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Nonparametric analysis of correlations in the binary black hole population with LIGO–Virgo–KAGRA data

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Formation channels of merging compact binaries imprint themselves on the distributions and correlations of their source parameters, but current understanding of this population is hindered by simplified parametric models. We overcome such limitations using PIXELPOP [Heinzl *et al.* (2024)]—our Bayesian nonparametric multidimensional population model. We analyze data from the first three LIGO–Virgo–KAGRA observing runs and make high resolution, minimally modeled measurements of the pairwise distributions of binary black hole masses, redshifts, and spins. We find no evidence that the mass spectrum evolves over redshift and show that such measurements are fundamentally limited by the detector horizon. We find support for correlations of the spin distribution with binary mass ratio and redshift, but at reduced significance compared to overly constraining parametric models. Confident data-driven conclusions about population-level correlations with flexible models like PIXELPOP will require more informative gravitational-wave catalogs.

Introduction.—From the first gravitational-wave (GW) detection of a merging black hole (BH) binary [1] to the catalog observed by the end of the third observing run (O3) [2–5] of the LIGO–Virgo–KAGRA collaboration (LVK) [6–8], our understanding of the underlying population has improved significantly [9–11]: the merger rate increases with redshift, consistent with cosmic star-formation history [12–15]; there is a continuum of masses with peaks at around $10M_{\odot}$ and $35M_{\odot}$, with most binaries having nearly equal masses [16–18]; and there are relatively fewer BHs with large spins [19–22]. Potential astrophysical correlations between source parameters have also been identified [23–27]. Despite this, the physical origins of these features are unclear, though there are a plethora of potential formation channels (e.g., Refs. [28–43]) with distinct GW signatures (e.g., Refs [44–54]). Interpreting GW data in light of these predictions could yield key astrophysical insights.

The above population constraints are made using a model for the merger rate density as a function of the binary source parameters, for which the combined catalog of GW events can be used to infer a Bayesian posterior [55–58]. Due to large theoretical uncertainties [59], a major difficulty is choosing appropriate models. Perhaps the most common approach due to its simplicity is to parametrize the rate density with basic functions (such as power laws and normal distributions) and infer their parameters (e.g., Refs. [11, 13, 16, 60–63]). However, such choices can be overly constraining and lead to missed features in the inferred population or, conversely, features driven by the strong model assumptions rather than data [64–66]. An alternative approach is to use nonparametric models in order to impose any assump-

tions as weakly as possible [11, 67–75]. Such flexible models are better able to fit complicated structures in the source distributions that may arise from astrophysical formation processes, though typically at the cost of larger uncertainties, higher computational cost, and lack of interpretability. Moreover, it is challenging to extend these methods to simultaneously probe multiple parameter dimensions and their correlations with high fidelity [11, 66, 74, 76–78].

In Ref. [79] we developed PIXELPOP—a Bayesian nonparametric multidimensional population model for inference on correlated parameter distributions, such as those that may result from the astrophysical formation of compact binaries. PIXELPOP works by densely binning the joint space of binary source parameters and inferring the comoving merger rate density in each bin. The only assumption made is a weak smoothing prior that couples each bin to its nearest neighbors—the conditional autoregressive (CAR) prior. The computational efficiency of the CAR model allows us to dramatically increase its resolution compared to similar nonparametric methods [11, 67, 74]—using $\mathcal{O}(10^4)$ bins compared to $\mathcal{O}(10^2)$ —and thus provide a much more complete view of GW populations while making fewer assumptions about them.

In this Letter, we analyze the LVK catalog of binary BH mergers with PIXELPOP, including the 69 binary BH events with false-alarm rates $< 1\text{ yr}^{-1}$ in the third GW transient catalog (GWTC-3) [5, 11]. We specifically target the joint distributions of: (1) the heavier (primary) BH mass m_1 and redshift z , (2) binary mass ratio $0 < q \leq 1$ and effective aligned spin $\chi_{\text{eff}} \in (-1, 1)$ [80], and (3) χ_{eff} and z (when not inferred with PIXELPOP we fit the mass, redshift, and spin distributions with the parametric POWER LAW + PEAK mass model, POWER LAW redshift model, and DEFAULT spin model from Ref. [11]). We use the publicly available [81, 82] parameter-estimation

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samples [83–86] (Overall, PrecessingSpinIMRHM, and Mixed for events from GWTC-1, 2, and 3, respectively) and sensitivity estimates [87] to compute the GW population likelihood, following a Monte Carlo approximation [11, 56, 57] and penalizing the likelihood in regions of high approximation uncertainty [88–90]. Full details of PIXELPOP and its analysis methods are available in Heinzel *et al.* [79]. We perform analyses on additional parameter pairs in the Supplemental Material.

