Thermal false vacuum decay around black holes

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On thermal false vacuum decay around black holes

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In flat space and at finite temperature, there are two regimes of false vacuum decay in quantum field theory. At low temperature, the decay proceeds through thermally-assisted tunneling described by periodic Euclidean solutions — bounces — with non-trivial time dependence. On the other hand, at high temperature the bounces are time-independent and describe thermal jumps of the field over the potential barrier. We argue that only solutions of the second type are relevant for false vacuum decay catalyzed by a black hole in equilibrium with thermal bath. The argument applies to a wide class of spherical black holes, including d-dimensional AdS/dS-Schwarzschild black holes and Reissner–Nordström black holes sufficiently far from criticality. It does not rely on the thin-wall approximation and applies to multi-field scalar theories.

I. INTRODUCTION

False vacuum decay in field theory has been subject of extensive research, see [1] for a review. In flat spacetime at zero temperature, the transition from the false to true vacuum proceeds via tunneling through the potential barrier separating the vacua. In the semiclassical approximation, the tunneling is described by bounce—regular solution of the field equations that satisfies vacuum boundary conditions. The latter are conveniently formulated in the purely imaginary (Euclidean) time τ [2, 3].

At finite temperature T, the tunneling is described by thermal bounce which is periodic in τ with the period $1/T$ [4, 5]. The thermal bounces depend on τ below certain critical temperature, $T < T_c$, in which case they are also called 'periodic instantons'. At higher temperatures the bounces degenerate into τ-independent solution. Every constant-τ slice of this solution coincides with sphaleron (or critical bubble), which is the saddle-point configuration at the top of the barrier separating the false and true vacua. This picture admits a natural physical interpretation. While at $T < T_c$ the vacuum decay proceeds via quantum tunneling from an excited state, at $T > T_c$ thermal fluctuations of the field are strong enough to drive the system over the barrier classically. Unless T is very large, the decay rate is exponentially suppressed,

$$\Gamma_{\text{decay}} \sim \mathrm{e}^{-B_\lambda} \ , \tag{1}$$

and the suppression factor B_λ equals the Euclidean action of the theory evaluated on the bounce solution. At $T > T_c$ this is simply given by the sphaleron energy divided by T.

We illustrate this on fig. 1 where we plot B_λ in the 2-dimensional theory of the real scalar field φ with the potential

$$V = \frac{m^2\varphi^2}{2} - \frac{\mathrm{g}_4\varphi^4}{4} \ , \tag{2}$$

where $\mathrm{g}_4 > 0$.[1] In this theory the periodic instantons smoothly merge to the sphaleron branch at $T_c \sim m$. Though the τ-independent bounces are still solutions of the field equation at $T < T_c$, their Euclidean action is larger than that of the periodic instantons. One can understand that they are not relevant for vacuum decay by counting the number of their negative modes — linearly independent perturbations reducing their Euclidean action. A relevant bounce, being a saddle point of the action, must have exactly one such mode [2, 3]. On the other hand, the τ-independent bounces at $T < T_c$ have at least two. One of them corresponds to the change of the sphaleron size and exists also at $T > T_c$. Another one appears only at $T < T_c$ and corresponds to the deformation into periodic instanton.

In general, the transition between periodic instantons and τ-independent bounces can be discontinuous, as e.g. in models admitting the thin-wall approximation in spacetime with $d > 2$ dimensions [6]. Still, the presence

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[1] The model does not have a stable true vacuum since the potential is unbounded from below at $\varphi \to \infty$. This does not affect our discussion.
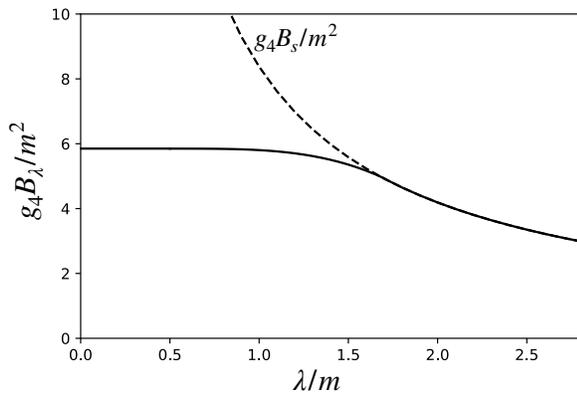
FIG. 1. Exponential suppression of false vacuum decay in 2-dimensional flat spacetime at finite temperature $T \equiv \lambda/(2\pi)$. The dashed line shows the suppression B_s due to sphalerons. We consider the scalar field theory with the potential (2).

of extra negative modes around τ-independent bounces at low temperature persists and provides a clear signal that these solutions are irrelevant for flat-space vacuum decay [1].

