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Phys. Rev. D **106**, 123524 — Published 30 December 2022

DOI: [10.1103/PhysRevD.106.123524](https://doi.org/10.1103/PhysRevD.106.123524)

Tilted non-spatially-flat inflation

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(Dated: December 2, 2022)

We construct non-linear inflaton potential energy densities that describe not-necessarily very-slowly-rolling closed and open inflation models, and compute tilted primordial spatial inhomogeneity power spectra that follow from quantum mechanical fluctuations during inflation in these models. **Earlier non-flat inflation model power spectra computations assumed an inflaton potential energy density with a linear slope that resulted in very-slow-roll during inflation and untilted power spectra.** These **new** tilted power spectra differ from those that have previously been used to study cosmological data in non-flat cosmological models.

I. INTRODUCTION

If general relativity provides an adequate description of gravity on cosmological scales — and there is no strong evidence indicating otherwise — dark energy is the dominant contributor to the current cosmological energy budget and powers the observed late-time accelerating cosmological expansion. Earlier on, prior to a redshift $z \sim 0.75$, non-relativistic (cold dark and baryonic) matter was the dominant contributor to the energy budget and was responsible for the observed earlier-time decelerating cosmological expansion. The simplest model consistent with these observations is the flat Λ CDM model [1], the current “standard” cosmological model. In this model spatial hypersurfaces are chosen to be flat and the dark energy is the cosmological constant Λ , with the next biggest contributor to the current cosmological energy budget being cold dark matter (CDM). For reviews see Refs. [2].

The flat Λ CDM model is consistent with a variety of observational constraints, including cosmic microwave background (CMB) anisotropy observations [3], baryon acoustic oscillation (BAO) data [4], Hubble parameter ($H(z)$) measurements [5], and Type Ia supernova (SNIa) apparent magnitude observations [6]. The standard model is also consistent with more recent constraints from probes of the intermediate redshift Universe, that include data between $z \sim 2.3$ of the highest redshift BAO observations and $z \sim 1100$ of the CMB data. However, the intermediate redshift data constraints are not yet as restrictive as the lower-redshift BAO, $H(z)$, and SNIa ones, nor as restrictive as the higher-redshift CMB anisotropy ones. These intermediate-redshift constraints include those from HII galaxy apparent magnitude versus redshift data [7], angular size as a function of redshift measurements [8], quasar X-ray and UV flux observations [9], and gamma-ray burst data [10].

While most current measurements are not inconsistent with the spatially-flat Λ CDM standard model, they also do not rule out mildly curved spatial hypersurfaces or,

weakly varying in time and space, dynamical dark energy. Near-future measurements are anticipated to provide significantly more restrictive constraints that should help distinguish between the options and better determine cosmological parameter values [11].

There are however some suggestions of inconsistencies between observations and the standard flat Λ CDM model. For example, differences between (model-dependent) measurements of the Hubble constant H_0 could be an indication of a problem with the standard model.¹ An early median statistics estimate, [12], $H_0 = 68 \pm 2.8 \text{ km s}^{-1} \text{ Mpc}^{-1}$, is consistent with a number of more recent H_0 measurements made using a variety of methods [13], including from CMB anisotropy data, $H_0 = 67.36 \pm 0.54 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [3]. On the other hand, some local measurements of the expansion rate favor a value significantly larger than the CMB one, $H_0 = 73.2 \pm 1.3 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [14].² Similar issues affect measurements of other parameters but the difference between H_0 measurements is the most significant. **For reviews of these issues see Refs. [2].**

Given such potential inconsistencies, and given the improving quality and amount of data, there is significant interest in studying cosmologies that have a free parameter or two more than the flat Λ CDM model. A widely considered option is dynamical dark energy that mildly varies in time and space. Scalar field dynamical dark energy (ϕ CDM) models are a popular example [16]. Allowing for non-zero spatial curvature is another option now under study. For recent discussions of observational constraints on spatial curvature and dark energy dynamics from a variety of different data sets, see Refs. [17, 18].³ For recent discussions of non-flat cosmological models, see Refs. [20].

CMB anisotropy data provide the most restrictive con-

¹ For recent reviews see Ref. [2].

² We note that some other local expansion rate determinations of H_0 are slightly lower and have larger error bars [15].

³ Compared to the cosmological constant, dynamical dark energy density evolves more similarly to spatial curvature energy density and this results in weaker constraints on both new parameters when compared to the case when either only non-zero spatial curvature or only dark energy density dynamics is assumed [19].

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straints on cosmological models. To use these data to constrain cosmological parameters of a model requires knowing the primordial power spectrum of spatial inhomogeneities as a function of wavenumber for the model. In the inflation scenario [21, 22] quantum mechanical zero-point fluctuations in the inflaton field during inflation generate the spatial inhomogeneities [23, 24]. If inflation lasts for a long time spatial curvature is redshifted away to insignificance (and this is by far the most commonly considered case). In this case, if the inflaton slowly rolls down a relatively-flatter inflaton potential energy density the cosmological scale factor grows exponentially in time (this is spatially-flat de Sitter inflation) and the resulting primordial power spectrum is close to scale invariant, [25], with very little tilt. It is possible to increase the power spectral tilt by choosing an inflaton potential energy density that causes the inflaton to evolve more rapidly during inflation and makes the scale factor grow only as a power of time (this is spatially-flat power-law inflation) [26–28].

Gott [29] generalized inflation to the open cosmological model. In this open-bubble inflation model a spatially-open bubble nucleates and the interior inflates for only a limited time so as to not redshift away all spatial curvature. If necessary, an earlier, pre open-bubble nucleation, epoch of less-limited spatially-flat inflation can be used to produce spatial homogeneity. Alternately, a slow enough open-bubble nucleation process might ensure that the interior of the open bubble is sufficiently spatially homogeneous.

Hawking’s prescription for the initial quantum state of the universe [30] — that the functional integral include only those field configurations which are regular on the Euclidean section — suggests that the universe nucleated as a closed de Sitter Lanczos (inflation) model on the Lorentzian section [30, 31]. The equator of the Euclidean (de Sitter Lanczos) four sphere is identified with the waist of the Lorentzian de Sitter Lanczos hyperboloid, which is where the nucleation occurs [30, 31].⁴ A slow enough nucleation process might ensure a sufficiently spatially homogeneous closed inflating de Sitter Lanczos model. Again, inflation can occur for only a limited amount of time so as to not redshift away all spatial curvature.⁵

To compute the power spectrum of spatial inhomogeneities generated by quantum fluctuations during inflation in a given model requires the solution of the spatial inhomogeneities linear perturbation equations and a set of initial conditions. Spatially flat, open, and closed de Sitter spacetimes have as large a symmetry group as Minkowski spacetime; spatially-flat power-law inflation and the non-flat inflation models we study here have less symmetry than the flat, open, and closed de Sitter inflation models. The initial condition prescription we uti-

lize here give inflaton scalar field two-point functions in flat and closed de Sitter inflation models that have the symmetries of those spacetimes, [24, 31, 37]. This initial condition prescription has also been used to compute the primordial power spectrum in the spatially-flat power-law inflation model [27, 28], giving a tilted primordial power spectrum that is in agreement with the result of Ref. [26]. All these inflation model spacetimes, as well as the two we consider in this paper, are conformally flat, i.e., when expressed in terms of conformal time their line-elements are proportional to the Minkowski spacetime line-element.

The initial condition prescription we use here is that during inflation, for large wavenumbers, inside the horizon, at early time, the conformally rescaled inflaton scalar field modes, as a function of conformal time, should be quantum-mechanically normalized simple harmonic oscillators.

In the open-bubble inflation model this initial condition prescription [38] results in a late-time energy density inhomogeneity power spectrum [39, 41] that is the generalization to the open inflation case [40]⁶ of the scale-invariant spectrum of the flat model [25]. In a variant of the open-bubble inflation model, that includes an initial epoch of spatially-flat de Sitter inflation, applying the initial condition prescription in the first epoch and following the computation through the bubble nucleation process, results in a primordial power spectrum that is observationally indistinguishable from that of the case when the initial conditions are applied inside the second, open-bubble epoch, [42, 43].⁷

This initial condition prescription has also been used in a computation of the primordial power spectrum in the closed de Sitter inflation model [31, 37]. In this case this initial condition prescription is equivalent to Hawking’s [30] prescription of only including field configurations regular on the Euclidean section [31]. It also leads to a de Sitter invariant ground state inflaton scalar field two-point correlation function [31]. The wavenumber dependence of the resulting primordial power spectrum is the generalization of the scale-invariant spectrum in the spatially-flat case [25] to the closed universe [45].⁸

The open and closed inflation model primordial power spectra computations of Refs. [37, 39] were done in models that had an inflaton (Φ) potential energy density $\propto (1 - \epsilon\Phi)$, where ϵ is a small constant. These are very-slow-roll models and so the resulting power spectra are

⁴ For variants of this picture see Refs. [32–35].

⁵ For a discussions of the nucleation of open and closed cosmological models, see Refs. [36] and references therein.

⁶ Lyth and Stewart [40] also use this initial condition prescription in their more approximate computation of the wavenumber dependence of the power spectrum

⁷ For other early papers on open inflation, see Refs. [44].

⁸ There are other computations of primordial spectra in the closed de Sitter inflation model [34, 46–50], using different initial conditions compared to our prescription. We emphasize that the initial conditions we use in the closed de Sitter case results in a scalar field perturbation two-point function that is de Sitter invariant [31].

untilted on small scales for infinitesimal ϵ . When these power spectra were used in a non-flat Λ CDM model analysis of the Planck 2015 CMB anisotropy data [51] it was found that these data favored closed spatial geometry [52, 53], even in combination with BAO, $H(z)$, SNIa, and other non-CMB data, where these data jointly favored about a 1% spatial curvature energy density contribution to the cosmological energy budget at 5σ significance [54].

A more correct analysis of the CMB anisotropy data requires tilted open and closed model primordial power spectra. Such spectra would be generated by quantum mechanical inflaton fluctuations in open and closed inflation models with non-linear inflaton potential energy densities, unlike the linear potential energy density function assumed in the analyses of Refs. [37, 39]. Pending such a computation, data analyses have been performed utilizing the primordial power spectra of Refs. [37, 39] multiplied by k^{n-1} , where k is the wavenumber and n is the power spectral index (with $n = 1$ being the scale-invariant case in the flat model) [55]. Using this phenomenological primordial power spectrum to define a tilted non-flat Λ CDM model for the analysis of CMB anisotropy data, in this model the Planck 2018 data [3] favors positive spatial curvature contributing about 1% to the cosmological energy budget at 1.6σ , but when BAO data are added to the mix the result is consistent with flat spatial hypersurfaces [3]; a similar result was originally found from the Planck 2015 data [51]. It is of interest to determine whether the power spectra [55] used in these analyses [3, 51] can be generated by inflaton quantum fluctuations in non-flat non-very-slow-roll inflation models that are closed but very close to flat, deviating from flatness at only the $\sim 1\%$ level. A recent numerical study in closed inflation models that computes power spectra generated for a few different initial conditions finds that it is possible to generate spectra of the form assumed in Refs. [3, 51], [56], at least in the closed case.