Primary mass and redshift.—The observed primary mass distribution spans roughly two orders of magnitude, so we place 100 bins uniformly over both $\ln m_1$ from $m_1 = 2M_\odot$ to $m_1 = 100M_\odot$ and redshift from $z = 0$ to $z = 1.9$ (10^4 bins total). In Fig. 1 we show the comoving merger rate density $\mathcal{R}(m_1; z)$ inferred from GWTC-3 over m_1 and z ; note that $\mathcal{R}(m_1; z)$ is a function of redshift z through the comoving volume element, but is not a density in z . A representative average over the posterior uncertainty is given by the median rate bins in the central panel. One dimensional slices of the joint distribution and their associated uncertainties are displayed in the upper and right-hand panels.

We observe peaks at $m_1 \approx 10M_\odot$ and $m_1 \approx 35M_\odot$, consistent with previous parametric and nonparametric results [11, 16, 18, 68, 73, 91, 92]. For $z \in [0, 0.2]$, the merger rate density integrated over $m_1/M_\odot \in [8, 15]$ is higher than when integrated over $m_1/M_\odot \in [8, 100]$ at almost 100% posterior credibility. The secondary peak has somewhat lower significance in the same redshift interval; we quantify this with the local integrated rate density over $m_1/M_\odot \in [25, 40]$ and $m_1/M_\odot \in [15, 50]$, where the former is higher than the latter at 72% credibility.

Within the posterior uncertainty we find no evidence for structure beyond these two peaks. But unlike for parametric mass models, the inferred merger rate does not decrease at small masses and the uncertainty grows as there is insufficient information in the catalog to constrain the merger rate with the flexibility of PIXELPOP. This is to be expected because at low masses the detector horizon—the redshift at which a source is not detectable, given by the white curve¹ in Fig. 1—excludes most of the mass range. In the absence of GW events, inference in this region of parameter space is dominated by the CAR prior, similar to other nonparametric models [73, 94]. In contrast, the merger rate decreases at large masses and low redshifts where sources should be detectable; having not detected such sources implies a low astrophysical merger rate.

There is a mild preference for an increasing merger rate at small redshifts $z \lesssim 0.2$, consistent with parametric models [11, 13–15]. For masses $m_1/M_\odot \in [8, 100]$

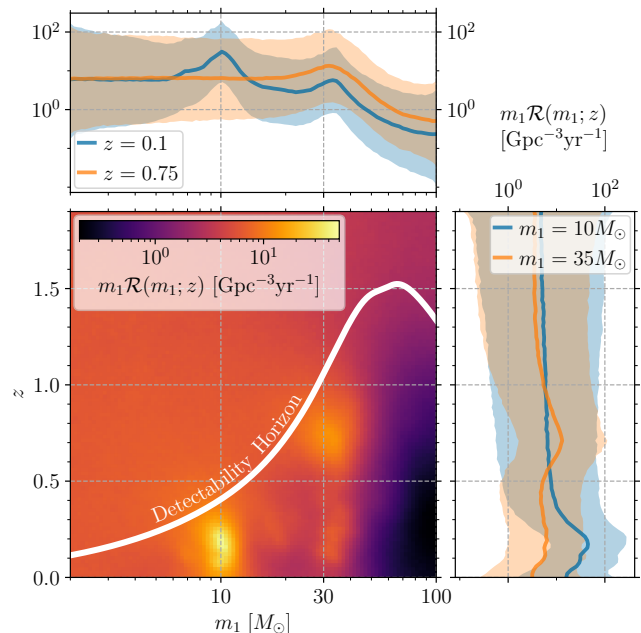


FIG. 1. Comoving merger rate density $\mathcal{R}(\ln m_1; z) = m_1 \mathcal{R}(m_1; z)$ as a function of primary mass m_1 and redshift z , inferred from GWTC-3. The central panel shows the posterior median, with lower to higher rates colored black to yellow, respectively. The white curve defines the detection horizon above which a binary BH merger is not detectable. The upper horizontal panel shows $\mathcal{R}$ as a function of m_1 at two fixed redshifts, $z = 0.1$ (blue) and $z = 0.75$ (orange). The solid line is the posterior median while the shaded region encloses the 90% symmetric credible region. Similarly, the right-hand vertical panel shows $\mathcal{R}$ as a function of z at the two peaks in the merger rate of $m_1 = 10M_\odot$ (blue) and $m_1 = 35M_\odot$ (orange).