The goal of the present paper is to understand how the above picture of thermal vacuum decay is affected by the curved metric of a black hole (BH). It is known that a BH catalyzes the decay providing a nucleation site for bubbles of the true vacuum [7–11]. Various aspects of this effect and its possible relevance for the instability of the electroweak vacuum of the Standard Model have been discussed in [12–26]. Assume that a BH with Bekenstein–Hawking temperature T_{BH} is surrounded by thermal bath with the same temperature. This equilibrium state is known as the Hartle–Hawking vacuum [27], and its decay can be studied in the Euclidean time formalism.[2] For a spherically symmetric BH in d dimensions this leads to the theory on a Euclidean manifold $\mathcal{M}^2 \times \mathcal{S}^{d-2}$, where $\mathcal{S}^{d-2}$ is the $(d-2)$-dimensional sphere, and $\mathcal{M}^2$ has geometry of a cigar with compactified Euclidean time τ playing the role of the angular variable and the radial coordinate covering the region outside the horizon [27]. This picture corresponds to the Euclidean partition function in the Hartle–Hawking vacuum, and the tunneling solutions are saddles of this partition function.

Previous studies give rise to the following puzzle. Analysis of $O(3)$-symmetric configurations in 4d Schwarzschild background using the thin-wall approximation yields only τ-independent solutions of the sphaleron type, with no analogs of periodic instantons [9]. Similarly, still working within the thin-wall approximation but with the gravitational back-reaction taken

into account, Refs. [11, 14] found that the dominant contribution to the vacuum decay probability around a static BH is always provided by the sphaleron. The absence of periodic instantons is surprising, and one may wonder if it is peculiar to the thin-wall regime.

We provide evidence that this is not the case. Namely, we prove that for a general multi-scalar theory and a wide class of spherically symmetric BHs in d dimensions the τ-independent bounces centered on the BH have exactly one $O(d-1)$-symmetric negative mode at any T_{BH}. This makes existence of any $O(d-1)$-symmetric periodic instantons around the BH unlikely.[3] Throughout our analysis we neglect the back-reaction of the scalar field on the metric.

The paper is organized as follows. We first develop the intuition for the proof on an example of 2-dimensional dilaton BH in sec. II. The proof is then extended to general d-dimensional BHs in sec. III. We discuss our results in sec. IV. We work in the system of units $\hbar = c = G = 1$.

II. DILATON BLACK HOLE IN TWO DIMENSIONS

As discussed in [24], only the exterior region of a BH is relevant for the computation of the vacuum decay probability. In this region the metric of a 2-dimensional dilaton BH [29] can be written in the form,

$$ds^2 = \Omega(x)(-dt^2 + dx^2) ,\qquad(3)$$

where $-\infty < x < \infty$ and

$$\Omega = \frac{1}{1 + e^{-2\lambda x}} .\qquad(4)$$

The parameter λ is related to the BH temperature, $\lambda \equiv 2\pi T_{\mathrm{BH}}$. The horizon is located at $x \to -\infty$; in the opposite limit the metric is asymptotically flat. We analytically continue the metric to the Euclidean time by replacing $t \mapsto -i\tau$. The Euclidean time is compactified with the period $2\pi/\lambda$ to ensure the absence of conical singularity at $x = -\infty$.

