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Phys. Rev. D **83**, 014020 — Published 26 January 2011

DOI: [10.1103/PhysRevD.83.014020](https://doi.org/10.1103/PhysRevD.83.014020)

# Uncovering the triple Pomeron vertex from Wilson line formalism

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We compute the triple Pomeron vertex from the Wilson line formalism, including both planar and non-planar contributions, and get perfect agreement with the result obtained in the Extended Generalized Logarithmic Approximation based on reggeon calculus.

PACS numbers:

## I. INTRODUCTION

It is well known that the Froissart bound [1] for total cross-section,  $\sigma_{tot} \leq \pi/m_\pi^2 \ln^2 s$  [2], is violated in perturbation theory within the Leading Logarithmic Approximation (LLA). The hard Pomeron exchange obtained in the LLA is described by the BFKL equation [3–6]. This violation persists also at the Next to Leading Logarithmic Approximation (NLLA). Although strictly speaking valid only for hadronic observables, and not for external virtual states such as  $\gamma^*$ , there is a common belief that this bound should be satisfied within any reasonable perturbatively resummed scheme. This was the starting point of various lines of research, which led to various unitarization and saturation models in which the triple Pomeron vertex is a key building block.

The *Generalized Leading Log Approximation*, which takes into account any *fixed* number  $n$  of  $t$ -channel exchanged reggeons, leads to the Bartels, Jaroszewicz, Kwiecinski, Praszalowicz (BJKP) equation [7–10]. The BJKP equation can be reformulated as a 2-dimensional quantum mechanical problem with  $n$  sites, each one corresponding to one of the (gluonic)  $t$ -channel reggeons (in the coordinate space), with time  $\sim \ln s$ . The underlying hamiltonian is holomorphically separable and invariant under global conformal transformations [11, 12]. In the large  $N_c$  limit, this 2-dimensional quantum mechanical model greatly simplifies, and turns out to be integrable [12–16]. The singlet color bound states of an even number  $n$  of  $t$ -channel reggeons have the quantum numbers of the Pomeron (with  $P = C = +1$ ), while for  $n$  odd, such bound states contribute both to Pomeron and Odderon (with  $P = C = -1$ ) exchange. For Odderon, the obtained trajectory belong to two types of solutions [17–22]. The first one satisfies  $\alpha_0 < 1$ , and when summing with respect to  $n$ , it is expected that the whole series, although divergent, could have a critical behavior with an Odderon intercept  $\alpha_0 = 1$ . However, these bound states decouple from Born impact factors. They couple to photon impact factor only through non trivial color states, at least of quadrupole type, which are therefore suppressed by  $1/N_c^2$  powers. In contrast, it is possible to exhibit a second *critical* solution,  $\alpha_0 = 1$  [20], which couples to Born impact factors. These peculiar solutions can also be obtained either from the perturbative Regge approach [23] or from the dipole model, see Ref. [24].

In comparison to the previous approach, the *Extended Generalized Leading Log Approximation* (EGLLA) [25–29], in which the number of reggeon in  $t$ -channel is not conserved, satisfies full unitarity (in all sub-channel) and leads to an *effective 2-d field theory* realizing the Gribov idea of Reggeon field theory [30] in QCD (for a pedagogical review on this approach see Ref. [29]). In the framework of EGLLA, the simplest new building block (with singlet sub-channels) is the triple Pomeron vertex [26–28, 31]. The conformal properties of this vertex allow one to relate it to the conformal blocks of an underlying (still unknown) conformal field theory, and using bootstrap properties, it was possible to evaluate this vertex [32]. The Pomeron vertex contains two contributions: a planar one, and a non-planar one, which is suppressed by a factor of  $1/N_c^2$  with respect to the planar one:

$$V^{1\mathbb{P} \rightarrow 2\mathbb{P}} = V_{\text{planar}}^{1\mathbb{P} \rightarrow 2\mathbb{P}} + V_{\text{non-planar}}^{1\mathbb{P} \rightarrow 2\mathbb{P}}. \quad (1)$$

The dipole model [33, 34], [35–38], equivalent to the BFKL equation at LLA [38, 39], is based on the description of the wave function of an onium state in terms of color dipole degrees of freedom in the 't Hooft limit and at large

s. This wave function satisfies a non linear evolution equation. In this approach the vertex of  $1 \rightarrow 2$  dipoles [40–42] is equivalent to the planar part of the triple Pomeron vertex derived through the ELLA approach. The non-planar part of the triple Pomeron vertex cannot be derived from the dipole model since this approach relies on the large  $N_c$  limit which suppresses all non-planar contributions.

Since it became clear that at high-energy (Regge limit) non-linear effects dominates the dynamics of the scattering processes, non-linear evolution equations were derived. One of these equations is the Balitsky-Kovchegov (BK) equation, derived first by Balitsky [43–46] in the Wilson line formalism, and then by Kovchegov [47, 48] in the dipole model.

The Wilson line formalism is an operator language. It is based on the concept of factorization of the scattering amplitude in rapidity space and on the extension to high-energy (Regge limit) of the Operator Product Expansion (OPE) technique, which was known before only at moderate energy (Bjorken limit) as an expansion in terms of local operators or in terms of light-ray operators. In Deep Inelastic Scattering (DIS) off a hadron at high-energy, the matrix elements made of Wilson line operators appearing in the OPE, describe the non perturbative part of the process, and their evolution in rapidity is related to the evolution of the structure function of the target. In order to find the evolution equation one may use the background field technique. The Wilson-line operators evolve with respect to rapidity according to the Balitsky equation, which reduces to the BK equation in the large  $N_c$  limit. The BK equation describes the so called fan diagrams neglecting all non-planar contributions, while the Balitsky equation gives one the possibility to describe the fan diagrams including also the non-planar contributions.

A similar method to the background field technique is given in Ref. [49], where the authors studied the propagation of a fast moving particle, and showed that resumming the emission of soft gluons from this source and neglecting its recoil (eikonal approximation), one may obtain the BFKL equation.

The Color Glass Condensate (CGC) [50–58] (for a review, see Ref. [59]) is another available way to describe high-energy scattering processes. This approach is similar to the Wilson line formalism described above, and the corresponding evolution equation is the Jalilian-Marian, Iancu, MacLerran, Weigert, Leonidov, and Kovner (JIMWLK) equation. Indeed, the Balitsky equation, as already mentioned above, is not a closed evolution equation, but it contains an infinite set of evolution equations which goes by the name of Balitsky-hierarchy. The JIMWLK equation is equivalent to this hierarchy of evolution equations thus leading to the acronym B-JIMWLK equation.

In all of these approaches, the description of the scattering of two probes involves the computation of the interaction of one Wilson loop describing one probe with the field of the other. Note that within the dipole model, a similar approach, based on the computation of the scattering phase of a dipole in the field emitted by a fast moving object (involving color structures as well as multicolor states), was suggested in Ref. [60].