beyond the prior-dominated regime, the integrated comoving rate density is higher over $z \in [0.1, 0.2]$ than $z \in [0, 0.1]$ at 67% credibility. At $m_1 = 10M_\odot$, the comoving merger rate increases from $z = 0$ to $z = 0.2$ at 73% credibility. With monotonic parametric models [11, 13], a positive slope at low redshifts enforces the same at higher redshifts, whereas PIXELPOP does not make this requirement. There appears to be another redshift mode at $z \approx 0.7$ for masses within the detection horizon, coincident with the onset of a plateau in the merger rate inferred in Ref. [73]. At $m_1 = 35M_\odot$, the comoving merger rate increases from $z = 0$ to $z = 0.7$ at 79% credibility. Above this redshift the posterior uncertainty increases as the majority of the parameter space lies beyond the detector horizon where the CAR model prefers broad distributions.

Beyond this, we do not confidently identify any correlations in the mass–redshift parameter space, in agreement with Ref. [74, 95]. While the $m_1 \approx 35M_\odot$ peak extends out to the $z \approx 0.7$ mode and beyond, we cannot conclude that the mass spectrum evolves over redshift as suggested

¹ We compute this as the maximum redshift at which a source has optimal signal-to-noise ratio > 8 in a LIGO–Virgo detector network with O3-like sensitivity [93].

in Ref. [96] because the $m_1 \approx 10M_\odot$ mode is limited by the detector horizon. We cannot exclude the possibility that the lower-mass peak has the same redshift extent as the higher-mass peak.

Mass ratio and effective spin.—The effective spin χ_{eff} is the mass-weighted average of the two BH spin components along the binary orbital angular momentum; it is large and positive (negative) for binaries with spinning BHs aligned (antialigned) with the orbital angular momentum, and closer to zero for small or misaligned BH spins. Callister *et al.* [23] originally discovered evidence for an anticorrelation between mass ratios q and effective spins χ_{eff} —i.e., binaries with larger χ_{eff} tend to have more unequal masses—with a strongly parameterized model. This result was corroborated by Refs. [11, 71, 97, 98] with additional models, though with reduced confidence in regions of uninformative data [71]. However, mechanisms that could produce this correlation are not well understood [44, 47, 48, 99–104] and the models used make strong assumptions about the form of the correlation. Instead, we quantify the evidence for this correlation and constrain its form with minimal assumptions by fitting the merger rate density $\mathcal{R}(\chi_{\text{eff}}, q; z)$ with 100 bins over both χ_{eff} and q using PIXELPOP.

In Fig. 2, we show the inferred comoving merger rate density evaluated at a redshift of $z = 0.2$, corresponding to the peak in Fig. 1. The merger rate is highest for near-equal masses and $\chi_{\text{eff}} \approx 0$ and decreases towards unequal masses and $|\chi_{\text{eff}}| > 0$, though the data do not require the global maximum to be $q = 1$ and $\chi_{\text{eff}} = 0$. PIXELPOP does not enforce that the merger rate vanishes as $q \rightarrow 0$ (unlike common parametric models [11]) and there is also support for negative χ_{eff} , though the tails of the distributions again become dominated by the CAR prior (cf. Ref. [73]). The bulk of the population has $q > 0.6$ and in this region the integrated merger rate is higher for $\chi_{\text{eff}} \in [0, 0.5]$ than $\chi_{\text{eff}} \in [-0.5, 0]$ at 91.7% posterior credibility, indicating that component BH spins tend to prefer alignment with the orbital angular momentum [11]. Specifically, the merger rate density at $z = 0.2$ is $11.3^{+5.8}_{-3.8} \text{ Gpc}^{-3}\text{yr}^{-1}$ for positive spins $\chi_{\text{eff}} \in [0, 0.5]$ (median and 90% credible interval) but $7.4^{+4.8}_{-3.1} \text{ Gpc}^{-3}\text{yr}^{-1}$ for negative $\chi_{\text{eff}} \in [-0.5, 0]$, in both cases taking $q \in [0.6, 1]$.