Consider a scalar field φ with the potential $V(\varphi)$ embedded in this metric.[4] The Euclidean action for φ reads,

$$S = \int d\tau dx \left\{ \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial \tau}\right)^2 + \Omega V(\varphi) \right\} .\qquad(5)$$

We assume that $V(\varphi)$ has a local metastable minimum $V = 0$ at $\varphi = 0$. Then the equation of motion for φ has a time-independent solution $\varphi_s(x)$ — sphaleron — which interpolates between $\varphi = 0$ at $x \to +\infty$ and a
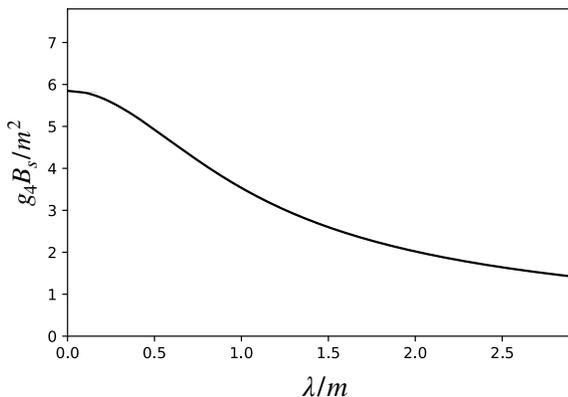

FIG. 2. Exponential suppression of sphalerons in the background of 2-dimensional dilaton black hole in equilibrium with thermal bath for the theory with the potential (2).

nonzero value at the horizon. It describes the saddle point of the energy barrier separating the false and true vacua. Note that the equation for φ contains the metric function $\Omega(x)$ and hence φ_s parametrically depends on the BH temperature λ.

The sphaleron gives rise to a τ-independent bounce. The first indication that this bounce dominates the vacuum decay at any λ comes from considering its Euclidean action B_s. For the model with the potential (2) it is plotted in fig. 2 which must be contrasted with fig. 1 in flat space. At high temperature the behaviour of the sphaleron suppression is qualitatively similar in both cases. However, at low temperature it is dramatically different. Instead of diverging at small λ as its flat-space counterpart, B_s in the case of BH smoothly increases towards a finite value B_0 equal to the suppression of the flat-space vacuum bounce. This is indeed the expected behaviour of the false vacuum decay rate at low temperature.

To see what happens in more detail, let us focus on the near-horizon region, $x < 0$ and $\lambda|x| \gg 1$. The conformal factor is then approximated as $\Omega \approx \mathrm{e}^{2\lambda x}$. We substitute this expression for Ω into (3) and make the following change of variables

$$\mathcal{T} = \frac{1}{\lambda}\mathrm{e}^{\lambda x}\sin\lambda\tau , \quad X = \frac{1}{\lambda}\mathrm{e}^{\lambda x}\cos\lambda\tau . \qquad (6)$$

This brings the line element to the form $\mathrm{d}s^2 = \mathrm{d}\mathcal{T}^2 + \mathrm{d}X^2$. Thus, close to the BH horizon, the spacetime geometry is approximately flat. The coordinates (τ, x) define the (Euclidean) Rindler frame in this flat space. In the Lorentzian signature, the Rindler frame corresponds to uniformly accelerating observers in the Minkowski spacetime. In particular, the line of constant x represents a trajectory of an observer moving with the acceleration $\lambda\mathrm{e}^{-\lambda x}$. The BH vicinity looks like the Rindler wedge— the portion of the Minkowski spacetime accessible to the observer, separated from the rest of the world by the horizon.

Consider the flat-space vacuum bounce $\varphi_0(R)$, where $R = \sqrt{\mathcal{T}^2 + X^2}$, centered around the origin of $(\mathcal{T}, X)$.[5] The transformation (6) converts it into a static solution $\varphi_0\left(\mathrm{e}^{\lambda x}/\lambda\right)$. Thus, the decay of the Minkowski vacuum in the Rindler frame is mediated by a τ-independent bounce or sphaleron. This agrees with the fact that the Minkowski vacuum corresponds to a thermal state from the viewpoint of an accelerating observer [24, 31]. Note that though in Lorentzian signature the Rindler coordinates cover only half of the flat spatial slices, the Euclidean action of the bounce computed in these coordinates coincides with the full flat-space integration in the Cartesian coordinates and equals B_0. Returning to the BH sphaleron, its physical size is much smaller than λ^{-1} in the limit $\lambda \ll m$. Hence it lies entirely in the near-horizon region and coincides with the Rindler sphaleron, $\varphi_s(x) = \varphi_0\left(\mathrm{e}^{\lambda x}/\lambda\right)$. This explains the smooth limit of its suppression $B_s \to B_0$ at $\lambda \to 0$.