A further approach to the description of high-energy processes is based on an effective field theory. This formalism [61–64] provides the building blocks (reggeon-reggeon-gluon) necessary to the explicit computation of any type of diagram at this regime. However, the precise relationship between this effective theory and the ELLA approach has not been clarified yet, and explicit applications of this effective field theory is highly desirable [65–67].

The triple Pomeron vertex is the first non-trivial building block common to all the above approaches. It turns out that up to now, its exact expression, including planar and non-planar contribution, was derived only in the ELLA approach [28]. Using its conformal invariance [31], both the planar and the non-planar contributions of the vertex were computed in the coordinate space [68] (see Ref. [32] for explicit expressions in the  $SU(N_c)$  case).

So far, only the planar contribution was derived with formalisms different from the ELLA one. These are the dipole model [40–42] and the Wilson line formalism applied to diffractive processes [69].

The purpose of the present paper is to show that the exact expression of the triple Pomeron vertex (planar and non-planar contributions) can be easily derived through the Wilson line formalism not only for diffractive case but also for fan diagrams.

Indeed, this formalism was already used to derive several new and desirable results, and to confirm others which have been derived after many years of calculations. This include for example the NLO BK kernel in QCD [70] and in  $\mathcal{N}=4$  SYM [71] whose linearized version confirmed the NLO BFKL kernel in QCD [72–75] and in  $\mathcal{N}=4$  SYM [76]. Moreover, through Wilson line formalism it was possible to derive for the first time the conformal expression for the NLO BFKL kernel [77], the full amplitude at NLO in  $\mathcal{N}=4$  SYM for four scalar currents made of chiral-primary operator [78], and the analytic result for the NLO photon impact factor [79], relevant for phenomenology at high-energy, was recently derived.

It is clear that the Wilson line approach to the study of high-energy scattering processes opens the way to attack more difficult problems which have not been solved before despite the many efforts that have been devoted to them using different techniques. This includes for example multiple Pomeron or Odderon vertices, as well as sub-leading contributions.

## II. INTRODUCTION TO THE WILSON LINE FORMALISM

In this section we will give a brief introduction to the Wilson line formalism. This will set the notations which will be used for the derivation of the triple Pomeron vertex first in the diffraction case and then for fan diagrams which are important for unitarization.

The main tool which we will use is the OPE for high energies [43] of the  $T$ -product of two electromagnetic currents in terms of Wilson lines:

$$T j_\mu(x) j_\nu(y) = \int d^2 z_1 d^2 z_2 I_{\mu\nu}^{LO}(x, y; z_{1\perp}, z_{2\perp}) \text{Tr}\{\hat{U}(z_{1\perp}) \hat{U}^\dagger(z_{2\perp})\} + \dots \quad (2)$$

with the operator  $j_\mu(x) = \bar{\Psi}(x) \gamma_\mu \Psi(x)$ . This expansion is in terms of a coefficient function to be identified with the photon impact factor and a matrix element of two Wilson line operators, with

$$\hat{\mathcal{U}}(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}, \quad (3)$$

where the Wilson line is defined as usual by the operator

$$\hat{U}(x_\perp) = \text{P exp}\left\{ig \int_{-\infty}^{\infty} du p_1^\mu A_\mu(p_1 u + x_\perp)\right\}. \quad (4)$$

We use here the standard Sudakov decomposition  $k = \alpha p_1 + \beta p_2 + k_\perp$ , where  $p_1$  and  $p_2$  are light-like vectors defined in such a way that in a typical high-energy scattering process, the first projectile (referred as the “above” one) flies almost along  $p_1$  while the second one (resp. “below”) flies almost along  $p_2$ .

The operator  $\hat{\mathcal{U}}$  evolves according to the Balitsky equation

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x_\perp, y_\perp) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \frac{(x-y)_\perp^2}{(x-z)_\perp^2 (z-y)_\perp^2} [\hat{\mathcal{U}}(x_\perp, z_\perp) + \hat{\mathcal{U}}(y_\perp, z_\perp) - \hat{\mathcal{U}}(x_\perp, y_\perp) - \hat{\mathcal{U}}(x_\perp, z_\perp) \hat{\mathcal{U}}(z_\perp, y_\perp)], \quad (5)$$

where  $x_\perp, y_\perp, z_\perp$  are two-dimensional vector with Euclidean metric<sup>1</sup>. The  $\eta$  dependence of the operator  $\mathcal{U}$  enters as a regulator of the divergence by changing the slope of the Wilson line according to  $p_1 \rightarrow p_1 + e^{-2\eta} p_2$  in Eq. (4).

The BK equation [43, 47] is obtained from Eq. (5) at large  $N_c$  when the correlation function of the non-linear term  $\hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y)$  decouple in a product of two correlation functions  $\langle \hat{\mathcal{U}}(x, z) \rangle \langle \hat{\mathcal{U}}(z, y) \rangle$ . From now on we will use the short-hand notation  $\mathcal{U}(x, y) \equiv \mathcal{U}_{xy}$ .

## III. TRIPLE POMERON VERTEX FROM DIFFRACTION

### A. Diffraction within Keldysh formalism

We will derive in this section the triple Pomeron vertex for diffractive processes, using Keldysh formalism adapted to describe diffractive processes through functional integration [80]. The idea is to use the OPE for diffractive high-energy processes [69] in order to reproduce automatically the Cutkosky rules for the calculation of total cross-sections. One introduces two different fields, each of them living on one side of the cut, which results in three different propagators:  $\langle A^+ A^+ \rangle$ ,  $\langle A^+ A^- \rangle$ ,  $\langle A^- A^- \rangle$ .

Following [69, 81], we write the diffractive amplitude of the  $\gamma^* p \rightarrow p' + X$  process as

$$W^{\text{diff}} = \sum_{\text{flavors}} e_i^2 \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp, 0) \langle N | \text{Tr}\{\hat{W}^{\zeta=m^2/s}(k_\perp) \hat{W}^{\dagger, \zeta=m^2/s}(-k_\perp)\} | N \rangle, \quad (6)$$

where  $\hat{W}(k_\perp)$  is the Fourier transform of

$$\hat{W}(x_\perp) = \hat{V}^\dagger(x_\perp) \hat{U}(x_\perp), \quad \hat{W}^\dagger(x_\perp) = \hat{U}^\dagger(x_\perp) \hat{V}(x_\perp), \quad (7)$$

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<sup>1</sup> Hereafter, these variables will be denoted simply like  $x, y, z$ .