Here we consider open and closed inflation models with non-linear inflaton potential energy densities. In this paper we generalize the exponential potential energy density of the flat-space power-law inflation model [26–28] to inflaton potential energy densities that allow for not-necessarily very-slow-roll inflation in open and closed models. In the very-slow-roll limit these potential energy densities reduce to that $\propto (1 - \epsilon\Phi)$ used in the open and closed inflation models of Refs. [37, 39], while at large Φ they become the exponential potential energy density used in the spatially-flat power-law inflation model of Refs. [26–28]. Here we are interested in a non-very-slow-roll limit of these models, and potentially in the parameter-space range where they deviate from spatial flatness at the $\sim 1\%$ level.

Our computed primordial power spectra of spatial inhomogeneities — that result from quantum mechanical zero-point fluctuations during the inflation epoch in these tilted closed and open models — differ from power spec-

tra that have previously been used to analyze observational data in closed and open cosmological models. These power spectra have been used in the analyses of CMB anisotropy and other data, [57]. It is interesting that there appears to be some additional ambiguity in the form of non-flat inflation model power spectra, caused by the ambiguity in the form of the assumed non-flat inflation initial conditions, compared to what happens in the flat inflation case.

In Section II we summarize the background geometry of the closed and open models and the Einstein-scalar-field model equations of motion. For more detailed descriptions see Refs. [37, 39]. In Section III we determine the inflaton potential energy densities we use and solve the spatially homogeneous background equations of motion in the inflation epoch of the closed and open models. We solve the linear perturbation equations in Section IV, where we compute the late-time primordial power spectra during inflation in the closed and open models. We conclude in Section V. Appendix A list some results in the spatially-flat tilted inflation model, that we use for comparison to some of our results on smaller scales during inflation in the non-flat models when spatial curvature is unimportant. Appendices B and C describe primordial power spectra definitions and conventions in the flat, closed, and open models.

II. TECHNICAL PRELIMINARIES

A. Spatially homogeneous background geometries

The positive spatial curvature (closed) FLRW model has line element

$$\begin{aligned} ds^2 &= dt^2 - a^2(t) H_{ij}(\vec{x}) dx^i dx^j \\ &= dt^2 - a^2(t) [d\chi^2 + \sin^2(\chi) \{d\theta^2 + \sin^2(\theta) d\phi^2\}], \end{aligned} \quad (1)$$

where $a(t)$ is the cosmological scale factor, $H_{ij}(\vec{x})$ is the metric on the closed spatial hypersurfaces, the ‘radial’ coordinate $0 \leq \chi < \pi$, and θ, ϕ are the usual angular coordinates on the two-sphere. The square of the distance between two points, (t, χ, θ, ϕ) and $(t, \chi', \theta', \phi')$, is

$$\sigma^2 = 2a^2(t) [-1 + \cos(\gamma_3)], \quad (2)$$

$$\cos(\gamma_3) = \cos(\chi)\cos(\chi') + \sin(\chi)\sin(\chi')\cos(\gamma_2), \quad (3)$$

where γ_2 is the usual angle between the two points (θ, ϕ) and (θ', ϕ') on the two-sphere,

$$\cos(\gamma_2) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi'). \quad (4)$$

The negative spatial curvature (open) FLRW model has line element

$$\begin{aligned} ds^2 &= dt^2 - a^2(t) H_{ij}(\vec{x}) dx^i dx^j \\ &= dt^2 - a^2(t) [d\chi^2 + \sinh^2(\chi) \{d\theta^2 + \sin^2(\theta) d\phi^2\}], \end{aligned} \quad (5)$$

where $H_{ij}(\vec{x})$ is now the metric on the open spatial hypersurfaces, and χ ($0 \leq \chi < \infty$), θ , and ϕ are defined above. The square of the distance between two points, (t, χ, θ, ϕ) and $(t, \chi', \theta', \phi')$, is

$$\sigma^2 = 2a^2(t) [1 - \cosh(\gamma_3)], \quad (6)$$

$$\cosh(\gamma_3) = \cosh(\chi)\cosh(\chi') - \sinh(\chi)\sinh(\chi')\cos(\gamma_2), \quad (7)$$

and γ_2 is defined in Eq. (4).

B. Einstein-scalar-field model conventions

The Einstein-scalar-field action, for metric tensor $g_{\mu\nu}$ and inflaton Φ , is

$$S = \frac{m_p^2}{16\pi} \int dt d^3x \sqrt{-g} \left[-R + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} V(\Phi) \right]. \quad (8)$$

Here $m_p = G^{-1/2}$ is the Planck mass and V is the scalar field potential energy density. Varying, we find the inflation and gravitation equations of motion,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) + \frac{1}{2} V'(\Phi) = 0, \quad (9)$$

$$R_{\mu\nu} = \frac{8\pi}{m_p^2} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \quad (10)$$

where a prime denotes a derivative with respect to Φ and T is the trace of the stress-energy tensor

$$T_{\mu\nu} = \frac{m_p^2}{16\pi} \left[\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \{ g^{\lambda\rho} \partial_\lambda \Phi \partial_\rho \Phi - V(\Phi) \} \right]. \quad (11)$$

To derive the equations of motion for the spatially homogeneous background fields and for the spatial inhomogeneities, we linearize Eqs. (9) – (11) about an open or closed FLRW model and a spatially homogeneous scalar field. We work in synchronous gauge, with line element

$$ds^2 = dt^2 - a^2(t) [H_{ij}(\vec{x}) - h_{ij}(t, \vec{x})] dx^i dx^j, \quad (12)$$

where the background metric on the closed [open] spatial hypersurfaces, H_{ij} , is given in Eq. (1) [Eq. (5)], and the metric perturbations are denoted by h_{ij} . The expansion for the scalar field is

$$\Phi(t, \vec{x}) = \Phi_b(t) + \phi(t, \vec{x}), \quad (13)$$

where Φ_b and ϕ are the spatially homogeneous and inhomogeneous parts of the inflaton field (the inflaton perturbation ϕ should not be confused with the angular variable ϕ of Sec. II A).

III. TILTED CLOSED AND OPEN INFLATION MODELS AND SPATIALLY HOMOGENEOUS BACKGROUND SOLUTIONS

The Einstein-scalar-field model equations of motion for the spatially homogeneous background fields, derived in Sec. III.A of Ref. [39] for the open case and in Sec. III.A of Ref. [37] for the closed case, are

$$\ddot{\Phi}_b + 3\frac{\dot{a}}{a}\dot{\Phi}_b + \frac{1}{2}V'(\Phi_b) = 0, \quad (14)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{12} \left[\dot{\Phi}_b^2 + V(\Phi_b) \right] + \frac{\kappa^2}{a^2}, \quad (15)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\dot{\Phi}_b^2 + \frac{1}{12}V(\Phi_b), \quad (16)$$

where an overdot denotes a derivative with respect to time and $\kappa^2 = +1(-1)$ for open (closed) spatial hypersurfaces.

Motivated by the power-law expansion inflation model in the spatially-flat case, [26–28], for suitable scalar field potential energy densities, discussed below, the background Friedmann equation (15) during closed or open inflation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{Q}{a^q} + \frac{\kappa^2}{a^2}, \quad (17)$$

where Q and q are constants and $0 < q < 2$ for inflation. In the closed case where $\kappa^2 = -1$, we require, at the waist at $t = t_i$, the initial condition $\dot{a}(t_i) = 0$, so the right hand side of Eq. (17) must also vanish at the waist, which results in $Qa_i^p = 1$ where $a_i = a(t_i)$ and $p = 2 - q$. In the closed case the inflation model includes only the $t \geq t_i$ part of the spacetime.

The integral of Eq. (17) is

$$\sqrt{\kappa^2}(t - t_0) = a {}_2F_1(1/2, 1/p; 1 + 1/p; -Qa^p/\kappa^2) - a_0 {}_2F_1(1/2, 1/p; 1 + 1/p; -Qa_0^p/\kappa^2), \quad (18)$$

where ${}_2F_1$ is the Gauss hypergeometric function, see Ch. 15 of Ref. [58], and $a_0 = a(t = t_0)$ is the constant of integration.

In the open case, this is

$$t = a {}_2F_1(1/2, 1/p; 1 + 1/p; -Qa^p) + \text{constant}, \quad (19)$$

and in the closed case, where t_i is at the waist,

$$t - t_i = ia {}_2F_1(1/2, 1/p; 1 + 1/p; Qa^p) - iQ^{-1/p} \sqrt{\pi} \frac{\Gamma(1 + 1/p)}{\Gamma(1/2 + 1/p)}, \quad (20)$$

where Γ is the Euler Gamma function.

In the flat limit where $\kappa^2 \rightarrow 0$, or $Qa^p/\kappa^2 \rightarrow \infty$, Eq. (18) becomes $a \propto (t - t_0)^{2/q}$, the usual flat-space tilted power-law inflation result, see Eq. (2.5) of Ref. [27]. In the $q \rightarrow 0$ limit Eqs. (19) and (20) reduce

to the correct open and closed slow-roll untilted de Sitter inflation relations, $a(t) = \sinh(\sqrt{Q}(t - t_0))/\sqrt{Q}$ and $a(t) = \cosh(\sqrt{Q}(t - t_0))/\sqrt{Q}$, see Eq. (4.10) of Ref. [39] and Eq. (105) of Ref. [37].

In the open case, conformal time

$$\tilde{t} = \frac{1}{p\sqrt{\kappa^2}} \ln \left[\frac{\sqrt{Qa^p/\kappa^2 + 1} - 1}{\sqrt{Qa^p/\kappa^2 + 1} + 1} \right], \quad (21)$$

where $-\infty \leq \tilde{t} \leq 0$, while in the closed case, conformal time

$$\tilde{t} = \frac{2}{p\sqrt{-\kappa^2}} \tan^{-1} \left[\sqrt{Qa^p/(-\kappa^2) - 1} \right], \quad (22)$$

where $0 \leq \tilde{t} \leq \pi/p$.

With the scalar field potential energy density in the open case,

$$V(\Phi) = \frac{2(6-q)Q}{(\kappa^2/Q)^{q/p} [\sinh \{p\Phi/\sqrt{8q}\}]^{2q/p}} \quad (23)$$

and in the closed case,

$$V(\Phi) = \frac{2(6-q)Q}{(-\kappa^2/Q)^{q/p} [\cosh \{p\Phi/\sqrt{8q}\}]^{2q/p}}, \quad (24)$$

the homogeneous part of the scalar field equation of motion (14) and the Friedmann equation (15)⁹ are satisfied by, in the open case,

$$a = (\kappa^2/Q)^{1/p} \left[\sinh \left\{ p\Phi_b/\sqrt{8q} \right\} \right]^{2/p}, \quad (25)$$

and in the closed case,

$$a = (-\kappa^2/Q)^{1/p} \left[\cosh \left\{ p\Phi_b/\sqrt{8q} \right\} \right]^{2/p}. \quad (26)$$

In the closed case we have used the initial condition $\Phi_b(t_i) = 0$.

