarding probable subtantially improved statistics, we anticipate competitive estimated sensitivities of 10^{-7} GeV on various combinations of $\alpha(\overline{a}^w_{\rm eff})_X$ and $\alpha(\overline{a}^w_{\rm eff})_{Y+Z}$, and a sensitivity of 10^{-7} on $(\overline{c}^n)_{(TJ)}$, where the notation of Eq. (148) is used and the dependence on the coefficients $\overline{s}_{\mu\nu}$ has been omitted for simplicity.

Ranging to artificial satellites with orbit orientations different from that of the Moon can yield sensitivity to additional independent linear combinations of $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$. Typically, the reach of satellite ranging is expected to be about an order of magnitude less than lunar laser ranging. Other possibilities for gravitational tests of Lorentz violation include ranging to objects orbiting bodies other than the Earth. For example, the time variation $G_N^{-1}dG_N/dt$ of the Newton gravitational constant has been constrained by ranging data to the Viking landers on Mars, to the Mariner 9 spacecraft orbiting Mars, and to other bodies including the Moon [128]. These studies primarily seek secular changes in the gravitational force. Although secular changes in coupling constants can result from Lorentz violation [129], the signals of interest in the present context are periodic. Reanalysis of existing data to seek periodic effects in $G_N^{-1}dG_N/dt$ would yield sensitivities to Lorentz violation estimated to be somewhat less than lunar laser ranging but involving different combinations of $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$.

We conclude this subsection with some comments about the coordinate location R^J of the center of mass of an Earth-satellite system. Boost invariance normally ensures this location is fixed, but the presence of Lorentz violation means it can be time dependent, although the effect may be unobservable via laser ranging. Neglecting the effects of other bodies and working at PNO(3), the Lorentz-violating contributions to the equation of motion for the center of mass of the Earth-satellite system can be written as the sum

$$\ddot{R}^J \supset \ddot{R}^J_{\overline{a}_{\rm eff}} + \ddot{R}^J_{\overline{c}} + \ddot{R}^J_{\overline{s}} \tag{179}$$

of contributions from $(\overline{a}^w_{\rm eff})_\mu$, $(\overline{c}^w)_{\mu\nu}$, and $\overline{s}_\mu$. Explicitly, we find

$$\ddot{R}^J_{\overline{a}_{\rm eff}} = \sum_w \frac{2G_N n_5^w \eta^{JK}\alpha(\overline{a}^w_{\rm eff})_K v_L r^L}{r^3}, \tag{180}$$

which contains only PNO(3) effects involving the internal motion of the system. The second term in Eq. (179) is

$$\begin{aligned} \ddot{R}^J_{\overline{c}} &= \sum_w \frac{2G_N m_1 m_2 m^w}{M^2 r^3} \\ &\quad \times \Big[\tfrac{1}{2} n_2^w (\overline{c}^w)_{TT} r^J + n_2^w \eta^{JK} (\overline{c}^w)_{KL} r^L \\ &\qquad + \eta^{JK} (\overline{c}^w)_{(TK)} (n_2^w V_L + n_5^w v_L) r^L \\ &\qquad + (\overline{c}^w)_{(TK)} (n_2^w V^K + n_5^w v^K) r^J \\ &\qquad + (\overline{c}^w)_{(TK)} (n_2^w V^J + n_5^w v^J) r^K \Big]. \end{aligned} \tag{181}$$

The first two terms are at PNO(2) and reflect the modification of the effective Newton inertial mass in the presence of nonzero $\overline{c}_{\mu\nu}$, while the remaining terms are at PNO(3). The ones proportional to V^J arise as a result of the system boost in the Sun-centered frame, and those proportional to v^J are due to the internal motion of the system. The last term in Eq. (179) is

$$\begin{aligned} \ddot{R}^J_{\overline{s}} &= \frac{G_N m_1 m_2}{M r^3} \Big[3\eta^{JK} \overline{s}_{TK} v_L r^L - \overline{s}_{TK} v^J r^K \\ &\qquad - \overline{s}_{TK} v^K r^J + 3\overline{s}_{TK} r^J v_L \hat{r}^K \hat{r}^L \Big], \end{aligned} \tag{182}$$

which again consists only of PNO(3) effects proportional to the internal motion of the system. Note that all these contributions introduce an oscillatory motion for the center of mass, and their presence is required by momentum conservation.

B. Perihelion precession

The presence of nonzero coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ for Lorentz violation leads to corrections to the motion of a test body in a gravitational field. These corrections can be calculated from the equation of motion (78) and from the expressions for the coefficient and metric fluctuations given in Sec. IV C. In this subsection, we determine the effect of nonzero $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ on the perihelion precession for planetary orbits. We follow the treatment of Ref. [7], which obtains the perihelion shift arising from nonzero SME coefficients $\overline{s}_{\mu\nu}$. Our notation matches that of Table XII and Eq. (174) in Sec. X A, with the labels 1 and 2 representing the planet and Sun, respectively.

The derivation of the perihelion precession used here relies on the method of osculating elements [130], in which the instantaneous motion of the planet is treated as part of an ellipse. The ellipse is characterized using the standard Kepler orbital elements, and the motion of the planet is described by specifying them as a function of time. The relevant orbital elements in the present case are the angle ω between the line of ascending nodes and the semimajor axis of the ellipse, the longitude Ω of the ascending node, and the inclination i with respect to the ecliptic. These are specified in the reference coordinate system, which can be taken as the Sun-centered frame for the planetary orbits considered here. More generally, the reference frame is related to the Sun-centered frame by a rotation and possibly a boost, as discussed in Sec. V E 5 of Ref. [7]. The physical quantity relevant for the perihelion precession is the change per period $\Delta\tilde{\omega}$ of the perihelion angle $\tilde{\omega}$ with respect to the equinox. In terms of the basic orbital elements, $\tilde{\omega}$ can be expressed as

$$\tilde{\omega} = \omega + \Omega \cos i. \tag{183}$$

For the cases of interest here, the angle i can be assumed small.

The secular changes in the orbital elements arising from $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ can be obtained by considering the relative acceleration of the planet and the Sun, which has the form

$$\begin{aligned}\frac{d^2r^j}{dt^2} &= -\frac{G_N}{r^3}\sum_w \Big[M + 2n_3^w\alpha(\overline{a}^w_{\rm eff})_0 r^j + n_1^w m^w(\overline{c}^w)_{00} r^j \\ &\quad -2\eta^{jk}n_7^w m^w(\overline{c}^w)_{(kl)}r^l + 2n_2^w\alpha(\overline{a}^w_{\rm eff})_k\eta^{jk}v_l r^l \\ &\quad -2n_2^w\alpha(\overline{a}^w_{\rm eff})_k v^k r^j - 2n_6^w m^w(\overline{c}^w)_{(0k)}v^k r^j \\ &\quad +2\eta^{jk}(n_6^w - 2n_8^w)m^w(\overline{c}^w)_{(0k)}v_l r^l \\ &\quad -2n_8^w m^w(\overline{c}^w)_{(0k)}v^j r^k\Big]. \end{aligned}\tag{184}$$

The unperturbed ellipse is given as the solution $\vec{r}_0$ of the Kepler-type equation

$$\frac{d^2r_0^j}{dt^2} = -\frac{G_N}{r^3}\sum_w \Big[M + 2n_3^w\alpha(\overline{a}^w_{\rm eff})_0 + n_1^w m^w(\overline{c}^w)_{00}\Big]r^j. \tag{185}$$

This shows that the frequency n and semimajor axis a of the unperturbed elliptic motion are related according to

$$n^2a^3 = G_N\sum_w \Big[M + 2n_3^w\alpha(\overline{a}^w_{\rm eff})_0 + n_1^w m^w(\overline{c}^w)_{00}\Big]. \tag{186}$$

Note that the right-hand side of this expression depends on the composition of the planet and the Sun.

The orientation of the orbit can be specified using three unit vectors $\vec{k}$, $\vec{P}$, and $\vec{Q}$. The first is chosen perpendicular to the orbit, the second points from the focus to the perihelion, and the third completes the orthonormal set. Their explicit form in terms of orbital elements is given in Eq. (116) of Ref. [7]. In terms of this basis set, the unperturbed elliptical orbit can be expressed as

$$\vec{r}_0 = \frac{a(1-e^2)}{1+e\cos f}(\vec{P}\cos f + \vec{Q}\sin f), \tag{187}$$

where e is the eccentricity and f is the true anomaly.

The perturbing acceleration a'^j consists of the terms in Eq. (184) that are absent from Eq. (185),

$$\mathrm{a}'^j = \frac{d^2r^j}{dt^2} - \frac{d^2r_0^j}{dt^2}. \tag{188}$$

The time dependence of the orbital elements can be extracted from this equation via the method of osculating elements. The general procedure is to insert the unperturbed solution (187) for $\vec{r}$ into the expression (188) for a'^j, to project the result as desired, and to integrate over the true anomaly.

To obtain the perihelion precession, the final results for the orbital elements ω and Ω must be combined according to Eq. (183). After some calculation, we obtain the expression

$$\begin{aligned}\Delta\tilde{\omega} &= 2\pi\sum_w \Big[\frac{(e^2-2\epsilon)}{Me^4}n_7^w m^w((\overline{c}^w)_{QQ} - (\overline{c}^w)_{PP}) \\ &\qquad -\frac{2na(e^2-\epsilon)}{e^3M\sqrt{1-e^2}}\big[(n_6^w - 2n_8^w)m^w(\overline{c}^w)_{(0Q)} \\ &\qquad +n_2^w\alpha(\overline{a}^w_{\rm eff})_Q\big]\Big] \end{aligned}\tag{189}$$

for the shift in the perihelion per orbit. Here, the subscripts P and Q on the coefficients for Lorentz violation indicate projections along the directions $\vec{P}$ and $\vec{Q}$, respectively. The quantity ϵ is the eccentricity function, defined by $\epsilon = 1 - \sqrt{1-e^2}$.

The result (189) reveals that the perihelion precession depends on the orbit orientation through the projections of the coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ along the directions $\vec{P}$ and $\vec{Q}$. Also, the factors scaling the coefficients in Eq.

(189) vary with the composition of the orbiting body. This means that the orbits of different planets or, more generally, different satellites are affected by different linear combinations of coefficients for Lorentz violation. It is therefore valuable to consider data from multiple systems so that independent measurements can be obtained.

To illustrate the sensitivities that can be achieved, we consider explicitly the perihelion precessions of Mercury and of the Earth. Substituting the relevant orbital data for the two planets into Eq. (189) in turn, taking the planetary mass as small compared to the solar mass $m_\odot$, and incorporating the results for the coefficients $\overline{s}_{\mu\nu}$ obtained in Eq. (190) of Ref. [7], we find the overall perihelion shifts $\dot{\hat{\omega}}_☿$ of Mercury and $\dot{\hat{\omega}}_\oplus$ of the Earth are given in units of arcseconds per century C by the expressions

$$\begin{aligned}
\dot{\hat{\omega}}_☿ &\approx \frac{7\times 10^{7\prime\prime}}{\mathrm{C}}\overline{s}_☿ \\
&\quad +\frac{1\times 10^{8\prime\prime}}{\mathrm{C}}\sum_w \Big(3\times 10^{-3}\frac{N^w_\odot}{m_\odot}(\overline{a}^w_{\rm eff})_☿ - \frac{N^w_☿ m^w}{m_☿}(\overline{c}^w)_☿\Big), \\
\dot{\hat{\omega}}_\oplus &\approx \frac{2\times 10^{7\prime\prime}}{\mathrm{C}}\overline{s}_\oplus \\
&\quad +\frac{4\times 10^{7\prime\prime}}{\mathrm{C}}\sum_w \Big(3\times 10^{-2}\frac{N^w_\odot}{m_\odot}(\overline{a}^w_{\rm eff})_\oplus - \frac{N^w_\oplus m^w}{m_\oplus}(\overline{c}^w)_\oplus\Big).
\end{aligned} \tag{190}$$

The combinations of coefficients for Lorentz violation appearing in these equations are defined as

$$\begin{aligned}
(\overline{a}^w_{\rm eff})_☿ &= \alpha(\overline{a}^w_{\rm eff})_Q, \\
(\overline{a}^w_{\rm eff})_\oplus &= \alpha(\overline{a}^w_{\rm eff})_Q, \\
(\overline{c}^w)_☿ &\approx [(\overline{c}^w)_{QQ}-(\overline{c}^w)_{PP}] - 6\times 10^{-3}(\overline{c}^w)_{(0Q)}, \\
(\overline{c}^w)_\oplus &\approx [(\overline{c}^w)_{QQ}-(\overline{c}^w)_{PP}] - 5\times 10^{-2}(\overline{c}^w)_{(0Q)}, \\
\overline{s}_☿ &\approx (\overline{s}_{PP}-\overline{s}_{QQ}) - 6\times 10^{-3}\overline{s}_{(0Q)}, \\
\overline{s}_\oplus &\approx (\overline{s}_{PP}-\overline{s}_{QQ}) - 5\times 10^{-2}\overline{s}_{(0Q)}.
\end{aligned} \tag{191}$$

Note that the subscripts P, Q here represent projections that differ for Mercury and the Earth.