and  $\hat{U}(x_\perp)$  now denotes the Wilson-line operator constructed from  $A^+$  fields while  $\hat{V}(x_\perp)$  denotes the same operator constructed from  $A^-$  fields:

$$\begin{aligned}\hat{U}(x_\perp) &= \text{P exp} \left\{ ig \int_{-\infty}^{\infty} du p_1^\mu A_\mu^+(p_1 u + x_\perp) \right\} \\ \hat{V}(x_\perp) &= \text{P exp} \left\{ ig \int_{-\infty}^{\infty} du p_1^\mu A_\mu^-(p_1 u + x_\perp) \right\}.\end{aligned}\quad (8)$$

In Eq. (6)  $m$  is a cut-off for collinear singularities (for example a non-zero quark mass), and the notation  $\zeta = e^{-2\eta}$  is used. In the case of diffractive processes, the operator

$$\hat{\mathcal{W}}(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{Tr} \{ \hat{W}(x_\perp) \hat{W}^\dagger(y_\perp) \} \quad (9)$$

evolves according to the same Balitsky equation (5), as [69]

$$\frac{d}{d\eta} \hat{\mathcal{W}}(x_\perp, y_\perp) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \frac{(x-y)_\perp^2}{(x-z)_\perp^2 (z-y)_\perp^2} \left\{ \hat{\mathcal{W}}(x, z) + \hat{\mathcal{W}}(z, y) - \hat{\mathcal{W}}(x, y) - \hat{\mathcal{W}}(x, z) \hat{\mathcal{W}}(z, y) \right\}. \quad (10)$$

In the RHS of this equation, the non-linear term should be interpreted as the splitting of a diffractive  $\mathbb{P}$ omeron defined by  $\langle \hat{\mathcal{W}}(x_\perp, y_\perp) \rangle$ . We now define  $\hat{\mathcal{V}}$  in a similar way as  $\hat{\mathcal{U}}$  in Eq.(3)

$$\hat{\mathcal{V}}(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{Tr} \{ \hat{V}(x_\perp) \hat{V}^\dagger(y_\perp) \}. \quad (11)$$

Our goal is now to extract from the non-linear part in Eq. (10) terms of the type  $\langle \hat{\mathcal{U}} \rangle \langle \hat{\mathcal{V}} \rangle$  which will be interpreted respectively as the  $\mathbb{P}$ omeron on the left (resp. right) of the cut. In order to do this, we should linearize this non-linear term up to two gluons accuracy<sup>2</sup>.

### B. Linearization of non-linear term

The idea of linearization consists in expanding the non linear term of Eq. (10) up to  $g^4$  and rewriting this result in terms of products of the type  $\mathcal{U} \mathcal{V}$ . The two gluon approximation means that each  $\mathcal{U}$  and  $\mathcal{V}$  should be approximated up to  $g^2$ , since each  $\mathbb{P}$ omeron is a singlet color object. The obtained result, whose detailed derivation is given in the Appendix, is:

$$\begin{aligned}& \text{Tr} \{ W_x W_z^\dagger - 1 \} \text{Tr} \{ W_z W_y^\dagger - 1 \} \stackrel{2g}{=} \mathcal{U}_{xz} \mathcal{V}_{zy} + \mathcal{U}_{yz} \mathcal{V}_{zx} + \frac{1}{N_c^2 - 1} [\mathcal{U}_{xy} - \mathcal{U}_{xz} - \mathcal{U}_{yz}] [\mathcal{V}_{xy} - \mathcal{V}_{zx} - \mathcal{V}_{zy}] \\ &= \frac{N_c^2}{N_c^2 - 1} \left[ \mathcal{U}_{xz} \mathcal{V}_{zy} + \mathcal{U}_{yz} \mathcal{V}_{zx} + \frac{1}{N_c^2} \left[ -\mathcal{U}_{xy} \mathcal{V}_{xy} + \mathcal{U}_{xz} \mathcal{V}_{xz} + \mathcal{U}_{zy} \mathcal{V}_{zy} + \mathcal{U}_{xy} (\mathcal{V}_{xy} - \mathcal{V}_{xz} - \mathcal{V}_{zy}) \right. \right. \\ & \left. \left. + \mathcal{V}_{xy} (\mathcal{U}_{xy} - \mathcal{U}_{xz} - \mathcal{U}_{zy}) \right] \right].\end{aligned}\quad (12)$$

### C. Projection on BFKL Green functions

In order to extract the triple  $\mathbb{P}$ omeron vertex, one has to define precisely how to factorize out each of the three  $\mathbb{P}$ omeron Green function from the three- $\mathbb{P}$ omeron correlator. This is achieved according to Fig. 1. We denote the above (below)  $\mathbb{P}$ omeron Green functions by  $\tilde{\Psi}'$  (resp.  $\tilde{\Psi}$ ). The vertex  $V^{1\mathbb{P} \rightarrow 2\mathbb{P}}$  is defined symbolically in the following way:

$$\langle \mathbb{P} \mathbb{P} \mathbb{P} \rangle = (\Delta \Delta \tilde{\Psi}') V^{1\mathbb{P} \rightarrow 2\mathbb{P}} \tilde{\Psi} \tilde{\Psi} \quad (13)$$

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<sup>2</sup> In the following, the notation  $\hat{\cdot}$  on operators will be removed for simplification.

where  $\Delta\tilde{\Delta}\tilde{\Psi}'$  is the amputated Pomeron Green function which is denoted as  $\tilde{\Psi}'_{\text{amp.}}$  in Fig. 1. This prescription is in accordance with standard definitions, as exhibited for example in Ref. [82], from which notations of Fig. 1 are inspired. We will now translate this definition of the triple Pomeron vertex in the shock-wave approach.

In the present treatment, we deal with colorless probes. These probes are dipoles, which respect the global conformal invariance of the BFKL equation. The dipole-dipole scattering, in the BFKL approximation, can then be presented as an elementary function of a conformal anharmonic ratio. This is the basis of the so-called Moebius representation of BFKL.

In the shock-wave analysis,  $\langle\mathcal{U}_{xy}\rangle$  describes the two-gluon non-amputated amplitude in the Moebius representation (the upper probe is a dipole with coordinate  $x$  and  $y$  referring respectively to the position of the quark and anti-quark pair), where the average is on the external field of a lower probe. The contact with the usual 4-gluon BFKL Green function in the Moebius representation can be made if one considers the specific case of a lower probe made of two Wilson lines in a color singlet state (a dipole), each of them having definite transverse coordinates.

The dynamics of the process is encoded in the dipole kernel (identical to the BK kernel) which acts on the coordinates  $x$  and  $y$  (the same situation appears for the evolution of multiple dipole densities in the dipole model, which is the starting point of the Kovchegov approach), while the averaging from below on a given probe does not affects this dynamics. In both the BFKL and the shock-wave pictures, this dynamics is encoded in a kernel which acts on non-amputated functions (this remains true also in the dipole picture).

