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Phys. Rev. D **111**, L081503 — Published 17 April 2025

DOI: [10.1103/PhysRevD.111.L081503](https://doi.org/10.1103/PhysRevD.111.L081503)

Universal phenomenological relations between spherical harmonic modes in non-precessing eccentric binary black hole merger waveforms

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(Dated: February 4, 2025)

Using publicly available numerical relativity (NR) simulations for non-spinning eccentric binary black hole (BBH) mergers, Ref [1] demonstrated that the eccentricity-induced modulations in the amplitudes and frequencies of different spherical harmonic modes are mutually consistent and can be modeled using a single time series modulation. We extend the validity of the results to all non-precessing binaries by using 83 NR simulations from the SXS, RIT, and MAYA catalogs for aligned-spin eccentric BBH mergers with mass ratios ranging from 1 : 1 to 1 : 4. Based on these phenomenological relations, we provide a framework named `gwNRXHME` to compute multi-modal eccentric non-precessing waveforms using two inputs: quadrupolar eccentric waveforms, and the corresponding multi-modal quasi-circular non-precessing waveforms. Furthermore, we compute an overall degree of departure in SXS, RIT, and MAYA NR data from these relations and find that SXS NR simulations generally adhere to these relations more strictly than RIT and MAYA data. We also show that these relations can offer a cost-effective way to filter out noisy higher-order spherical harmonic modes extracted from NR data. Our framework is publicly available through the `gwModels` package.

I. INTRODUCTION

The detection and subsequent characterization of gravitational wave (GW) signals (radiation) from merging compact binaries in the LIGO-Virgo-KAGRA (LVK) data [2–5] depend on the availability of accurate numerical relativity (NR) waveforms and computationally efficient waveform models that are tuned to (or built from) NR data. Different waveform models adopt various simplifying assumptions about the underlying binary system in order to reduce the problem’s dimensionality and facilitate the modeling process.

The gravitational waveform from a binary black hole (BBH) merger is expressed as a superposition of -2 spin-weighted complex-valued spherical harmonic modes with indices (ℓ, m) [6, 7]:

$$h(t, \theta, \phi; \lambda) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m}(t; \lambda) {}_{-2}Y_{\ell m}(\theta, \phi), \quad (1)$$

where h is the complexified waveform constructed from the individual polarizations, t represents time, θ and ϕ are angles on the merger’s sky, and λ is the set of intrinsic parameters that describe the binary. If we use geometric units in which time is measured in units of the total mass of the binary, for non-precessing binaries, we have $\lambda := \{q, \chi_1, \chi_2, e_{\text{ref}}, l_{\text{ref}}\}$, where $q := m_1/m_2$ is the mass ratio, with m_1 and m_2 being the masses of the larger and smaller black holes, respectively. The parameter χ_1 (χ_2) denotes the dimensionless spin magnitudes of the larger (smaller) black hole. Eccentricity is characterized by two parameters: e_{ref} , the eccentricity, and l_{ref} , the mean anomaly, estimated at a chosen reference time or frequency.

Multiple definitions of eccentricity and mean anomaly exist [8–12]; however, any consistent choice applied to all binaries will effectively characterize the eccentricity. We decompose each spherical harmonic mode $h_{\ell m}(t; \lambda)$ into a real-valued amplitude $A_{\ell m}(t)$ and phase $\phi_{\ell m}(t)$ such that:

$$h_{\ell m}(t; \lambda) = A_{\ell m}(t) e^{i\phi_{\ell m}(t)}. \quad (2)$$

We obtain the instantaneous frequency of each spherical harmonic mode as:

$$\omega_{\ell m}(t; \lambda) = \frac{d\phi_{\ell m}(t)}{dt}. \quad (3)$$

The orbital angular frequency of the binary is $\omega_{\text{orb}} = 0.5 \times \omega_{22}$, and the dimensionless frequency is given by $x = (M\omega_{\text{orb}})^{2/3}$. In this paper, we only focus on the positive m modes as the negative m modes are obtained by the symmetry relation: $h_{\ell m} = (-1)^{\ell} h_{\ell -m}^*$ where $*$ indicates complex conjugate (this breaks down in precessing binaries [13]).