The chemical composition of the Sun is believed to be over 70% hydrogen and about 27% helium by mass [131]. The factors in Eq. (190) that depend on the solar composition can therefore be estimated as $N^e_\odot/m_\odot = N^p_\odot/m_\odot \simeq 0.9$ GeV^{-1} and $N^n_\odot/m_\odot \simeq 0.1$ GeV^{-1}. As can be seen from Eq. (191), these factors suffice for placing approximate bounds on the coefficients $(\overline{a}^w_{\rm eff})_\mu$ from knowledge of the perihelion precessions. The composition of Mercury is believed to be about 70% iron and about 30% rocky material [131], so the analogous ratios for Mercury are roughly $N^e_☿/m_☿ = N^p_☿/m_☿ \simeq 0.4$ GeV^{-1} and $N^n_☿/m_☿ \simeq 0.6$ GeV^{-1}. For the Earth, using Ref. [46] and following the discussion of Sec. IV A 2, we find $N^e_\oplus/m_\oplus = N^p_\oplus/m_\oplus \approx N^n_\oplus/m_\oplus \simeq 0.5$ GeV^{-1}. However, for the approximate bounds obtained below on the coefficients $\overline{c}_{\mu\nu}$, it suffices that the composition-dependent factors for the planets are of order 10^{-1} GeV^{-1}.

We are now in a position to place constraints on some combinations of the coefficients $(\overline{a}^w_{\rm eff})_\mu$, $\overline{c}_{\mu\nu}$, and $\overline{s}_{\mu\nu}$ by adopting the established error bars in the existing data for perihelion shifts. These error bars are $0.043''$ C^{-1} for Mercury and $0.4''$ C^{-1} for the Earth [8, 132]. Taking the error bars to be upper bounds on the perihelion shifts in Eq. (190), we obtain the order-of-magnitude constraints

$$\begin{aligned}
&|\overline{s}_☿ + 10^{-3}[(\overline{a}^e_{\rm eff})_☿ + (\overline{a}^p_{\rm eff})_☿] + 10^{-4}(\overline{a}^n_{\rm eff})_☿ \\
&\quad -10^{-4}(\overline{c}^e)_☿ - 10^{-1}(\overline{c}^p)_☿ - 10^{-1}(\overline{c}^n)_☿| \lesssim 10^{-9}\ \text{GeV}, \\
&|\overline{s}_\oplus + 10^{-2}[(\overline{a}^e_{\rm eff})_\oplus + (\overline{a}^p_{\rm eff})_\oplus] + 10^{-3}(\overline{a}^n_{\rm eff})_\oplus \\
&\quad -10^{-4}(\overline{c}^e)_\oplus - 10^{-1}(\overline{c}^p)_\oplus - 10^{-1}(\overline{c}^n)_\oplus| \lesssim 10^{-8}\ \text{GeV}.
\end{aligned} \tag{192}$$

Assuming a model with nonzero coefficients $(\overline{a}^w_{\rm eff})_\mu$ only, this yields the approximate constraints

$$\begin{aligned}
|(\overline{a}^e_{\rm eff})_☿ + (\overline{a}^p_{\rm eff})_☿ + 0.1(\overline{a}^n_{\rm eff})_☿| &\lesssim 10^{-6}\ \text{GeV}, \\
|(\overline{a}^e_{\rm eff})_\oplus + (\overline{a}^p_{\rm eff})_\oplus + 0.1(\overline{a}^n_{\rm eff})_\oplus| &\lesssim 10^{-6}\ \text{GeV}.
\end{aligned} \tag{193}$$

Similarly, assuming a model with nonzero coefficients $(\overline{c}^w)_{\mu\nu}$ only and making use of existing limits on $\overline{c}_{\mu\nu}$ for protons and electrons [2], we obtain the approximate constraints

$$|(\overline{c}^n)_☿| \lesssim 10^{-8}, \qquad |(\overline{c}^n)_\oplus| \lesssim 10^{-7}. \tag{194}$$

A careful reanalysis of the existing data for multiple bodies in the solar system could yield sharper sensitivities.

The result (193) represents first constraints on the spatial coefficients $\alpha(\overline{a}^w_{\rm eff})_J$. A sense of the maximal attained sensitivity to the nine components in $\alpha(\overline{a}^w_{\rm eff})_J$ can be obtained by taking each component in turn to be the only nonzero one. Extracting these sensitivities requires the explicit form of the vectors $\vec{Q}_☿$ and $\vec{Q}_\oplus$. The relevant orbital elements in heliocentric coordinates are $\omega_☿ \simeq 29°$, $\Omega_☿ \simeq 48°$, $i_☿ \simeq 7°$ and $\omega_\oplus \simeq 103°$, $\Omega_\oplus = 0°$ $i_\oplus = 0°$. Converting to the Sun-centered frame using a counterclockwise rotation by $\eta \simeq 23.5°$ about the X axis yields

$$\begin{aligned}
\vec{Q}_☿ &\simeq -0.97\mathbf{e}_X + 0.15\mathbf{e}_Y + 0.18\mathbf{e}_Z, \\
\vec{Q}_\oplus &\simeq -0.97\mathbf{e}_X - 0.21\mathbf{e}_Y - 0.10\mathbf{e}_Z.
\end{aligned} \tag{195}$$

Taking each component $\alpha(\overline{a}^w_{\rm eff})_J$ as the only nonzero coefficient in turn yields the order-of-magnitude sensitivities

$$\begin{aligned}
|\alpha(\overline{a}^e_{\rm eff})_X| &\lesssim 10^{-6}\ \text{GeV}, \\
|\alpha(\overline{a}^e_{\rm eff})_Y|, |\alpha(\overline{a}^e_{\rm eff})_Z| &\lesssim 10^{-5}\ \text{GeV}, \\
|\alpha(\overline{a}^p_{\rm eff})_X| &\lesssim 10^{-6}\ \text{GeV}, \\
|\alpha(\overline{a}^p_{\rm eff})_Y|, |\alpha(\overline{a}^p_{\rm eff})_Z| &\lesssim 10^{-5}\ \text{GeV}, \\
|\alpha(\overline{a}^n_{\rm eff})_X| &\lesssim 10^{-5}\ \text{GeV}, \\
|\alpha(\overline{a}^n_{\rm eff})_Y|, |\alpha(\overline{a}^n_{\rm eff})_Z| &\lesssim 10^{-4}\ \text{GeV}.
\end{aligned} \tag{196}$$

These results are the maximal sensitivities achieved to date on the coefficients $\alpha(\overline{a}^w_{\rm eff})_J$.

XI. PHOTON TESTS

In this penultimate section, we consider searches for gravitational Lorentz violation involving the trajectories of photons. With the coordinate choice (23) adopted in this work, photons follow null geodesics. The signals of interest therefore arise from the modifications to the metric, which are associated with Lorentz-violating matter-gravity couplings of the source body and in certain cases also of the clocks and rods used for measurements.

Photon tests for Lorentz violation involving the coefficients $\overline{s}_{\mu\nu}$ in the pure-gravity sector of the SME have been studied in Refs. [7, 133]. Here, this analysis is extended to include the matter-sector coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$. The treatment and notation of Ref. [133] is adopted where possible. Some quantities relevant for the analysis are listed in Table XIII.

Table XIII. Notation for photon tests.

Quantity	Definition
$x^\mu_E = (t_E, \vec{r}_E)$	coordinates of event E
r_E	magnitude of $\vec{r}_E$
$x^\mu_P = (t_P, \vec{r}_P)$	coordinates of event P
r_P	magnitude of $\vec{r}_P$
$\vec{R} = \vec{r}_P - \vec{r}_E$	zeroth-order light trajectory
$\hat{R} = \vec{R}/R$	unit vector along $\vec{R}$
$R = \lvert\vec{R}\rvert$	magnitude of $\vec{R}$
$b^j = r^j_P - \hat{R}^j \vec{r_P}\cdot\hat{R}$	impact-parameter vector
b	magnitude of $\vec{b}$
$m^{\rm S}$	mass of source body
$l_P = \vec{r_P}\cdot\hat{R}$	λ at P
$-l_E = \vec{r_E}\cdot\hat{R}$	λ at E
τ_E	proper time of E
τ_P	proper time of P
$u^\mu_E = dx^\mu_E/d\tau_E$	4-velocity of E
$u^\mu_P = dx^\mu_P/d\tau_P$	4-velocity of P
$\vec{v} = d\vec{r}_E/dt$	3-velocity of E
$\vec{w} = d\vec{r}_P/dt$	3-velocity of P
ν_E	frequency at E
ν_P	frequency at P

In what follows, we consider various effects on a light signal as it travels from an emission event E to a spacetime point P located near a massive body. The light path can be specified parametrically as $x^\mu = x^\mu(\lambda)$, where λ is the path parameter. The wave 4-vector p^μ of the ray tangent to the path is

$$p^\mu = \frac{dx^\mu}{d\lambda}, \tag{197}$$

and it obeys the conditions

$$\begin{aligned} \frac{dx^\mu}{d\lambda} &= -\Gamma^\mu{}_{\alpha\beta} p^\alpha p^\beta, \\ p^\mu p^\nu g_{\mu\nu} &= 0. \end{aligned} \tag{198}$$

The wave 4-vector can be linearized as

$$p^\mu = \overline{p}^\mu + \delta p^\mu, \tag{199}$$

where the first term is the zeroth-order wave vector and the second term contains gravitational corrections. Our interest here lies in the O(1,1) contributions to δp^μ. The basic procedure is to insert the modifications (84) and (88) of the metric arising from matter-sector effects into the general expressions obtained in Ref. [133]. We consider in turn Lorentz-violating contributions to the Shapiro time delay, to the gravitational Doppler shift, to the gravitational redshift, and to the null redshift, and we compare the results to the effective mass of a gravitational source as measured by orbital tests. We also offer some comments about the implications of the results for various experiments.

A. Shapiro time delay

In this subsection, we obtain the Lorentz-violating modifications to the Shapiro time delay of a light signal as it passes from a source to a detector in the presence of a massive body such as the Sun. The one-way time delay $t_P - t_E$ can be determined by integrating δp^μ along the path and applying the null condition,

$$t_P - t_E = R + \tfrac{1}{2}\int_{-l_E}^{l_P} h_{\mu\nu}\overline{p}^\mu \overline{p}^\nu d\lambda. \tag{200}$$

Inserting the Lorentz-violating metric modifications (84) and (88) and integrating, we find the delay can be written in the form

$$\begin{aligned} t_P - t_E &= R + (t_P - t_E)_{\rm GR} \\ &\quad + (t_P - t_E)_{\overline{a}_{\rm eff},\overline{c}} + (t_P - t_E)_{\overline{s}}. \end{aligned} \tag{201}$$

Here, R is the zeroth-order time difference. The second term is the standard GR contribution, which at O(0,1) and PNO(2) takes the form

$$(t_P - t_E)_{\rm GR} = 2G_N m^{\rm S} \ln\left(\frac{r_E + r_P + R}{r_E + r_P - R}\right). \tag{202}$$

The third term of Eq. (201) consists of contributions from Lorentz-violating matter-gravity couplings associated with the source body S. At O(1,1) and PNO(2), these contributions are

$$\begin{aligned} &(t_P - t_E)_{\overline{a}_{\rm eff},\overline{c}} = \\ &\qquad 2G_N m^{\rm S}\Big(\frac{\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0 + \frac{\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_j \hat{R}^j + (\overline{c}^{\rm S})_{00}\Big) \\ &\qquad\quad \times \ln\left(\frac{r_E + r_P + R}{r_E + r_P - R}\right) \\ &\qquad - G_N \alpha((\overline{a}^{\rm S}_{\rm eff})_0 + (\overline{a}^{\rm S}_{\rm eff})_j \hat{R}^j)\left(\frac{l_E}{r_E} + \frac{l_P}{r_P}\right) \\ &\qquad - G_N \alpha(\overline{a}^{\rm S}_{\rm eff})_k b^k \left(\frac{r_E - r_P}{r_E r_P}\right). \end{aligned} \tag{203}$$

The final term in Eq. (201) arises from gravitational Lorentz violation involving the coefficient $\overline{s}_{\mu\nu}$ and is given in Ref. [133].