Now, on one side, the BFKL Green function is non-amputated, both from below and from above. Indeed, at Born order it simply reduces to the product of two propagators. Its Moebius representation is obtained by Fourier transform and then by a “subtraction“, which is needed to enforce its vanishing for equal upper or lower coordinates [82]. This can be obtained directly when computing the elementary dipole-dipole scattering amplitude in the two gluon exchange approximation [39]). On the other side, in both the shock-wave and dipole formalisms, the operator  $\mathcal{U}$  or the dipole densities correspond to amputated quantities from the point of view of the below probe.

Indeed, to get a scattering amplitude one should convolute these amputated Green function to the below probe, thus restoring the gluonic propagators, as it is indeed done in these two formalisms. In the Wilson line formalism the fields are not contracted with the probe from below (no propagators). In the dipole model, when computing the scattering of two onia, in the frame where the below onium is almost at rest, one first evaluates the dipole content of the above probe. Then, the restoration of these gluonic propagators appears when convoluting this dipole density with the elementary dipole-dipole scattering amplitude between an internal dipole constituent of the above probe and the below dipole.

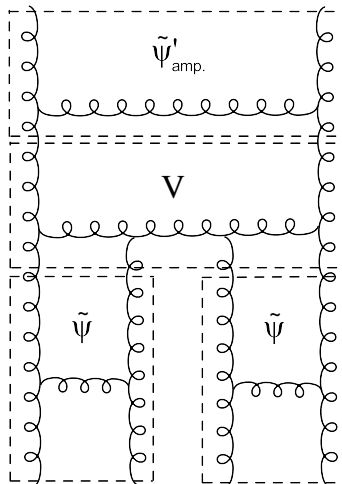


Figure 1: A typical diagram contributing to the triple Pomeron vertex  $V$ , with three 4-gluon Green functions (denoted as  $\tilde{\Psi}$  and  $\tilde{\Psi}'$  in the Moebius representation) at  $g^2$  order. This exhibits explicitly the amputation of the above Green function from below.

### D. Projection on conformal three-point functions

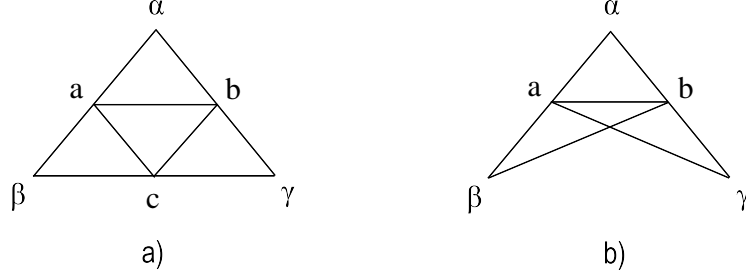


Figure 2: a) Configuration of planar diagrams. b) Configuration of non-planar diagrams.

In the Moebius representation the Hamiltonian of the BFKL equation acts as an integral operator, according to

$$\pi H^{BFKL} f(x, y) = \int d^2 z \frac{(x-y)^2}{(x-z)^2 (y-z)^2} [f(x, y) - f(x, z) - f(z, y)] . \quad (14)$$

Based on the conformal invariance of this BFKL hamiltonian, one can diagonalize it using the following set of eigenfunctions

$$E_{h\bar{h}}(\rho_{10}, \rho_{20}) \equiv \left( \frac{\rho_{12}}{\rho_{10} \rho_{20}} \right)^h \left( \frac{\bar{\rho}_{12}}{\bar{\rho}_{10} \bar{\rho}_{20}} \right)^{\bar{h}} , \quad (15)$$

where the conformal weights  $h$  and  $\bar{h}$  are given by

$$h = \frac{1+n}{2} + i\nu, \quad \bar{h} = 1 - h^* = \frac{1-n}{2} + i\nu . \quad (16)$$

and where the notation  $\rho_{ij} = \rho_i - \rho_j$  is used. The eigenvalue equation then reads

$$-\frac{\bar{\alpha}_s}{2} H_{xy}^{BFKL} E_{h\bar{h}} = \omega(h) E_{h\bar{h}} , \quad (17)$$

with  $\bar{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}$ , and where  $\omega(h)$  is given by

$$\omega(h) = 2 \bar{\alpha}_s \text{Re}[\psi(1) - \psi(h)] . \quad (18)$$

The corresponding BFKL Green function then reads, in the Mellin space  $\omega$  conjugated to  $s$ ,

$$G_\omega^{\text{BFKL}}(x, y, x', y') = \sum_h \left| \frac{h - \frac{1}{2}}{h(h-1)} \right|^2 \frac{1}{\omega - \omega(h)} \int d^2 \rho_0 E_{h\bar{h}}(x - \rho_0, y - \rho_0) E_{h\bar{h}}^*(x' - \rho_0, y' - \rho_0) . \quad (19)$$

Let us now turn to the shock-wave formalism. In the BFKL approximation, the average of the  $\mathcal{U}$  operator on a given probe (denoted as B) can be written as

$$\langle B | \mathcal{U}_{xy}^{(\omega)} | B \rangle = \sum_\alpha \langle x, y | \alpha \rangle \langle \alpha | B \rangle \frac{1}{\omega - \omega(h_\alpha)} \left| \frac{h_\alpha - \frac{1}{2}}{h_\alpha(h_\alpha - 1)} \right|^2 , \quad (20)$$

where the notation  $|\alpha\rangle$  is used in order to label both the center of mass  $\rho_\alpha$  and the conformal weights  $(h_\alpha, \bar{h}_\alpha)$ . Note that in Eq. (20) we make an identification of the evaluation of the operator  $\mathcal{U}$  on the state B through functional integration on the LHS, with quantum mechanical notations appearing on the RHS. Here  $|x, y\rangle$  denotes an upper dipole,  $x$  and  $y$  being the coordinates of the two corresponding Wilson line. The relationship with the 4-reggeon BFKL Green function  $G^{\text{BFKL}}(x, y, x', y')$  in the Moebius representation is obtained when choosing the lower state B to be a dipole of coordinates  $(x', y')$ , for which (19) reads, in the above notations,

$$\langle \text{dipole}(x', y') | \mathcal{U}_{xy}^{(\omega)} | \text{dipole}(x', y') \rangle = \sum_\alpha \langle x, y | \alpha \rangle \langle \alpha | x', y' \rangle \frac{1}{\omega - \omega(h_\alpha)} \left| \frac{h_\alpha - \frac{1}{2}}{h_\alpha(h_\alpha - 1)} \right|^2 , \quad (21)$$

leading to the following natural identification:

$$E_{h_\alpha, \bar{h}_\alpha}(x - \rho_\alpha, y - \rho_\alpha) = \langle x, y | \alpha \rangle, \quad E_{h_\alpha, \bar{h}_\alpha}^*(x' - \rho_\alpha, y' - \rho_\alpha) = \langle \alpha | x', y' \rangle. \quad (22)$$

We denote the quantum numbers of the upper Pomeron by  $\alpha$ , while the lower left Pomeron (resp. lower right) is labelled by  $\beta$  (resp.  $\gamma$ ) (see Fig. 2). We are interested here in the extraction of the triple Pomeron vertex. For that purpose, based on the factorised form (21) of the BFKL Green function as a series of products of conformal blocks, the two lower Pomerons are described through  $\langle x, z | \beta \rangle$  and  $\langle z, y | \gamma \rangle$  while the above one is described through  $\langle \alpha | x, y \rangle$ . Now, using the property that

$$(E_{h, \bar{h}}(x - \rho, y - \rho))^* = E_{1-h, 1-\bar{h}}(x - \rho, y - \rho), \quad (23)$$

we will not make any distinction between the above and the two below Pomerons, which will all be described by a set of  $E_{h, \bar{h}}$  functions, as it is done for example in Ref.[32] to which we would like to compare our final results.