Years of effort have resulted in accurate waveform models for quasi-circular binaries [14–48], but modeling radiation from eccentric binaries is still in its nascent stage [38, 48–72]. Often, simple phenomenological relations observed in gravitational waveforms (either in the amplitudes or phases of different modes [16], or between precessing and non-precessing waveforms [19, 25]) greatly aid the development of fast yet accurate waveform models. Recently, using publicly available NR simulations obtained from the SXS and RIT catalog, Ref. [1] provided a simple phenomenological relation between the amplitudes and instantaneous frequencies of different spherical harmonic modes in eccentric non-spinning binaries (i.e. with $\chi_1 = \chi_2 = 0$), thereby effectively simplifying the process of waveform modeling for these binaries. Ref. [1] first quantifies the effect of eccentricity on different spherical harmonic modes in the eccentric waveform $h_{\ell m}(t; \lambda)$ by computing modulations with respect to the corresponding non-eccentric

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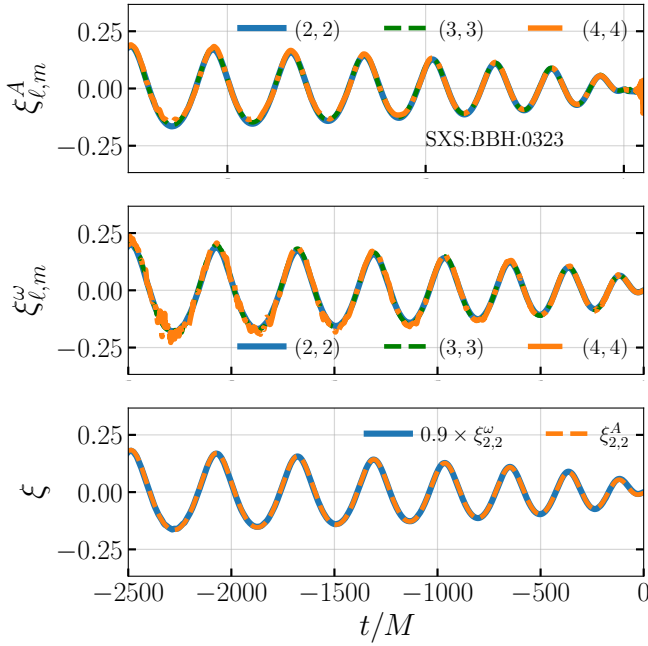


Figure 1. We show the eccentric modulations in amplitudes $\xi_{\ell,m}^A$ (upper panel) and in frequencies $\xi_{\ell,m}^\omega$ (middle panel) for three representative modes: (2, 2) (blue) (3, 3) (green) and (4, 4) (orange) for a binary with mass ratio $q = 1.22$, spin on the primary $\chi_1 = 0.33$, spin on the secondary $\chi_2 = -0.44$ and eccentricity $e_{\text{ref}} = 0.193$. We extract these modulations from the eccentric NR simulation SXS:BBH:0323 and the corresponding circular simulation SXS:BBH:0318. In the lower panel, we demonstrate that these two modulations are related by a factor of $K = 0.9$ (obtained through a phenomenological fit provided in Ref. [1]).

waveform $h_{\ell m}(t; \lambda^0)$. Here, $\lambda^0 := \{q, \chi_1, \chi_2, e_{\text{ref}} = 0, l_{\text{ref}} = 0\}$. Ref. [1] defines the eccentric frequency modulation as:

$$\xi_{\ell m}^\omega(t; \lambda) = b_{\ell m}^\omega \frac{\omega_{\ell m}(t; \lambda) - \omega_{\ell m}(t; \lambda^0)}{\omega_{\ell m}(t; \lambda^0)}, \quad (4)$$

and the eccentric amplitude modulation as:

$$\xi_{\ell m}^A(t; \lambda) = b_{\ell m}^A \frac{2}{\ell} \frac{A_{\ell m}(t; \lambda) - A_{\ell m}(t; \lambda^0)}{A_{\ell m}(t; \lambda^0)}, \quad (5)$$

where $b_{\ell m}^\omega = b_{\ell m}^A = 1$. Ref. [1] then empirically noted that (i) the amplitude modulations $\xi_{\ell m}^A(t; \lambda)$ are consistent for all the modes used; (ii) the frequency modulations $\xi_{\ell m}^\omega(t; \lambda)$ obtained from different modes are all consistent; and (iii) the amplitude modulations and frequency modulations are related by:

$$\xi_{\ell m}^A(t; \lambda) = B \xi_{\ell m}^\omega(t; \lambda), \quad (6)$$

where the scaling factor $B = 0.9$. Note that while $\xi_{\ell m}^\omega(t; \lambda)$ does not have any mode dependent pre-factor, the pre-factor in $\xi_{\ell m}^A(t; \lambda)$ depends on the ℓ value of the spherical harmonic mode. In this paper, we use 83 publicly available NR simulations for non-precessing eccentric BBH mergers (see Table I, Table II and Table III) from three different NR catalogs, SXS catalog (<https://data.black-holes.org/waveforms/catalog.html>) [55, 73, 74], RIT catalog