In the untilted, slow-roll, small q limit, both scalar field potential energy densities, Eqs. (23) and (24), become $\propto (1 - \epsilon\Phi)$, where $\epsilon = \sqrt{q/2}$, which are the potential energy densities used in untilted very-slow-roll open [39] and closed [37] inflation model computations. However, the $a(\Phi_b)$ equations (25) and (26) appear to not behave sensibly at $q = 0$ and so it appears that these models do not make sense at $q = 0$. For small q the potential energy densities, Eqs. (23) and (24), change only slowly with Φ and at $q = 0$ they are flat. At $q = 0$ the scalar field will not move if is initially at rest, but $a(t)$ grows, resulting in a breakdown of Eqs. (25) and (26) at $q = 0$.

In the limit that Φ is large, both scalar field potential energy densities, Eqs. (23) and (24), become $\propto \exp(-\sqrt{q/2}\Phi)$, which is the potential energy density used in the standard flat-space tilted inflation model [26–28].

IV. LINEAR SCALAR PERTURBATIONS DURING INFLATION

A. Synchronous gauge linear scalar perturbation equations

The scalar parts of the synchronous gauge inflation-epoch linear perturbation equations in spatial momentum space are derived in Secs. II and III.A of Refs. [37, 39] for the open and closed cases.

With $-A(A+2)$, integer $A = 0, 1, 2 \dots$ ($A \geq 2$ modes are physical), being the closed model spatial Laplacian eigenvalue, and $-(A^2+1)$, $A > 0$, being the open model spatial Laplacian eigenvalue, we define $k^2 = A^2 + 1$ and $\bar{k}^2 = A^2 + 4$ for the open case and $k^2 = A(A+2)$ and $\bar{k}^2 = (A-1)(A+3)$ for the closed case.

The linear scalar perturbation equations for the spatial momentum space scalar field $\phi(A, t)$, trace of the metric perturbation $h(A, t)$ (the perturbation to the size of the proper volume element), and the trace-free part of the metric perturbation $\mathcal{H}(A, t)$ (the shearing perturbation of the volume element) modes are

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{k^2}{a^2}\phi + \frac{1}{2}V''(\Phi_b)\phi = \frac{1}{2}\dot{h}\dot{\Phi}_b, \quad (27)$$

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} = 2\dot{\Phi}_b\dot{\phi} - \frac{1}{2}V'(\Phi_b)\phi, \quad (28)$$

$$\dot{\mathcal{H}} = \frac{k^2}{k^2} \left[\frac{3}{2}\dot{\Phi}_b\phi - \dot{h} \right], \quad (29)$$

$$\begin{aligned} \ddot{h} + 6\frac{\dot{a}}{a}\dot{h} + \frac{(\kappa^2 + \bar{k}^2)}{a^2}h + \ddot{\mathcal{H}} + 3\frac{\dot{a}}{a}\dot{\mathcal{H}} \\ + \frac{(\kappa^2 + \bar{k}^2)}{a^2}\mathcal{H} + \frac{3}{2}V'(\Phi_b)\phi = 0, \end{aligned} \quad (30)$$

$$\ddot{\mathcal{H}} + 3\frac{\dot{a}}{a}\dot{\mathcal{H}} - \frac{k^2}{3a^2}\mathcal{H} - \frac{k^2}{3a^2}h = 0. \quad (31)$$

B. Synchronous gauge linear scalar perturbation solutions

Using Eqs. (17) and (23)–(26), Eqs. (27) and (28) can be re-expressed as

$$\begin{aligned} [Qa^p + \kappa^2] \frac{d^2\phi}{da^2} + \frac{1}{2a} [(8-q)Qa^p + 6\kappa^2] \frac{d\phi}{da} + \frac{k^2}{a^2}\phi + \\ \frac{(6-q)}{4a^2} [2qQa^p + (2+q)\kappa^2]\phi = \frac{\sqrt{2qQ}}{2a^{q/2}} [Qa^p + \kappa^2]^{1/2} \frac{dh}{da}, \end{aligned} \quad (32)$$

and

$$\begin{aligned} [Qa^p + \kappa^2] \frac{d^2h}{da^2} + \frac{1}{2a} [(6-q)Qa^p + 4\kappa^2] \frac{dh}{da} = \\ \frac{2\sqrt{2qQ}}{a^{q/2}} [Qa^p + \kappa^2]^{1/2} \frac{d\phi}{da} + \frac{(6-q)\sqrt{2qQ}}{2a^{(2+q)/2}} [Qa^p + \kappa^2]^{1/2} \phi. \end{aligned} \quad (33)$$

⁹ More precisely, Eqs. (25) and (26) reduce the Friedmann equation (15) to Eq. (17) which is solved by Eqs. (19) and (20).

These can be combined to give a third order equation for ϕ

$$\begin{aligned} & [Qa^p + \kappa^2] \frac{d^3\phi}{da^3} + \frac{1}{2a} [2(8-q)Qa^p + (10+q)\kappa^2] \frac{d^2\phi}{da^2} \\ & + \frac{1}{4a^2} [(48-10q-q^2)Qa^p + (12-q)(2+q)\kappa^2 + 4k^2] \frac{d\phi}{da} \\ & + \frac{q}{8a^3} [2p(6-q)Qa^p + (6-q)(2+q)\kappa^2 + 4k^2] \phi \\ & = 0. \end{aligned} \quad (34)$$

Changing variables from ϕ and a to f and x where $\phi = f/a^{q/2}$ and $x = Qa^p/\kappa^2$, this equation becomes

$$\begin{aligned} & x^2(x+1) \frac{d^2Z}{dx^2} + \left[\frac{11}{2}x^2 + 4x \right] \frac{dZ}{dx} + \\ & \left[\frac{(40-41q+10q^2)}{2p^2}x + \frac{(11-8q+2q^2+k^2/\kappa^2)}{p^2} \right] Z = 0, \end{aligned} \quad (35)$$

where $Z(x) = df/dx$. The general solution $Z(x)$ of Eq. (35) can be expressed in terms of Gauss hypergeometric functions and this can be integrated once with respect to x to get $f(x)$ and so $\phi(a)$.

Defining A_{\pm} , B , D , and G_{\pm} ,

$$4pA_{\pm} = 3p - 2W \pm (2+q), \quad (36)$$

$$pB = p - W, \quad (37)$$

$$pD = p + W, \quad (38)$$

$$4pG_{\pm} = 3p + 2W \pm (2+q), \quad (39)$$

where

$$W = \sqrt{-8-4q+q^2-4k^2/\kappa^2}, \quad (40)$$

the scalar field solution of the linear perturbation equations is

$$\begin{aligned} a^{q/2}\phi &= \tilde{c} \\ &+ \tilde{c}_+ x^{(B-2)/2} {}_3F_2(A_+, A_-, B/2-1; B, B/2; -x) \\ &+ \tilde{c}_- x^{(D-2)/2} {}_3F_2(G_+, G_-, D/2-1; D, D/2; -x). \end{aligned} \quad (41)$$

Here \tilde{c} and \tilde{c}_{\pm} are constants of integration, the \tilde{c} solution is a gauge solution corresponding to the remnants of time translation invariance in synchronous gauge, and ${}_3F_2$ is a generalized hypergeometric function, see Ch. IV of Ref. [59] and Ch. 16 of Ref. [60].

In terms of x , Eq. (33) is

$$\begin{aligned} & x^2(x+1) \frac{d^2h}{dx^2} + \left[\frac{(8-3q)}{2p}x^2 + \frac{(3-q)}{p}x \right] \frac{dh}{dx} = \\ & \frac{2\sqrt{2q}}{p} x^{3/2}(x+1)^{1/2} \frac{d\phi}{dx} + \frac{(6-q)\sqrt{2q}}{2p^2} x^{1/2}(x+1)^{1/2} \phi. \end{aligned} \quad (42)$$

Using Eq. (41) to evaluate the right hand side of this equation, it can be solved to give the trace of the metric perturbation solution

$$\begin{aligned} h &= c_2 - 2c_1(-1)^{1/p} \sqrt{x+1} {}_2F_1(1/2, 1+1/p; 3/2; x+1) \\ &+ \frac{3\sqrt{2q}}{p} \left(\frac{Q}{\kappa^2} \right)^{q/2p} \tilde{c}(-1)^{1/p} \sqrt{x+1} \\ &\quad \times {}_2F_1(1/2, 1/p; 3/2; x+1) \\ &+ \frac{3\sqrt{2q}}{2p} \left(\frac{Q}{\kappa^2} \right)^{q/2p} \tilde{c}_+ \int dx \frac{x^{(B-2)/2-1/p}}{\sqrt{x+1}} \\ &\quad \times \{ {}_3F_2(A_+, A_-, B/2-1; B, B/2; -x) \\ &\quad + \frac{(B-2)}{3B} {}_3F_2(A_+, A_-, B/2; B, B/2+1; -x) \} \\ &+ \frac{3\sqrt{2q}}{2p} \left(\frac{Q}{\kappa^2} \right)^{q/2p} \tilde{c}_- \int dx \frac{x^{(D-2)/2-1/p}}{\sqrt{x+1}} \\ &\quad \times \{ {}_3F_2(G_+, G_-, D/2-1; D, D/2; -x) \\ &\quad + \frac{(D-2)}{3D} {}_3F_2(G_+, G_-, D/2; D, D/2+1; -x) \}, \end{aligned} \quad (43)$$

where the integrals can be done (and expressed as infinite series) but this is not of use to us and so these series are not recorded here. In Eq. (43) \tilde{c} and \tilde{c}_{\pm} are the constants in Eq. (41) and c_1 and c_2 are constants of integration, with c_2 corresponding to a gauge mode.