The two-dimensional posterior median in Fig. 2 suggests a tail in the distribution toward $(q, \chi_{\text{eff}}) \approx (0.25, 0.25)$. In the region $q \in [0.2, 0.4]$, we find a higher rate density for $\chi_{\text{eff}} \in [0, 0.5]$ than $\chi_{\text{eff}} \in [-0.5, 0]$ at 66% credibility when accounting for the posterior uncertainty. To further test for the aforementioned q – χ_{eff} correlation, we compute the Spearman rank correlation coefficient ρ [105, 106]—which is positive (negative) when there is an increasing (decreasing) monotonic correlation—for $\mathcal{R}(\chi_{\text{eff}}, q)$, restricting to $q \in [0.2, 1]$ and $\chi_{\text{eff}} \in [-0.5, 0.5]$ to avoid prior-dominated regions. Whereas Ref. [11] identifies a negative correlation between q and χ_{eff} at 97.5%

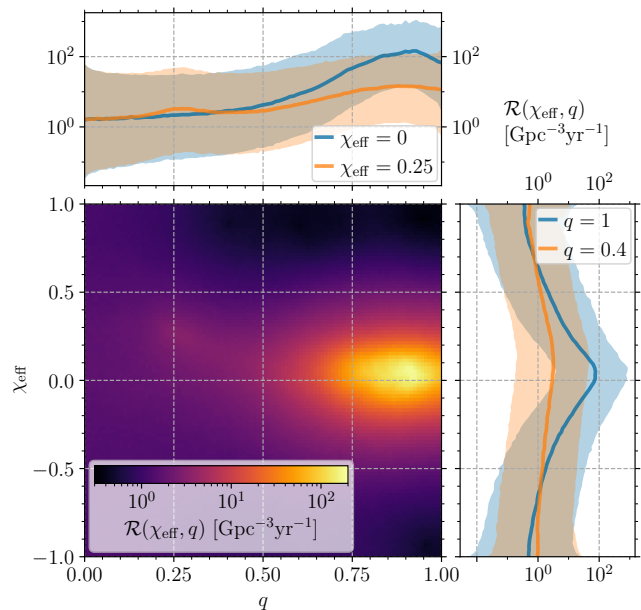


FIG. 2. Joint comoving merger rate density $\mathcal{R}(\chi_{\text{eff}}, q; z = 0.2)$ as a function of effective spin χ_{eff} and binary mass ratio q , evaluated at a fixed redshift $z = 0.2$. The central panel displays the joint two-dimensional posterior median. The merger rate density evaluated at $\chi_{\text{eff}} = 0$ (blue) and $\chi_{\text{eff}} = 0.25$ (orange) is shown as a function of q in the upper panel, and evaluated at $q = 1$ (blue) and $q = 0.4$ (orange) as a function of χ_{eff} in the right-hand panel. For the one-dimensional slices the bold line gives the posterior median while the shaded region encloses the 90% symmetric credible region.

significance, we find $\rho = 0.023^{+0.281}_{-0.270}$ and no significant evidence for a correlation. We stress, however, that the PIXELPOP posterior does not exclude such a correlation and that this result is not at odds with the results of strongly parametrized models [11, 23, 71, 97] when used to analyze the same GW catalog—if one mandates a functional form that allows a global monotonic trend then the data prefer an anticorrelation, but this is neither required nor excluded by the more flexible PIXELPOP model that allows a much broader class of nontrivial correlations.

Redshift and effective spin.—Biscoveanu *et al.* [24] found evidence for an astrophysical broadening of the χ_{eff} distribution with increasing redshift, assuming a linear relationship. This was confirmed by Heinzl *et al.* [71] with a more flexible model for the correlation, but in both cases a simple Gaussian parametrization was made for the underlying population. We remove these assumptions by using PIXELPOP to infer the comoving merger rate density $\mathcal{R}(\chi_{\text{eff}}; z)$ jointly over 100 bins in each of χ_{eff} and z .

We show the PIXELPOP posterior in Fig. 3. The merger rate may evolve differently over redshift depending on the effective spin: for $\chi_{\text{eff}} = 0$ the median rate density is initially higher at lower redshifts than $\chi_{\text{eff}} = 0.25$, increases from $z = 0$ to $z \approx 0.25$, then drops at $z \approx 0.5$