We can infer one more lesson from considering the Rindler wedge. In Minkowski space centering bounce at the origin is not the only possibility. We can as well translate it to an arbitrary point. In particular, take the bounce centered at $\mathcal{T} = 0$, $X = X_0$ and denote it by φ_{X_0}. Switching to the Rindler frame converts it to a τ-dependent periodic instanton which nevertheless has exactly the same Euclidean action B_0 as the sphaleron. In other words, the Rindler space possesses a degenerate family of periodic Euclidean solutions—the 'Rindler valley'—parameterized by X_0, with the limit $X_0 = 0$ corresponding to the sphaleron. In the vicinity of the point $X_0 = 0$, the periodic instantons from the valley are obtained from the sphaleron by a dipole perturbation

$$\varphi_{X_0} \approx \varphi_0 + X_0\cos\theta\frac{\mathrm{d}\varphi_0}{\mathrm{d}R} , \qquad (7)$$

where $\theta = \mathrm{arctg}\,\mathcal{T}/X$. The deviation of the BH metric from Rindler tilts the valley. The direction of the tilt is then crucial. If it is towards $X_0 = 0$, the sphaleron is less suppressed than any of the would-be periodic instantons. And vice versa, a tilt away from $X_0 = 0$ would lead to periodic instantons being less suppressed that the sphaleron. We are going to see that the first option is realized, ruling out periodic instantons connected to sphaleron.

We presently prove that the τ-independent bounce has only a single negative mode. Introducing polar coordinates

$$\sigma = \lambda^{-1}\mathrm{arcsh}(\mathrm{e}^{\lambda x}) , \quad \theta = \lambda\tau , \qquad (8)$$

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[5] See [30] for the proof of $O(2)$-symmetry of the vacuum bounce in 2 dimensions.

the action (5) takes the form

$$S = \int_0^{2\pi} \mathrm{d}\theta \int_0^\infty \frac{\mathrm{d}\sigma \, \mathrm{th}\,\lambda\sigma}{\lambda} \left\{ \frac{1}{2} \left(\frac{\partial\varphi}{\partial\sigma} \right)^2 \right.$$
$$\left. + \frac{\lambda^2}{2\,\mathrm{th}^2\,\lambda\sigma} \left(\frac{\partial\varphi}{\partial\theta} \right)^2 + V(\varphi) \right\} . \qquad (9)$$

Note that the radial coordinate σ is chosen to measure the geodesic distance to the horizon which ensures that the term $\frac{1}{2}(\partial\varphi/\partial\sigma)^2$ appears in the action with the same coefficient as the potential term. The sphaleron has only radial dependence and satisfies the equation,

$$-\frac{\mathrm{d}^2\varphi_s}{\mathrm{d}\sigma^2} - \frac{2\lambda}{\mathrm{sh}\,2\lambda\sigma} \frac{\mathrm{d}\varphi_s}{\mathrm{d}\sigma} + V'(\varphi_s) = 0 , \qquad (10)$$

where prime means derivative of the potential with respect to the field. Due to the rotational invariance, the eigenmodes of perturbations around φ_s decompose into multipole sectors with the angular dependence $\mathrm{e}^{\pm in\theta}$. A radial eigenfunction $\chi_\mu(\sigma)$ corresponding to the eigenvalue μ in the n-th sector obeys a Schrödinger-type equation

$$H_{\mathrm{eff},n}\chi_\mu = \mu\,\chi_\mu \qquad (11)$$

with the effective Hamiltonian

$$H_{\mathrm{eff},n} = -\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} - \frac{2\lambda}{\mathrm{sh}\,2\lambda\sigma} \frac{\mathrm{d}}{\mathrm{d}\sigma} + U_{\mathrm{eff},n}(\sigma) \qquad (12)$$

and

$$U_{\mathrm{eff},n}(\sigma) = V''\big(\varphi_s(\sigma)\big) + \frac{n^2\lambda^2}{\mathrm{th}^2\,\lambda\sigma} . \qquad (13)$$

Note that the Hamiltonian (12) is Hermitian with respect to the positive measure $\int\mathrm{d}\sigma\,\lambda^{-1}\,\mathrm{th}\,\lambda\sigma$ and the potential (13) is positive at $\sigma \to \infty$. Hence, negative modes ($\mu < 0$) of $H_{\mathrm{eff},n}$, if any, form a discrete spectrum. We will take them to be normalized.