In terms of x , Eq. (29) is

$$\frac{d\mathcal{H}}{dx} = \frac{k^2}{\bar{k}^2} \left[\frac{3\sqrt{2q}}{2p} \frac{\phi}{\sqrt{x(x+1)}} - \frac{dh}{dx} \right]. \quad (44)$$

Using Eqs. (41) and (43) to evaluate the right hand side of this equation, it can be solved to give the trace-free metric perturbation solution

$$\begin{aligned} & \frac{\bar{k}^2}{k^2} \mathcal{H} = \\ & c_3 + 2c_1(-1)^{1/p} \sqrt{x+1} {}_2F_1(1/2, 1+1/p; 3/2; x+1) \\ & - \frac{\sqrt{2q}}{2p} \left(\frac{Q}{\kappa^2} \right)^{q/2p} \tilde{c}_+ \frac{(B-2)}{B} \int dx \frac{x^{(B-2)/2-1/p}}{\sqrt{x+1}} \\ & \quad \times {}_3F_2(A_+, A_-, B/2; B, B/2+1; -x) \\ & - \frac{\sqrt{2q}}{2p} \left(\frac{Q}{\kappa^2} \right)^{q/2p} \tilde{c}_- \frac{(D-2)}{D} \int dx \frac{x^{(D-2)/2-1/p}}{\sqrt{x+1}} \\ & \quad \times {}_3F_2(G_+, G_-, D/2; D, D/2+1; -x), \end{aligned} \quad (45)$$

where the integrals can be done (and expressed as infinite series) but this not of use to us and so these series are not recorded here. In Eq. (45) \tilde{c}_{\pm} and c_1 are the constants in Eqs. (41) and (43), and c_3 is a constant of integration corresponding to a gauge mode.

Using the solutions given in Eqs. (41), (43) and (45), the left-hand sides of Eqs. (30) and (31) are proportional. Requiring they vanish results in the relation

$$c_1 = \sqrt{\frac{2}{q}} \frac{1}{p} \left(\frac{Q}{\kappa^2} \right)^{q/2p} \frac{\bar{k}^2}{\kappa^2} \tilde{c}. \quad (46)$$

We are unable to analytically establish that the coefficients of \tilde{c}_\pm — the gauge-invariant contributions — vanish in these equations, as they must. However, we are convinced that they indeed do vanish since, as discussed below, the complete numerical solution (for given sets of parameter values) of the linear perturbation equations results in power spectra that are in very good agreement with the analytic power spectra we have derived. Additionally, we show below that on smaller scales when spatial curvature is unimportant, the primordial power spectra in these tilted non-flat inflation models are identical to the primordial power spectrum in the tilted flat model of Refs. [26–28], where in the computation of Ref. [27] it was shown that the corresponding coefficients of \tilde{c}_\pm vanished in the corresponding equations.

C. Gauge-invariant variables solutions

For scalar perturbations there are two independent gauge-invariant variables, invariant under the remnants of general coordinate invariance in synchronous gauge [61]. We choose these to be

$$\Delta_\Phi = \frac{1}{\dot{\Phi}_b^2 + V(\Phi_b)} \left[2\dot{\Phi}_b\dot{\phi} + V'(\Phi_b)\phi + 6\frac{\dot{a}}{a}\dot{\Phi}_b\phi \right], \quad (47)$$

$$A_\Phi = \frac{1}{\dot{\Phi}_b^2 + V(\Phi_b)} \left[2\dot{\Phi}_b\dot{\phi} + V'(\Phi_b)\phi - \dot{\Phi}_b^2(h + \mathcal{H}) \right]. \quad (48)$$

Another useful gauge-invariant combination is

$$\mathcal{R}_\Phi = \frac{\dot{\Phi}_b^2 + V(\Phi_b)}{6\dot{\Phi}_b^2} [\Delta_\Phi - A_\Phi]. \quad (49)$$

During inflation, from the ϕ , h , and \mathcal{H} solutions of the previous sub-section, the gauge-invariant variables are

$$\begin{aligned} \Delta_\Phi &= \frac{p\sqrt{2q}}{12} \left(\frac{Q}{\kappa^2} \right)^{q/2p} \sqrt{x+1} \\ &\times \left[(B-2)\tilde{c}_+x^{(B-2)/2-1/p} {}_2F_1(A_+, A_-; B; -x) \right. \\ &\quad \left. + (D-2)\tilde{c}_-x^{(D-2)/2-1/p} {}_2F_1(G_+, G_-; D; -x) \right], \end{aligned} \quad (50)$$

and

$$\begin{aligned} A_\Phi &= -\frac{q}{6} \left(\frac{Q}{\kappa^2} \right)^{q/2p} \sqrt{x+1} \\ &\times \left[\frac{6}{\sqrt{2q}} \tilde{c}_+x^{(B-2)/2-1/p} \right. \\ &\quad \times {}_3F_2(A_+, A_-, B/2-1; B, B/2; -x) \\ &\quad + \frac{6}{\sqrt{2q}} \tilde{c}_-x^{(D-2)/2-1/p} \\ &\quad \times {}_3F_2(G_+, G_-, D/2-1; D, D/2; -x) \\ &\quad - \left(\frac{3q}{2} \frac{\kappa^2}{k^2} x + 1 \right) \frac{p}{\sqrt{2q}} (B-2)\tilde{c}_+x^{(B-2)/2-1/p} \\ &\quad \times {}_2F_1(A_+, A_-; B; -x) \\ &\quad - \left(\frac{3q}{2} \frac{\kappa^2}{k^2} x + 1 \right) \frac{p}{\sqrt{2q}} (D-2)\tilde{c}_-x^{(D-2)/2-1/p} \\ &\quad \times {}_2F_1(G_+, G_-; D; -x) \\ &\quad - \frac{3}{2} \frac{\kappa^2}{k^2} \sqrt{2q} \frac{(B-2)}{B} \tilde{c}_+x^{B/2-1/p} \\ &\quad \times {}_3F_2(A_+, A_-, B/2; B, B/2+1; -x) \\ &\quad - \frac{3}{2} \frac{\kappa^2}{k^2} \sqrt{2q} \frac{(D-2)}{D} \tilde{c}_-x^{D/2-1/p} \\ &\quad \left. \times {}_3F_2(G_+, G_-, D/2; D, D/2+1; -x) \right]. \end{aligned} \quad (51)$$

We assume that the gauge-invariant solutions, those proportional to \tilde{c}_\pm , obey Eqs. (30) and (31), and use these and Eq. (46) to simplify the expression for A_Φ to that given in Eq. (51), see the discussion around Eq. (46).

From the expressions for Δ_ϕ and A_ϕ above,

$$\begin{aligned} \mathcal{R}_\Phi &= \left(\frac{Q}{\kappa^2} \right)^{q/2p} \sqrt{x+1} \\ &\times \left[\frac{1}{\sqrt{2q}} \tilde{c}_+x^{(B-2)/2-1/p} \right. \\ &\quad \times {}_3F_2(A_+, A_-, B/2-1; B, B/2; -x) \\ &\quad + \frac{1}{\sqrt{2q}} \tilde{c}_-x^{(D-2)/2-1/p} \\ &\quad \times {}_3F_2(G_+, G_-, D/2-1; D, D/2; -x) \\ &\quad - \frac{\sqrt{2qp}}{8} \frac{\kappa^2}{k^2} (B-2)\tilde{c}_+x^{B/2-1/p} \\ &\quad \times {}_2F_1(A_+, A_-; B; -x) \\ &\quad - \frac{\sqrt{2qp}}{8} \frac{\kappa^2}{k^2} (D-2)\tilde{c}_-x^{D/2-1/p} \\ &\quad \times {}_2F_1(G_+, G_-; D; -x) \\ &\quad - \frac{\sqrt{2q}}{4} \frac{\kappa^2}{k^2} \frac{(B-2)}{B} \tilde{c}_+x^{B/2-1/p} \\ &\quad \times {}_3F_2(A_+, A_-, B/2; B, B/2+1; -x) \\ &\quad - \frac{\sqrt{2q}}{4} \frac{\kappa^2}{k^2} \frac{(D-2)}{D} \tilde{c}_-x^{D/2-1/p} \\ &\quad \left. \times {}_3F_2(G_+, G_-, D/2; D, D/2+1; -x) \right]. \end{aligned} \quad (52)$$

D. Initial conditions and constants of integration

Defining $\hat{c}_\pm = \tilde{c}_\pm \sqrt{\kappa^2/Q}$, and ignoring the first, gauge-dependent, term on the right hand side of Eq. (41), a more convenient form of the inflaton field perturbation solution is

$$a\phi = \hat{c}_+ x^{(B-1)/2} {}_3F_2(A_+, A_-, B/2 - 1; B, B/2; -x) + \hat{c}_- x^{(D-1)/2} {}_3F_2(G_+, G_-, D/2 - 1; D, D/2; -x). \quad (53)$$

We assume as an initial condition that the conformally-rescaled scalar field perturbation ($a\phi$) is in the conformal time harmonic oscillator ground state, when non-flat inflation initiated, on small length scales. I.e., we require

$$\lim_{A \rightarrow \infty} a\phi(A, t) = \left(\frac{16\pi}{m_p^2} \right)^{1/2} \frac{e^{-iA\tilde{t}}}{\sqrt{2A}}, \quad (54)$$

where conformal time \tilde{t} is defined, for the open and closed models, in Eqs. (21) and (22). See Refs. [27, 28, 31, 37–39] for discussions of such initial conditions in a variety of flat and non-flat inflation models. These initial conditions result in de Sitter invariant inflaton two-point correlation functions in the very-slow-roll flat and closed de Sitter inflation models. The non-flat open and closed not-necessarily very-slow-roll inflation models which we apply them to here have less symmetry than the flat and closed de Sitter inflation models and so it is possible that our assumption here of the absence of an additional subdominant at large A correction, that cannot be determined from Eq. (54), and that might be important at small A , might not be justified in the non-flat not-necessarily very-slow-roll inflation models (such corrections do not contribute in the flat and closed de Sitter inflation models).¹⁰

We have not been able to analytically compute the large A asymptotic limit of Eq. (53).¹¹ However, if we set $\hat{c}_+ = 0$ in both non-flat models, and choose for the closed model

$$\hat{c}_- = \left(\frac{16\pi}{m_p^2} \right)^{1/2} \frac{2^{-2A/p} (-1)^{A/p}}{\sqrt{2A}}, \quad (55)$$

and for the open model

$$\hat{c}_- = \left(\frac{16\pi}{m_p^2} \right)^{1/2} \frac{i 2^{-i2A/p}}{\sqrt{2A}}, \quad (56)$$

it may numerically be shown in both non-flat models that the initial condition of Eq. (54) is satisfied, [62]. Additionally, we show below that on smaller scales when spatial curvature is unimportant, the primordial power

spectra in these tilted non-flat inflation models (which depend on these expressions for \hat{c}_\pm), are identical to the primordial power spectrum in the tilted flat inflation model of Refs. [26–28], where in Ref. [27] the corresponding asymptotic limits were computed.

E. $P_{\mathcal{R}}$ at late time during inflation in the open and closed models

Here we record expressions for the power spectra at late time during inflation in the open and closed tilted inflation models. The power spectrum definition and conventions are given in Appendices B and C.

