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On QFT in Curved Spacetime from Quantum Gravity: proper WKB decomposition of the gravitational component

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Starting from a re-analysis of previous work, we construct the proper low energy quantum field theory (QFT) limit of a full quantum gravity theory in the Born-Oppenheimer approach. We separate the gravitational sector into a classical background, given by a vacuum diagonal Bianchi I cosmology, and its quantum perturbations represented by the two graviton degrees of freedom; we further include quantum matter in the form of a test scalar field. We then implement a Born-Oppenheimer separation, where the gravitons and matter play the role of "slow" and "fast" quantum components respectively, and perform a Wentzel-Kramers-Brillouin (WKB) expansion in a Planckian parameter. The functional Schrödinger evolution for matter is recovered after averaging over quantum gravitational effects, provided that a condition is imposed on the gravitons' wave functional. Such a condition fixes the graviton dynamics and is equivalent to the purely gravitational Wheeler-DeWitt constraint imposed in previous approaches. The main accomplishment of the present work is to clarify that QFT in curved spacetime can be recovered in the low energy limit of quantum gravity only after averaging over the graviton degrees of freedom, in the spirit of effective field theory. Furthermore, it justifies *a posteriori* the implementation of the gravitational Wheeler-DeWitt equation on the "slow" gravitons' wave functional rather than assuming its validity *a priori*.

I. INTRODUCTION

One of the most striking differences between the gravitational field and other fundamental forces is that, as a consequence of its geometrical nature, the former is an "environment" interaction [1–3]. This peculiarity of the gravitational field is particularly evident when we attempt a canonical quantization of geometrodynamics [4–7]. In fact, the Hamiltonian vanishes and the quantum evolution appears to be frozen, leading to the so-called "problem of time" in quantum gravity [7, 8]. This feature is not altered by the introduction of matter fields, in the presence of which the full gravity-matter Hamiltonian is vanishing. The simple observation that the pure gravity Hamiltonian is no longer zero suggests the possible role of matter as a clock for the gravitational field evolution [9–15]. However, quantum field theory on curved spacetime (QFT-CS) is an established theory [16–18] which led to a number of intriguing and robust predictions, such as the Unruh effect [19] and the Hawking effect [20]. It is then natural to ask how QFT-CS, which relies on a notion of time, can be recovered from a full timeless quantum gravity theory including matter in the appropriate low-energy limit.

This question was first approached in [21], where a notion of Tomonaga "bubble time" was introduced. A more robust and physically well-grounded proposal was discussed in [22] using a WKB expansion [23, 24] in $\hbar$ at zeroth and first order (see also [25]). In [22] the notion of time arises from the matter wave functional's dependence on the quasi-classical gravitational field, which in turn depends at zeroth order on the label time of the spacetime slicing. The same notion of time was adopted in [26] (see also [27, 28]), where the expansion in a Planckian parameter was considered up to the order where quantum gravity corrections to QFT naturally emerge. The main merit of [26]was to stress how, at such order, a Born-Oppenheimer (B-O) [29, 30] separation between the behavior of the "slow" gravitational variables and the "fast" matter is affected by the serious problem of non-unitarity (see [28, 31–35] for possible solutions to this puzzle).

In this letter, we re-evaluate the validity of some of the assumptions made in [22, 26]; our analysis reformulates on more solid physical grounds the problem of recovering QFT-CS at low energies using a WKB approach. We consider a mini-superspace model—a Bianchi I vacuum cosmology—with a quantum free scalar field. One major difference (with the analyses [22, 26]) is that we identify a "slow" quantum component in the gravitational sector, represented by independent graviton degrees of freedom. Different from [22], we do not impose *a priori* a separate Wheeler-DeWitt (WDW) equation for the gravitational component only, but rather justify it by using the typical gauge invariance of the B-O formulation [26] to have QFT-CS hold in the appropriate limit. The result of our analysis is that the matter dynamics is obtained after averaging over the graviton degrees of freedom, as one would expect in the context of an effective field theory on a quasi-classical background.

This letter is structured as follows. In Section II we motivate the need for a reformulation of the semiclas-

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sical approach to quantum gravitational corrections by outlining four basic conceptual points. In Section III we introduce the model of our interest, whose dynamics is studied in the perturbative WKB scheme in Section IV, comparing also with the gravitational WDW equation. Concluding remarks follow in Section V .