In typical time-delay measurements, an observer emits a light signal at E that is reflected at the spacetime point P and subsequently detected by the observer at E'. The round-trip coordinate travel time Δt, which is related to the measured proper time $\Delta\tau_E$ by the factor $d\tau_E/dt$, can be written to O(1,1) and PNO(2) as

$$\Delta t = 2R(1+v^2-\vec{v}\cdot\hat{R})+(\Delta t)_{\rm GR}+(\Delta t)_{\overline{a}_{\rm eff},\overline{c}}+(\Delta t)_{\overline{s}}. \tag{204}$$

The zeroth-order term in this expression incorporates Lorentz-violating corrections to the trajectory of the emitter, which here can depend on particle species. These can in principle be determined by modeling the relevant orbits along the lines of the treatment in Sec. X and Ref. [7]. The second term in Eq. (204) contains the leading GR corrections,

$$\Delta t_{\rm GR} = 4G_N m^{\rm S} \ln\left(\frac{r_E+r_P+R}{r_E+r_P-R}\right). \tag{205}$$

The third term in Eq. (204) contains the leading contributions from nonzero $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$,

$$\begin{aligned}\Delta t_{\overline{a}_{\rm eff},\overline{c}} &= 4G_N m^{\rm S}\left(\frac{\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0+(\overline{c}^{\rm S})_{00}\right)\\ &\qquad\times\ln\left(\frac{r_E+r_P+R}{r_E+r_P-R}\right)\\ &\qquad -2G_N\alpha(\overline{a}^{\rm S}_{\rm eff})_0\left(\frac{l_E}{r_E}+\frac{l_P}{r_P}\right).\end{aligned} \tag{206}$$

The last term of Eq. (204) contains corrections involving the coefficient $\overline{s}_{\mu\nu}$ and is given in Ref. [133]. Note that contributions from the coefficients $(\overline{a}_{\rm eff})_j$ and $\overline{s}_{0j}$ cancel in the round-trip expression, a result that can be traced to the parity-odd nature of the corresponding Lorentz-violating operators. Note also that the time-delay signal changes over two relevant time scales, the conjunction time b/v and the typically longer orbital time r/v, which enables separation of the zeroth-order and gravitational effects.

The dominant Lorentz-violating corrections to Δt are proportional to the logarithm in Eq. (206). The primary effect of the Lorentz-violating matter-gravity couplings is therefore to scale the factor of $G_N m^{\rm S}$ in the usual GR time delay (205). The scaling can be interpreted as an effective value $(G_N M)_{\rm TD}$ for the source body relevant for time-delay tests,

$$(G_N M)_{\rm TD} = G_N m^{\rm S}\left(1+\frac{\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0+(\overline{c}^{\rm S})_{00}+\overline{s}_{00}\right). \tag{207}$$

This scaling is unobservable in time-delay tests alone. However, we show in what follows that other tests can yield different effective values of $G_N m^{\rm S}$, so suitable comparisons can reveal signals for Lorentz violation. This prospect is considered in Sec. XI E below.

B. Gravitational Doppler shift

When light passes near a massive body, it suffers a frequency shift as well as a time delay. In this and the subsequent subsections, we consider the corrections to the frequency shift due to the matter-sector coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$.

The relevant quantity is the ratio of frequencies observed at the two events E and P,

$$\frac{\nu_P}{\nu_E} = \frac{(u^\mu p_\mu)_P}{(u^\nu p_\mu)_E}. \tag{208}$$

At PNO(3), this can be written as

$$\frac{\nu_P}{\nu_E} = \sqrt{\frac{1-v^2}{1-w^2}}\left(\frac{1-\vec{w}\cdot\hat{R}}{1-\vec{v}\cdot\hat{R}}\right)\left[1+\left(\frac{\nu_P}{\nu_E}\right)_g\right]. \tag{209}$$

Here, the term labeled g contains gravitational effects involving both the Doppler shift and the redshift,

$$\left(\frac{\nu_P}{\nu_E}\right)_g = \left(\frac{\nu_P}{\nu_E}\right)_{\rm DS}+\left(\frac{\nu_P}{\nu_E}\right)_{\rm RS}. \tag{210}$$

This subsection treats the gravitational Doppler shift, while the redshift effects are discussed in the next subsection.

Corrections to the gravitational Doppler shift $(\nu_P/\nu_E)_{\rm DS}$ depending on the coefficients $(\overline{a}_{\rm eff})_\mu$, $\overline{c}_{\mu\nu}$, and $\overline{s}_{\mu\nu}$ can be obtained by inserting into Eq. (31) of Ref. [133] the modifications to the metric from Eqs. (84) and (88), along with those due to $\overline{s}_{\mu\nu}$ given in Ref. [7]. Near conjunction, we find that the dominant effects take the form

$$\begin{aligned}\left(\frac{\nu_P}{\nu_E}\right)_{\rm DS} &\approx \frac{4G_N m^{\rm S}}{b}\Big(1+\frac{\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0+\frac{\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_j\hat{R}^j\\ &\qquad +(\overline{c}^{\rm S})_{00}\Big)\frac{db}{dt}+\left(\frac{\nu_P}{\nu_E}\right)_{{\rm DS},\overline{s}},\end{aligned} \tag{211}$$

where the last term contains the contributions from $\overline{s}_{\mu\nu}$ found in Ref. [133].

Typical searches measure the round-trip frequency shift,

$$\begin{aligned}\left(\frac{\delta\nu}{\nu}\right)_{\rm DS} &= \frac{8G_N m^{\rm S}}{b}\left(1+\frac{\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0+(\overline{c}^{\rm S})_{00}\right)\frac{db}{dt}\\ &\qquad +\left(\frac{\delta\nu}{\nu}\right)_{{\rm DS},\overline{s}}.\end{aligned} \tag{212}$$

Note that the effects from parity-odd operators again cancel. The coefficients $(\overline{a}_{\rm eff})_0$, $\overline{c}_{00}$, and $\overline{s}_{00}$ associated with isotropic Lorentz violation in the chosen inertial frame act to scale the factor $G_N m^{\rm S}$ in the usual expression for the gravitational Doppler shift, leading to an effective value $(G_N M)_{\rm DS}$ given by

$$(G_N M)_{\rm DS} = G_N m^{\rm S}\left(1+\frac{\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0+(\overline{c}^{\rm S})_{00}+\overline{s}_{00}\right). \tag{213}$$

The scaling (213) is unobservable in Doppler-shift tests alone. This result for $(G_N M)_{\rm DS}$ is identical in form to that of the time-delay value $(G_N M)_{\rm TD}$ in Eq. (207).

C. Gravitational redshift

The Lorentz-violating contributions to the term $(\nu_P/\nu_E)_{\rm RS}$ in Eq. (210) for the gravitational redshift can be viewed as subdominant to the time delay or Doppler shift because they occur at the slow time scale. However, in dedicated redshift measurements, the Lorentz-violating gravitational redshift can appear as the dominant effect. In this subsection, we discuss Lorentz-violating modifications to the usual gravitational redshift and effects in null-redshift tests.

To place in context the results in this subsection, we note that clocks can be used to perform three distinct types of gravitational tests that are often convolved in the literature under the term 'redshift tests.' The first type, which measures the traditional gravitational redshift, involves two clocks held at different gravitational potentials whose frequency is compared using light or some other signal passing between them. This type of test is discussed in Sec. XI C 1 below. The second type of test is called a null-redshift test, and it involves monitoring the frequencies of two clocks of different composition as they move together through the gravitational potential. This is discussed in Sec. XI C 2. The third kind of test involves synchronizing two clocks and then moving one of them around a closed path in the gravitational potential. The signal in this case is the accumulated phase difference between the clocks. An example of this 'twin-paradox' redshift test is the free-fall gravimeter measurement with interferometers discussed in Sec. VII D. These three kinds of tests produce related signals in GR. However, they can yield distinct sensitivities in a more general context such as the SME, as is demonstrated in what follows.

1. *Modified redshift*

The term $(\nu_P/\nu_E)_{\rm RS}$ in Eq. (210) for the gravitational redshift can be understood as the product

$$\left(\frac{\nu_P}{\nu_E}\right)_{\rm RS} = \left(\frac{dt}{d\tau_P}\right)\left(\frac{d\tau_E}{dt}\right), \tag{214}$$

of the factors relating proper and coordinate times for the clocks at the two points E and P. Each factor is determined by the dispersion relation for the corresponding clock, which depends on coefficients for Lorentz violation via its material composition and on the Lorentz violation associated with the gravitational field.

For simplicity in what follows, we assume the sending and receiving clocks are identical. This eliminates the need to consider O(1,0) effects, which have been sought in numerous clock-comparison experiments performed with both clocks at the same gravitational potential [2]. To the order at which we work, the redshift can then be expanded as

$$\left(\frac{\nu_P}{\nu_E}\right)_{\rm RS} = \left(\frac{\nu_P}{\nu_E}\right)^{(0,1)}_{\rm RS} + \left(\frac{\nu_P}{\nu_E}\right)^{(1,1)\rm S}_{\rm RS} + \left(\frac{\nu_P}{\nu_E}\right)^{(1,1)\rm T}_{\rm RS}, \tag{215}$$

where the term at O(0,1) is the conventional redshift, the term at O(1,1) labeled by S contains Lorentz-violating corrections from the gravitational source, and the last term labeled by T involves O(1,1) contributions from the clocks. For our present purposes, it suffices to work at PNO(2).

For an ideal clock, the Lorentz-violating contributions to the first two terms in Eq. (215) can be calculated by inserting into the usual redshift equation the modifications (84) and (88) to the metric from the coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$, along with the corrections from Ref. [7] involving the coefficients $\overline{s}_{\mu\nu}$. This gives

$$\left(\frac{\nu_P}{\nu_E}\right)^{(0,1)}_{\rm RS} + \left(\frac{\nu_P}{\nu_E}\right)^{(1,1)\rm S}_{\rm RS} = \sqrt{\frac{1-(h^{(0,1)}_{00})_E-(h^{(1,1)}_{00})_E}{1-(h^{(0,1)}_{00})_P-(h^{(1,1)}_{00})_P}}. \tag{216}$$

Expanding to PNO(2) and keeping leading-order terms in Lorentz violation, we obtain the conventional PNO(2) result,

$$\left(\frac{\nu_P}{\nu_E}\right)^{(0,1)}_{\rm RS} = G_N m^{\rm S}\left(\frac{r_e-r_p}{r_e r_p}\right),$$

together with the correction

$$\left(\frac{\nu_P}{\nu_E}\right)^{(1,1)\rm S}_{\rm RS} = G_N m^{\rm S}\Big(\frac{2\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0+(\overline{c}^{\rm S})_{00}\Big)\left(\frac{r_e-r_p}{r_e r_p}\right) + \left(\frac{\nu_P}{\nu_E}\right)_{{\rm RS},\overline{s}}. \tag{217}$$

The last term contains the contributions from $\overline{s}_{\mu\nu}$ given in Ref. [133].

For the remaining term in Eq. (215), the situation is more complicated because the clock frequency must be calculated directly and typically depends on the structure and composition of the clock. Moreover, although our interest is at O(1,1), all three of the perturbative contributions O(1,0), O(0,1), O(1,1) must be treated due to the appearance of cross terms in the calculation. For convenience, we can express the last term in Eq. (215) in the form

$$\left(\frac{\nu_P}{\nu_E}\right)^{(1,1)\rm T}_{\rm RS} = G_N m^{\rm S}\xi_{\rm clock}\left(\frac{r_e-r_p}{r_e r_p}\right), \tag{218}$$

where $\xi_{\rm clock}$ is a function of the coefficients for Lorentz violation associated with the clock. If the clock's ticking rate is set by its inertial properties, as is the case for most atomic clocks, then $\xi_{\rm clock}$ can be expected to depend on the coefficients $(\overline{c}^w)_{\mu\nu}$. If the clock's ticking rate depends intrinsically on the local gravitational acceleration, as occurs for a pendulum clock, then $\xi_{\rm clock}$ can be expected to depend on the coefficients $(\overline{a}^w_{\rm eff})_\mu$. In general, the value of $\xi_{\rm clock}$ can depend on both sets of coefficients,

$$\xi_{\rm clock} = \xi_{\rm clock}\big((\overline{a}^w_{\rm eff})_\mu, (\overline{c}^w)_{\mu\nu}\big). \tag{219}$$

The key point is that different clocks have different $\xi_{\rm clock}$ according to the details of their construction and flavor content.