We now fix our notation for the internal degrees of freedom of the vertex, based on the non-linear term of Eq. (10). Let us equate the coordinates  $x, y$  and  $z$  with  $\rho_a, \rho_b$  and  $\rho_c$  respectively. It is convenient to identify the component of the dipole with definite conformal weight with  $\mathcal{U}$ , according to

$$\begin{aligned} \mathcal{U}_{xy} &\rightarrow E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{b\beta}), & \mathcal{U}_{xz} &\rightarrow E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{c\beta}) \\ \mathcal{V}_{xy} &\rightarrow E_{h_\gamma \bar{h}_\gamma}(\rho_{a\gamma}, \rho_{b\gamma}), & \mathcal{V}_{yz} &\rightarrow E_{h_\gamma \bar{h}_\gamma}(\rho_{b\gamma}, \rho_{c\gamma}). \end{aligned} \quad (24)$$

Now, since we are in the Moebius representation, for each of the three Pomeron which are here involved, the Green function is the sum of products of conformal block with even conformal spin  $n$ . This is due to the fact that

$$G_{\text{Moebius}}^{\text{BFKL}}(x, y, x', y') \propto G^{\text{BFKL}}(x, y, x', y') + G^{\text{BFKL}}(y, x, x', y'), \quad (25)$$

as discussed in Ref. [39] (see Eqs. (2.50) and (2.51), as well as Eq. (B.12) in the Born approximation). Therefore, since  $n = h + \bar{h}$  is even, and using the property

$$E_{h, \bar{h}}(x - \rho, y - \rho) = (-1)^{h+\bar{h}} E_{h, \bar{h}}(y - \rho, x - \rho), \quad (26)$$

one can make the following identification

$$\mathcal{U}_{ab} = \mathcal{U}_{ba} = E_{h_\alpha \bar{h}_\alpha}(\rho_{a\alpha}, \rho_{b\alpha}) = E_{h_\alpha \bar{h}_\alpha}(\rho_{b\alpha}, \rho_{a\alpha}) \quad (27)$$

where  $\mathcal{U}_{ab} = \mathcal{U}_{ba}$  in the two-gluon approximation. Identifications similar to Eq. (27) are done also for  $\mathcal{U}_{bc}$ ,  $\mathcal{V}_{ab}$  etc...

Since the upper Pomeron is amputated, one needs to evaluate the effect of the amputation on a conformal block. This is obtained using the following identity:

$$\Delta_a \Delta_b E_{h_\alpha \bar{h}_\alpha}(\rho_{a\alpha}, \rho_{b\alpha}) = 16 h_\alpha (h_\alpha - 1) \bar{h}_\alpha (\bar{h}_\alpha - 1) E_{h_\alpha \bar{h}_\alpha}(\rho_{a\alpha}, \rho_{b\alpha}) \frac{1}{\rho_{ab}^2 \bar{\rho}_{ab}^2}. \quad (28)$$

The three-pomeron vertex is now defined, according to Eq. (12), as

$$\begin{aligned} V^{1\mathbb{P} \rightarrow 2\mathbb{P}} &= \frac{1}{2} \int d^2 \rho_a d^2 \rho_b d^2 \rho_c \, 16 h_\alpha (h_\alpha - 1) \bar{h}_\alpha (\bar{h}_\alpha - 1) E_{h_\alpha \bar{h}_\alpha}(\rho_{a\alpha}, \rho_{b\alpha}) \frac{1}{\rho_{ab}^2 \bar{\rho}_{ab}^2} \frac{|\rho_{ab}|^2}{|\rho_{ac}|^2 |\rho_{bc}|^2} \left[ \mathcal{U}_{ac} \mathcal{V}_{cb} + \mathcal{U}_{bc} \mathcal{V}_{ca} \right. \\ &\quad \left. + \frac{1}{N_c^2} \left[ -\mathcal{U}_{ab} \mathcal{V}_{ab} + \mathcal{U}_{ac} \mathcal{V}_{ac} + \mathcal{U}_{cb} \mathcal{V}_{cb} + \mathcal{U}_{ab} (\mathcal{V}_{ab} - \mathcal{V}_{ac} - \mathcal{V}_{cb}) + \mathcal{V}_{ab} (\mathcal{U}_{ab} - \mathcal{U}_{ac} - \mathcal{U}_{cb}) \right] \right]. \end{aligned} \quad (29)$$

Using property (27), the planar part in Eq.(29) can be rewritten as

$$\begin{aligned} V_{\text{planar}}^{1\mathbb{P} \rightarrow 2\mathbb{P}} &= 16 h_\alpha (h_\alpha - 1) \bar{h}_\alpha (\bar{h}_\alpha - 1) \int d^2 \rho_a d^2 \rho_b d^2 \rho_c \, E_{h_\alpha \bar{h}_\alpha}(\rho_{b\alpha}, \rho_{a\alpha}) \frac{1}{|\rho_{ab}|^2 |\rho_{ac}|^2 |\rho_{bc}|^2} E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{c\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{c\gamma}, \rho_{b\gamma}). \end{aligned} \quad (30)$$

Let us now consider the non-planar part of the vertex. Using the hermiticity property of  $H^{\text{BFKL}}$  (14) when acting on the product of the two below Green functions  $\mathcal{U}_{xy} \mathcal{V}_{xy}$  and acting on the above amputated Green function we have,

$$\begin{aligned} &\pi \int d^2 \rho_a d^2 \rho_b d^2 \rho_c \, E_{h_\alpha \bar{h}_\alpha}(\rho_{a\alpha}, \rho_{b\alpha}) [H_{ab}^{\text{BFKL}} \mathcal{U}_{ab} \mathcal{V}_{ab}] \\ &= \pi \int d^2 \rho_a d^2 \rho_b d^2 \rho_c \, [H_{ab}^{\text{BFKL}} E_{h_\alpha \bar{h}_\alpha}(\rho_{a\alpha}, \rho_{b\alpha})] E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{b\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{a\gamma}, \rho_{b\gamma}) \\ &= \pi \int d^2 \rho_a d^2 \rho_b \, E_{h_\alpha \bar{h}_\alpha}(\rho_{a\alpha}, \rho_{b\alpha}) E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{b\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{a\gamma}, \rho_{b\gamma}) \left[ -\frac{2}{\bar{\alpha}_s} \omega(h_\alpha) \right] \end{aligned} \quad (31)$$