II. MOTIVATIONS FOR A NEW SCHEME

We now motivate our analysis by re-evaluating some aspects of the proposal developed in [22] (see also [26, 36, 37]). First, we observe that in [22] the separation between a quasi-classical background system and a "small" quantum one was pursued without taking into account the physical nature of the variables. Here, in analogy with [21, 26], we consider a quasi-classical spacetime described by variables h_A (with $A = 1, ..., n$) and a matter sector described by variables q_r (with $r = 1, ..., m$). Both in the present discussion and in the concrete application below, we focus on a mini-superspace model. The gravitational component of the superHamiltonian in the Arnowitt-Deser-Misner (ADM) formalism [38] takes the form [22]

$$H^g \equiv G_{AB}\, p^A p^B + V(h_A) = 0\,, \tag{1}$$

where p^A are the momenta conjugate to h_A. Both the mini-supermetric G^{AB} and the potential term V are functions of h_A, the latter due to the non-vanishing spatial curvature. The quantum matter component of the superHamiltonian H^Q depends on the matter degrees of freedom q_r as well as on the gravitational variables h_A.

In [22] a B-O separation of the quasi-classical and quantum wave functionals is implemented, in which the former is the "slow" and the latter the "fast" component of the coupled system, based on the scale separation $\langle H^Q\rangle/\langle H^g\rangle \sim \hbar$, where $\langle\cdot\rangle$ denotes the expectation value on the respective wave functionals. The total wave functional of the gravity+matter system is decomposed as

$$\Psi(h_A, q_r) = A(h_A)\, e^{iS(h_A)/\hbar}\, \chi(h_A, q_r)\,, \tag{2}$$

where the amplitude A and the function S are real, and χ is associated to quantum matter.

Promoting the two superHamiltonian terms to canonical operators $\hat{H}^g$ and $\hat{H}^Q$, the system is quantized *à la* Dirac by imposing the following constraints:

$$\left(\hat{H}^g + \hat{H}^Q\right)\Psi = 0\,, \tag{3}$$

$$\hat{H}^g\, A\, e^{iS/\hbar} = 0. \tag{4}$$

where (4) states that the gravitational component independently satisfies its own WDW equation. Combining via a WKB expansion the zeroth and first order in $\hbar$, eqs. (3),(4) take the form

$$G_{AB}\frac{\partial S}{\partial h_A}\frac{\partial S}{\partial h_B} + V(h_A) = 0\,, \tag{5}$$

$$G_{AB}\frac{\partial}{\partial h_A}\left(A^2\frac{\partial S}{\partial h_B}\right) = 0\,, \tag{6}$$

$$i\hbar\partial_t\chi = N\hat{H}^Q\chi\,, \tag{7}$$

where N is the lapse function, i.e. $dt_s = N(t)dt$, being t_s the synchronous time. The time derivative in eq. (7) is defined as

$$\partial_t\chi \equiv 2NG_{AB}\frac{\partial S}{\partial h_A}\frac{\partial\chi}{\partial h_B} = \dot{h}_A\frac{\partial}{\partial h_A}\chi\,, \tag{8}$$

where in the second equality we made use of the Hamilton's equation obtained by variating with respect to p_A (here a dot denotes differentiation with respect to label time). Eq. (5) is of order $\hbar^0$ and corresponds to the Hamilton-Jacobi equation for the classical limit of gravity. Both eq. (6) and (7) are obtained[1] at order $\hbar$; the former arises from the gravitational WDW equation, while the latter yields the desired QFT dynamics for quantum matter, recovered by simply combining an expansion in $\hbar$ with the B-O separation.