Using Eqs. (52) and (55), and setting $\hat{c}_+ = 0$, we find at late time during inflation in the closed model

$$\sqrt{|P_{\mathcal{R}}|} = \left(\frac{16\pi}{m_p^2} \right)^{1/2} Q^{1/p} \frac{(2+q)p}{\sqrt{\pi q}} \left| -1 + \frac{W}{p} \right| \times \frac{2^{-(6-4q+2A-W)/p}}{\sqrt{A}(A-1)(A+3)} \left| \frac{\Gamma(1+W/p) \Gamma((2+q)/(2p))}{\Gamma((2+W)/p)} \right|, \quad (57)$$

where

$$W = \sqrt{-8 - 4q + q^2 + 4A(A+2)}, \quad (58)$$

while Eqs. (52) and (56), and setting $\hat{c}_+ = 0$, give at late time during inflation in the open model

$$\sqrt{|P_{\mathcal{R}}|} = \left(\frac{16\pi}{m_p^2} \right)^{1/2} Q^{1/p} \frac{(2+q)p}{\sqrt{\pi q}} \left| -1 + \frac{W}{p} \right| \times \frac{2^{-(6-4q)/p}}{\sqrt{A}(A^2+4)} \left| \frac{\Gamma(1+W/p) \Gamma((2+q)/(2p))}{\Gamma((2+W)/p)} \right|, \quad (59)$$

where

$$W = \sqrt{-12 - 4q + q^2 - 4A^2}, \quad (60)$$

and in both cases the scalar spectral power-law index $n = (2-3q)/(2-q)$. In the closed case, for fixed parameter values, the power spectrum of Eq. (57) agrees very well with power spectra computed numerically in the model described by the potential energy density of Eq. (24), [62]. **For representative plots of these and other non-flat inflation model power spectra, see Fig. 1 of Ref. [57].**

In the large A limit where $W = 2A$ ($2iA$) in the tilted closed (open) case, the $P_{\mathcal{R}}$ expressions in Eqs. (57) and (59) reduce identically to the flat-space tilted inflation model expression of Eq. (C1). This shows that on small scales at late times during tilted non-flat inflation, when spatial curvature is not important, the tilted closed and open model primordial power spectra are identical to the tilted flat model primordial power spectrum. **This is a useful consistency test of our analyses, including the initial conditions we have used here. This is because there is somewhat less uncertainty about initial conditions in the flat inflation model compared to the closed inflation model, as well as**

¹⁰ I thank A. Guth for emphasizing this point.

¹¹ Except for the \hat{c}_- term in the closed case where the result agrees with the expression given in Eq. (55) below.

possibly even the open inflation model. Given that potentially only a tiny $\sim 1\%$ deviation from flatness is what is of interest, it is reassuring that this initial conditions consistency test is passed.

V. CONCLUSION

We have extended from the very-slow-roll, untilted, linear inflaton potential energy density open and closed inflation models of Refs. [37, 39] to not-necessarily very-slowly-rolling, tilted, non-linear inflaton potential energy density open and closed inflation models. We have determined power spectra for quantum-mechanically produced spatial inhomogeneities in these models. These power spectra can be used to characterize spatial inhomogeneities in closed and open inflation models, and have been used in analyses of CMB anisotropy and other data, [57]. They differ from those that have previously been used for this purpose, [3, 51]. The power spectra of Refs. [3, 51] can also be generated by quantum fluctuations during (so far, only in closed) inflation, assuming different initial conditions, [56].

Recent hints of observational tension with a few predictions of the spatially-flat Λ CDM model provides motivation for studying non-flat cosmological models as well as other alternatives. Moreover, even if space is flat, to properly establish this from CMB anisotropy data requires use of consistent non-flat cosmological models — such as those constructed here — and the primordial power spectra in these models, and in Ref. [56] for the closed case. It might be significant that in the non-flat inflation model there appears to be additional freedom in the form of the inflation generated power spectrum, compared to the spatially-flat case.

ACKNOWLEDGMENTS

I acknowledge very valuable discussions with A. Guth, M. H. Namjoo, C.-G. Park, and J. de Cruz. This work was supported in part by DOE grant DE-SC0011840.

Appendix A: \mathcal{R}_Φ in the spatially-flat tilted inflation model

In the spatially-flat tilted inflation model of Refs. [26–28], the scalar field potential energy density during inflation is

$$V(\Phi) = \left(\frac{6-q}{3}\right) \frac{16\pi}{m_p^2} \rho_\Phi^{(0)} \exp\left[-\sqrt{\frac{q}{2}}(\Phi - \Phi^{(0)})\right], \quad (\text{A1})$$

where $\Phi^{(0)}$ and $\rho_\Phi^{(0)}$ are the scalar field and the scalar field energy density during inflation at scale factor a_0 . The scalar field energy density during inflation is

$$\rho_\Phi = \rho_\Phi^{(0)} \left(\frac{a_0}{a}\right)^q. \quad (\text{A2})$$

During inflation, the gauge-invariant

$$\mathcal{R}_\Phi = -\frac{1}{6} \left(\frac{q}{2}\right)^{(5-q)/q} \frac{aH^{(10-q)/(2q)}}{k} \quad (\text{A3})$$

$$\times \left[c_+ H_{\nu+1}^{(1)} \left(\frac{2k}{paH}\right) + c_- H_{\nu+1}^{(2)} \left(\frac{2k}{paH}\right) \right],$$

where H is the Hubble parameter, $\nu = (2+q)/(2p)$, $H_{\nu+1}^{(i)}$ are Hankel functions, and from the initial conditions the constant of integration $c_- = 0$ and

$$c_+ = \quad (\text{A4})$$

$$\left(\frac{16\pi}{m_p^2}\right)^{1/2} \frac{k}{2} \left(\frac{q\pi}{p}\right)^{1/2} \left(a_0 M^{2/q}\right)^{-5/2} e^{i(\nu-1/2)\pi/2},$$

where

$$M = \frac{q}{2} \left(\frac{8\pi}{3m_p^2} \rho_\Phi^{(0)}\right)^{1/2}. \quad (\text{A5})$$

Appendix B: Relation between P and \mathcal{P} in flat, open, and closed models

In this Appendix we define two power spectra we use and relate them to the two-point function in spatial momentum space. In this Appendix we do not explicitly indicate the time dependence of the fields.

In the flat model, defining the Fourier expansion of a position space field

$$\zeta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \zeta(\vec{k}), \quad (\text{B1})$$

and

$$\langle \zeta(\vec{k}) \zeta^*(\vec{k}') \rangle = P_\zeta(k) (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'), \quad (\text{B2})$$

where $P_\zeta(k)$ is the power spectrum and $k = |\vec{k}|$, we have

$$\langle \zeta(\vec{x}) \zeta^*(\vec{x}') \rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{i(\vec{k}\cdot\vec{x} - \vec{k}'\cdot\vec{x}')} \langle \zeta(\vec{k}) \zeta^*(\vec{k}') \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x} - \vec{x}')} P_\zeta(k). \quad (\text{B3})$$

Setting $\vec{x} = \vec{x}'$, we have

$$\langle |\zeta(\vec{x})|^2 \rangle = \int \frac{d^3k}{(2\pi)^3} P_\zeta(k) = \int_0^\infty \frac{dk}{k} \frac{k^3 P_\zeta(k)}{2\pi^2}, \quad (\text{B4})$$

and so define the power spectrum

$$\mathcal{P}_\zeta(k) = \frac{k^3 P_\zeta(k)}{2\pi^2} \quad (\text{B5})$$

which gives the power in a logarithmic wavenumber interval.

In the open model, defining

$$\zeta(\vec{\Omega}) = \int_0^\infty dA \sum_{BC} Z_{ABC}(\vec{\Omega}) \zeta(A), \quad (\text{B6})$$

where B and C are ‘magnetic’ integral indices, Z_{ABC} is defined in Eq. (2.9) of Ref. [39], and

$$\langle \zeta(A)\zeta^*(A') \rangle = P_\zeta(A)\delta_{A-A'}\delta_{B,B'}\delta_{C,C'}, \quad (\text{B7})$$

we have

$$\begin{aligned} & \langle \zeta(\vec{\Omega})\zeta^*(\vec{\Omega}') \rangle \\ &= \int dA \int dA' \sum_{BC} \sum_{B'C'} Z_{ABC}(\vec{\Omega})Z_{A'B'C'}^*(\vec{\Omega}') \langle \zeta(A)\zeta^*(A') \rangle \\ &= \int dA \sum_{BC} Z_{ABC}(\vec{\Omega})Z_{ABC}^*(\vec{\Omega}')P_\zeta(A) \\ &= \int_0^\infty \frac{dA}{A} \frac{A^3 P_\zeta(A)}{(2\pi)^{3/2}} \frac{P_{iA-1/2}^{-1/2}(\cosh(\gamma_3))}{\sqrt{\sinh(\gamma_3)}}, \end{aligned} \quad (\text{B8})$$

where we have used Eq. (2.11) of Ref. [39], P_ν^μ is the associated Legendre function of the first kind, and $\cosh(\gamma_3)$ is defined in Eq. (7). Setting $\vec{\Omega} = \vec{\Omega}'$ and using Eq. (A.3) of Ref. [38], we have

$$\langle |\zeta(\vec{\Omega})|^2 \rangle = \int_0^\infty \frac{dA}{A} \frac{A^3 P_\zeta(A)}{2\pi^2}, \quad (\text{B9})$$

and so define

$$\mathcal{P}_\zeta(A) = \frac{A^3 P_\zeta(A)}{2\pi^2}. \quad (\text{B10})$$

In the closed model, defining

$$\zeta(\vec{\Omega}) = \sum_{A=2}^\infty \sum_{BC} Y_{ABC}(\vec{\Omega})\zeta(A), \quad (\text{B11})$$

where B and C are ‘magnetic’ integral indices, Y_{ABC} is defined in Eq. (9) of Ref. [37], and

$$\langle \zeta(A)\zeta^*(A') \rangle = P_\zeta(A)\delta_{A,A'}\delta_{B,B'}\delta_{C,C'}, \quad (\text{B12})$$

we have

$$\begin{aligned} & \langle \zeta(\vec{\Omega})\zeta^*(\vec{\Omega}') \rangle \\ &= \sum_{A=2}^\infty \sum_{A'=2}^\infty \sum_{BC} \sum_{B'C'} Y_{ABC}(\vec{\Omega})Y_{A'B'C'}^*(\vec{\Omega}') \langle \zeta(A)\zeta^*(A') \rangle \\ &= \sum_{A=2}^\infty \sum_{BC} Y_{ABC}(\vec{\Omega})Y_{ABC}^*(\vec{\Omega}')P_\zeta(A) \\ &= \sum_{A=2}^\infty \frac{1}{A+1} \frac{(A+1)^3 P_\zeta(A)}{(2\pi)^{3/2}} \frac{P_{A-1/2}^{-1/2}(\cos(\gamma_3))}{\sqrt{\sin(\gamma_3)}}, \end{aligned} \quad (\text{B13})$$

where we have used Eq. (11) of Ref. [37], and $\cos(\gamma_3)$ is defined in Eq. (3). Setting $\vec{\Omega} = \vec{\Omega}'$ and using Eq. (3.9.2 (8)) on page 163 of Ref. [59], we have

$$\langle |\zeta(\vec{\Omega})|^2 \rangle = \sum_{A=2}^\infty \frac{1}{A+1} \frac{(A+1)^3 P_\zeta(A)}{2\pi^2}, \quad (\text{B14})$$