Similarly, for the other two terms of the non-planar part we have

$$\begin{aligned}
& \int d^2 \rho_c \frac{|\rho_{ab}|^2}{|\rho_{ac}|^2 |\rho_{bc}|^2} \left[ \mathcal{U}_{ab} (\mathcal{V}_{ab} - \mathcal{V}_{ac} - \mathcal{V}_{cb}) + \mathcal{V}_{ab} (\mathcal{U}_{ab} - \mathcal{U}_{ac} - \mathcal{U}_{cb}) \right] = \pi \mathcal{U}_{ab} [H_{ab}^{BFKL} \mathcal{V}_{ab}] + \pi \mathcal{V}_{ab} [H_{ab}^{BFKL} \mathcal{U}_{ab}] \\
& = \pi \left[ E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{b\beta}) H_{ab}^{BFKL} E_{h_\gamma \bar{h}_\gamma}(\rho_{a\gamma}, \rho_{b\gamma}) + E_{h_\gamma \bar{h}_\gamma}(\rho_{a\gamma}, \rho_{b\gamma}) H_{ab}^{BFKL} E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{b\beta}) \right] \\
& = -\pi \frac{2}{\bar{\alpha}_s} [\omega(h_\beta) + \omega(h_\gamma)] E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{b\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{a\gamma}, \rho_{b\gamma})
\end{aligned} \tag{32}$$

Thus, using once more property (27), the non-planar part reads

$$\begin{aligned}
V_{\text{non-planar}}^{1\mathbb{P} \rightarrow 2\mathbb{P}} & = -\frac{2\pi}{N_c^2} \int d^2 \rho_a d^2 \rho_b 16 h_\alpha (h_\alpha - 1) \bar{h}_\alpha (\bar{h}_\alpha - 1) E_{h_\alpha \bar{h}_\alpha}(\rho_{b\alpha}, \rho_{a\alpha}) \\
& \times \frac{1}{|\rho_{ab}|^4} \text{Re}\{\psi(1) + \psi(h_\alpha) - \psi(h_\beta) - \psi(h_\gamma)\} E_{h_\beta \bar{h}_\beta}(\rho_{b\beta}, \rho_{a\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{b\gamma}, \rho_{a\gamma}).
\end{aligned} \tag{33}$$

Finally, putting together the non-planar part Eq.(33) and the planar one Eq.(30), the triple Pomeron vertex is

$$\begin{aligned}
V^{1\mathbb{P} \rightarrow 2\mathbb{P}} & = \\
& \int d^2 \rho_a d^2 \rho_b 16 h_\alpha (h_\alpha - 1) \bar{h}_\alpha (\bar{h}_\alpha - 1) E_{h_\alpha \bar{h}_\alpha}(\rho_{b\alpha}, \rho_{a\alpha}) \left[ \int d^2 \rho_c \frac{1}{|\rho_{ab}|^2 |\rho_{ac}|^2 |\rho_{bc}|^2} E_{h_\beta \bar{h}_\beta}(\rho_{a\beta}, \rho_{c\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{c\gamma}, \rho_{b\gamma}) \right. \\
& \left. - \frac{2\pi}{N_c^2} \frac{1}{|\rho_{ab}|^4} \text{Re}\{\psi(1) + \psi(h_\alpha) - \psi(h_\beta) - \psi(h_\gamma)\} E_{h_\beta \bar{h}_\beta}(\rho_{b\beta}, \rho_{a\beta}) E_{h_\gamma \bar{h}_\gamma}(\rho_{b\gamma}, \rho_{a\gamma}) \right].
\end{aligned} \tag{34}$$

We now compare our result (34) with the one obtained through the reggeon approach of Ref. [28], and which is written explicitly for  $SU(N_c)$  and for arbitrary conformal weights in Ref. [32]. Up to a global normalization factor, related to the convention used to define the triple Pomeron vertex, Eq.(34) agrees with Eq. (2.1) of Ref. [32].

#### IV. FAN DIAGRAM APPROACH

The diffractive case, discussed in the previous section, implied that the upper Pomeron was at  $t = 0$ . In this section instead, we will show how to obtain the triple pomeron vertex without using the Keldysh formalism. This will allow us to obtain the triple Pomeron vertex for fan diagrams which means that every Pomeron is now at arbitrary  $t$ . Therefore, we need to work with the Balitsky equation (5) which we rewrite here for convenience

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \frac{(x - y)_\perp^2}{(x - z)_\perp^2 (z - y)_\perp^2} [\hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(y, z) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y)].$$

It is easy to see that at large  $N_c$  limit the correlation function  $\langle \mathcal{U}(x_\perp, z_\perp) \mathcal{U}(z_\perp, y_\perp) \rangle$  decouples to the product of two correlation functions  $\langle \mathcal{U}(x_\perp, z_\perp) \rangle \langle \mathcal{U}(z_\perp, y_\perp) \rangle$  and this non-linear term is interpreted as the splitting of one Pomeron into two Pomerons. In this way the Balitsky equation with the truncation of the hierarchy reduces to the BK equation; its non linear term coincides exactly with the planar part of the triple Pomeron vertex [82]. The triple Pomeron vertex takes the following form

$$V_{\text{BK}}^{1\mathbb{P} \rightarrow 2\mathbb{P}} \propto \int d^2 z \frac{(x - y)_\perp^2}{(x - z)_\perp^2 (z - y)_\perp^2} \langle \mathcal{U}_{xz} \rangle \langle \mathcal{U}_{zy} \rangle. \tag{35}$$

Our aim is now to extract from the non-linear term  $\mathcal{U}_{xz} \mathcal{U}_{zy}$ , not only the planar contribution to the triple Pomeron vertex but also the non-planar one. We then show that the result so obtained coincides with the one we obtained for diffractive processes. We will adopt the following procedure. First, we consider the correlation function of four Wilson lines i.e.  $\langle \mathcal{U}_{xz} \mathcal{U}_{zy} \rangle$ , we then apply the two gluon approximation to them and finally we rewrite the contributions thus obtained in terms of decoupled correlation function of the type  $\langle \mathcal{U}_{xz} \rangle \langle \mathcal{U}_{zy} \rangle$ . This method is technically very similar to the one used in Sec. III, the details of which are given in the Appendix. We first expand each Wilson line operator. In what follows, we use the shorthand notation  $U_x \approx U_x^{(0)} + U_x^{(1)} + U_x^{(2)} + \dots$ , with  $U^{(0)}$  being the zeroth order term of the expansion,  $U_x^{(1)}$  the first order term, and so on. Thus, we have

$$\begin{aligned}
& N_c^2 \langle \mathcal{U}_{xz} \mathcal{U}_{zy} \rangle \\
& \approx \langle \text{Tr}\{1 - (U_x^{(0)} + U_x^{(1)} + U_x^{(2)})(U_z^{(0)\dagger} + U_z^{(1)\dagger} + U_z^{(2)\dagger})\} \text{Tr}\{1 - (U_z^{(0)} + U_z^{(1)} + U_z^{(2)})(U_y^{(0)\dagger} + U_y^{(1)\dagger} + U_y^{(2)\dagger})\} \rangle.
\end{aligned} \tag{36}$$