Now we are ready to outline four ambiguous points of the approach [22] being the main motivations for the present study.

i) The variables h_A do not represent a set of classical gravitational degrees of freedom, because a quantum amplitude $A(h_A)$ is retained at order $\hbar$. Qualitatively, we could write $h_A = h^0_A(t) + \delta h_A$, where $h^0_A(t)$ account for the classical gravitational degrees of freedom (with the dependence on the label time t determined by the Hamilton's equations), while δh_A represent quantum corrections of order $\hbar$ to some suitable power. Thus the time differentiation (8) should be defined employing derivatives with respect to h^0_A only, rather than the full quantum variable h_A.

ii) This also implies that δh_A are independent degrees of freedom with respect to $h^0_A(t)$. Therefore, a description of their dynamics is necessary. This is readily understood if we remember that the small metric perturbations of an isotropic universe (whose only degree of freedom is given by the cosmic scale factor a) have two scalar, two vector and two tensor components, both at a classical and at a quantum level. These degrees of freedom are independent from a and are different in number and morphology from the small quantum fluctuations δa.

[1] Unlike [22], here we adopted the "natural" operator ordering (functions of h_A always on the left of the corresponding derivatives). This choice, also discussed in [26], has no deep physical implications on the conceptual paradigm.

iii) Eqs. (6) and (7) both live at the same order in $\hbar$ and their separation relies on the assumption that it is *a priori* possible to impose the gravitational WDW constraint independently. However, this assumption does not have a physical motivation in the analysis of [22], and is inconsistent with a pure B-O approximation, because it violates its typical gauge invariance. In fact, the B-O method separates the whole system into a slow and a fast component, with the wave functional (2). Thus, if we multiply the quantum matter wave functional χ by a phase depending on h_A, the state is invariant provided that we multiply the gravitational component by an inverse phase. This gauge symmetry is broken if we separately impose the gravitational constraint, so that such a procedure appears rather ambiguous.

iv) The functional Schrödinger equation (7) is not the right one for quantum matter on a classical curved spacetime, since the matter wave functional χ depends on the quantum fluctuations of the background δh_A. This dependence, which has been implicitly neglected in [22], is problematic for the purpose of recovering QFT-CS.

We would like to remark that the difficulties i), ii) and iv) are present also in the analysis [26], while iii) is not, because the equation for the quantum gravitational amplitude $A(h_A)$ is obtained via a gauge condition (see [34] for a comparison of the two approaches [22] and [26]).

With these motivations, we now reformulate the problem in a Bianchi I cosmological background, obtaining the correct QFT-CS limit without imposing the gravitational constraint and after averaging over quantum gravitational effects.

III. MINISUPERSPACE MODEL

Starting from point i) of the previous section, we take the classical cosmological background to be a vacuum diagonal Bianchi I model, which is a homogeneous and spatially flat geometry (the simplest case of the Bianchi classification [2, 39]). The advantage of this choice over a Friedmann-Lemaître-Robertson-Walker (FLRW) model (e.g. in [27, 40–43]) is that, being a vacuum geometry, no scalar or vector perturbations are present [1, 3].

In the Misner variables α, β_+ and β_- [1, 44], the line element reads

$$ds^2 = -N^2(t)dt^2 + e^{\alpha}(e^{\beta})_{ij}dx^i dx^j, \qquad (9)$$

where $\beta \equiv diag\{\beta_+ + \sqrt{3}\beta_- \, , \, \beta_+ - \sqrt{3}\beta_- \, , \, -2\beta_-\}$ is a diagonal traceless matrix. The Misner variables depend on the label time t only; α corresponds to the logarithmic volume of the universe, while β_+ and β_- represent the spatial anisotropies. The supermomentum constraint is identically satisfied and the superHamiltonian is

$$H^I(\alpha(t), \beta_\pm(t)) = \frac{4}{3M} e^{-\frac{3}{2}\alpha} \left(-p_\alpha^2 + p_+^2 + p_-^2\right), \qquad (10)$$

where $M = c/32\pi G = cm_{pl}^2/4\hbar$ is a Planckian order parameter of dimension mass over length (G being the Newton constant and m_{Pl} being the reduced Planck mass).