(the factor of $1/(A+1) = 1/\nu$ is chosen for consistency with the definition of Ref. [55], see below) and so define

$$\mathcal{P}_\zeta(A) = \frac{(A+1)^3 P_\zeta(A)}{2\pi^2}. \quad (\text{B15})$$

Appendix C: $\mathcal{P}_\mathcal{R}$ and $\mathcal{P}_\mathcal{R}$ at late time in the flat exponential potential and the open and closed linear potential inflation models, and comparison to CAMB and CLASS input power spectra

In the flat model of Refs. [26–28], at late time during inflation, using the results of Appendix A and Eq. (9.1.9) of Ref. [58], we have

$$P_\mathcal{R} = Ck^{n-4}, \quad (\text{C1})$$

where $n = (2-3q)/(2-q)$ and the proportionality constant

$$C = \frac{4}{m_p^2 q} \Gamma^2\left(\frac{6-q}{2p}\right) p^{4/p} \left(\frac{8\pi}{3m_p^2} \rho_\Phi^{(0)} a_0^q\right)^{2/p}, \quad (\text{C2})$$

so, from Eq. (B5),

$$\mathcal{P}_\mathcal{R} = \frac{C}{2\pi^2} k^{n-1}. \quad (\text{C3})$$

In the limit of small $q = 2\epsilon^2$, Eq. (C1) reduces to

$$\mathcal{P}_\mathcal{R} = \frac{2\pi h^2}{m_p^2 \epsilon^2} k^{-3}, \quad (\text{C4})$$

where $h^2 = 4\pi(6-q)\rho_\Phi^{(0)}/(9m_p^2)$.

In the open linear scalar field potential energy density model of Ref. [39], with $V(\Phi) = 12h^2[1 - \epsilon\Phi]$ where h is a constant, at late times during inflation

$$P_\mathcal{R} = \frac{2\pi h^2}{m_p^2 \epsilon^2} \frac{1}{A(A^2+1)}, \quad (\text{C5})$$

so

$$\mathcal{P}_\mathcal{R} = \frac{h^2}{\pi m_p^2 \epsilon^2} \frac{A^2}{A^2+1}. \quad (\text{C6})$$

In the closed linear scalar field potential energy density model of Ref. [37], at late times during inflation

$$P_\mathcal{R} = \frac{2\pi h^2}{m_p^2 \epsilon^2} \frac{1}{A(A+1)(A+2)}, \quad (\text{C7})$$

so

$$\mathcal{P}_\mathcal{R} = \frac{h^2}{\pi m_p^2 \epsilon^2} \frac{(A+1)^2}{A(A+2)}. \quad (\text{C8})$$

In the limit $A \gg 1$, Eqs. (C5) and (C7) become

$$\mathcal{P}_\mathcal{R} = \frac{2\pi h^2}{m_p^2 \epsilon^2} A^{-3}, \quad (\text{C9})$$

which is identical to the expression in Eq. (C4). As expected, on small scales at late times during non-flat inflation when spatial curvature is unimportant, the very-slow-roll closed and open inflation model primordial power spectra are identical to the very-slow-roll flat inflation model primordial power spectrum.

The CAMB and CLASS input power spectra are defined in Ref. [55]. The definition in their Eq. (3.23) is

a little unusual given the $(2\pi)^3$ in their Eq. (3.24), but the normalization of the power spectrum is an adjustable parameter to be determined by fitting to data.

Comparing Ref. [55] Eqs. (3.25) and (3.26) in the flat case, we see that their Eq. (3.26) is identical to Eq. (C3) above.

Reference [63] defines the negative of the eigenvalue of the spatial Laplacian to be $k^2/|K|$ in their Eq. (1.11). Here $K = -H_0^2\Omega_{k0}$ is negative (positive) for open (closed) spatial hypersurfaces and Ω_{k0} is the current value of the spatial curvature density parameter. In the second line below Eq. (3.4) of Ref. [55] (where they define q in terms of k) they define another wavenumber $\nu = q/\sqrt{|K|}$. It can be seen that in the open model their ν is identical to the A we use here while in the closed model their ν is identical to the $A + 1$ we use here.

Reference [55] uses an unusual but now standard convention. They define $\mathcal{P}_{\mathcal{R}} = A_s k^{n-1}$ to be the flat space expression, see their Eq. (3.26). To make this clear, in what follows, we put a superscript FS on this and write

$\mathcal{P}_{\mathcal{R}}^{\text{FS}} = A_s k^{n-1}$. What we call $\mathcal{P}_{\mathcal{R}}$ they refer to as

$$\tilde{\mathcal{P}}_{\mathcal{R}}(\nu) = \frac{\nu^2}{\nu^2 - \hat{K}} \mathcal{P}_{\mathcal{R}}^{\text{FS}}. \quad (\text{C10})$$

Here $\hat{K} = K/|K|$ which is $-1(+1)$ for open (closed) hypersurfaces. The above expression can be derived from their Eq. (3.29) by changing variables from q to ν . That our $\mathcal{P}_{\mathcal{R}}$ is their $\tilde{\mathcal{P}}_{\mathcal{R}}$ follows from their Eq. (3.28) by rewriting $d\nu \nu^2$ as $(d\nu/\nu) \nu^3$ and combining the ν^3 with the last factor in the integrand and then comparing to the flat-space expression in their Eq. (3.25).

Setting $n = 1$ in the Ref. [55] non-flat expressions, as we want to compare to the linear scalar field potential energy density inflation expressions above, we find in the open case the Ref. [55] formula is

$$\mathcal{P}_{\mathcal{R}} \propto \frac{A^2}{A^2 + 1}, \quad (\text{C11})$$

that is in agreement with Eq. (C6) above, and in the closed case the Ref. [55] formula is

$$\mathcal{P}_{\mathcal{R}} \propto \frac{(A + 1)^2}{A(A + 2)}, \quad (\text{C12})$$

which agrees with Eq. (C8) above.

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- [1] P. J. E. Peebles, *Astrophys. J.* **284**, 439 (1984).
- [2] E. Di Valentino, et al., *Class. Quant. Grav.* **38**, 153001 (2021) [arXiv:2103.01183]; L. Perivolaropoulos and F. Skara, *New Astron. Rev.* **95**, 101659 (2022) [arXiv:2105.05208]; E. Abdalla, et al., *J. High Energy Astrophys.* **34**, 49 (2022) [arXiv:2203.06142], and references therein.
- [3] Planck Collaboration, *Astron. Astrophys.* **641**, A6 (2020) [arXiv:1807.06209], and references therein.
- [4] eBOSS Collaboration, *Phys. Rev. D* **103**, 083533 (2021) [arXiv:2007.08991], and references therein.
- [5] O. Farooq, F. R. Madiyar, S. Crandall, and B. Ratra, *Astrophys. J.* **835**, 26 (2017) [arXiv:1607.03537], and references therein.
- [6] D. M. Scolnic, et al., *Astrophys. J.* **859**, 101 (2018) [arXiv:1710.00845], and references therein.
- [7] D. Mania and B. Ratra, *Phys. Lett. B* **715**, 9 (2012) [arXiv:1110.5626]; R. Chávez, et al., *Mon. Not. R. Astron. Soc.* **442**, 3565 (2014) [arXiv:1405.4010]; A. L. González-Morán, et al., *Mon. Not. R. Astron. Soc.* **487**, 4669 (2019) [arXiv:1906.02195], *Mon. Not. R. Astron. Soc.* **505**, 1441 (2021) [arXiv:2105.04025]; S. Cao, J. Ryan, and B. Ratra, *Mon. Not. R. Astron. Soc.* **497**, 3191 (2020) [arXiv:2005.12617], *Mon. Not. R. Astron. Soc.* **509**, 4745 (2022) [arXiv:2109.01987]; S. Cao, J. Ryan, N. Khadka, and B. Ratra, *Mon. Not. R. Astron. Soc.* **501**, 1520 (2021) [arXiv:2009.12953], and references therein.
- [8] S. Cao, et al., *Astron. Astrophys.* **606**, A15 (2017) [arXiv:1708.08635]; J. Ryan, Y. Chen, and B. Ratra, *Mon. Not. R. Astron. Soc.* **488**, 3844 (2019) [arXiv:1902.03196]; S. Cao, J. Ryan, N. Khadka, and B. Ratra, *Mon. Not. R. Astron. Soc.* **501**, 1520 (2021) [arXiv:2009.12953]; X. Zheng, et al., *Sci. China Phys. Mech. Astron.* **64**, 259511 (2021) [arXiv:2103.07139]; S. Cao, J. Ryan, and B. Ratra, *Mon. Not. R. Astron. Soc.* **509**, 4745 (2022) [arXiv:2109.01987], and references therein.
- [9] G. Risaliti and E. Lusso, *Astrophys. J.* **815**, 33 (2015) [arXiv:1505.07118], *Nat. Astron.* **3**, 272 (2019) [arXiv:1811.02590]; N. Khadka and B. Ratra, *Mon. Not. R. Astron. Soc.* **492**, 4456 (2020) [arXiv:1909.01400], *Mon. Not. R. Astron. Soc.* **497**, 263 (2020) [arXiv:2004.09979], *Mon. Not. R. Astron. Soc.* **502**, 6140 (2021) [arXiv:2012.09291], *Mon. Not. R. Astron. Soc.* **510**, 2753 (2022) [arXiv:2107.07600]; E. Lusso, et al., *Astron. Astrophys.* **642**, A150 (2020) [arXiv:2008.08586]; O. Luongo, et al., *Phys. Rev. D* **105**, 103510 (2022) [arXiv:2108.13228]; M. Rezaei, J. Solà Peracaula, and M. Malekjani, *Mon. Not. R. Astron. Soc.* **509**, 2593 (2022) [arXiv:2108.06255]; M. Dainotti, et al., *Astrophys. J.* **931**, 106 (2022) [arXiv:2203.12914]; V. Petrosian, J. Singal, and S. Mutchnick, *Astrophys. J.* **935**, L19 (2022) [arXiv:2205.07981], and references therein.
- [10] J. S. Wang, F. Y. Wang, K. S. Cheng, and Z. G. Dai, *Astron. Astrophys.* **585**, A68 (2016) [arXiv:1509.08558]; F. Fana Dirirsa, et al., *Astrophys. J.* **887**, 13 (2019) [arXiv:1910.07009]; M. Demianski, E. Piedipalumbo, D. Sawant, and L. Amati, *Mon. Not. R. Astron. Soc.* **506**, 903 (2021) [arXiv:1911.08228]; N. Khadka and B. Ratra, *Mon. Not. R. Astron. Soc.* **499**, 391 (2020) [arXiv:2007.13907]; S. Cao, J. Ryan, N. Khadka, and