At this point we want to rewrite eq. (36) as a sum of decoupled correlation functions like  $\langle \mathcal{U}_{xz} \rangle \langle \mathcal{U}_{zy} \rangle$  such that when we apply the 2-gluon approximation to them we get back the Eq. (36). Since  $U^{(0)} = 1$ , the only possible contraction in order to produce terms of the type  $\langle \mathcal{U}_{xz} \rangle \langle \mathcal{U}_{zy} \rangle$  are between  $U^{(1)}$  terms or between  $U^{(2)}$  terms. Contraction between terms of order higher than 2 would result in remaining multiplicative terms which are not color singlet (these terms are of the type  $\text{Tr } U^{(1)}$  which vanish). Other terms involving contraction of gluon fields at the same coordinate will clearly not contribute: terms like  $\mathcal{U}_{zz}$  vanish. Finally, the expansion of each Wilson line is needed only up to second order. So, it is then easy to see that

$$\langle \mathcal{U}_{xz} \mathcal{U}_{zy} \rangle \approx \frac{2g}{2(N_c^2 - 1)} \left\{ 2\langle \mathcal{U}_{xz} \rangle \langle \mathcal{U}_{zy} \rangle + \frac{1}{N_c^2} \left[ 2\langle \mathcal{U}_{xy} \rangle (\langle \mathcal{U}_{xy} \rangle - \langle \mathcal{U}_{xz} \rangle - \langle \mathcal{U}_{yz} \rangle) + \langle \mathcal{U}_{zy} \rangle \langle \mathcal{U}_{zy} \rangle + \langle \mathcal{U}_{xz} \rangle \langle \mathcal{U}_{xz} \rangle - \langle \mathcal{U}_{xy} \rangle \langle \mathcal{U}_{xy} \rangle \right] \right\}. \quad (37)$$

We can immediately recognize in Eq. (38) the planar contribution  $\langle \mathcal{U}_{xz} \rangle \langle \mathcal{U}_{zy} \rangle$  which coincides with the non-linear term in the BK equation and with the planar part of the diffractive triple Pomeron vertex obtained in the previous section (cf. Eq. (47)). The terms proportional to  $N_c^{-2}$  are instead the non-planar contributions which are suppressed in the limit of  $N_c \rightarrow \infty$  and which are, therefore, obtained only from Balitsky equation and not from BK equation.

We now want to compare the two above approaches: the one based on diffractive processes, Eq. (29), and the other one based on the fan diagram approach, Eq. (38). The first obtained result is a particular case of the second one, since it was derived for the splitting Pomeron at  $t = 0$ . Indeed, the second one can be obtained when identifying  $\mathcal{U}$  with  $\mathcal{V}$  since in the fan diagram case one cannot distinguish between the two produced Pomerons. Let us first consider the planar contribution. In the diffractive approach, the obtained structure (see Eq. (12))

$$\frac{N_c^2}{N_c^2 - 1} [\mathcal{U}_{ac} \mathcal{V}_{cb} + \mathcal{V}_{ac} \mathcal{U}_{cb}] \quad (38)$$

is an operator which should be contracted with an external set of Pomeron states of quantum numbers denoted by  $|\beta\rangle$  and  $|\gamma\rangle$  (which as already stated above describes both the conformal weight and the center of mass coordinate of the Pomeron state). At this stage, in the case of the diffractive amplitude, these two Pomeron states are distinguishable (one is at the left of the cut while the other one is at the right), and after using the symmetry of the integrand under the replacement  $a \leftrightarrow b$ , the net result reads symbolically

$$2 \frac{N_c^2}{N_c^2 - 1} \mathcal{U}_{ac} |\beta\rangle \mathcal{V}_{cb} |\gamma\rangle. \quad (39)$$

Now, one can make the identification of the  $\mathcal{U}_{ij} |\beta\rangle$  and  $\mathcal{V}_{ij} |\beta\rangle$  states, leading to the final result

$$2 \frac{N_c^2}{N_c^2 - 1} \mathcal{U}_{ac} |\beta\rangle \mathcal{U}_{cb} |\gamma\rangle. \quad (40)$$

On the other hand, from the fan diagram approach, one has (see first term of Eq. (38)), using the same overall normalization as in Eq. (38),

$$\frac{N_c^2}{2(N_c^2 - 1)} 2 \mathcal{U}_{ac} \mathcal{U}_{cb} |\beta\rangle |\gamma\rangle = \frac{N_c^2}{N_c^2 - 1} 2 (\mathcal{U}_{ac} |\beta\rangle) (\mathcal{U}_{cb} |\gamma\rangle) \quad (41)$$

where the factor of 2 on the RHS of Eq. (41) is due to the two possible contractions. This shows explicitly that both, the planar result obtained from the general fan diagram case and the one obtained from a continuation of the  $t = 0$  diffractive case in Keldysh formalism, are in agreement. The proof for the non-planar case follows the same line of thinking.

## V. CONCLUSION

In this paper we have shown that the triple Pomeron vertex, including the planar and the non-planar contribution, can be obtained very easily within Wilson line formalism. In Ref. [69] this was already done for the case of diffractive processes using Keldysh formalism, but the result obtained there was only for the planar part of the vertex. In section III, we have shown how to compute also the non-planar contribution of the diffractive triple Pomeron vertex from Wilson line formalism. To this end we considered the generalization of the Balitsky equation for diffractive process,

and using the linearization procedure and the 2-gluon approximation, we have extracted the desired subleading term in  $N_c$  of the vertex.

In section IV we have extended the result of section III to the more generic case of fan diagrams, where the Pomeron, which split to two other ones, does not need to be at  $t = 0$ . We then showed that the triple pomeron vertex for fan diagrams is the same as the one obtained in the diffractive case.