According to point ii), we describe the gravitational fluctuations via tensor perturbations only, as guaranteed by the choice of the vacuum Bianchi I model. Thus the "slow" quantum degrees of freedom δh^A correspond to gravitons and are independent from the classical background. In the Mukhanov-Sasaki (M-S) formalism [45–47], the tensor perturbations can be described via the gauge-invariant variables $v_{\mathbf{k}}^{\lambda}$ in Fourier space (λ identifies the two polarization states). For the Bianchi I model [48], the corresponding Hamiltonian (where $N = e^{\alpha}$ in the conformal time η gauge) is

$$NH^{(v^\lambda)} = \sum_{\mathbf{k},\lambda} \frac{1}{2} \left[-\partial^2_{v^\lambda_{\mathbf{k}}} + \omega_k^2(\eta)(v^\lambda_{\mathbf{k}})^2 + \mathcal{V}_{\lambda,\bar{\lambda}}\right]. \qquad (11)$$

Here each mode $\mathbf{k}, \lambda$ behaves as a time-dependent harmonic oscillator with $\omega_k^2(\eta) = k^2 - z_\lambda''/z_\lambda$, being $z_\lambda(\eta, k_i)$ a function of the background metric and $' \equiv \partial_\eta$. The interaction potential $\mathcal{V}_{\lambda,\bar{\lambda}}$ depends on the shear tensor $\sigma_{ij} = \frac{1}{2}(e^{\beta})'_{ij}$ of the background metric and expresses the mixing of the two polarization modes $(\lambda, \bar{\lambda})$ which takes place due to the anisotropies [48] even at the classical level [49], differently from isotropic settings. There is no mixing between scalar and tensor perturbations because we are neglecting the backreaction of the scalar field on the metric (see [50–53] for perturbations in a Bianchi I universe coupled to matter).

We consider a free test scalar field as the "fast" quantum matter sector (e.g. the inflaton field), whose Hamiltonian in the M-S formalism takes the form

$$NH^{(\phi)} = \sum_{\mathbf{k}} \frac{1}{2} \left[-\partial^2_{\phi_{\mathbf{k}}} + \nu_k^2(\eta) \, (\phi_{\mathbf{k}})^2\right]. \qquad (12)$$

Here, each Fourier mode corresponds to a time-dependent harmonic oscillator with $\nu_k^2(\eta) = k^2 - (e^{\alpha})''/e^{\alpha}$.

The WDW equation for the full model is

$$\hat{H}\Psi = \left(\hat{H}^I + \hat{H}^{(v^\lambda)} + \hat{H}^{(\phi)}\right)\Psi = 0\,, \qquad (13)$$

and the wave functional Ψ is assumed to be separable in a B-O scheme as

$$\Psi = \psi_g(\alpha, \beta_\pm, v^\lambda_{\mathbf{k}}) \, \chi_m(\phi_{\mathbf{k}}; \alpha, \beta_\pm, v^\lambda_{\mathbf{k}})\,. \qquad (14)$$

This factorization follows from the assumed difference in energy scale between the matter and the gravitational sectors; furthermore, ψ_g is independent of the matter variables $\phi_{\mathbf{k}}$ because we assume that the fast quantum sector has a negligible backreaction on the gravitational one. Given the separation (14), the WDW equation (13) is invariant under the transformation:

$$\psi_g \to \psi_g e^{-\frac{i}{\hbar}\theta}\,, \quad \chi_m \to e^{\frac{i}{\hbar}\theta}\chi_m\,, \qquad (15)$$

where the phase $\theta = \theta(\alpha, \beta_\pm, v_{\mathbf{k}}^\lambda)$ depends on the gravitational variables only.

As in point iii), we will not require the gravitational sector to satisfy the gravitational constraint *a priori*. The gravitons' evolution will instead be derived on physical grounds by requiring the correct QFT dynamics to arise in the appropriate limit and exploiting the gauge invariance (15).

IV. WKB ANALYSIS OF THE DYNAMICS

We can now apply the WKB perturbative scheme to our model. We use $1/M$ as the expansion parameter, where M is the (large) Planckian parameter in eq. (10). This allows us to consistently separate the gravity and matter sectors, in analogy with [26–28, 31, 34, 35, 41]. We emphasize that the (semiclassical) WKB expansion in the Planck constant $\hbar$ used in [22] is equivalent to the one used here (see [34] for a detailed discussion on this point).