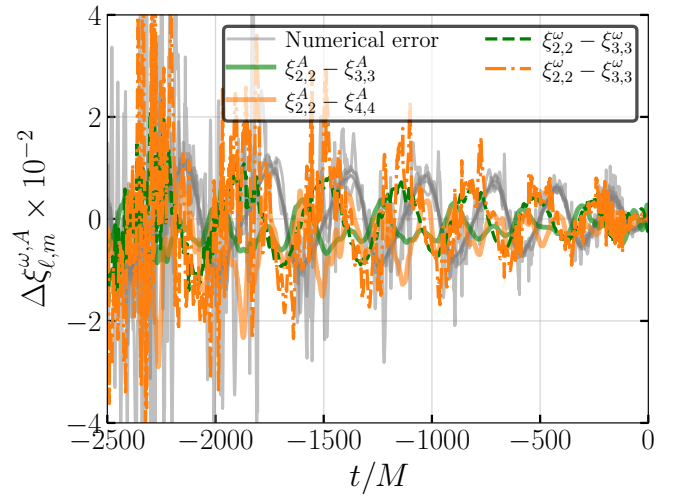


Figure 2. We show the differences between eccentric amplitude modulations $\xi_{\ell,m}^A$ (solid lines; cf. Eq.(5)) and frequency modulations $\xi_{\ell,m}^\omega$ (dashed lines; cf. Eq.(4)) obtained from different modes (and shown in Figure 1) for a binary with mass ratio $q = 1.22$, spin on the primary $\chi_1 = 0.33$, spin on the secondary $\chi_2 = -0.44$ and eccentricity $e_{\text{ref}} = 0.193$. We extract these modulations from the eccentric NR simulation SXS:BBH:0323 and the corresponding circular simulation SXS:BBH:0318. Additionally, we provide an estimate of numerical errors in both eccentric amplitude modulations and frequency modulations using the highest two resolutions of NR data (grey lines). We find that the differences between the modulations obtained from different modes are comparable to the numerical error in NR.

(<https://ccrpages.rit.edu/RITCatalog/> [75, 76], and MAYA catalog (<https://cgp.ph.utexas.edu/waveform> [77]), which employ different codes to solve the Einstein equations, to probe the validity of the aforementioned relations in aligned-spin binaries.

SXS:BBH:0108	SXS:BBH:1355	SXS:BBH:1366
SXS:BBH:0319	SXS:BBH:1356	SXS:BBH:1367
SXS:BBH:0320	SXS:BBH:1357	SXS:BBH:1368
SXS:BBH:0321	SXS:BBH:1358	SXS:BBH:1371
SXS:BBH:0322	SXS:BBH:1359	SXS:BBH:1372
SXS:BBH:0323	SXS:BBH:1364	SXS:BBH:1373
SXS:BBH:1149	SXS:BBH:1365	SXS:BBH:1169

Table I. SXS NR simulations [55] used in this work.

II. QUASI-UNIVERSAL PHENOMENOLOGICAL RELATION

We first demonstrate the relations mentioned in Section I for an aligned-spin eccentric BBH merger waveform obtained from NR simulation SXS:BBH:0323 (Figure 1). This binary is characterized by mass ratio $q = 1.22$, spin on the primary $\chi_1 = 0.33$, spin on the secondary $\chi_2 = -0.44$ and eccentricity $e_{\text{ref}} = 0.193$ (as obtained from the simulation metadata pro-