Combining the above results, we see that the dominant Lorentz-violating effects for the gravitational redshift can be represented as an effective value $(G_N M)_{\rm RS}$ implementing a scaling of $G_N m^{\rm S}$, in parallel with the results for the time delay and the gravitational Doppler shift. We obtain

$$\begin{aligned}(G_N M)_{\rm RS} \;=\; G_N m^{\rm S}\Big(1 + \frac{2\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0 + (\overline{c}^{\rm S})_{00} + \tfrac{5}{3}\overline{s}_{00} \\ +\xi_{\rm clock}\Big).\end{aligned} \tag{220}$$

This represents an unobservable scaling in any particular redshift test, but comparing redshift tests performed with different clocks could yield access to differences in $\xi_{\rm clock}$. Moreover, the result for $(G_N M)_{\rm RS}$ differs from both the time-delay value $(G_N M)_{\rm TD}$ in Eq. (207) and the Doppler-shift value $(G_N M)_{\rm DS}$ in Eq. (213), so comparing results from different tests could yield independent sensitivities to $(\overline{a}^{\rm S}_{\rm eff})_0$ that are inaccessible in other searches with ordinary matter. This prospect is considered in Sec. XI E.

We remark in passing that for certain special models the observable redshift effects in $(G_N M)_{\rm RS}$ may be hidden in WEP tests. A simple example is provided by the isotropic parachute model discussed in Sec. IX B. By virtue of Eq. (169), the effective inertial and gravitational masses in this model are equal for a test body made of ordinary matter, so no signals are observable in WEP tests. However, the presence of nonzero $(\overline{c}^w)_{\mu\nu}$ implies a nonzero rescaling of $(G_N M)_{\rm RS}$, which is observable by comparing to $(G_N M)_{\rm TD}$ or $(G_N M)_{\rm DS}$. Signals from this model could also arise in the null-redshift tests discussed in Sec. XI C 2 below.

We conclude this subsection with an illustrative calculation of $\xi_{\rm clock}$ for a simplified clock based on transitions between the Bohr levels of hydrogen, for which we determine $(\nu_P/\nu_E)^{(1,1)\rm T}_{\rm RS}$ and $\xi_{\rm clock} \equiv \xi_{\rm H,Bohr}$ assuming both the clocks and the gravitational source are at rest. This calculation is straightforward due to the spherical symmetry and the zero velocity, and also because a simple match exists between the zeroth-order hamiltonian $h^{(0,0)}$ and the kinetic contributions to the sum $h^{(1,0)} + h^{(0,1)} + h^{(1,1)}$ of the perturbative corrections presented in Sec. III C. By matching these expressions, we find that the kinetic portion of the hamiltonian in the presence of gravity and Lorentz violation can be obtained from the zeroth-order one by the following simple replacements for the proton and electron mass:

$$\begin{aligned}\frac{1}{m^p} &\;\to\; \frac{1}{m^p}\left(1 - \tfrac{3}{2}h_{00} + \tfrac{5}{3}(\overline{c}^p)_{00} + \tfrac{13}{6}(\overline{c}^p)_{00}h_{00}\right), \\ \frac{1}{m^e} &\;\to\; \frac{1}{m^e}\left(1 - \tfrac{3}{2}h_{00} + \tfrac{5}{3}(\overline{c}^e)_{00} + \tfrac{13}{6}(\overline{c}^e)_{00}h_{00}\right).\end{aligned} \tag{221}$$

Also, the source term in the Maxwell equations is corrected by the vierbein determinant e, and the result can be obtained by a simple replacement for the proton charge,

$$q^p \to \frac{q^p}{e} \approx q^p(1 - h_{00}). \tag{222}$$

It follows that the calculation of interest can be directly performed by implementing the above replacements in the standard result for the Bohr energy levels. This yields

$$\begin{aligned}E \;\to\; E\Big(&1 - \tfrac{1}{2}h_{00} \\ &+\frac{1}{m^p + m^e}[m^p(\overline{c}^e)_{00} + m^e(\overline{c}^p)_{00}](\tfrac{5}{3} - \tfrac{1}{2}h_{00})\Big).\end{aligned} \tag{223}$$

The modification (218) to the gravitational redshift is therefore given by

$$\begin{aligned}\left(\frac{\nu_P}{\nu_E}\right)^{(1,1)\rm T}_{\rm RS} \;&=\; -\frac{2G_N m^{\rm S}}{3(m^p + m^e)}(m^p(\overline{c}^e)_{00} + m^e(\overline{c}^p)_{00}) \\ &\quad\times\left(\frac{r_e - r_p}{r_e r_p}\right)\end{aligned} \tag{224}$$

when the clock transitions are those of the Bohr levels of hydrogen. This implies the result

$$\xi_{\rm H,Bohr} = -\frac{2}{3(m^p + m^e)}(m^p(\overline{c}^e)_{00} + m^e(\overline{c}^p)_{00}). \tag{225}$$

The value of $\xi_{\rm clock}$ for a realistic clock can be obtained via calculation if the hamiltonian describing the clock is known.

2. Null redshift

Another Lorentz-violating signal can be accessed by comparing two clocks of different types as they explore the gravitational potential together. This type of measurement is called a null-redshift test [134].

Consider comparing the frequencies of two clocks A and B having different values $\xi_{\rm clock} = \xi_A$ and $\xi_{\rm clock} = \xi_B$ that are located at a point P with gravitational potential h^P_{00}. The frequency ratio is given by

$$\left(\frac{\nu_A}{\nu_B}\right)^P = \left[1 + \tfrac{1}{2}(\xi_A - \xi_B)h^P_{00}\right]\left(\frac{\nu_A^{(0)}}{\nu_B^{(0)}}\right), \tag{226}$$

where the superscript (0) denotes a frequency at a hypothetical zero gravitational potential $h_{\mu\nu} = 0$. This frequency ratio depends inseparably on the potential h^P_{00} at point P and the ratio in zero potential.

When the same two clocks are moved to a point Q at potential h^Q_{00}, the frequency ratio takes a new value. If the values ξ_A and ξ_B differ, then so do the frequency ratios at P and Q. The ratio of frequency ratios then shifts away from 1 and is given by

$$\left(\frac{\nu_A}{\nu_B}\right)^P \left(\frac{\nu_B}{\nu_A}\right)^Q = 1 - \tfrac{1}{2}(\xi_A - \xi_B)(h^Q_{00} - h^P_{00}). \quad (227)$$

The shift is an observable, and it depends on the difference $\Delta\xi_{AB} = \xi_A - \xi_B$ and also on the potential difference between P and Q.

For a gravitational source with $h_{00} = 2G_N m^{\rm S}/r$ at PNO(2), we obtain

$$\left(\frac{\nu_A}{\nu_B}\right)^P \left(\frac{\nu_B}{\nu_A}\right)^Q = 1 - G_N m^{\rm S} \Delta\xi_{AB} \frac{(r_P - r_Q)}{r_P r_Q}. \quad (228)$$

Unlike the other Lorentz-violating photon effects discussed here, all of which represent scalings of $G_N m^{\rm S}$, this result is a qualitative change from conventional gravity. It is also strictly a gravitational effect, vanishing in Minkowski spacetime.

Since the shift varies with spacetime position, it exhibits features analogous to violations of local position invariance, which have been the subject of numerous studies [8]. In the present case, these features arise from the Lorentz-violating flavor dependence of the clock material. Note also that the observable (228) contains the same information as the result of two separate redshift tests performed with different clocks but the same gravitational source. This can be verified by inspection of the effective value $(G_N M)_{\rm RS}$ in Eq. (220). Some relevant experiments are described in Sec. XI E.

D. Comparison to effective orbital mass

The preceding subsections reveal that the Lorentz-violating contributions to the Shapiro time delay, the gravitational Doppler shift, and the gravitational redshift are all controlled by the effective value of $G_N M$ for the gravitational source. In the context of the solar-system tests discussed in Sec. X, rescalings of $G_N M$ also occur but can be disregarded as unobservable. Here, we determine the effective value of $G_N M$ relevant to observations of orbiting bodies, $(G_N M)_{\rm OB}$.

For Lorentz violation involving the coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$, the secular changes in the orbital elements for the trajectory of an orbiting body are given by Eq. (184). The analogous result for the coefficients $\overline{s}_{\mu\nu}$ is given in Eq. (162) of Ref. [7]. Inspecting these equations, we can deduce the effective reduced mass of the source and test bodies and hence extract the effective value $(G_N M)_{\rm OB}$. Making no additional assumptions about the masses of the source and test bodies, we find

$$\begin{aligned}(G_N M)_{\rm OB} \;=\; G_N M \Big(1 &+ \frac{2\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0 + \frac{2\alpha}{m^{\rm T}}(\overline{a}^{\rm T}_{\rm eff})_0 + \tfrac{5}{3}\overline{s}_{00} \\ &+ \frac{m^{\rm S} - \frac{2}{3}m^{\rm T}}{M}(\overline{c}^{\rm S})_{00} + \frac{m^{\rm T} - \frac{2}{3}m^{\rm S}}{M}(\overline{c}^{\rm T})_{00}\Big).\end{aligned} \quad (229)$$

To obtain an expression that is more readily comparable to the effective values of $(G_N M)$ measured in photon tests, we note that $m^{\rm T} \ll m^{\rm S}$ under typical circumstances. The above result then reduces to

$$\begin{aligned}(G_N M)_{\rm OB} \;=\; G_N m^{\rm S} \Big(1 &+ \frac{2\alpha}{m^{\rm S}}(\overline{a}^{\rm S}_{\rm eff})_0 + (\overline{c}^{\rm S})_{00} + \tfrac{5}{3}\overline{s}_{00} \\ &+ \frac{2\alpha}{m^{\rm T}}(\overline{a}^{\rm T}_{\rm eff})_0 - \tfrac{2}{3}(\overline{c}^{\rm T})_{00}\Big).\end{aligned} \quad (230)$$

This expression for $(G_N M)_{\rm OB}$ contains a linear combination of coefficients for Lorentz violation that is independent of the three combinations $(G_N M)_{\rm TD}$, $(G_N M)_{\rm DS}$, and $(G_N M)_{\rm RS}$ obtained for photon tests. Some comments about tests with this result are provided in the next subsection.

E. Experiments

The above subsections show that each type of photon test of Lorentz symmetry is sensitive to an effective value of $G_N M$ that contains a combination of coefficients for Lorentz violation. The time-delay value $(G_N M)_{\rm TD}$ is given by Eq. (207), and it depends on the coefficients $\alpha(\overline{a}^{\rm S}_{\rm eff})_0$, $(\overline{c}^{\rm S})_{00}$, and $\overline{s}_{00}$. The gravitational Doppler shift involves the value $(G_N M)_{\rm DS}$ in Eq. (213) and involves the same combination of the three coefficients. The value $(G_N M)_{\rm RS}$ for the gravitational redshift is given by Eq. (220), which contains a different combination of coefficients and varies also with $\xi_{\rm clock}$. All three of these photon tests yield sensitivities differing from those in orbital tests, which involve the value $(G_N M)_{\rm OB}$ in Eq. (230) that depends also on the test-body coefficients $\alpha(\overline{a}^{\rm T}_{\rm eff})_0$ and $(\overline{c}^{\rm T})_{00}$. Note that no qualitatively new signals are involved in any of these cases, since the effects are merely scalings of established physics. In contrast, the null-redshift observable given in Eq. (227), which depends on the difference of clock quantities $\Delta\xi_{AB}$, represents a qualitative departure from conventional gravitational physics.

Comparisons of the time-delay value $(G_N M)_{\rm TD}$ or the Doppler-shift value $(G_N M)_{\rm DS}$ to the redshift value $(G_N M)_{\rm RS}$ for the same source body can be used to obtain sensitivity to combinations of coefficients for Lorentz violation. High-quality data for the time delay and the gravitational Doppler shift have been obtained by tracking the Cassini spacecraft [135] in the gravitational field of the Sun. Proposed missions such as the Astrodynamical Space Test of Relativity using Optical Devices (AS-

TROD) [136], the Mercury Orbiter Radio-science Experiment (MORE) [137], the Search for Anomalous Gravitation using Atomic Sensors (SAGAS) [138], and the Solar System Odyssey (SSO) [139] have the potential to improve these measurements using the Sun as the gravitational source, while the Beyond Einstein Advanced Coherent Optical Network (BEACON) [140] could sharpen results using the Earth as the gravitational source. Another relevant recent proposal involves the use of very-long-baseline interferometry (VLBI) [141] to measure the deflection of radio waves from distant sources by solar-system objects. The sensitivity of this measurement to Lorentz violation is likely to be comparatively weaker but may be offset by the enhanced access to independent coefficient combinations offered by multiple measurements and perhaps by access to anisotropic effects involving spatial components of $\overline{s}_{\mu\nu}$.