Expanding up to order M^0, the wave function (14) takes the form

$$\Psi = e^{\frac{i}{\hbar} M S_0}\, e^{\frac{i}{\hbar}\left(S_1 + \mathcal{O}\left(M^{-1}\right)\right)}\, e^{\frac{i}{\hbar}\left(Q_1 + \mathcal{O}\left(M^{-1}\right)\right)}\,, \tag{16}$$

where at leading order $S_0 = S_0(\alpha, \beta_\pm)$. The complex functions $S_n = S_n(v_{\mathbf{k}}^\lambda; \alpha, \beta_\pm)$ and $Q_n = Q_n(\phi_{\mathbf{k}}; \alpha, \beta_\pm, v_{\mathbf{k}}^\lambda)$ are associated to the tensor and scalar quantum components of the system, respectively, which must also depend on $\alpha, \beta_\pm$.[2] The WDW equation (13) applied to (16) can then be perturbatively examined at each order in $1/M$. At $\mathcal{O}(M)$ we obtain

$$\frac{4}{3} e^{-\frac{3}{2}\alpha} M \left(- \left(\partial_\alpha S_0\right)^2 + \left(\partial_+ S_0\right)^2 + \left(\partial_- S_0\right)^2 \right) = 0, \tag{17}$$

which is consistent with the classical Bianchi I solution

$$S_0 = k_+ \beta_+ + k_- \beta_- + k_\alpha \alpha \tag{18}$$

with $k_\alpha < 0$ corresponding to an expanding universe.

Let us now introduce the time differentiation operator as in (8) for $N = e^\alpha$, but now constructed using only derivatives with respect to the classical variables $\alpha, \beta_\pm$ (this way the issue i) introduced in Section II does not arise):

$$-i\hbar\partial_T = \frac{8}{3} e^{-\frac{1}{2}\alpha} \left(\partial_\alpha S_0\, \partial_\alpha + \partial_+ S_0\, \partial_+ + \partial_- S_0\, \partial_- \right). \tag{19}$$

Using (19) and (18), at $\mathcal{O}\left(M^0\right)$ we find

$$\begin{aligned} &-i\hbar(\partial_T e^{\frac{i}{\hbar}S_1}) e^{\frac{i}{\hbar}Q_1} - i\hbar(\partial_T e^{\frac{i}{\hbar}Q_1}) e^{\frac{i}{\hbar}S_1} \\ &\quad + \frac{1}{2}\sum_{\mathbf{k},\lambda} \left[\omega_k^2 (v_{\mathbf{k}}^\lambda)^2 + \mathcal{V}_{\lambda,\bar\lambda} - \partial^2_{v_{\mathbf{k}}^\lambda} \right] e^{\frac{i}{\hbar}(S_1+Q_1)} \\ &\quad + \frac{1}{2}\sum_{\mathbf{k}} \left[\nu_k^2 (\phi_{\mathbf{k}})^2 - \partial^2_{\phi_{\mathbf{k}}} \right] e^{\frac{i}{\hbar}(S_1+Q_1)} = 0. \end{aligned} \tag{20}$$

To address the problem of the dependence of the quantum matter wave functional on the graviton variables (see point iv) in Sec. II) and in the spirit of effective field theory, we average over quantum gravitational effects to recover QFT-CS, i.e. a functional Schrödinger equation for the quantum matter sector. To this end, we label $\Gamma_g = \exp(iS_1/\hbar)$ (which is ψ_g at order M^0 only) and multiply eq. (20) by the conjugate $\Gamma_g^* = \exp(-iS_1^*/\hbar)$, obtaining

$$\begin{aligned} &-i\hbar\partial_T \left(\Gamma_g^* \Gamma_g \chi\right) + i\hbar(\partial_T \Gamma_g^*)\, \Gamma_g \chi \\ &\quad + \frac{1}{2}\sum_{\mathbf{k},\lambda} \Big[\left(\omega_k^2\, (v_{\mathbf{k}}^\lambda)^2 \Gamma_g^* + \mathcal{V}_{\lambda,\bar\lambda} \Gamma_g^* - \partial^2_{v_{\mathbf{k}}^\lambda} \Gamma_g^* \right) \Gamma_g \chi \\ &\quad + \partial_{v_{\mathbf{k}}^\lambda} \left(2(\partial_{v_{\mathbf{k}}^\lambda} \Gamma_g^*) \Gamma_g \chi - \partial_{v_{\mathbf{k}}^\lambda} \left(\Gamma_g^* \Gamma_g \chi \right) \right) \Big] \\ &\quad + \frac{1}{2}\sum_{\mathbf{k}} \left[\nu_k^2\, (\phi_{\mathbf{k}})^2 - \partial^2_{\phi_{\mathbf{k}}} \right] \Gamma_g^* \Gamma_g \chi = 0\,, \end{aligned} \tag{21}$$