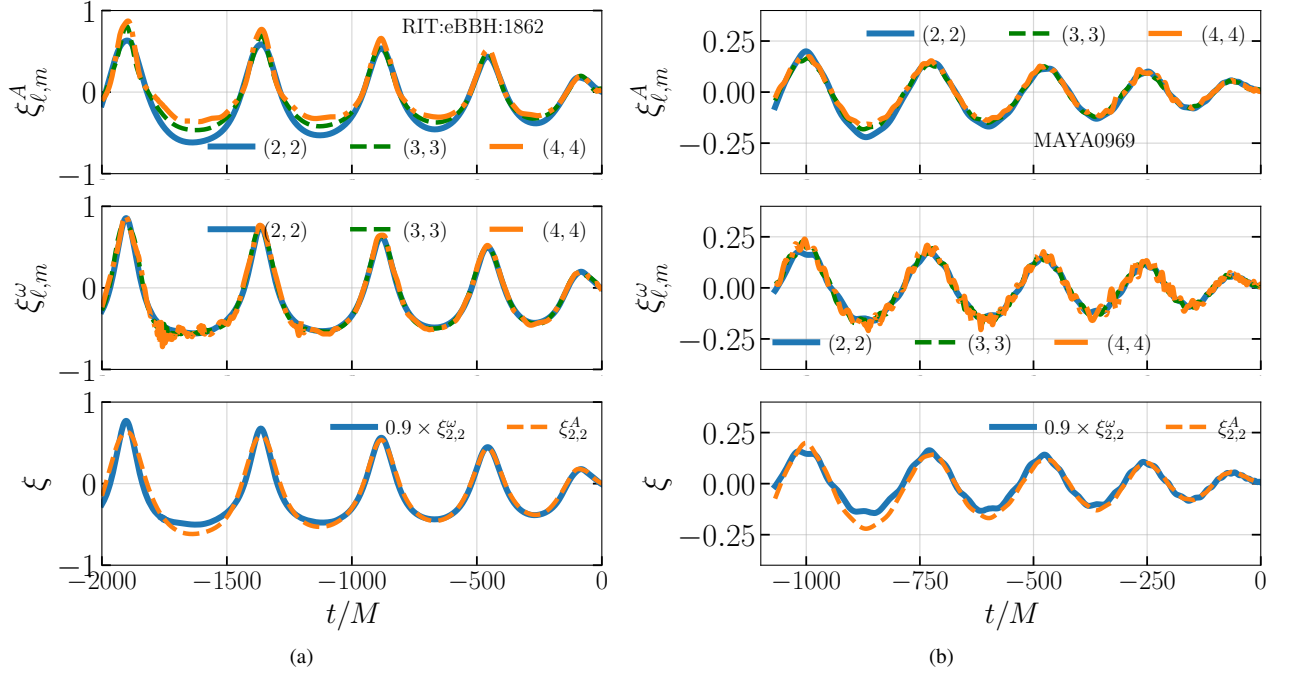


Figure 3. (a) We show the eccentric modulations in amplitudes $\xi_{\ell,m}^A$ (upper panel) and in frequencies $\xi_{\ell,m}^\omega$ (middle panel) for three representative modes: (2, 2) (blue) (3, 3) (green) and (4, 4) (orange) for a binary with mass ratio $q = 3$, spin on the primary $\chi_1 = -0.8$, spin on the secondary $\chi_2 = 0.0$ and eccentricity $e_{\text{ref}} = 0.36$ [75, 76]. We extract this modulations from the eccentric NR simulation RIT:eBBH:1862 and the corresponding circular waveform is obtained from NRHybSur3dq8 model. In the lower panel, we demonstrate that these two modulations are related by a factor of $K = 0.9$ (obtained through a phenomenological fit provided in Ref. [1]). (b) We show the same quantities for the eccentric NR simulation MAYA0969 characterized by mass ratio $q = 3$, spin on the primary $\chi_1 = 0.4$, spin on the secondary $\chi_2 = 0.4$ and eccentricity $e_{\text{ref}} = 0.112$ [77].

RIT:eBBH:1899	RIT:eBBH:1099	RIT:eBBH:1254
RIT:eBBH:1900	RIT:eBBH:1100	RIT:eBBH:1256
RIT:eBBH:1828	RIT:eBBH:1282	RIT:eBBH:1330
RIT:eBBH:1807	RIT:eBBH:1210	RIT:eBBH:1353
RIT:eBBH:1786	RIT:eBBH:1213	RIT:eBBH:1376
RIT:eBBH:1763	RIT:eBBH:1215	RIT:eBBH:1399
RIT:eBBH:1740	RIT:eBBH:1422	RIT:eBBH:1445
RIT:eBBH:1883	RIT:eBBH:1252	RIT:eBBH:1468
RIT:eBBH:1862	RIT:eBBH:1253	RIT:eBBH:1491
RIT:eBBH:1098		

Table II. RIT NR simulations [75, 76] used in this work.