Redshift tests permit sensitivities to effects controlled by $\xi_{\rm clock}$. These can be isolated either by comparing separate redshift tests performed with different clocks in the same gravitational source or more directly by null-redshift tests, in which the signal depends on the difference $\Delta\xi_{AB}$ between two clocks A, B and vanishes in the absence of gravity. The results of some investigations of local position invariance can be reinterpreted as measurements of $\Delta\xi_{AB}$. For example, a recent Earth-based test comparing a hydrogen maser with a Cs fountain [142] obtained a sensitivity that corresponds to the bound

$$|\xi_{\rm H} - \xi_{\rm Cs}| < (0.1 \pm 1.4) \times 10^{-6}, \qquad (231)$$

while another comparing a hydrogen maser with a cryogenic sapphire oscillator [143] yields the measurement

$$\xi_{\rm H} - \xi_{\rm CSO} = (-2.7 \pm 1.4) \times 10^{-4}. \qquad (232)$$

These results offer a benchmark for currently attainable sensitivities to $\xi_{\rm clock}$. The two experiments involve different clocks and hence likely different sensitivities to the coefficients $(\overline{c}^w)_{\mu\nu}$. Calculating the specific constraints on $(\overline{c}^w)_{\mu\nu}$ and possibly other coefficients for Lorentz violation from these and other tests is an interesting open project. Note that Earth-based searches of this type typically take advantage of the annual and diurnal variations in the gravitational potential of the Sun as experienced in the laboratory. In searches using the annual variation, it is challenging and perhaps impossible to disentangle gravitational effects of nonzero $\xi_{\rm clock}$ from other Lorentz-violating effects in Minkowski spacetime. However, diurnal searches can distinguish the two types of effects because the Minkowski-spacetime signals occur at the sidereal frequency instead. Note also that other clock-comparison tests normally viewed as sensitive to SME coefficients in Minkowski spacetime may also have sensitivity to $\xi_{\rm clock}$. One intriguing possibility is that suitable choices of clocks could separate effects from $(\overline{c}^e)_{TT}$ and $(\overline{c}^p)_{TT}$, which would then lead to independent sensitivities to $(\overline{a}^e_{\rm eff})_T$ and $(\overline{a}^p_{\rm eff})_T$, a result otherwise challenging to achieve.

Satellites carrying two different clocks offer interesting prospects for improved null-redshift searches for Lorentz violation. Since the attainable sensitivities improve with the gravitational potential difference according to Eq. (227), it is desirable to acquire elliptical orbits. The Space-Time Asymmetry Research (STAR) program [144] presently under development proposes to compare two different clocks on a satellite traveling in an elliptical orbit. This mission could improve sensitivities to $\xi_{\rm clock}$ by an order of magnitude or more relative to ground-based tests. Improved sensitivities may also be possible by comparing clocks aboard the proposed SAGAS spacecraft. Note also that experiments in highly elliptical orbits can be expected to have increased sensitivity to anisotropic effects on the redshift produced by $\overline{s}_{JK}$.

Provided effects due to $\xi_{\rm clock}$ are excluded, either through independent experiments or by using a clock with $\xi_{\rm clock} = 0$, then the dependence of $(G_N M)_{\rm RS}$ on $(\overline{a}^{\rm S}_{\rm eff})_T$ implies that measurements of the gravitational redshift can be compared with other photon tests performed with the same gravity source to obtain independent sensitivities to $(\overline{a}^w_{\rm eff})_T$. The Gravity Probe A (GPA) mission [145], which used the Earth as the gravity source, confirmed the conventional gravitational redshift to parts in 10^4. This result could eventually be combined with proposed time-delay or Doppler-shift measurements of the BEACON type to yield sensitivity to the coefficient $(\overline{a}^{\rm S}_{\rm eff})_T$ for the Earth. Improved tests of the gravitational redshift are also proposed for the Atomic Clock Ensemble in Space (ACES) [146], SAGAS, and STAR missions. With the Sun as the gravity source instead, the Galileo space probe obtained sensitivity to deviations for the gravitational redshift at the level of parts in 10^2 [147]. Given knowledge of $\xi_{\rm clock}$ for the Galileo clock, this result could be combined with the Cassini results to yield sensitivity to the coefficient $(\overline{a}^{\rm S}_{\rm eff})_T$ for the Sun. If feasible, a redshift test performed directly with Cassini would be of interest in this respect. Other gravitational sources could also be used. For example, the gravitational redshift was measured to parts in 10^2 using Saturn as the source during the flyby of Voyager [148]. Time-delay or Doppler-shift data could therefore permit sensitivity to the coefficient $(\overline{a}_{\rm eff})_T$ for Saturn.

Combinations of photon tests with measurements of the effective orbital mass are also of interest. In the limit of zero matter-sector Lorentz violation, the result (230) for $(G_N M)_{\rm OB}$ has been combined with Eq. (207) for $(G_N M)_{\rm TD}$ to extract sensitivity to $\overline{s}_{TT}$ [133]. However, with nonzero coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$, the effective value $(G_N M)_{\rm OB}$ involves properties of the test body as well as the source. Note that these appear in the familiar combination $\alpha(\overline{a}^{\rm T}_{\rm eff})_T - m^{\rm T}(\overline{c}^{\rm T})_{TT}/3$ discussed in Secs. VII and IX B. The WEP tests considered in this work constrain the degree to which this combination can differ between neutrons and neutral combinations of electrons and protons, but only through the indirect arguments involving binding energy described in Sec. VI B. In contrast, comparing $G_N M$ factors for measurements with orbiting bodies and for photon tests offers the opportunity to obtain direct sensitivity to $\alpha(\overline{a}^n_{\rm eff})_T - m^{\rm T}(\overline{c}^n)_{TT}/3$ and $\alpha(\overline{a}^{e+p}_{\rm eff})_T - m^{\rm T}(\overline{c}^{e+p})_{TT}/3$. Comparisons with sensitive gravimeters may also be of interest in this respect.

We remark in passing that proposed sensitive experiments to measure gravitational light bending, including the Laser Astrometric Test of Relativity (LATOR) [149] and the Space Interferometry Mission (SIM) [150], are likely to have signals affected by Lorentz violation. The attainable sensitivities can be expected to be similar to those discussed above, but the analysis of this possibility lies beyond our present scope.

XII. SUMMARY

This work studies the gravitational couplings of matter in the presence of Lorentz violation. The framework for the investigation is the fermion sector of the gravitationally coupled minimal SME in a post-newtonian expansion. Our primary goal is to develop a suitable methodology for searches for Lorentz and CPT violation that exploit the couplings of matter to gravity, incorporating in particular effects that are challenging or impossible to detect in Minkowski spacetime.

Section II presents the basic formalism for the work. The action for the gravity-matter system is given in Sec. II A, and the linearization procedure is outlined in Sec. II B. Some types of Lorentz violation are unobservable in principle. This issue is discussed in Sec. II C, which also fixes the coordinate choice (23) used in this work. The metric and coefficient fields for Lorentz violation can fluctuate about their background values, and the corresponding interactions must be incorporated in analyses of experiments. In Sec. II D, we develop general perturbative techniques to analyze these fluctuations. Two notions of perturbative order are introduced. One is denoted O(m,n) and tracks the orders in Lorentz violation and in gravity, while the other is denoted PNO(p) and tracks the post-newtonian order. The goal of this work is to investigate dominant terms involving Lorentz violation in gravity, which are at O(1,1).

Section III studies the quantum theory of the gravity-matter system. Starting from the field-theoretic action, we construct the relativistic quantum mechanics in the presence of gravitational fluctuations and Lorentz violation. Formulating the quantum theory for matter in the presence of gravitational fluctuations is a standard challenge. In Sec. III A, we present a solution to this problem via a field redefinition, which yields a hamiltonian that is hermitian with respect to the usual scalar product for wave functions. We then use this procedure in Sec. III B to extract the explicit form of the relativistic hamiltonian involving all coefficients for Lorentz violation in the minimal QED extension. The result forms the appropriate starting point for general investigations of Lorentz and CPT violation in matter-gravity couplings. To maintain a reasonable scope in this work, we subsequently specialize our focus to the study of spin-independent Lorentz-violating effects, which are governed by the coefficient fields $(a_{\rm eff})_\mu$, $c_{\mu\nu}$ and the metric fluctuation $h_{\mu\nu}$. The nonrelativistic quantum hamiltonian for this case is obtained in Sec. III C using the standard Foldy-Wouthuysen procedure.

Measurements of gravity-matter couplings typically are performed at the classical level. Section IV constructs the classical theory associated with the quantum-mechanical dynamics of matter involving nonzero $(a_{\rm eff})_\mu$, $c_{\mu\nu}$, and $h_{\mu\nu}$. The behavior of test and source bodies in the presence of Lorentz violation is the subject of Sec. IV A. Working from the action for a point particle, we provide expressions for the mass and for the effective coefficients for Lorentz violation for a test or source body, along with the effective action (76) describing the dynamics of the body. These results enable the derivation in Sec. IV B of the modified Einstein equation and the equation (78) for the trajectory of a test particle. To apply this equation in practice requires knowledge of the coefficient and metric fluctuations. In Sec. IV C, we develop a systematic methodology for calculating this information in perturbation theory and obtain general expressions for the coefficient and metric fluctuations to O(1,1) in terms of various gravitational potentials and the background coefficient values $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$.

To illustrate the application of the general formalism, we consider in Sec. V a specific class of bumblebee models, which are theories with a vector field driving spontaneous Lorentz breaking. The action for the bumblebee field B_μ is given in Sec. V A, where a match at the field-theoretic level to the general formalism of earlier sections is made and the coefficient fields $(a_{\rm eff})_\mu$ and $c_{\mu\nu}$ are identified in terms of B_μ and the metric. In Sec. V B, we explicitly solve the model at the relevant order in perturbation theory, extract the modified Einstein equation, and derive the equation for the trajectory of a test particle. The results are shown to match those obtained using the general formalism developed in the earlier sections.

The largest portion of the paper is devoted to a discussion of experiments and observations that can achieve sensitivity to the coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$. Section VI presents some general material broadly applicable to searches for Lorentz violation. Various choices of reference frame and their relationship to the canonical Sun-centered frame are discussed in Sec. VI A. Attainable sensitivities to the coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ in any measurement procedure are constrained by certain generic features. Section VI B considers some of these, including the role of binding energy in impeding or aiding the analysis of WEP tests for signals of Lorentz violation.

A major class of searches for Lorentz violation involves laboratory tests with ordinary neutral bulk matter, neutral atoms, and neutrons. Section VII treats this topic. The PNO(3) lagrangian describing the dynamics of a test body moving near the surface of the Earth in the presence of Lorentz violation is considered in Sec. VII A. Expressions are given in an Earth-centered frame and the transformation to the laboratory frame is outlined. The resulting description of laboratory signals for gravitational Lorentz violation includes effects from the matter-sector coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$ and ones from the gravity-sector coefficients $\overline{s}_{\mu\nu}$ obtained in Ref. [7]. It reveals that the gravitational force acquires tiny corrections both along and perpendicular to the usual free-fall trajectory near the surface of the Earth, while the effective inertial mass of a test body becomes a direction-dependent quantity. These effects can be sought in numerous laboratory experiments. Since the standard relationship between force and acceleration is modified, it is useful to distinguish tests measuring gravitational acceleration from ones comparing forces. In Sec. VII B, we consider free-fall gravimeter tests such as falling corner cubes and atom interferometry. Force-comparison gravimeter tests using equipment such as superconducting gravimeters are studied in Sec. VII C. An important potential signal for gravitational Lorentz violation arises from the flavor dependence of the effects, which implies signals in WEP tests. A variety of free-fall WEP tests is considered in Sec. VII D, while force-comparison WEP tests with a torsion pendulum are treated in Sec. VII E. For all the tests considered, the possible signals for Lorentz violation are decomposed according to their time dependence, and estimates of the attainable sensitivities are obtained.

Table XIV. Summary of actual and attainable sensitivities in past or present tests.