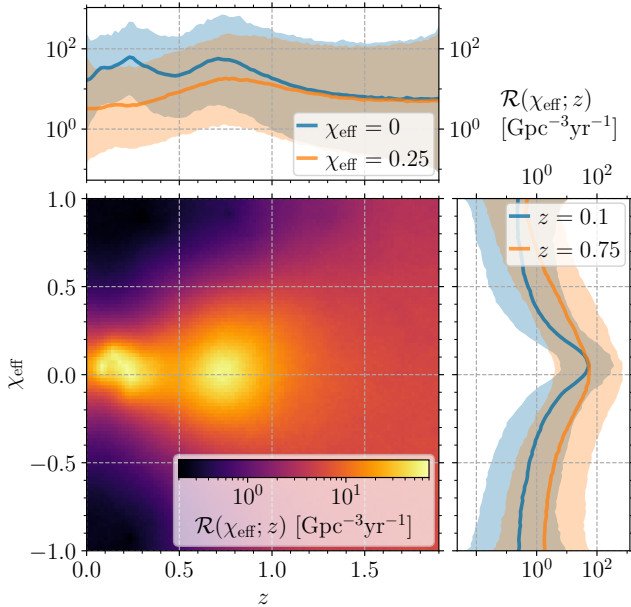


FIG. 3. Comoving merger rate density $\mathcal{R}(\chi_{\text{eff}}; z)$ as a function of effective spin χ_{eff} and redshift z . The central panel displays the joint two-dimensional posterior median. The merger rate density evaluated at $\chi_{\text{eff}} = 0$ (blue) and $\chi_{\text{eff}} = 0.25$ (orange) is shown as a function of z in the upper panel, and evaluated at $z = 0.1$ (blue) and $z = 0.75$ (orange) as a function of χ_{eff} in the right-hand panel. For the one-dimensional slices the bold line gives the posterior median while the shaded region encloses the 90% symmetric credible region.

before increasing again at $z \approx 0.7$; for $\chi_{\text{eff}} = 0.25$ the median rate density increases monotonically between $z = 0$ and $z \approx 0.7$; in both cases the rate density plateaus in the prior-dominated region $z \gtrsim 1$. While these results hold for the median inferred rate, this marginalizes over large uncertainties, within which a lack of redshift peaks is also possible.

We can confidently conclude that the merger rate significantly decreases for $|\chi_{\text{eff}}| \gtrsim 0.5$ at low redshifts where sources would be detectable but have not been detected, implying that the astrophysical merger rate must be low. This corresponds to the broadening of the χ_{eff} spin distribution visible in both the two-dimensional median posterior rate density and the one-dimensional slices in Fig. 3. There is a more pronounced peak just above $\chi_{\text{eff}} \approx 0$ at $z = 0.1$ compared to $z = 0.75$ where the overall rate is higher. We quantify this trend by computing the Spearman rank correlation coefficient ρ between the redshift and the width of the effective spin distribution—if ρ is positive, the χ_{eff} distribution broadens at larger redshifts. However, we must be careful not to mistake the natural broadening of the CAR model in uninformative regions of parameter space—such as higher redshifts—for an astrophysical correlation. For $z < 0.5$, we find a weak preference for broadening, with $\rho = 0.096^{+0.231}_{-0.240}$ and $\rho > 0$

at 75% credibility; cf. 98% in Refs. [24, 71]. Limiting to $z < 1$ instead yields $\rho = 0.164^{+0.267}_{-0.236}$ and $\rho > 0$ at a larger but still mild significance of 84%. The broadening of χ_{eff} over redshift appears to be a genuine astrophysical feature in the data, but with PIXELPOP we cannot make a definitive statement about the underlying nature of the population without more informative data.

Conclusions.—As the catalog of binary BH mergers observed with GW detectors grows, increasingly more information can be obtained about the underlying population. In particular, nontrivial structures and population-level parameter correlations may be identified and interpreted in terms of their astrophysical origins and formation channels, but only by using a suitable population model. Indeed, correlations have been identified in the population of binary BHs with simplified parametric models [23–25, 27, 71, 97, 98], but these risk misspecification and overconfident inference.

PIXELPOP [79] is a nonparametric multidimensional population model that imposes minimal assumptions on the form of the underlying merger rate and is computationally efficient. We used PIXELPOP to analyze bivariate source distributions with high resolution—primary BH masses and redshifts, binary mass ratios and effective spins, and effective spins and redshifts—using LVK binary BH detections [2–5]. We showed that: there is no evidence for an evolving BH mass spectrum over redshift and this measurement is currently prohibited by the detector horizon; and the GW data are consistent with a distribution of effective spins that is correlated with binary mass ratio and redshift, but that we cannot be confident in this hypothesis without strong model assumptions.

Flexible population inference methods such as Bayesian nonparametrics will be increasingly useful in the future as GW catalogs become more informative. They offer a valuable consistency check that strongly parameterized models are not overfitting to or extrapolating from observational data, as the shapes of distributions as well as correlations in their joint space may be complicated and impossible to model with simple parameterizations. Flexible models can also be used to account for unmodeled contributions to the GW population that would otherwise lead to systematically biased inference [107], or for cosmological constraints with astrophysics-agnostic assumptions about the source population [94, 108]. The ability of PIXELPOP to capture arbitrary correlations between source parameters make it a standout model for such GW population analyses.

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