MAYA0952	MAYA0916	MAYA0965
MAYA0953	MAYA0917	MAYA0966
MAYA0974	MAYA0918	MAYA0967
MAYA0975	MAYA0919	MAYA0968
MAYA0976	MAYA0941	MAYA0969
MAYA0977	MAYA0942	MAYA0985
MAYA0995	MAYA0943	MAYA0986
MAYA0996	MAYA0944	MAYA0987
MAYA0997	MAYA0945	MAYA0988
MAYA0998	MAYA0963	MAYA0989
MAYA0999	MAYA0964	MAYA0990
MAYA0991		

Table III. MAYA NR simulations [77] used in this work.

vided at <https://data.black-holes.org/waveforms/catalog.html>). The corresponding quasi-circular BBH merger simulation is SXS:BBH:0318. First, we show that the amplitude modulations $\xi_{\ell m}^A$ obtained from different spherical harmonic modes are consistent (Figure 1, upper panel). Second, the frequency modulations $\xi_{\ell m}^\omega$ in different spherical harmonic modes also match with each other (Figure 1, middle panel). Finally, amplitude and frequency modulations are related by the factor B (Figure 1, lower panel). For each spherical harmonic mode, we compute the differences between $\xi_{\ell m}^A(t; \lambda)$ and $\xi_{\ell m}^\omega(t; \lambda)$ computed from NR simulations with different numerical resolution. This

gives an estimate of the numerical errors in the simulations. For the highest resolution NR simulation, we then compute the differences in $\xi_{\ell m}^A(t; \lambda)$ obtained from the (2, 2), (3, 3) and (4, 4) modes. We do the same for $\xi_{\ell m}^\omega(t; \lambda)$. These differences quantify the degree of dispersion about the phenomenological relations. Figure 2 shows that where the differences between $\xi_{\ell m}^A(t; q, e_{\text{ref}})$ (and $\xi_{\ell m}^\omega(t; q, e_{\text{ref}})$) computed from different spherical harmonic modes closely follow the numerical errors in NR simulations.

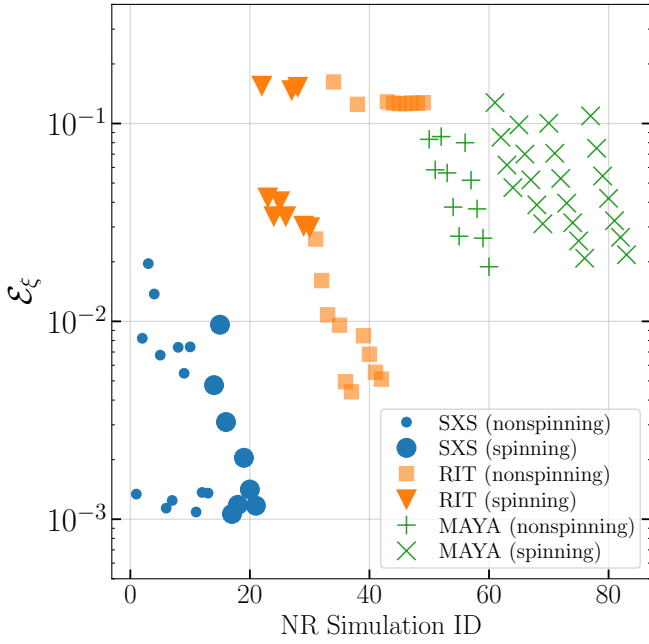


Figure 4. We show the overall degree of departures $\mathcal{E}_\xi$ (defined in Eq.(8)) associated with the quasi-universal relations (presented in Section I) observed in all non-precessing NR simulations considered in this work. Different markers are used for non-spinning and spinning waveforms from the SXS, RIT, and MAYA catalogs.

Next, we show the quasi-universal eccentric modulations obtained from one of the representative RIT NR BBH merger simulations: RIT:eBBH:1862, characterized by $q = 3$, $\chi_1 = -0.8$, $\chi_2 = 0.0$, and $e_{\text{ref}} = 0.36$ [75, 76], in Figure 3(a). The corresponding quasi-circular waveforms are obtained from NRHybSur3dq8 [46], an accurate reduced-order surrogate approximation of the aligned-spin SXS NR data. This choice was made because, for the non-precessing eccentric NR simulations in the RIT and MAYA catalogs, we could not always find a corresponding quasi-circular simulation. Figure 3(b) shows the same quantities observed in MAYA0969, a representative MAYA NR BBH merger simulations. This particular binary has the following parameters: $q = 3$, $\chi_1 = 0.4$, $\chi_2 = 0.4$, and $e_{\text{ref}} = 0.112$ [77]. It is important to stress that while it is possible to characterize these eccentric simulations using a common set of definitions in this paper, it is not necessary to understand the universal relations observed. We therefore provide the eccentricity values as noted in the respective NR catalog papers. The relations between the various $\xi_{\ell m}^A$ (and $\xi_{\ell m}^\omega$), as well as the relation between $\xi_{\ell m}^A$ and $\xi_{\ell m}^\omega$ are qualitatively apparent in the RIT and MAYA simulations as well (Figures 3(a), 3(b)), but overall the relations are not obeyed at the level that they are in the SXS simulation (Figure 1). Note that the modulations in Figures 3(a) and 3(b) show a higher level of numerical noise, which is more prominent for the (3, 3) and (4, 4) modes. The level of numerical noise in many of the other RIT and MAYA NR simulations considered in this paper is larger than what is shown in these figures.