Coefficient combinations	Gravimeter	Free-fall WEP	Force-comparison WEP	Solar system
$\alpha(\overline{a}^{e+p}_{\rm eff})_X$	$[10^{-7}$ GeV$]$	$[10^{-3}$ GeV$]$	$[10^{-7}$ GeV$]$	...
$\alpha(\overline{a}^{e+p}_{\rm eff})_{Y+Z}$	$[10^{-7}$ GeV$]$	$[10^{-3}$ GeV$]$	$[10^{-6}$ GeV$]$	...
$\alpha(\overline{a}^{e+p}_{\rm eff})_Y$	$[10^{-5}$ GeV$]$	...	$[10^{-7}$ GeV$]$	...
$\alpha(\overline{a}^{e+p}_{\rm eff})_Z$	$[10^{-5}$ GeV$]$	...	$[10^{-6}$ GeV$]$	...
$\alpha(\overline{a}^{e+p}_{\rm eff})_T$	...	10^{-7} GeV†	10^{-10} GeV†	...
$\alpha(\overline{a}^{n}_{\rm eff})_X$	$[10^{-7}$ GeV$]$	$[10^{-3}$ GeV$]$	$[10^{-7}$ GeV$]$	...
$\alpha(\overline{a}^{n}_{\rm eff})_{Y+Z}$	$[10^{-7}$ GeV$]$	$[10^{-3}$ GeV$]$	$[10^{-6}$ GeV$]$	...
$\alpha(\overline{a}^{n}_{\rm eff})_Y$	$[10^{-5}$ GeV$]$	...	$[10^{-7}$ GeV$]$	...
$\alpha(\overline{a}^{n}_{\rm eff})_Z$	$[10^{-5}$ GeV$]$	...	$[10^{-6}$ GeV$]$	...
$\alpha(\overline{a}^{n}_{\rm eff})_T$	...	10^{-7} GeV†	10^{-10} GeV†	...
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_X$	$[10^{-7}$ GeV$]$	$[10^{-4}$ GeV$]$	$[10^{-8}$ GeV$]$	$[10^{-6}$ GeV$]$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_{Y+Z}$	$[10^{-7}$ GeV$]$	$[10^{-4}$ GeV$]$	$[10^{-7}$ GeV$]$	$[10^{-6}$ GeV$]$
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_Y$	$[10^{-5}$ GeV$]$	...	$[10^{-8}$ GeV$]$	...
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_Z$	$[10^{-5}$ GeV$]$	...	$[10^{-7}$ GeV$]$	...
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_T - \frac{1}{3}m^p(\overline{c}^{e+p-n})_{TT}$	...	10^{-8} GeV‡	10^{-8} GeV‡	...
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_T - \frac{1}{3}m^p(\overline{c}^{e+p-n})_{TT} + (\frac{1}{2}\cos^2\chi - \frac{1}{6})m^n(\overline{c}^n)_Q$	...	10^{-8} GeV	...	...
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_T - \frac{1}{3}m^p(\overline{c}^{e+p-n})_{TT} - \frac{1}{6}m^n(\overline{c}^n)_Q$	...	...	10^{-11} GeV	...
$(\overline{a}^{e}_{\rm eff})_{\text{☿}} + (\overline{a}^{p}_{\rm eff})_{\text{☿}} + 0.1(\overline{a}^{n}_{\rm eff})_{\text{☿}}$	...	...	...	10^{-6} GeV†
$(\overline{a}^{e}_{\rm eff})_\oplus + (\overline{a}^{p}_{\rm eff})_\oplus + 0.1(\overline{a}^{n}_{\rm eff})_\oplus$	...	...	...	10^{-6} GeV†
$(\overline{c}^n)_{(TJ)}$	$[10^{-7}]$	$[10^{-4}]$	$[10^{-7}]$	$[10^{-6}]$
$(\overline{c}^n)_Q$	...	$10^{-8\ddagger}$	$10^{-8\ddagger}$	...
$(\overline{c}^n)_{\text{☿}}$	...	...	...	$10^{-8\dagger}$
$(\overline{c}^n)_\oplus$	...	...	...	$10^{-7\dagger}$

Section VIII considers satellite-based WEP tests, which offer interesting prospects for improved sensitivities to Lorentz violation. In this context, the signal for Lorentz violation is an anomalous time variation of the relative local acceleration between two test bodies of differing composition located on the satellite. We derive the frequency decomposition of the signal for Lorentz violation, and we consider idealized scenarios for several proposed satellite-based WEP tests. Based on the design reach of the missions, we estimate the sensitivities that could be achieved to various combinations of the matter-sector coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$.

Studies of the gravitational couplings of charged particles, antimatter, and second- and third-generation particles present distinct experimental challenges but can yield sensitivities to Lorentz and CPT violation that are otherwise difficult or impossible to achieve. Section IX addresses some of these possibilities, including charged-particle interferometry, ballistic tests with charged particles, gravitational experiments with antihydrogen, and signals in muonium free fall. For antihydrogen experiments, simple toy models are introduced to illustrate aspects of their discovery potential and to address attempts to place indirect limits on possible effects.

Traditional tests of gravity couplings to matter include observations of the motion of bodies within the solar system. Section X contains a discussion of the signals accessible via lunar and satellite laser ranging and via measurements of the precession of the perihelion of orbiting bodies. A reanalysis of existing data from lunar laser ranging could yield interesting sensitivities to some combinations of the matter-sector coefficients $(\overline{a}_{\rm eff})_\mu$ and $\overline{c}_{\mu\nu}$. We use the established advance of the perihelion for Mercury and for the Earth to obtain constraints on combinations of $(\overline{a}_{\rm eff})_\mu$, $\overline{c}_{\mu\nu}$, and $\overline{s}_{\mu\nu}$.

The interaction of photons with gravity offers a different arena in which to seek Lorentz and CPT violation. Section XI is devoted to this topic. We consider signals arising in measurements of the photon time delay, studies of the gravitational Doppler and redshifts, and comparisons of the behaviors of photons and massive bodies. A variety of existing and proposed experiments on spacecraft offer interesting prospects for these measurements.

Tables XIV and XV collect estimated sensitivities to the matter-sector coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$ obtained from many of the measurements discussed in this work. These tables disregard possible effects from the pure-gravity coefficients $\overline{s}_{\mu\nu}$ that could in principle be relevant to solar-system tests. Table XIV concerns existing data, while Table XV tabulates future prospects. One result omitted from these tables is the generalization (121) of the constraint obtained in Ref. [11] using data from force-comparison WEP tests with a torsion pendulum. In Sec. VII E, multiple datasets are combined to separate this constraint into the two limits (153), and both of these are included in Table XIV instead.

Table XV. Summary of attainable sensitivities in future tests.

Coefficient combinations	Free-fall gravimeter	Free-fall WEP	Satellite WEP	Solar system
$\alpha(\overline{a}^{e+p}_{\rm eff})_X$	10^{-10} GeV	10^{-10} GeV	10^{-11} GeV	...
$\alpha(\overline{a}^{e+p}_{\rm eff})_{Y+Z}$	10^{-10} GeV	10^{-10} GeV	10^{-11} GeV	...
$\alpha(\overline{a}^{e+p}_{\rm eff})_Y$	10^{-8} GeV	10^{-8} GeV	10^{-9} GeV	...
$\alpha(\overline{a}^{e+p}_{\rm eff})_Z$	10^{-8} GeV	10^{-8} GeV	10^{-9} GeV	...
$\alpha(\overline{a}^{e+p}_{\rm eff})_T$	...	10^{-14} GeV†	10^{-15} GeV†	...
$\alpha(\overline{a}^{n}_{\rm eff})_X$	10^{-10} GeV	10^{-10} GeV	10^{-11} GeV	...
$\alpha(\overline{a}^{n}_{\rm eff})_{Y+Z}$	10^{-10} GeV	10^{-10} GeV	10^{-11} GeV	...
$\alpha(\overline{a}^{n}_{\rm eff})_Y$	10^{-8} GeV	10^{-8} GeV	10^{-9} GeV	...
$\alpha(\overline{a}^{n}_{\rm eff})_Z$	10^{-8} GeV	10^{-8} GeV	10^{-9} GeV	...
$\alpha(\overline{a}^{n}_{\rm eff})_T$	...	10^{-14} GeV†	10^{-15} GeV†	...
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_X$	10^{-10} GeV	10^{-11} GeV	10^{-12} GeV	10^{-7} GeV
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_{Y+Z}$	10^{-10} GeV	10^{-11} GeV	10^{-12} GeV	10^{-7} GeV
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_Y$	10^{-8} GeV	10^{-9} GeV	10^{-10} GeV	...
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_Z$	10^{-8} GeV	10^{-9} GeV	10^{-10} GeV	...
$\alpha(\overline{a}^{e+p-n}_{\rm eff})_T - \frac{1}{3}m^p(\overline{c}^{e+p-n})_{TT}$	...	10^{-15} GeV	10^{-16} GeV	...
$(\overline{c}^n)_{(TJ)}$	10^{-10}	10^{-11}	10^{-12}	10^{-7}
$(\overline{c}^n)_Q$	...	10^{-15}	10^{-16}	...

The formalism and the analytical results for gravitational signals of Lorentz violation presented in this work apply to the nonzero matter-sector coefficients $(\overline{a}^w_{\rm eff})_\mu$, $(\overline{c}^w)_{\mu\nu}$ and in some cases also to the gravity-sector coefficients $\overline{s}_{\mu\nu}$. Comparatively little is known about the coefficients $(\overline{a}^w_{\rm eff})_\mu$, and scenarios exist in which they could be countershaded, having large values while still escaping notice in searches to date [11]. However, nongravitational measurements have already yielded impressive sensitivities to various components of $(\overline{c}^w)_{\mu\nu}$ [2]. The estimated attainable sensitivities to $(\overline{c}^w)_{\mu\nu}$ derived in this work are therefore primarily restricted to components of $(\overline{c}^n)_{\mu\nu}$, for which existing constraints are weaker. Tables XIV and XV reflect these facts, containing mostly entries for combinations of the coefficients $(\overline{a}^w_{\rm eff})_\mu$ along with some results for $(\overline{c}^w)_{\mu\nu}$.

Table XIV summarizes actual sensitivities or estimated attainable ones using data from past or present measurements. The table is based on the calculations presented in this work and includes only sensitivities below parts in 10^2. Each entry in the first column of this table represents a linear combination of coefficients that is accessible in principle via existing searches. Each of the other four columns contains our estimates for sensitivities that could be achieved in the listed class of tests, expressed to the nearest order of magnitude. Values in these four columns that are shown without brackets represent order-of-magnitude sensitivities implied by our present analysis to the modulus of the coefficient combination displayed. Values appearing in brackets in the table represent our estimate of sensitivities that could in principle be obtained from a suitable reanalysis of existing data. An obelisk (†) following a value indicates a limit attainable under the assumption that either $(\overline{a}^w_{\rm eff})_\mu$ or $(\overline{c}^w)_{\mu\nu}$ is negligibly small or vanishes. A diesis (‡) indicates a sensitivity that is attained by combining data from two different classes of experiments, and this sensitivity is placed in each of the two corresponding columns in the table.

Table XV contains future attainable sensitivities to the moduli of various combinations of the matter-sector coefficients $(\overline{a}^w_{\rm eff})_\mu$ and $(\overline{c}^w)_{\mu\nu}$, as estimated in previous sections of this work. The structure of this table is similar to that of Table XIV. The listed entries are based on the best design reach and are given to the nearest order of magnitude. For each class of search, we assume enough measurements have been performed to achieve the maximum number of independent sensitivities. The reader is cautioned that for certain coefficients a single measurement cannot attain the indicated sensitivity, but instead only a linear combination of coefficients with multipliers controlled by composition and orientation factors. Note that elsewhere in this work the convention is to display values of future sensitivities in braces, but this convention is suppressed in Table XV because all entries are of this type. Note also that further improvements in theoretical techniques and experimental design in all types of searches, including ones not listed in Table XV such as exotic gravitational tests or photon tests, are expected to yield additional interesting prospects for future attainable sensitivities.

Taken together, Tables XIV and XV reveal excellent prospects for using matter-gravity couplings to seek effects of Lorentz violation. The opportunities for measuring the countershaded coefficients $(\overline{a}^w_{\rm eff})_\mu$ at sensitive levels are of particular interest in this context, as these coefficients typically cannot be detected in nongravitational searches. Indeed, the spatial components of $(\overline{a}^w_{\rm eff})_\mu$ remain essentially unconstrained to date. The tests proposed here can be performed with existing or near-future technology, and they offer a promising new arena for searches for signals from the Planck scale.

Acknowledgments

This work was supported in part by the Department of Energy under grant DE-FG02-91ER40661 and by the Indiana University Center for Spacetime Symmetries.

[1] V.A. Kostelecký and S. Samuel, Phys. Rev. D **39**, 683 (1989); V.A. Kostelecký and R. Potting, Nucl. Phys. B **359**, 545 (1991).

[2] *Data Tables for Lorentz and CPT Violation,* V.A. Kostelecký and N. Russell, 2010 edition, Rev. Mod. Phys., in press [arXiv:0801.0287v3].