First, note that in the monopole sector ($n = 0$) the sphaleron has exactly one negative mode. Indeed, monopole perturbations correspond to static configurations whose Euclidean action is given simply by their energy times $2\pi/\lambda$. But the sphaleron is, by definition, the saddle-point of the static energy functional and hence the minimum of the energy in all but one direction.

Next we show that the spectrum in the dipole sector ($n = 1$) is strictly positive. This is the key part of the proof. We take the derivative of the sphaleron equation (10) with respect to σ and obtain,

$$-\frac{\mathrm{d}^2\dot\varphi_s}{\mathrm{d}\sigma^2} - \frac{2\lambda}{\mathrm{sh}\,2\lambda\sigma} \frac{\mathrm{d}\dot\varphi_s}{\mathrm{d}\sigma} + \tilde{U}_{\mathrm{eff}}(\sigma)\dot\varphi_s = 0 , \qquad (14)$$

where $\dot\varphi_s \equiv \mathrm{d}\varphi_s/\mathrm{d}\sigma$ and we have introduced a new effective potential

$$\tilde{U}_{\mathrm{eff}}(\sigma) = U_{\mathrm{eff},1}(\sigma) - \lambda^2\,\mathrm{th}^2\,\lambda\sigma . \qquad (15)$$

This implies that $\dot\varphi_s$ is a zero mode of the new Hamiltonian

$$\tilde{H}_{\mathrm{eff}} = -\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} - \frac{2\lambda}{\mathrm{sh}\,2\lambda\sigma} \frac{\mathrm{d}}{\mathrm{d}\sigma} + \tilde{U}_{\mathrm{eff}}(\sigma) . \qquad (16)$$

The sphaleron is a monotonic function of σ, as can be easily shown using the analogy between eq. (10) and the equation of motion of a unit-mass particle moving with the friction in the potential $-V(\varphi_s)$ [2]. This means that $\dot\varphi_s$ does not cross zero and, hence, it is the ground state of $\tilde{H}_{\mathrm{eff}}$. But $H_{\mathrm{eff},1}$ differs from $\tilde{H}_{\mathrm{eff}}$ by an addition of a positive potential $\lambda^2\,\mathrm{th}^2\,\lambda\sigma$, and therefore, its discrete eigenvalues are strictly larger than those of $\tilde{H}_{\mathrm{eff}}$.[6] Thus, the ground state energy of $H_{\mathrm{eff},1}$ is strictly positive.

Clearly, the last result also implies the positivity of the spectrum in higher multipoles since $U_{\mathrm{eff},n}(\sigma) > U_{\mathrm{eff},1}(\sigma)$ for $n > 1$. Thus, we conclude that the only negative mode of the τ-independent bounce resides in the monopole sector. This completes the proof.

The above proof is straightforwardly generalized to the theory of several scalar fields φ^i. Proceeding as before, one obtains two Hamiltonians: $H_{\mathrm{eff},1}$ for the perturbations in the dipole sector, and $\tilde{H}_{\mathrm{eff}}$ arising upon differentiation of the sphaleron equation. The difference between $H_{\mathrm{eff},1}$ and $\tilde{H}_{\mathrm{eff}}$ is diagonal in the space of fields and is manifestly positive. Hence, the eigenvalues of $H_{\mathrm{eff},1}$ are strictly larger than those of $\tilde{H}_{\mathrm{eff}}$. On the other hand, the latter has the zero mode $\dot\varphi_s^i$. The only thing that needs to be proven is that $\dot\varphi_s^i$ is the ground state of $\tilde{H}_{\mathrm{eff}}$. Consider the low-temperature limit in which, as we know, the sphaleron is obtained from the flat vacuum bounce $\varphi_0^i(\sigma)$ by the Rindler transformation. Since φ_0^i is a tunneling solution, it has a single negative mode in the monopole sector, and no negative modes in the dipole sector. Also $\tilde{H}_{\mathrm{eff}} = H_{\mathrm{eff},1}$ at $\lambda = 0$. Hence, $\dot\varphi_s^i$ is the ground state of $\tilde{H}_{\mathrm{eff}}$ at small temperature and, by continuity, it remains to be the ground state at all temperatures.