- B. Ratra, *Mon. Not. R. Astron. Soc.* **501**, 1520 (2021) [arXiv:2009.12953]; N. Khadka, O. Luongo, M. Muccino, and B. Ratra, *J. Cosmol. Astropart. Phys.* **2109**, 042 (2021) [arXiv:2105.12692]; F. Y. Wang, J. P. Hu, G. Q. Zhang, and Z. G. Dai, *Astrophys. J.* **924**, 97 (2022) [arXiv:2106.14155]; J. P. Hu, F. Y. Wang, and Z. G. Dai, *Mon. Not. R. Astron. Soc.* **507**, 730 (2021) [arXiv:2107.12718]; O. Luongo and M. Muccino, *Galaxies* **9**, 77 (2021) [arXiv:2110.14408]; S. Cao, N. Khadka, and B. Ratra, *Mon. Not. R. Astron. Soc.* **510**, 2928 (2022) [arXiv:2110.14840]; S. Cao, M. Dainotti, and B. Ratra, *Mon. Not. R. Astron. Soc.* **512**, 439 (2022) [arXiv:2201.05245], *Mon. Not. R. Astron. Soc.* **516**, 1386 (2022) [arXiv:2204.08710]; Y. Liu, et al., *Astrophys. J.* **931**, 50 (2022) [arXiv:2203.03178], *Astrophys. J.* **935**, 7 (2022) [arXiv:2207.00455]; M. Dainotti, et al., *Mon. Not. R. Astron. Soc.* **514**, 1828 (2022) [arXiv:2203.15538]; X. D. Jia, et al., *Mon. Not. R. Astron. Soc.* **516**, 2575 (2022) [arXiv:2208.09272], and references therein.
- [11] Simons Observatory Collaboration, *J. Cosmol. Astropart. Phys.* **1902**, 056 (2019) [arXiv:1808.07445]; M. Vargas-Magaña, D. Brooks, M. Levi, and G. Tarle, arXiv:1901.01581; Euclid Collaboration, *Astron. Astrophys.* **642**, A191 (2020) [arXiv:1910.09273], and references therein.
- [12] G. Chen and B. Ratra, *Publ. Astron. Soc. Pacific* **123**, 1127 (2011) [arXiv:1105.5206]; for earlier median statistics estimates see J. R. Gott, M. S. Vogeley, S. Podariu, and B. Ratra, *Astrophys. J.* **549**, 1 (2001) [arXiv:astro-ph/0006103], and G. Chen, J. R. Gott, and B. Ratra, *Publ. Astron. Soc. Pacific* **115**, 1269 (2003) [arXiv:astro-ph/0308099]. Also see E. Calabrese, M. Archidiacono, A. Melchiorri, and B. Ratra, *Phys. Rev. D* **86**, 043520 (2012) [arXiv:1205.6753].
- [13] C. G. Park and B. Ratra, *Astrophys. Space Sci.* **364**, 134 (2019) [arXiv:1809.03598]; A. Domínguez et al., *Astrophys. J.* **885**, 137 (2019) [arXiv:1903.12097]; A. Cuceu, J. Farr, P. Lemos, and A. Font-Ribera, *J. Cosmol. Astropart. Phys.* **1910**, 044 (2019) [arXiv:1906.11628]; H. Zeng and D. Yan, *Astrophys. J.* **882**, 87 (2019) [arXiv:1907.10965]; N. Schöneberg, J. Lesgourgues, and D. C. Hooper, *J. Cosmol. Astropart. Phys.* **1910**, 029 (2019) [arXiv:1907.11594]; W. Lin and M. Ishak, *J. Cosmol. Astropart. Phys.* **2105**, 009 (2021) [arXiv:1909.10991]; K. Blum, E. Castorina, and M. Simonović, *Astrophys. J.* **892**, L27 (2020) [arXiv:2001.07182]; M.-Z. Lyu, B. S. Haridasu, M. Viel, and J.-Q. Xia, *Astrophys. J.* **900**, 160 (2020) [arXiv:2001.08713]; O. H. E. Philcox, M. M. Ivanov, M. Simonović, and M. Zaldarriaga, *J. Cosmol. Astropart. Phys.* **2005**, 032 (2020) [arXiv:2002.04035]; X. Zhang and Q.-G. Huang, *Phys. Rev. D* **103**, 043513 (2021) [arXiv:2006.16692]; S. Birrer et al., *Astron. Astrophys.* **643**, A165 (2020) [arXiv:2007.02941]; P. Denzel, J. P. Coles, P. Saha, and L. L. R. Williams, *Mon. Not. R. Astron. Soc.* **501**, 784 (2021) [arXiv:2007.14398]; L. Pogosian, G.-B. Zhao, and K. Jedamzik, *Astrophys. J.* **904**, L17 (2020) [arXiv:2009.08455]; S. S. Boruah, M. J. Hudson, and G. Lavaux, *Mon. Not. R. Astro. Soc.* **507**, 2697 (2021) [arXiv:2010.01119]; Y. J. Kim, J. Kang, M. G. Lee, and I. S. Jang, *Astrophys. J.* **905**, 104 (2020) [arXiv:2010.01364]; D. Harvey, *Mon. Not. R. Astro. Soc.* **498**, 2871 (2020) [arXiv:2011.09488]; S. Cao, J. Ryan, and B. Ratra, *Mon. Not. R. Astro. Soc.* **504**, 300 (2021) [arXiv:2101.08817], *Mon. Not. R. Astron. Soc.* **509**, 4745 (2022) [arXiv:2109.01987]; Q. Wu, G.-Q. Zhang, and F.-Y. Wang, *Mon. Not. R. Astron. Soc.* **515**, L1 (2022) [arXiv:2108.00581]; S. Cao and B. Ratra, *Mon. Not. R. Astro. Soc.* **513**, 5686 (2022) [arXiv:2203.10825], and references therein.
- [14] A. G. Riess, et al., *Astrophys. J.* **934**, L7 (2022) [arXiv:2112.04510].
- [15] B. R. Zhang, et al., *Mon. Not. R. Astro. Soc.* **471**, 2254 (2017) [arXiv:1706.07573]; S. Dhawan, S. W. Jha, and B. Leibundgut, *Astron. Astrophys.* **609**, A72 (2017) [arXiv:1707.00715]; D. Fernández Arenas, et al., *Mon. Not. R. Astro. Soc.* **474**, 1250 (2018) [arXiv:1710.05951]; M. Rameez and S. Sarkar, *Class. Quant. Grav.* **38**, 154005 (2021) [arXiv:1911.06456]; W. L. Freedman, et al., *Astrophys. J.* **891**, 57 (2020) [arXiv:2002.01550]; L. Breuval, et al., *Astron. Astrophys.* **643**, A115 (2020) [arXiv:2006.08763]; G. Efstathiou, arXiv:2007.10716; N. Khetan, et al., *Astron. Astrophys.* **647**, A72 (2021) [arXiv:2008.07754]; W. L. Freedman, *Astrophys. J.* **919**, 16 (2021) [arXiv:2106.15656], and references therein.
- [16] P. J. E. Peebles and B. Ratra, *Astrophys. J.* **325**, L17 (1988); B. Ratra and P. J. E. Peebles, *Phys. Rev. D* **37**, 3406 (1988); A. Pavlov, S. Westmoreland, K. Saaidi, and B. Ratra, *Phys. Rev. D* **88**, 123513 (2013) [arXiv:1307.7399].
- [17] O. Farooq, D. Mania, and B. Ratra, *Astrophys. Space Sci.* **357**, 11 (2015) [arXiv:1308.0834]; Y. Chen, et al., *Astrophys. J.* **829**, 61 (2016) [arXiv:1603.07115]; A. Rana, D. Jain, S. Mahajan, and A. Mukherjee, *J. Cosmol. Astropart. Phys.* **1703**, 028 (2017) [arXiv:1611.07196]; J. Ooba, B. Ratra, and N. Sugiyama, *Astrophys. J.* **864**, 80 (2018) [arXiv:1707.03452]; H. Yu, B. Ratra, and F.-Y. Wang, *Astrophys. J.* **856**, 3 (2018) [arXiv:1711.03437]; C.-G. Park and B. Ratra, *Astrophys. J.* **882**, 158 (2019) [arXiv:1801.00213]; J.-J. Wei, *Astrophys. J.* **868**, 29 (2018) [arXiv:1806.09781]; DES Collaboration, *Phys. Rev. D* **99**, 123505 (2019) [arXiv:1810.02499]; E.-K. Li, M. Du, and L. Xu, *Mon. Not. R. Astro. Soc.* **491**, 4960 (2019) [arXiv:1903.11433]; W. Handley, *Phys. Rev. D* **100**, 123517 (2019) [arXiv:1907.08524]; C.-Q. Geng, Y.-T. Hsu, L. Yin, and K. Zhang, *Chinese Phys. C* **44**, 105104 (2020) [arXiv:2002.05290]; G. Efstathiou and S. Gratton, *Mon. Not. R. Astro. Soc.* **496**, L91 (2020) [arXiv:2002.06892]; E. Di Valentino, A. Melchiorri, and J. Silk, *Astrophys. J.* **908**, L9 (2021) [arXiv:2003.04935]; Y. Yang and Y. Gong, *Mon. Not. R. Astro. Soc.* **504**, 3092 (2021) [arXiv:2007.05714]; J. A. Agudelo Ruiz, J. C. Fabris, A. M. Velasquez-Toribio, and I. L. Shapiro, *Grav. Cosmol.* **26**, 316 (2020) [arXiv:2007.12636]; KiDS Collaboration, *Astron. Astrophys.* **649**, A88 (2021) [arXiv:2010.16416]; S. Vagnozzi, A. Loeb, and M. Moresco, *Astrophys. J.* **908**, 84 (2021) [arXiv:2011.11645]; R. Arjona and S. Nesseris, *Phys. Rev. D* **103**, 103539 (2021) [arXiv:2103.06789]; S. Dhawan, J. Alsing, and S. Vagnozzi, *Mon. Not. R. Astro. Soc.* **506**, L1 (2021) [arXiv:2104.02485]; J. E. Gonzalez, M. Benetti, R. von Marttens, and J. Alcaniz, *J. Cosmol. Astropart. Phys.* **2111**, 060 (2021) [arXiv:2104.13455]; F. Renzi, N. B. Hogg, and A. W. Giarè, *Mon. Not. R. Astron. Soc.* **513**, 4004 (2022) [arXiv:2112.05701]; C.-Q. Geng, Y.-T. Hsu, and J.-R. Lu, *Astrophys. J.* **926**, 74 (2022) [arXiv:2112.10041]; J.-J. Wei and F. Melia, *Astrophys. J.* **928**, 165 (2022) [arXiv:2202.07865]; P.