[3] V.A. Kostelecký, Phys. Rev. D **69**, 105009 (2004).

[4] J.B.R. Battat, J.F. Chandler, and C.W. Stubbs, Phys. Rev. Lett. **99**, 241103 (2007).

[5] K.-Y. Chung, S-w. Chiow, S. Herrmann, S. Chu, and H. Müller, Phys. Rev. D **80**, 016002 (2009); H. Müller, S.-w. Chiow, S. Herrmann, S. Chu, and K.-Y. Chung, Phys. Rev. Lett. 100, 031101 (2008).

[6] W.M. Jensen, S.M. Lewis, and J.C. Long, in V.A. Kostelecký, ed., *CPT and Lorentz Symmetry IV*, World Scientific, Singapore, 2008; J.M. Overduin, *ibid.*

[7] Q.G. Bailey and V.A. Kostelecký, Phys. Rev. D **74**, 045001 (2006).

[8] C.M. Will, *Theory and Experiment in Gravitational Physics*, Cambridge University Press, Cambridge, 1993.

[9] V.A. Kostelecký and R. Potting, Phys. Rev. D **51**, 3923 (1995).

[10] O.W. Greenberg, Phys. Rev. Lett. **89**, 231602 (2002).

[11] V.A. Kostelecký and J.D. Tasson, Phys. Rev. Lett. **102**, 010402 (2009).

[12] L. Carbone, H. Panjwani, C.C. Speake, T.J. Quinn and C.J. Collins, in T. Damour, R.T. Jantzen, and R. Ruffini, eds., *Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity*, World Scientific, Singapore, 2010.

[13] Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960); J. Goldstone, Nuov. Cim. **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[14] R. Bluhm and V.A. Kostelecký, Phys. Rev. D **71** 065008 (2005).

[15] R. Bluhm *et al.*, Phys. Rev. D **77**, 065020 (2008).

[16] R. Utiyama, Phys. Rev. **101**, 1597 (1956); T.W.B. Kibble, J. Math. Phys. **2**, 212 (1961).

[17] V.A. Kostelecký, N. Russell, and J.D. Tasson, Phys. Rev. Lett. **100**, 111102 (2008).

[18] D. Colladay and V.A. Kostelecký, Phys. Rev. D **55**, 6760 (1997); Phys. Rev. D **58**, 116002 (1998).

[19] D. Colladay and P. McDonald, J. Math. Phys. **43**, 3554 (2002).

[20] M.S. Berger and V.A. Kostelecký, Phys. Rev. D **65**, 091701(R) (2002).

[21] V.A. Kostelecký and M. Mewes, Phys. Rev. D **66**, 056005 (2002).

[22] Q.G. Bailey and V.A. Kostelecký, Phys. Rev. D **70**, 076006 (2004).
[23] B. Altschul, J. Phys. A **39** 13757 (2006).
[24] R. Lehnert, Phys. Rev. D **75**, 041301 (2007).
[25] V.A. Kostelecký and M. Mewes, Phys. Rev. D **80**, 015020 (2009); Ap. J. Lett. **689**, L1 (2008).
[26] C. Armendariz-Picon, A. Diez-Tejedor, and R. Penco, JHEP **1010**, 079 (2010).
[27] V.A. Kostelecký and R. Potting, Gen. Rel. Grav. **37**, 1675 (2005); Phys. Rev. D **79**, 065018 (2009).
[28] S.M. Carroll, H. Tam, and I.K. Wehus, Phys. Rev. D **80**, 025020 (2009).
[29] N. Arkani-Hamed, H.-C. Cheng, M. Luty, and J. Thaler, JHEP **0507**, 029 (2005).
[30] B. Altschul *et al.*, Phys. Rev. D **81**, 065028 (2010).
[31] Y. Nambu, Prog. Theor. Phys. Suppl. Extra 190 (1968); J.D. Bjorken, Ann. Phys. **24**, 174 (1963); P.G.O. Freund, Acta Phys. Austriaca **14**, 445 (1961); W. Heisenberg, Rev. Mod. Phys. **29**, 269 (1957); P.A.M. Dirac, Proc. R. Soc. Lon. **A209**, 291, (1951).
[32] Z. Berezhiani and O.V. Kancheli, arXiv:0808.3181; P. Kraus and E.T. Tomboulis Phys. Rev. D **66**, 045015 (2002); D. Atkatz, Phys. Rev. D **17**, 1972 (1978); H.C. Ohanian, Phys. Rev. **184**, 1305 (1969); P.R. Phillips, Phys. Rev. **146**, 966 (1966).
[33] I.L. Shapiro, Phys. Rep. 357, 113 (2002).
[34] B.R. Heckel, E.G. Adelberger, C.E. Cramer, T.S. Cook, S. Schlamminger, and U. Schmidt, Phys. Rev. D **78**, 092006 (2008).
[35] S. Schlamminger, K.-Y. Choi, T.A. Wagner, J.H. Gundlach, and E.G. Adelberger, Phys. Rev. Lett. **100**, 041101 (2008).
[36] R. Bluhm *et al.*, Phys. Rev. D **57**, 3932 (1998).
[37] V.A. Kostelecký and C.D. Lane, J. Math. Phys. **40**, 6245 (1999).
[38] V.A. Kostelecký and R. Lehnert, Phys. Rev. D **63**, 065008 (2001).
[39] L. Parker, Phys. Rev. D **22**, 1922 (1980).
[40] X. Huang and L. Parker, Phys. Rev. D **79**, 024020 (2009).
[41] B.R. Heckel, C.E. Cramer, T.S. Cook, E.G. Adelberger, S. Schlamminger, and U. Schmidt, Phys. Rev. Lett. **97** 021603 (2006).
[42] L.L. Foldy and S.A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[43] B. Goncalves, Y.N. Obukhov, and I.L. Shapiro, Phys. Rev. **80**, 125034 (2009).
[44] V.A. Kostelecký and N. Russell, Phys. Lett. B **693**, 443 (2010).
[45] V.A. Kostelecký and C.D. Lane, Phys. Rev. D **60**, 116010 (1999).
[46] C.J. Allègre, J.-P. Poirier, E. Humler, and A.W. Hofmann, Earth Planet. Sci. Lett. **134**, 515 (1995).
[47] L.I. Schiff, Am. J. Phys. **28**, 340 (1960).
[48] The breaking of general coordinate invariance is studied in M.M. Anber, U. Aydemir, and J.F. Donoghue, Phys. Rev. D **81**, 084059 (2010).
[49] V.A. Kostelecký and S. Samuel, Phys. Rev. D **40**, 1886 (1989); Phys. Rev. Lett. **63**, 224 (1989).
[50] M.D. Seifert, Phys. Rev. D **81**, 065010 (2010); Phys. Rev. D **79**, 124012 (2009); J. Alfaro and L.F. Urrutia, Phys. Rev. D **81**, 025007 (2010); J.L. Chkareuli, C.D. Froggatt, and H.B. Nielsen, Nucl. Phys. B **821**, 65 (2009); C. Armendariz-Picon and A. Diez-Tejedor, JCAP **0912**, 018 (2009).
[51] S.M. Carroll, T.R. Dulaney, M.I. Gresham, and H. Tam, Phys. Rev. D **79**, 065011 (2009); R. Bluhm, N.L. Gagne, R. Potting, and A. Vrublevskis, Phys. Rev. D **77**, 125007 (2008); M.D. Seifert, Phys. Rev. D **76**, 064002 (2007).
[52] G. Leibbrandt, Rev. Mod. Phys. **59**, 1067 (1987).
[53] KTeV Collaboration, H. Nguyen, in V.A. Kostelecký, ed., *CPT and Lorentz Symmetry II*, World Scientific, Singapore, 2002 [hep-ex/0112046]; A. Di Domenico, KLOE Collaboration, J. Phys. Conf. Ser. **171**, 012008 (2009); FOCUS Collaboration, J.M. Link *et al.*, Phys. Lett. B **556**, 7 (2003); BaBar Collaboration, B. Aubert *et al.*, Phys. Rev. Lett. **100**, 131802 (2008); hep-ex/0607103.
[54] V.A. Kostelecký, Phys. Rev. Lett. **80**, 1818 (1998); Phys. Rev. D **61**, 016002 (2000); Phys. Rev. D **64**, 076001 (2001); V.A. Kostelecký and R. Van Kooten, Phys. Rev. D, Rapid Communications, in press [arXiv:1007.5312].
[55] MINOS Collaboration, P. Adamson *et al.*, Phys. Rev. Lett. **101**, 151601 (2008); LSND Collaboration, L.B. Auerbach *et al.*, Phys. Rev. D **72**, 076004 (2005); M.D. Messier (SK), in V.A. Kostelecký, ed., *CPT and Lorentz Symmetry II*, World Scientific, Singapore, 2005; V.A. Kostelecký and M. Mewes, Phys. Rev. D **69**, 016005 (2004); Phys. Rev. D **70**, 031902 (2004); Phys. Rev. D **70**, 076002 (2004); T. Katori *et al.*, Phys. Rev. D **74**, 105009 (2006); V. Barger, D. Marfatia, and K. Whisnant, Phys. Lett. B **653**, 267 (2007); J.S. Díaz *et al.*, Phys. Rev. D **80**, 076007 (2009).
[56] Q.G. Bailey, Phys. Rev. D **82**, 065012 (2010).
[57] I. Marson and J.E. Faller, J. Phys. E **19**, 22 (1986).
[58] A. Peters, K.Y. Chung, and S. Chu, Nature **400**, 849 (1999); Metrologia **38**, 25 (2001).
[59] T.L. Gustavson, A. Landragin, and M.A. Kasevich, Class. Quantum. Grav. **17**, 2385 (2000); T.L. Gustavson, P. Bouyer, and M.A. Kasevich, Phys. Rev. Lett. **78**, 2046 (1997).
[60] P. Story and C. Cohen-Tannoudji, J. Phys. II France **4**, 1999 (1994).
[61] Special forms of spin-dependent Lorentz violation in matter interferometers have been discussed in J. Audretsch, U. Bleyer, and C. Lämmerzahl, Phys. Rev. A **47**, 4632 (1993).
[62] J.M. McGuirk, G.T. Foster, J.B. Fixler, M.J. Snadden, and M.A. Kasevich, Phys. Rev. A **65**, 033608 (2002).
[63] N. Yu, J.M. Kohel, J.R. Kellogg, and L. Maleki, Appl. Phys. B **84**, 647 (2006).
[64] B. Canuel *et al.*, Phys. Rev. Lett. **97**, 010402 (2006).
[65] S. Dimopoulos, P.W. Graham, J.M. Hogan, and M.A. Kasevich, Phys. Rev. Lett. **98**, 111102 (2007); Phys. Rev. D **78**, 042003 (2008).
[66] H. Kaiser *et al.*, Physica B **385-386**, 1384 (2006).
[67] R.J. Warburton and J.M. Goodkind, Astrophys. J. **208**, 881 (1976).
[68] S. Shiomi, arXiv:0902.4081.
[69] A.A. Geraci, S.J. Smullin, D.M. Weld, J. Chiaverini, and A. Kapitulnik, Phys. Rev. D **78**, 022002 (2008); R.S. Decca, D. López, E. Fischbach, G.L. Klimchitskaya, D.E. Krause, and V.M. Mostepanenko, Phys. Rev. D **75**, 077101 (2007); D.J. Kapner, T.S. Cook, E.G. Adelberger, J.H. Gundlach, B.R. Heckel, C.D. Hoyle, and H.E. Swanson, Phys. Rev. Lett. **98**, 021101 (2007);