where $\chi = \exp(iQ_1/\hbar)$ depends also on the $v_{\mathbf{k}}^\lambda$. We can eliminate such a dependence by integrating over the $v_{\mathbf{k}}^\lambda$, thus considering an "average effect" of the gravitons. In doing so, we assume that the wave functionals satisfy appropriate boundary conditions such that

$$\int \prod_{\mathbf{k},\lambda} dv_{\mathbf{k}}^\lambda \sum_{\mathbf{k},\lambda} \partial_{v_{\mathbf{k}}^\lambda} \left(2(\partial_{v_{\mathbf{k}}^\lambda} \Gamma_g^*) \Gamma_g \chi - \partial_{v_{\mathbf{k}}^\lambda} \left(\Gamma_g^* \Gamma_g \chi \right) \right) \tag{22}$$

vanishes. In order to recover the desired Schrödinger dynamics, we now use the gauge freedom (15) to impose the following condition on Γ_g

$$\begin{aligned} &\Gamma_g \Big[i\hbar\partial_T \Gamma_g^* + \frac{1}{2}\sum_{\mathbf{k},\lambda} \left(\omega_k^2 (v_{\mathbf{k}}^\lambda)^2 + \mathcal{V}_{\lambda,\bar\lambda} - \partial^2_{v_{\mathbf{k}}^\lambda} \right) \Gamma_g^* \Big] \\ &\quad = 0\,. \end{aligned} \tag{23}$$

This is possible, provided that the equation

$$\begin{aligned} &\frac{1}{2\hbar}\sum_{\mathbf{k},\lambda} \left[-i\partial^2_{v_{\mathbf{k}}^\lambda}\theta + \hbar^{-1}(\partial_{v_{\mathbf{k}}^\lambda}\theta)^2 - i(\partial_{v_{\mathbf{k}}^\lambda}\theta)\partial_{v_{\mathbf{k}}^\lambda}(\ln\Gamma_g^*) \right] \\ &\quad - \partial_T\theta = \frac{1}{2}\sum_{\mathbf{k},\lambda} \left[\omega_k^2 (v_{\mathbf{k}}^\lambda)^2 + \mathcal{V}_{\lambda,\bar\lambda} - \partial^2_{v_{\mathbf{k}}^\lambda} \right] \Gamma_g^* \\ &\quad - i\hbar\partial_T(\ln\Gamma_g^*) \end{aligned} \tag{24}$$

has a solution.[3] Eqs. (21) and (23) then guarantee that

[2] Here S_n and Q_n are in general complex functions, whereas in (2) the exponent S is real-defined and an amplitude A is explicitly extracted.

[3] It is understood that the boundary condition (22) is imposed in the specific gauge set by (23).

the "averaged" quantum matter wave functional

$$\widetilde{\Theta}(\phi_{\mathbf{k}};\alpha,\beta_+,\beta_-)=\int\prod_{\mathbf{k},\lambda}dv^{\lambda}_{\mathbf{k}}\,\Gamma^*_g\,\Gamma_g\,e^{\frac{i}{\hbar}Q_1} \tag{25}$$

satisfies the functional Schrödinger equation

$$i\hbar\,\partial_T\widetilde{\Theta}=\frac{1}{2}\sum_{\mathbf{k}}\left[\nu_k^2(\phi_{\mathbf{k}})^2-\partial^2_{\phi_{\mathbf{k}}}\right]\widetilde{\Theta}=N\hat{H}^{(\phi)}\widetilde{\Theta}\,, \tag{26}$$

therefore recovering QFT-CS on average. We remark that (23) fixes the independent dynamics of gravitons, so the issue ii) is also resolved.