III. BLACK HOLES IN d DIMENSIONS

We are ready to address the case of a spherically symmetric BH in $d > 2$ dimensions. Consider the metric

$$\mathrm{d}s^2 = f(r)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Sigma_{d-2}^2 , \qquad (17)$$

where $\mathrm{d}\Sigma_{d-2}^2$ is the line element on the $(d-2)$-dimensional unit sphere. We do not specify $f(r)$ at this point, assuming only that it has horizon at $r = r_h$, where $f(r_h) = 0$, $f'(r_h) > 0$, and that it is positive at $r > r_h$. We analyze explicitly the theory of a single real scalar

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[6] To prove this, consider a continuous deformation of $\tilde{H}_{\mathrm{eff}}$ into $H_{\mathrm{eff},1}$ by monotonically increasing the potential and use $\delta\mu = \langle\chi_\mu|\delta H|\chi_\mu\rangle$.

field; the multi-field generalization is straightforward, as in the 2d case studied above.

A theory with a metastable vacuum has a sphaleron solution that inherits the $O(d-1)$ symmetry of the metric. We want to prove that, for a broad class of BHs, it has a single negative mode within the sector of $O(d-1)$-symmetric configurations. For such configurations the Euclidean action reads,

$$
S = \mathcal{A}_{d-2} \int \mathrm{d}\tau \mathrm{d}r\, r^{d-2} \left\{ \frac{f}{2} \left(\frac{\partial \varphi}{\partial r} \right)^2 + \frac{1}{2f} \left(\frac{\partial \varphi}{\partial \tau} \right)^2 + V(\varphi) \right\}
$$
$$
= \frac{2\mathcal{A}_{d-2}}{f'(r_h)} \int \mathrm{d}\theta \mathrm{d}\sigma\, r^{d-2} \sqrt{f} \left\{ \frac{1}{2} \left(\frac{\partial \varphi}{\partial \sigma} \right)^2 \right.
$$
$$
\left. + \frac{(f'(r_h))^2}{8f} \left(\frac{\partial \varphi}{\partial \theta} \right)^2 + V(\varphi) \right\},
$$
(18)

where $\mathcal{A}_{d-2} = 2\pi^{\frac{d-1}{2}}/\Gamma(\frac{d-1}{2})$ is the surface area of a unit $(d-2)$-dimensional sphere, and in the second line we introduced the geodesic coordinates

$$
\sigma = \int \frac{\mathrm{d}r}{\sqrt{f(r)}}, \qquad \theta = \frac{f'(r_h)}{2}\tau.
$$
(19)

Note that σ is set to be zero at the horizon, $\sigma(r_h) = 0$, and θ is an angular variable with period 2π ensuring the regularity of the Euclidean metric (17) at the horizon.

As in 2 dimensions, it is sufficient to focus on the dipole perturbations of the form $\chi_\mu(\sigma)\cos\theta$. The radial function satisfies the equation

$$
-\frac{\mathrm{d}^2\chi_\mu}{\mathrm{d}\sigma^2} - \frac{\mathrm{d}\ln(r^{d-2}\sqrt{f})}{\mathrm{d}\sigma} \frac{\mathrm{d}\chi_\mu}{\mathrm{d}\sigma} + U_{\mathrm{eff},1}(\sigma)\chi_\mu = \mu\,\chi_\mu \quad (20)
$$

with the effective potential

$$
U_{\mathrm{eff},1} = V''(\varphi_s) + \frac{(f'(r_h))^2}{4f}.
$$
(21)

On the other hand, by taking the σ-derivative of the sphaleron equation

$$
-\frac{\mathrm{d}^2\varphi_s}{\mathrm{d}\sigma^2} - \frac{\mathrm{d}\ln(r^{d-2}\sqrt{f})}{\mathrm{d}\sigma} \frac{\mathrm{d}\varphi_s}{\mathrm{d}\sigma} + V'(\varphi_s) = 0, \quad (22)
$$

we obtain

$$
-\frac{\mathrm{d}^2\dot\varphi_s}{\mathrm{d}\sigma^2} - \frac{\mathrm{d}\ln(r^{d-2}\sqrt{f})}{\mathrm{d}\sigma} \frac{\mathrm{d}\dot\varphi_s}{\mathrm{d}\sigma} + \tilde{U}_{\mathrm{eff}}(\sigma)\dot\varphi_s = 0. \quad (23)
$$