Finally, we compute the degree of departure from the pre-

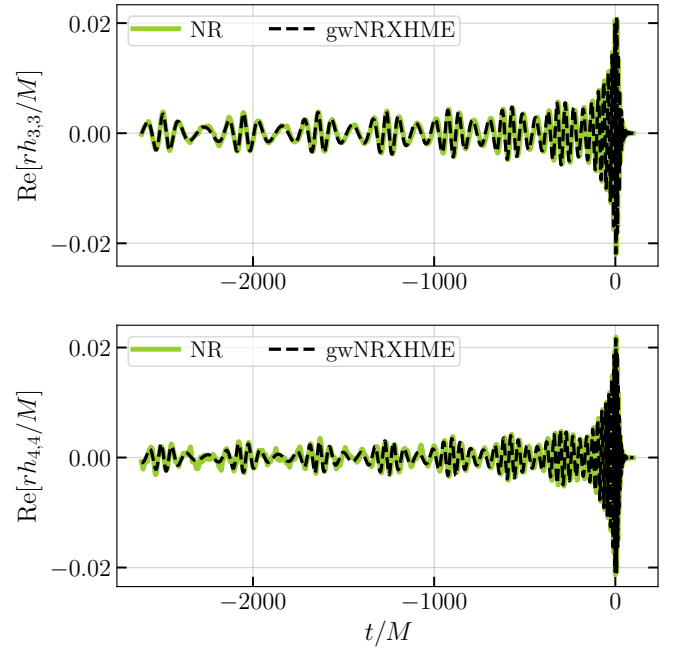


Figure 5. We show the eccentric higher-order spherical harmonic modes (black dashed lines) obtained from the gwNRXHME framework (available at <https://github.com/tousifislam/gwModels>) and corresponding NR data from SXS:BBH:0324 simulation (green solid lines). We obtain gwNRHME predictions by applying eccentric modulations (computed from the (2, 2) mode of SXS:BBH:0324 data) on quasi-circular spherical harmonic modes obtained from SXS:BBH:0318. We find that gwNRXHME predictions are visually indistinguishable from NR.

sented phenomenological relations, from Eq.(4) to Eq.(6), for all NR simulations considered in this work. We use the relative L_2 -norm to quantify the degree of departure. For two given time-series $s_1(t)$ and $s_2(t)$, the relative L_2 -norm is defined as (Eq.21 of Ref. [44]):

$$\mathcal{E}(s_1, s_2) = \frac{1}{2} \frac{\int_{t_{\min}}^{t_{\max}} |s_1(t) - s_2(t)|^2 dt}{\int_{t_{\min}}^{t_{\max}} |s_1(t)|^2 dt}, \quad (7)$$

where $t_{\min}$ and $t_{\max}$ denote the initial and final times, respectively.

To ensure that our estimates for the degree of departure account for (i) differences in $\xi_{\ell m}^A$ estimated from different modes, (ii) differences in $\xi_{\ell m}^\omega$ estimated from different modes, and (iii) differences in the proposed relation between $\xi_{\ell m}^A$ and $\xi_{\ell m}^\omega$, we define an overall degree of departure $\mathcal{E}_\xi$ associated with the quasi-universal relations, presented in Section I, as follows:

$$\mathcal{E}_1 = \mathcal{E}(\xi_{22}^A, \xi_{33}^A), \quad (8a)$$

$$\mathcal{E}_2 = \mathcal{E}(\xi_{22}^A, \xi_{44}^A) \quad (8b)$$

$$\mathcal{E}_3 = \mathcal{E}(\xi_{22}^\omega, \xi_{33}^\omega). \quad (8c)$$

$$\mathcal{E}_4 = \mathcal{E}(\xi_{22}^\omega, \xi_{44}^\omega). \quad (8d)$$

$$\mathcal{E}_5 = \mathcal{E}(\xi_{22}^A, B_{22}^\omega). \quad (8e)$$

$$\mathcal{E}_\xi = \frac{\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5}{5}. \quad (8f)$$

These five errors are not really independent of each other, so we define the net error as an average of these five errors to give a sense of the typical error due to one of those terms. For equal mass binaries, odd m modes are zero due to symmetry in the system. We therefore do not compute $\mathcal{E}_3$ and $\mathcal{E}_5$ in those cases. The overall degree of departure is also modified accordingly as: $\mathcal{E}_\xi = \frac{\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_4}{3}$.