Since, as we have shown in the present paper, the Wilson line formalism allows one to re-derive very easily results that have been obtained after non trivial and lengthy calculation, we plan to use it to study other, still unknown and highly desirable results. For example, it will be interesting to compute the vertex for  $\mathbb{P} \rightarrow 3\mathbb{P}$  [83],  $\mathbb{P} \rightarrow \mathbb{O}\mathbb{O}$  [84],  $\mathbb{O} \rightarrow \mathbb{P}\mathbb{O}$  (so far unknown), or more generally  $n\mathbb{P} \rightarrow m\mathbb{P}$  (inaccessible at the moment through reggeon calculus techniques). These non trivial building blocks will be relevant in order to identify the unknown underlying effective theory for high-energy scattering processes. This study is in progress.

### Acknowledgments

We warmly thank I. Balitsky for many inspiring discussions and comments. We thank G. P. Korchemsky for claryfying us the derivation of formula (2.1) in Ref. [32]. We also thank J. Bartels, L. N. Lipatov, S. Munier, B. Pire and G. P. Vacca for discussions. This work is partly supported by the ANR-06-JCJC-0084 and by the Polish Grant N202 249235. G. A. C. and L. S. thank the Institute for Nuclear Theory at the University of Washington for its hospitality and the Department of Energy for partial support during the completion of this work. G. A. C. thanks Lawrence Berkeley National Laboratory for support at the last stage of this work.

### Appendix

Below we present some details of the linearization procedure which we used to extract the planar and non-planar part of the triple Pomeron vertex.

Let us consider the non-linear term in Eq. (10)

$$N_c^2 \text{Tr}\{\mathcal{W}(x, z)\mathcal{W}(z, y)\} = \text{Tr}\{W_x W_z^\dagger - 1\} \text{Tr}\{W_z W_y^\dagger - 1\} = \text{Tr}\{V_z V_x^\dagger U_x U_z^\dagger - 1\} \text{Tr}\{V_y V_z^\dagger U_z U_y^\dagger - 1\}. \quad (42)$$

Our aim is to extract from this expression the contribution of two non-interacting Pomerons: one Pomeron built by  $A^+$ -fields and the other one built by  $A^-$ -fields. We now approximate Wilson line operators  $U$  and  $V$  up to linear terms in  $A^+$  and  $A^-$  fields (here we use the short-hand notation  $iA_x^+ \equiv ig \int_0^1 du p_1^\mu A_\mu^+(p_1 u + x_\perp)$  and similarly for  $iA_x^-$ )

$$\begin{aligned} N_c^2 \text{Tr}\{\mathcal{W}(x, z)\mathcal{W}(z, y)\} \\ \approx \text{Tr}\{(1 + iA_z^-)(1 - iA_x^-)(1 + iA_x^+)(1 - iA_z^+) - 1\} \text{Tr}\{(1 + iA_y^-)(1 - iA_z^-)(1 + iA_z^+)(1 - iA_y^+) - 1\}. \end{aligned} \quad (43)$$

From each trace of Eq. (43) we keep only terms to second order since we are working in the 2-gluon approximation. Thus, we have

$$\begin{aligned} N_c^2 \text{Tr}\{\mathcal{W}(x, z)\mathcal{W}(z, y)\} \approx & \text{Tr}\{A_z^- A_x^- + A_x^+ A_z^+ - A_z^- A_x^+ + A_z^+ A_x^- + A_x^- A_z^+ - A_x^+ A_z^-\} \\ & \times \text{Tr}\{A_y^- A_z^- - A_y^- A_z^+ + A_y^+ A_z^+ + A_z^- A_z^+ - A_z^- A_y^+ - A_z^+ A_y^+\}. \end{aligned} \quad (44)$$

The next step of the linearisation procedure up to 2 gluon accuracy consists in keeping in the product of the two traces in (44) only terms which involve two  $A^+$  fields with different coordinates and two  $A^-$  fields with different coordinates. Thus, we obtain that

$$\begin{aligned} N_c^2 \text{Tr}\{\mathcal{W}(x, z)\mathcal{W}(z, y)\} \stackrel{2g}{\approx} & \text{Tr}\{A_x^- A_z^-\} \text{Tr}\{A_z^+ A_y^+\} + \text{Tr}\{A_x^+ A_z^+\} \text{Tr}\{A_y^- A_z^-\} + \text{Tr}\{A_z^- A_x^-\} \text{Tr}\{A_y^- A_z^+\} \\ & - \text{Tr}\{A_z^- A_x^+\} \text{Tr}\{A_y^- A_z^+\} + \text{Tr}\{A_z^+ A_x^+\} \text{Tr}\{A_y^- A_z^-\} - \text{Tr}\{A_x^- A_z^+\} \text{Tr}\{A_y^- A_z^+\} + \text{Tr}\{A_x^+ A_z^+\} \text{Tr}\{A_y^- A_z^-\} \\ & + \text{Tr}\{A_x^- A_z^+\} \text{Tr}\{A_z^- A_y^+\} - \text{Tr}\{A_x^- A_z^+\} \text{Tr}\{A_z^- A_y^+\} - \text{Tr}\{A_x^- A_z^+\} \text{Tr}\{A_z^+ A_y^+\} + \text{Tr}\{A_x^- A_z^+\} \text{Tr}\{A_z^+ A_y^+\}. \end{aligned} \quad (45)$$

We now have to rewrite Eq. (45) in terms of product of traces involving only  $A^+$  and  $A^-$  fields, so we obtain

$$\begin{aligned} N_c^2 \text{Tr}\{\mathcal{W}(x, z)\mathcal{W}(z, y)\} \stackrel{2g}{\approx} & \text{Tr}\{A_x^- A_z^-\} \text{Tr}\{A_z^+ A_y^+\} + \text{Tr}\{A_x^+ A_z^+\} \text{Tr}\{A_y^- A_z^-\} \\ & + \frac{1}{N_c^2 - 1} [\text{Tr}\{A_x^+ A_y^+\} - \text{Tr}\{A_x^+ A_z^+\} - \text{Tr}\{A_y^+ A_z^+\}] [\text{Tr}\{A_x^- A_y^-\} - \text{Tr}\{A_x^- A_z^-\} - \text{Tr}\{A_y^- A_z^-\}]. \end{aligned} \quad (46)$$

The final step is to write Eq. (46) in terms of the original operators  $\mathcal{U}$  and  $\mathcal{V}$  such that when we apply to them the 2-gluon approximation we get back Eq. (46). So, we have

$$N_c^2 \text{Tr}\{\mathcal{W}(x, z)\mathcal{W}(z, y)\} \approx N_c^2 \left( \mathcal{U}_{xz}\mathcal{V}_{zy} + \mathcal{U}_{yz}\mathcal{V}_{zx} + \frac{1}{N_c^2 - 1} [\mathcal{U}_{xy} - \mathcal{U}_{xz} - \mathcal{U}_{yz}] [\mathcal{V}_{xy} - \mathcal{V}_{zx} - \mathcal{V}_{zy}] \right) \quad (47)$$

which is the linearization we used in Eq. (12).

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