At this point, it is worth briefly discussing the relationship between our analysis and standard QFT on curved space-time [16, 17]. In that approach, at the 1-loop order of approximation, the semiclassical background metric is sourced by the expectation values associated with the quantum components:

$$G^{(0)}_{\mu\nu}=\frac{8\pi G}{c^4}\left(\langle T^{(m)}_{\mu\nu}\rangle+\langle t^{(g)}_{\mu\nu}\rangle\right)\,, \tag{27}$$

where $G^{(0)}_{\mu\nu}$ is the Einstein tensor, while $T^{(m)}_{\mu\nu}$ and $t^{(g)}_{\mu\nu}$ denote the energy-momentum tensors of the (renormalized) quantum matter and graviton contributions, respectively. The last two are in principle of the same order, although the graviton term is often neglected in QFT applications [16]. In our WKB approach, both backreaction terms are $1/M$-suppressed at leading order [54] and the background is therefore a purely classical vacuum solution described by (17), i.e. the Bianchi I spacetime.

The backreaction of the fast (matter) component on the slow one does arise at the next order in the general B-O scheme, in the form of an expectation value of the matter Hamiltonian[4], and it was considered in [31]. For another formulation of the quantum matter backreaction on the Bianchi I cosmology, see [55]. This contribution can be removed from the equation governing the matter dynamics and included instead in the gauge condition (23) specifying the gravitons' dynamics by a phase rescaling of both the matter and the gravitational wave functions [34]. In our analysis we neglected such an expectation value in the gauge condition based on the assumed separation of energy scales between gravitons and matter, but its inclusion does not alter the final result, which is the recovery of QFT-CS in the appropriate low-energy limit.

A. Comparison with gravitational WDW equation

Let us now analyze the WKB dynamics arising when separately imposing the gravitational WDW constraint (as in [22]). In the conformal time gauge, this equation reads

$$\left(\hat{H}^I+\hat{H}^{(v^\lambda)}\right)^\dagger\psi^*_g=\left[\frac{4}{3M}e^{-\frac{3}{2}\alpha}\left(-p^2_\alpha+p^2_++p^2_-\right)^\dagger\right.$$
$$\left.+\frac{1}{2}e^{-\alpha}\sum_{\mathbf{k},\lambda}\left(-\partial^2_{v^\lambda_{\mathbf{k}}}+\omega^2_k(v^\lambda_{\mathbf{k}})^2+\mathcal{V}_{\lambda,\bar{\lambda}}\right)^\dagger\right]\psi^*_g=0 \tag{28}$$

where $\psi^*_g=\exp\left(-i(MS^*_0+S^*_1)/\hbar\right)$. At $\mathcal{O}\left(M^0\right)$ and using the Hamilton-Jacobi solution (18) for S_0, which is real-valued, we obtain

$$-\frac{8}{3}e^{-\frac{3}{2}\alpha}\left(\partial_\alpha S_0\partial_\alpha+\partial_+S_0\partial_++\partial_-S_0\partial_-\right)S^*_1\,e^{-\frac{i}{\hbar}S^*_1}$$
$$+\frac{1}{2}e^{-\alpha}\sum_{\mathbf{k},\lambda}\left[-i\hbar^{-1}\,\partial^2_{v^\lambda_{\mathbf{k}}}S^*_1+\hbar^{-2}\left(-\partial_{v^\lambda_{\mathbf{k}}}S^*_1\right)^2\right. \tag{29}$$
$$\left.+\left(\omega^2_k\,(v^\lambda_{\mathbf{k}})^2\right)^\dagger+\mathcal{V}^\dagger_{\lambda,\bar{\lambda}}\right]e^{-\frac{i}{\hbar}S^*_1}=0\,.$$

From Eq. (19), this reduces to

$$-i\hbar\partial_T\Gamma^*_g=\frac{1}{2}\sum_{\mathbf{k},\lambda}\left(-\partial^2_{v^\lambda_{\mathbf{k}}}+\omega^2_k(v^\lambda_{\mathbf{k}})^2+\mathcal{V}_{\lambda,\bar{\lambda}}\right)^\dagger\Gamma^*_g\,. \tag{30}$$

The terms on the right-hand side are Hermitian for each mode $\mathbf{k},\lambda$ separately, so Eq. (30) multiplied by Γ_g coincides with the condition (23). Thus, the gravitons' dynamics imposed by selecting the gauge (23) is equivalent to the one following from the gravitational constraint. In other words, requiring on phenomenological grounds that the quantum matter sector follows the Schrödinger dynamics implies that the gravitons' wave functional must satisfy (30).