J.C. Long, H.W. Chan, A.B. Churnside, E.A. Gulbis, M.C.M. Varney, and J.C. Price, Nature **421**, 922; T.J. Quinn, C.C. Speake, S.J. Richman, R.S. Davis, and A. Picard, Phys. Rev. Lett. **87**, 111101 (2001).
[70] K. Kuroda and N. Mio, Phys. Rev. D **42**, 3903 (1990).
[71] T.M. Niebauer, M.P. McHugh, and J.E. Faller, Phys. Rev. Lett. **59**, 609 (1987).
[72] S. Fray, C.A. Diez, T.W. Hänsch, and M. Weitz, Phys. Rev. Lett. **93**, 240404 (2004).
[73] R.D. Reasenberg, in V.A. Kostelecký, ed., *CPT and Lorentz Symmetry II*, World Scientific, Singapore, 2005.
[74] V. Iafolla, S. Nozzoli, E.C. Lorenzini, I.I. Shapiro, and V. Milyukov, Class. Quantum Grav. **17**, 2327 (2000).
[75] H. Dittus and C. Mehls, Class. Quantum Grav. **18**, 2417 (2001).
[76] H. Marion *et al.*, Phys. Rev. Lett. **90**, 150801 (2003).
[77] M.K. Oberthaler, Nucl. Instr. and Meth. B **192**, 129 (2002).
[78] R.D. Reasenberg and J.D. Phillips, Class. Q. Grav. **27**, 095005 (2010).
[79] L. Koester, Phys. Rev. D **14**, 907 (1976).
[80] V.F. Sears, Phys. Rev. D **25**, 2023 (1982).
[81] Y. Su, B.R. Heckel, E.G. Adelberger, J.H. Gundlach, M. Harris, G.L. Smith, and H.E. Swanson, Phys. Rev. D **50**, 3614 (1994).
[82] L.-S. Hou, W.-T. Ni, and Y.-C.M. Li, Phys. Rev. Lett. **90**, 201101 (2003); R. Bluhm and V.A. Kostelecký, Phys. Rev. Lett. **84**, 1381 (2000).
[83] S. Shiomi, Phys. Rev. D **78**, 042001 (2008).
[84] For reviews of space-based tests of relativity see, for example C. Lämmerzahl, C.W.F. Everitt, and F.W. Hehl, eds., *Gyros, Clocks, Interferometers ...: Testing Relativistic Gravity in Space*, Springer, Berlin, 2001.
[85] R. Bluhm *et al.*, Phys. Rev. Lett. **88**, 090801 (2002); Phys. Rev. D **68**, 125008 (2003).
[86] P. Touboul, M. Rodrigues, G. Métris, and B. Tatry, Comptes Rendus de l'Académie des Sciences, Series IV, **2**, 1271 (2001).
[87] T.J. Sumner *et al.*, Adv. Space Res. **39**, 254 (2007).
[88] A.M. Nobili *et al.*, Exp. Astron. **23**, 689 (2009).
[89] G. Amelino-Camelia *et al.*, Exp. Astron. **23**, 549 (2009).
[90] B. Altschul, Phys. Rev. D **75**, 023001 (2007).
[91] G.L. Comandi *et al.*, Rev. Sci. Instrum. **77**, 034501 (2006).
[92] F. Hasselbach and M. Nicklaus, Phys. Rev. A **48**, 143 (1993); R. Neutze and F. Hasselbach, Phys. Rev. A **58**, 557 (1998).
[93] B. Neyenhuis, D. Christensen, and D.S. Durfee, Phys. Rev. Lett. **99**, 200401 (2007).
[94] L.I. Schiff and M.V. Barnhill, Phys. Rev. **151**, 1067 (1967); A.J. Dessler, F.C. Michel, H.E. Rorschach, and G.T. Trammell, Phys. Rev. **168**, 737 (1968); C. Herring, Phys. Rev. **171**, 1361 (1968); L.I. Schiff, Phys. Rev. B **1**, 4649 (1970).
[95] F.S. Witteborn and W.M. Fairbank, Phys. Rev. Lett. **19**, 1049 (1967); Rev. Sci. Instrum. **48**, 1 (1977).
[96] T. Goldman and M.M. Nieto, Phys. Lett. B **112**, 437 (1982).
[97] M.H. Holzscheiter *et al.*, Nucl. Phys. A **558**, 709c (1993).
[98] T.W. Darling, F. Rossi, G.I. Opat, and G.F. Moorhead, Rev. Mod. Phys. **64**, 237 (1992).
[99] L.S. Brown and G. Gabrielse, Rev. Mod. Phys. **58**, 233 (1986).
[100] V. Lagomarsino, V. Lia, G. Manuzio, and G. Testera, Phys. Rev. A **50**, 977 (1994); V. Lagomarsino, G. Manuzio, G. Testera, and M.H. Holszscheiter, Hyperfine Int. **100**, 153 (1996).
[101] PS210 Collaboration, G. Baur *et al.*, Phys. Lett. B **368**, 251 (1996).
[102] E862 Collaboration, G. Blanford, D.C. Christian, K. Gollwitzer, M. Mandelkern, C.T. Munger, J. Schultz, and G. Zioulas, Phys. Rev. Lett. **80**, 3037 (1998).
[103] ATHENA Collaboration, M. Amoretti *et al.*, Nature **419**, 456 (2002); ATRAP Collaboration, G. Gabrielse *et al.*, Phys. Rev. Lett. **89**, 213401 (2002).
[104] ALPHA Collaboration, G. Andresen *et al.*, Phys. Rev. Lett. **98**, 023402 (2007); ATRAP Collaboration, G. Gabrielse *et al.*, Phys. Rev. Lett. **98**, 113002 (2007).
[105] R. Bluhm *et al.*, Phys. Rev. Lett. **82**, 2254 (1999).
[106] B. Juhász and E. Widmann, Hyperfine Int. **193**, 305 (2009).
[107] G. Gabrielse, Hyperfine Int. **44**, 349 (1988); N. Beverini, V. Lagomarsino, G. Manuzio, F. Scuri, and G. Torelli, Hyperfine Int. **44**, 357 (1988); R. Poggiani, Hyperfine Int. **76**, 371 (1993).
[108] T.J. Phillips, Hyperfine Int. **109**, 357 (1997); AGE Collaboration, A.D. Cronin *et al.*, *Letter of Intent: Antimatter Gravity Experiment (AGE) at Fermilab,* February 2009.
[109] J. Walz and T.W. Hänsch, Gen. Rel. Grav. **36**, 561 (2004); P. Pérez, L. Liszkay, B. Mansoulié, J.M. Rey, A. Mohri, Y. Yamazaki, N. Kuroda, and H.A. Torii, *Letter of Intent to the CERN-SPSC,* November 2007.
[110] F.M. Huber, E.W. Messerschmid, and G.A. Smith, Class. Quantum Grav. **18**, 2457 (2001).
[111] AEGIS Collaboration, A. Kellerbauer *et al.*, Nucl. Instr. Meth. B **266**, 351 (2008).
[112] M.M. Nieto and T. Goldman, Phys. Rep. **205**, 221 (1991).
[113] V.A. Kostelecký, unpublished (2003).
[114] P. Morrison, Am. J. Phys. **26**, 358 (1958).
[115] L.I. Schiff, Phys. Rev. Lett. **1**, 254 (1958), Proc. Natl. Acad. Sci. **45**, 69 (1959).
[116] R. Jackiw and V.A. Kostelecký, Phys. Rev. Lett. **82**, 3572 (1999); M. Pérez-Victoria, JHEP **0104**, 032 (2001); V.A. Kostelecký, C.D. Lane, and A.G.M. Pickering, Phys. Rev. D **65**, 056006 (2002); V.A. Kostelecký and A.G.M. Pickering, Phys. Rev. Lett. **91**, 031801 (2003); B. Altschul, Phys. Rev. D **69**, 125009 (2004); Phys. Rev. D **70**, 101701 (2004); B. Altschul and V.A. Kostelecký, Phys. Lett. B **628**, 106 (2005); H. Belich, T. Costa-Soares, M.M. Ferreira, and J.A. Helayel-Neto, Eur. Phys. J. C **42**, 127 (2005); T. Mariz, J.R. Nascimento, E. Passos, R.F. Ribeiro, and F.A. Brito, JHEP **0510**, 019 (2005); G. de Berredo-Peixoto and I.L. Shapiro, Phys. Lett. B **642**, 153 (2006); P. Arias, H. Falomir, J. Gamboa, F. Méndez, and F.A. Schaposnik, Phys. Rev. D **76**, 025019 (2007); D. Colladay and P. McDonald, Phys. Rev. D **75**, 105002 (2007); Phys. Rev. D **77**, 085006 (2008); Phys. Rev. D **79**, 125019 (2009); M. Gomes, T. Mariz, J.R. Nascimento, E. Passos, A.Yu. Petrov, and A.J. da Silva, Phys. Rev. D **78**, 025029 (2009); D. Anselmi, Ann. Phys. **324**, 874 (2009); Ann. Phys. **324**, 1058 (2009).
[117] M.L. Good, Phys. Rev. **121**, 311 (1961).
[118] P. Wolf, F. Chapelet, S. Bize, and A. Clairon, Phys. Rev. Lett. **96**, 060801 (2006).

[119] B. Altschul, Phys. Rev. D **82**, 016002 (2010).
[120] V.A. Kostelecký and M. Mewes, in preparation.
[121] G.W. Bennett *et al.*, Muon g–2 Collaboration, Phys. Rev. Lett. **100**, 091602 (2008); B. Altschul, Astropart. Phys. **28**, 380 (2007); V.W. Hughes *et al.*, Phys. Rev. Lett. **87**, 111804 (2001); R. Bluhm *et al.*, Phys. Rev. Lett. **84**, 1098 (2000).
[122] K. Kirch, arXiv:physics/0702143.
[123] B. Lesche, Gen. Rel. Grav. **21**, 623 (1989).
[124] D0 Collaboration, V.M. Abazov *et al.*, Phys. Rev. D **82**, 032001 (2010).
[125] O. Bertolami *et al.*, Phys. Lett. B **395**, 178 (1997); G. Lambiase, Phys. Rev. D **72**, 087702 (2005); J.M. Carmona, J.L. Cortés, A. Das, J. Gamboa, and F. Méndez, Mod. Phys. Lett. **A21**, 883 (2006); S.M. Carroll and J. Shu, Phys. Rev. D **73**, 103515 (2006).
[126] J.G. Williams, S.G. Turyshev, and H.D. Boggs, Phys. Rev. Lett. **93**, 261101 (2004).
[127] T.W. Murphy *et al.*, Pub. Astron. Soc. Pac. **120**, 20 (2008).
[128] J. Müller and L. Biskupek, Class. Quantum Grav. **24**, 4533 (2007); R.W. Hellings *et al.*, Phys. Rev. Lett. **51**, 1609 (1983).
[129] V.A. Kostelecký, R. Lehnert, and M.J. Perry, Phys. Rev. D **68**, 123511 (2003).
[130] V.A. Brumberg, *Essential Relativistic Celestial Mechanics,* Adam Hilger, Bristol, 1991.
[131] P. Moore, *The Data Book of Astronomy*, Institute of Physics Publishing, Bristol, 2000.
[132] C.M. Will, Living Rev. Relativity **4**, 4 (2001) [gr-qc/0510072].
[133] Q.G. Bailey, Phys. Rev. D **80**, 044004 (2009).
[134] C.M. Will, Phys. Rev. D **10**, 2330 (1974); J.P. Turneaure, C.M. Will, B.F. Farrell, E.M. Mattison, and R.F.C. Vessot, Phys. Rev. D **27** , 1705 (1983).
[135] B. Bertotti, L. Iess, and P. Tortora, Nature **425**, 374 (2003).
[136] T. Appourchaux *et al.*, Exp. Astron. **23**, 491 (2009).
[137] L. Iess and S. Asmar, Int. J. Mod. Phys. D **16**, 2117 (2007).
[138] P. Wolf *et al.*, Exp. Astron. **23**, 651 (2009); S. Reynaud, C. Salomon, and P. Wolf, Space Sci. Rev. **148**, 233 (2009).
[139] B. Christophe *et al.*, Exper. Astron. **23**, 529 (2009).
[140] S.G. Turyshev, B. Lane, M. Shao, and A. Girerd, Int. J. Mod. Phys. D **18**, 1025 (2009).
[141] S.B. Lambert and C. Le Poncin-Lafitte, Astron. Astrophys. **499**, 331 (2009).
[142] N. Ashby, T.P. Heavner, S.R. Jefferts, T.E. Parker, A.G. Radnaev, and Y.O. Dudin, Phys. Rev. Lett. **98**, 070802 (2007).
[143] M.E. Tobar, P. Wolf, S. Bize, G. Santarelli, and V. Flambaum, Phys. Rev. D **81**, 022003 (2010).
[144] R. Byer, *Space-Time Asymmetry Research*, Stanford University proposal, January 2008.
[145] R.F.C. Vessot *et al.*, Phys. Rev. Lett. **45**, 2081 (1980).
[146] L. Cacciapuoti and C. Salomon, Eur. Phys. J. Spec. Top. **172**, 57 (2009).
[147] T.P. Krisher, D.D. Morabito, and J.D. Anderson, Phys. Rev. Lett. **70**, 2213 (1993).
[148] T.P. Krisher, J.D. Anderson, and J.K. Campbell, Phys. Rev. Lett. **64**, 1322 (1990).
[149] S.G. Turyshev and M. Shao, Int. J. Mod. Phys. D **16**, 2191 (2007).
[150] S.C. Unwin *et al.*, Pub. Astron. Soc. Pacific **120**, 38 (2008).