As before, we have defined $\dot\varphi_s \equiv \mathrm{d}\varphi_s/\mathrm{d}\sigma$ and

$$
\tilde{U}_{\mathrm{eff}} = V''(\varphi_s) - \frac{\mathrm{d}^2\ln(r^{d-2}\sqrt{f})}{\mathrm{d}\sigma^2}
$$
$$
= V''(\varphi_s) + \frac{(d-2)f}{r^2} - \frac{(d-2)f'}{2r} - \frac{f''}{2} + \frac{(f')^2}{4f},
$$
(24)

where f' stands for the derivative of f with respect to r. We observe that $\dot\varphi_s$ is the ground state[7] with zero

energy of the Hamiltonian $\tilde{H}_{\mathrm{eff}}$ corresponding to the potential (24). Hence, eq. (20) does not have any negative eigenmodes if $U_{\mathrm{eff},1} > \tilde{U}_{\mathrm{eff}}$, or equivalently if

$$
D(r) \equiv \frac{(f'(r_h))^2 - (f')^2}{4}
$$
$$
- \frac{(d-2)f^2}{r^2} + \frac{(d-2)ff'}{2r} + \frac{ff''}{2} > 0
$$
(25)

at $r > r_h$. This inequality provides a general sufficient condition for the absence of non-monopole negative modes on the τ-independent bounce. Note that it involves only the properties of the metric, but not of the scalar potential.

We now show that the condition (25) is satisfied for several common BH metrics. We start with the Schwarzschild metric, $f(r) = 1 - 2M/r^{d-3}$. Instead of computing $D(r)$ directly, it is easier to find its derivative,

$$
D'(r) = \frac{2(d-2)}{r^3} \left(1 - \frac{(d-1)M}{r^{d-3}} \right)^2.
$$
(26)

We see that the latter is positive. Since $D(r_h) = 0$, this implies positivity of $D(r)$ everywhere outside the horizon. The argument extends to the Schwarzschild-(anti) de Sitter metric,[8] $f(r) = 1 - 2M/r^{d-3} \pm r^2/l^2$. Remarkably, l^2 drops out of $D'(r)$ which is still given by (26).

Next consider the Reissner–Nordström BH with

$$
f(r) = 1 - \frac{2M}{r^{d-3}} + \frac{Q^2}{r^{2(d-3)}},
$$
(27)

where M and Q are BH mass and charge, respectively. Inequality (25) is satisfied outside the horizon as long as $|Q|$ is small enough. For example, in 4 dimensions a numerical analysis yields the bound $Q^2 < Q_*^2$, where $Q_*^2 \simeq 0.833\,M^2$.

On the other hand, a nearly critical BH provides a notable violation of the condition (25). Computing $D(r)$ for $Q = M$ we obtain,

$$
D(r) = -\frac{d-2}{r^2} \left(1 - \frac{M}{r^{d-3}} \right)^4 < 0.
$$
(28)

Recalling that $D(r)$ sets the difference between $U_{\mathrm{eff},1}$ and $\tilde{U}_{\mathrm{eff}}$, we conclude that in this case the τ-independent bounce *does* have a negative dipole mode. Hence, for a nearly critical BH, it is not a valid solution for the false vacuum decay and we expect existence of periodic instantons centered on the BH.

We focused above on $O(d-1)$-symmetric configurations. Our results do not exclude existence of extra negative modes which break this symmetry. In fact, presence of such negative modes is expected for large enough

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[7] $\dot\varphi_s$ does not have nodes as long as $\varphi_s(\sigma)$ is monotonic. A sufficient condition for this is the positivity of the "friction" term in eq. (22), $\mathrm{d}\ln(r^{d-2}\sqrt{f})/\mathrm{d}\sigma > 0$.

[8] The Schwarzschild-de Sitter metric has, apart from the BH horizon, a cosmological horizon. We consider here a thermal ensemble in equilibrium with the former, but not the latter.