V. DISCUSSION AND CONCLUSIONS

Our analysis was motivated by some misleading points (presented in Sec. II) of the WKB formulation developed in [22, 26]. These works investigated how to obtain the standard quantum dynamics of a "small" (or matter) subsystem from the full WDW equation of such degrees of freedom coupled to quasi-classical (or gravitational) ones, in the limit $\hbar\to 0$ (or $1/M\to 0$). The basic ambiguity of [22, 26] is related to the presence of a quantum correction δh_A to the classical background degrees of freedom $h^0_A(t)$ (here we considered a homogeneous diagonal Bianchi I cosmology). In the original analysis of [22], the existence of this quantum correction was implicitly assumed, as it is clear from the presence of a quantum amplitude $A(h_A)$ computed at first order in $\hbar$; a similar feature is found in the analysis of [26] where the expansion parameter is taken to be $1/M$.

In order to address the observations and consequential difficulties listed in points i)-iv) of Section II, we separated *ab initio* the Bianchi I classical background from its

[4] Note that, since we work under the assumption that gravitons are part of the "slow" component in the BO approximation, they do not give any contribution to the average over the fast sector.

first order quantum perturbations. Since our background is a vacuum geometry, we restricted our analysis to tensorial perturbations, described by graviton variables. We demonstrated that the functional Schrödinger equation for the matter sector is correctly recovered after averaging over quantum gravitational effects. To obtain this result, predicted by low energy phenomenology, we had to fix a gauge from (15) on the gravitons' sector, whose dynamics corresponds to the one dictated by the gravitational WDW equation only. The possibility to independently impose such constraint was one of the starting assumptions in [22], although not sufficiently motivated. Since the graviton dynamics cannot be regarded as a gauge-dependent feature, the present study justifies *a posteriori* and on physical grounds the assumption that the gravitational constraint simultaneously holds. In [22], however, such condition would no longer correspond to a gauge choice, simply because the gauge symmetry was broken from the very beginning.

Apart from the works [22, 26], other related analyses reconstruct a Schrödinger dynamics of a subsystem starting from a quantum gravity framework *à la* B-O. For instance, in [21] a Tomonaga-Schwinger equation for quantum matter was constructed; that approach is similar to the one discussed here, but the gravitational field was treated as purely classical. Our separation of the gravitational degrees of freedom into classical and quantum ones could be implemented in the same scheme, with the expected resulting picture being equivalent to our final outcome. In [33], the non-unitarity issue of the original study [27, 41] on quantum cosmological perturbations is addressed. The authors construct a suitable inner product, in the spirit of a gauge fixing approach to the definition of the time variable, and recover a Schrödinger dynamics for the scalar and tensor perturbations including quantum gravity corrections. Since we limited our attention to the first two expansion orders (where the non-unitarity issue does not arise), there is no direct overlap between this work and the achievements of [33]. However, it would be interesting to combine the ideas of the present paper, in particular the average on the quantum gravitational degrees of freedom, with the approach used in [33] to calculate higher order quantum gravity corrections. In this respect, it is worthwhile to clarify that the tensor fluctuations in [27, 33, 41] are treated on the same level as the matter degrees of freedom (i.e. as a fast contribution in the B-O scheme), whereas in our approach the gravitons are separated in energy scale from matter (i.e. they belong to the slow component).

The present study should be regarded as a starting point for future developments of the present approach, where the expansion is performed up to the next orders in the WKB parameter. Constructing the time variable as discussed above, it is natural to expect the non-unitarity problems analyzed in [26, 28, 34] to still arise at $\mathcal{O}\left(M^{-1}\right)$. However, the situation can be different for distinct choices of the time coordinates, see e.g. [21, 35, 56, 57]. In fact, our study clarifies how the B-O approximation for the low energy dynamics of quantum matter is recovered only after adequate separation of the gravitational degrees of freedom into a main classical background plus small quantum fluctuations.

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