We show the overall degree of departure $\mathcal{E}_\xi$ computed for all non-precessing NR simulations in Figure 4. Different markers are used for non-spinning and spinning waveforms from the SXS, RIT, and MAYA catalogs. The spinning RIT data exhibit larger overall degree of departures compared to the non-spinning cases. For non-spinning cases, the overall degree of departures in RIT data are somewhat comparable to those in SXS data. However, SXS NR simulations generally adhere to these relations more strictly than RIT and MAYA data. For the SXS data, the overall degree of departures associated with these universal relations mostly lies between 10^{-3} and 10^{-2} , whereas overall errors for the MAYA simulations range between 10^{-2} and 10^{-1} . For the RIT simulations, errors range from 0.004 to 0.2. Motivated by the level of numerical scatter in the empirically calculated modulations in Figures 3(a) and 3(b), we conjecture that the larger departures from the phenomenological relations in the RIT and MAYA simulations may be related to the numerical errors in the simulations, but this fact requires further investigation. Note that some of the SXS NR simulations too exhibit increased level of numerical noises - mostly in the higher order modes. These simulations corresponds to the largest overall errors. These errors provide quantitative and qualitative evidences in support of the existence of a quasi-universal relation between spherical harmonic modes in non-precessing eccentric BBH waveforms. Furthermore, it provides a new benchmark to compare eccentric NR simulations obtained from different catalog and codes.

III. IMPLICATIONS

Our results have several interesting implications. First, it extends the validity of the quasi-universal relations observed earlier between modes in non-spinning eccentric BBH merger waveforms [1] to cases involving aligned-spin and anti-aligned-spin configurations. Our results show that these relations are observed, with varying accuracy, in NR data obtained from three different catalogs. Second, as a consequence, the results provide a novel method for comparing current and future eccentric NR catalogs by assessing how well they adhere to these quasi-universal relations.

Third, as noted in Ref. [1], these relations greatly simplify the modeling challenges in eccentric binaries. They indicate that one can combine dominant quadrupolar mode of a non-precessing eccentric (NR) waveform $h_{\ell m}(t; \lambda)$ with the corresponding quasi-circular non-precessing multi-modal (NR) waveform $h_{\ell m}(t; \lambda^0)$ to obtain multi-modal non-precessing eccentric waveforms. It offers an alternate and simpler approach to filter out the otherwise noisy higher-order modes in eccentric NR simulations. For this, one has to simply exploit Eq.(4), Eq.(5) and Eq.(6). Ref. [1] has presented such a frame-

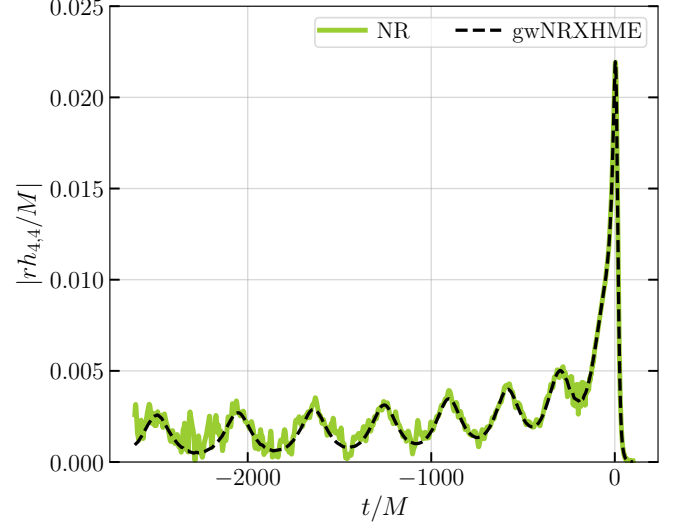


Figure 6. We demonstrate that gwNRXHME framework (black dashed line; available at <https://github.com/tousifislam/gwModels>) can be used to provide clean eccentric higher-order spherical harmonic modes. We illustrate this for SXS:BBH:0324 (green), characterized by a mass ratio $q = 1.22$, spins $\chi_1 = 0.33$ and $\chi_2 = -0.4$ and eccentricity $e_{\text{ref}} = 0.13$ [74]. While the amplitude of the $(\ell, m) = (4, 4)$ mode obtained from NR is quite noisy, the gwNRXHME prediction (obtained by combining $(2, 2)$ mode eccentric modulation from SXS:BBH:0324 simulation with quasi-circular SXS:BBH:0318 simulation) is clean and goes right through the NR data.

work named gwNRHME for the non-spinning eccentric binaries. Our result shows that the framework can be easily extended for the non-precessing binaries. We call our new framework gwNRXHME.