BHs. Indeed, consider the equation for a mode which is τ-independent, but has non-trivial dependence on the angles Σ_{d-2},

$$-\frac{\mathrm{d}^2\chi_\mu}{\mathrm{d}\sigma^2} - \frac{\mathrm{d}\ln(r^{d-2}\sqrt{f})}{\mathrm{d}\sigma}\frac{\mathrm{d}\chi_\mu}{\mathrm{d}\sigma} + \left(V''(\varphi_s) + \frac{L^2}{r^2}\right)\chi_\mu = \mu\,\chi_\mu\,. \tag{29}$$

Here $-L^2$ is the eigenvalue of the Laplacian on a unit $(d-2)$-dimensional sphere. We know that for $L^2 = 0$ this equation has a negative eigenvalue $\mu_- < 0$. Then, since r is bounded from below by r_h, the equations with non-zero L^2 also have negative modes if $L^2/r_h^2 < |\mu_-|$. This instability has an intuitive explanation. For large r_h the symmetric sphaleron looks like a shell stretched on the BH horizon and locally has almost flat geometry. Not surprisingly, it is unstable with respect to inhomogeneous perturbations. Note that in this regime the sphaleron suppression is proportional to the horizon area $\propto r_h^{d-2}$ and exceeds the suppression of flat-space bounces. Since large BHs have low temperature $T_{\mathrm{BH}} \lesssim r_h^{-1}$, we conclude that low-temperature decays are dominated by bounces away from the BH. This is an important difference of the $d > 2$ case from the theory in 2 dimensions.

IV. DISCUSSION

We have shown for a large class of BHs that the τ-independent $O(d-1)$-symmetric bounce centered on them has a single negative mode in the sector of $O(d-1)$-symmetric perturbations. This negative mode is τ-independent. Though this does not strictly rule out a branch of $O(d-1)$-symmetric periodic instantons completely disconnected from the sphaleron, we believe this possibility to be unlikely. Asymmetric, but still τ-independent, negative modes do exist for large (and cold) BHs. In this case, however, the bounce centered on the BH is more suppressed than the bounce in the asymptotically flat region and is irrelevant for vacuum decay.

Our result has two implications. On the technical side, it simplifies the analysis of BH catalysis of false vacuum decay in thermal bath by sparing the task of looking for periodic instantons. The latter requires solving nonlinear elliptic partial differential equations, or a system thereof for multi-field theories. By contrast, the static sphaleron depends only on the radial coordinate and is found as a solution of ordinary differential equations. This simplification can be useful in various contexts where one deals with thermal ensembles containing BHs, such as e.g. cosmology [18–20] or holography [32–35].

From the physical perspective, the dominance of τ-independent bounce suggests to interpret false vacuum decay around the BH as driven by over-barrier transitions at any T_{BH}. This appears consistent with the fact that, from the viewpoint of the asymptotic observer, the temperature in the vicinity of the BH is blue-shifted and exceeds T_c close enough to the horizon leading to enhanced thermal fluctuations. However, the clear distinction between quantum tunneling and over-barrier transitions is absent in curved spacetime. This is known from the study of Coleman–De Luccia [36] and Hawking–Moss instantons [37] in de Sitter which can be interpreted either as vacuum bounces in inflationary coordinates [38], or as thermal sphaleron transitions in the static patch [39]. Another example is provided by the relation between Minkowski and Rindler bounces [24, 31, 40] mentioned in sec. II. Similarly, the static bounce in Schwarzschild metric becomes time-dependent in a different reference frame, such as e.g. Kruskal coordinates, and in this frame corresponds to a vacuum bounce. Thus, the BH spacetime gives one more example that the question whether the vacuum decay is due to quantum or thermal fluctuations has no covariant meaning.

A notable exception to our proof is provided by nearly critical Reissner–Nordström BH, in which case we expect existence of periodic instantons centered on the BH. An intuitive explanation why such BHs are special comes from considering an exactly critical BH. This has zero temperature and the state around it appears as vacuum even for a static observer. The true bounce in this case must be localized in the Euclidean time, as in Minkowski space, whereas the τ-independent bounce has infinite suppression.

In this paper we focused on thermal ensemble and did not consider non-equilibrium situations, such as a BH emitting Hawking radiation into empty space (Unruh vacuum). It will be interesting to see if the approach of this work can be used to get insight into the structure of complex bounces describing false vacuum decay in this case [24, 25]. Finally, it would be interesting to extend our analysis by including gravitational back-reaction of the bounce on the metric.

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