We first define an common modulation parameter $\xi(t) := \xi_{22}^A(t; \lambda)$. The amplitude of higher order modes is given as:

$$A_{\ell m}^{\text{gwNRXHME}}(t; \lambda) = A_{\ell m}(t; \lambda^0) \left[\frac{\ell}{2} \xi(t) + 1 \right]. \quad (9)$$

The frequency of different spherical harmonic modes is then simply:

$$\omega_{\ell m}^{\text{gwNRXHME}}(t; \lambda) = \omega_{\ell m}(t; \lambda^0) \left[\frac{\xi(t)}{B} + 1 \right]. \quad (10)$$

Integrating the frequency, we get the phase of each mode:

$$\phi_{\ell m}^{\text{gwNRXHME}}(t; \lambda) = \phi_0 + \int \omega_{\ell m}^{\text{gwNRXHME}}(t; \lambda) dt, \quad (11)$$

where $\phi_0 = \phi_{\ell m}(t; \lambda^0)$ is the integration constant. Finally, We obtain complex time-series for each mode as:

$$h_{\ell m}^{\text{gwNRXHME}}(t; \lambda) = A_{\ell m}^{\text{gwNRXHME}}(t; \lambda) e^{i \phi_{\ell m}^{\text{gwNRXHME}}(t; \lambda)}. \quad (12)$$

We demonstrate this framework using non-precessing eccentric NR simulation SXS:BBH:0324 and corresponding quasi-circular NR simulation SXS:BBH:0318. Both simulations are for $q = 1.22$, $\chi_1 = 0.33$ and $\chi_2 = -0.44$. Simulation

SXS:BBH:0324 is characterized by eccentricity of $e_{\text{ref}} = 0.13$ measured at a reference frequency of $\omega = 0.017351$ [74]. In Figure 5, we show the eccentric higher-order spherical harmonic modes (black dashed lines) obtained from the gwNRXHME framework and corresponding NR data from SXS:BBH:0324 simulation (green solid lines). We find that gwNRXHME predictions are visually indistinguishable from NR. In Figure 6, we focus on the (4, 4) mode amplitude. We illustrate that while the NR data is noisy, gwNRXHME prediction for the (4, 4) mode exhibits noise-free amplitude and accurately tracks the noisy NR data.

As an immediate next step, we can use the gwNRXHME framework to combine existing multi-modal quasi-circular non-precessing waveform models, such as NRHybSur3dq8, IMRPhenomTHM, or SEOBNRv5HM, with an existing quadrupolar mode eccentric waveform model to develop accurate multi-modal non-precessing eccentric waveform models. This approach will significantly reduce the complexity and time required to build multi-modal non-precessing eccentric waveform models. We would like to mention that such exercise has already been performed for non-spinning eccentric binaries in Ref. [78] using gwNRHME framework. We should also note that these phenomenological relations have so far been tested only for non-precessing BBH merger scenarios. It is necessary to study their validity in precessing eccentric scenarios. However, few NR simulations for precessing eccentric BBH mergers

exist, making it difficult to perform such tests. Therefore, dedicated NR simulations for precessing eccentric BBH mergers are needed, which we leave for future work.

ACKNOWLEDGMENTS

We are grateful to the SXS collaboration, RIT NR group and MAYA collaboration for maintaining publicly available catalog of NR simulations which has been used in this study. We thank Scott Field, Gaurav Khanna, Deborah Ferguson, Vijay Varma, Chandra Kant Mishra, Prayush Kumar and Saul Teukolsky for helpful discussions, and Shrobona Ghosh for comments and suggestions on the initial draft. This research was supported in part by the National Science Foundation under Grant No. NSF PHY-2309135 and the Simons Foundation (216179, LB). TV acknowledges support from NSF grants 2012086 and 2309360, the Alfred P. Sloan Foundation through grant number FG-2023-20470, the BSF through award number 2022136, and the Hellman Family Faculty Fellowship. Use was made of computational facilities purchased with funds from the National Science Foundation (CNS-1725797) and administered by the Center for Scientific Computing (CSC). The CSC is supported by the California NanoSystems Institute and the Materials Research Science and Engineering Center (MRSEC; NSF DMR 2308708) at UC Santa Barbara.

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