- Mukherjee and N. Banerjee, *Phys. Rev. D* **105**, 063516 (2022) [arXiv:2202.07886]; A. Glanville, C. Howlett, and T. Davis, *Mon. Not. R. Astro. Soc.* **517**, 3087 (2022) [arXiv:2205.05892]; S. Cao, et al., *Mon. Not. R. Astro. Soc.* **516**, 1721 (2022) [arXiv:2205.15552]; P.-J. Wu, J.-Z. Qi, and X. Zhang, arXiv:2209.08502, and references therein.
- [18] Z. Zhai, M. Blanton, A. Slosar, and J. Tinker, *Astrophys. J.* **850**, 183 (2017) [arXiv:1705.10031]; J. Ooba, B. Ratra, and N. Sugiyama, *Astrophys. J.* **869**, 34 (2018) [arXiv:1710.03271], *Astrophys. J.* **866**, 68 (2018) [arXiv:1712.08617], *Astrophys. Space Sci.* **364**, 176 (2019) [arXiv:1802.05571]; C.-G. Park and B. Ratra, *Astrophys. Space Sci.* **364**, 82 (2019) [arXiv:1803.05522], *Astrophys. J.* **868**, 83 (2018) [arXiv:1807.07421]; A. Sangwan, A. Tripathi, and H. K. Jassal, arXiv:1804.09350; J. Ryan, S. Doshi, and B. Ratra, *Mon. Not. R. Astro. Soc.* **480**, 759 (2018) [arXiv:1805.06408]; J. Solà Peracaula, A. Gómez-Valent, and J. de Cruz Pérez, *Phys. Dark Univ.* **25**, 100311 (2019) [arXiv:1811.03505]; A. Singh, A. Sangwan, and H. K. Jassal, *J. Cosmol. Astropart. Phys.* **1904**, 047 (2019) [arXiv:1811.07513]; L. A. Ureña-López and N. Roy, *Phys. Rev. D* **102**, 063510 (2020) [arXiv:2007.08873]; S. Sinha and N. Banerjee, *J. Cosmol. Astropart. Phys.* **2104**, 060 (2021) [arXiv:2010.02651]; J. P. Johnson, A. Sangwan, and S. Shankaranarayanan, *J. Cosmol. Astropart. Phys.* **2201**, 024 (2022) [arXiv:2102.12367]; N. Khadka, et al., *Mon. Not. R. Astro. Soc.* **508**, 4722 (2021) [arXiv:2106.11136], *Mon. Not. R. Astro. Soc.* **513**, 1985 (2022) [arXiv:2112.00052], *Mon. Not. R. Astro. Soc.* **515**, 3729 (2022) [arXiv:2205.05813]; T. Xu, Y. Chen, L. Xu, and S. Cao, *Phys. Dark Univ.* **36**, 101023 (2022) [arXiv:2109.02453]; J. de Cruz Pérez, J. Solà Peracaula, A. Gómez-Valent, and C. Moreno-Pulido, arXiv:2110.07569; J. F. Jesus, et al., *J. Cosmol. Astropart. Phys.* **2211**, 037 (2022) [arXiv:2112.09722]; A. Adil, A. Albrecht, and L. Knox, arXiv:2207.10235, and references therein.
- [19] Early work on CMB anisotropy (and other) data include R. Aurich and F. Steiner, *Mon. Not. Roy. Astron. Soc.* **334**, 735 (2002) [arXiv:astro-ph/0109288]; *Phys. Rev. D* **67**, 123511 (2003) [arXiv:astro-ph/0212471]; *Int. J. Mod. Phys. D* **13**, 123 (2004) [arXiv:astro-ph/0302264]; J. L. Crooks, et al., *Astropart. Phys.* **20**, 361 (2003) [arXiv:astro-ph/0305495]; K. Ichikawa and T. Takahashi, *Phys. Rev. D* **73**, 083526 (2006) [arXiv:astro-ph/0511821]; *J. Cosmol. Astropart. Phys.* **0702**, 001 (2007) [arXiv:astro-ph/0612739]; *J. Cosmol. Astropart. Phys.* **0804**, 027 (2008) [arXiv:0710.3995]; E. L. Wright, arXiv:astro-ph/0603750; K. Ichikawa, M. Kawasaki, T. Sekiguchi, and T. Takahashi, *J. Cosmol. Astropart. Phys.* **0612**, 005 (2006) [arXiv:astro-ph/0605481]; Y. Wang and P. Mukherjee, *Phys. Rev. D* **76**, 103533 (2007) [arXiv:astro-ph/0703780].
- [20] S. Baumgartner and J. Yoo, *J. Cosmol. Astropart. Phys.* **2208**, 077 (2022) [arXiv:2205.12973]; S. Anselmi, et al., arXiv:2207.06547; R. Jimenez, et al., arXiv:2210.10102.
- [21] A. H. Guth, *Phys. Rev. D* **23**, 347 (1981).
- [22] K. Sato, *Mon. Not. Roy. Astron. Soc.* **195**, 467 (1981), *Phys. Lett.* **99B**, 66 (1981); D. Kazanas, *Astrophys. J.* **241**, L59 (1980).
- [23] S. W. Hawking, *Phys. Lett.* **115B**, 295 (1982); A. A. Starobinsky, *Phys. Lett.* **117B**, 175 (1982); A. H. Guth and S.-Y. Pi, *Phys. Rev. Lett.* **49**, 1110 (1982); J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, *Phys. Rev. D* **28**, 679 (1983).
- [24] W. Fischler, B. Ratra and L. Susskind, *Nucl. Phys. B* **259**, 730 (1985).
- [25] E. R. Harrison, *Phys. Rev.* **D1**, 2726 (1970); P. J. E. Peebles and J. T. Yu, *Astrophys. J.* **162**, 815 (1970); Ya. B. Zel'dovich, *Mon. Not. R. Astron. Soc.* **160**, 1P (1972).
- [26] F. Lucchin and S. Matarrese, *Phys. Rev. D* **32**, 1316 (1985).
- [27] B. Ratra, *Phys. Rev. D* **45**, 1913 (1992).
- [28] B. Ratra, *Phys. Rev. D* **40**, 3939 (1989).
- [29] J. R. Gott, *Nature* **295**, 304 (1982).
- [30] S. W. Hawking, *Nucl. Phys. B* **239**, 257 (1984).
- [31] B. Ratra, *Phys. Rev.* **D31**, 1931 (1985).
- [32] A. Linde, *Nucl. Phys. B* **372**, 421 (1992); A. Linde and A. Mezhlumian, *Phys. Rev. D* **52**, 6789 (1995) [arXiv:astro-ph/9506017]; A. Linde, *J. Cosmol. Astropart. Phys.* **0305**, 002 (2003) [arXiv:astro-ph/0303245].
- [33] S. Gratton, A. Lewis, and N. Turok, *Phys. Rev. D* **65**, 043513 (2002) [arXiv:astro-ph/0111012].
- [34] A. Lasenby and C. Doran, *Phys. Rev. D* **71**, 063502 (2005) [arXiv:astro-ph/0307311].
- [35] L. Gordon, B.-F. Li, and P. Singh, *Phys. Rev. D* **103**, 046016 (2021) [arXiv:2010.04738]; M. Motaharfard and P. Singh, *Phys. Rev. D* **104**, 106006 (2021) [arXiv:2102.09578].
- [36] S. Céspedes, S. P. de Alwis, F. Muia, and F. Quevedo, *Phys. Rev. D* **104**, 026013 (2021) [arXiv:2011.13936], arXiv:2112.11650.
- [37] B. Ratra, *Phys. Rev. D* **96**, 103534 (2017) [arXiv:1707.03439].
- [38] B. Ratra, *Phys. Rev. D* **50**, 5252 (1994).
- [39] B. Ratra, and P. J. E. Peebles, *Phys. Rev. D* **52**, 1837 (1995).
- [40] D. H. Lyth and E. D. Stewart, *Phys. Lett. B* **252**, 336 (1990).
- [41] B. Ratra and P. J. E. Peebles, *Astrophys. J.* **432**, L5 (1994).
- [42] M. Bucher, A. S. Goldhaber and N. Turok, *Phys. Rev. D* **52**, 3314 (1995) [arXiv:hep-ph/9411206]; K. Yamamoto, M. Sasaki, and T. Tanaka, *Astrophys. J.* **455**, 412 (1995) [arXiv:astro-ph/9501109].
- [43] K. M. Górski, et al., *Astrophys. J. Suppl.* **114**, 1 (1998) [arXiv:astro-ph/9608054].
- [44] M. Kamionkowski, B. Ratra, D. N. Spergel, and N. Sugiyama, *Astrophys. J.* **434**, L1 (1994) [arXiv:astro-ph/9406069]; D. H. Lyth and A. Woszczyna, *Phys. Rev. D* **52**, 3338 (1995) [arXiv:astro-ph/9501044]; K. M. Górski, B. Ratra, N. Sugiyama, and A. J. Banday, *Astrophys. J.* **444**, L65 (1995) [arXiv:astro-ph/9502034]; A. Linde, *Phys. Lett. B* **351**, 99 (1995) [arXiv:hep-th/9503097]; A. Linde and A. Mezhlumian, *Phys. Rev. D* **52**, 6789 (1995) [arXiv:astro-ph/9506017]; B. Ratra, A. J. Banday, K. M. Górski, and N. Sugiyama, *Astrophys. J.* **481**, 22 (1997) [arXiv:astro-ph/9512148]; B. Ratra, et al., *Astrophys. J.* **517**, 549 (1999) [arXiv:astro-ph/9901014], and references therein.
- [45] M. White and D. Scott, *Astrophys. J.* **459**, 415 (1996) [arXiv:astro-ph/9508157]; A. A. Starobinsky, arXiv:astro-ph/9603075.
- [46] E. Massó, S. Mohanty, A. Nautiyal, and G. Zsem-

- binszki, Phys. Rev. D **78**, 043534 (2008) [arXiv:astro-ph/0609349].
- [47] A.A. Asgari and A.H. Abbassi, Eur. Phys. J. C **75**, 544 (2015) [arXiv:1501.07089].
- [48] B. Bonga, B. Gupt, and N. Yokomizo, J. Cosmol. Astropart. Phys. **1610**, 031 (2016) [arXiv:1605.07556].
- [49] W. Handley, Phys. Rev. D **100**, 123517 (2019) [arXiv:1907.08524]; A. Thavanesan, D. Werth, and W. Handley, Phys. Rev. D **103**, 023519 (2021) [arXiv:2009.05573]; L. T. Hergt, et al., Phys. Rev. D **106**, 063529 (2022) [arXiv:2205.07374].
- [50] C. Kiefer and T. Vardanyan, Gen. Rel. Grav. **54**, 30 (2022) [arXiv:2111.07835].
- [51] Planck Collaboration, Astron. Astrophys. **594**, A13 (2016) [arXiv:1502.01589].
- [52] J. Ooba, B. Ratra, and N. Sugiyama, Astrophys. J. **864**, 80 (2018) [arXiv:1707.03452].
- [53] C.-G. Park and B. Ratra, Astrophys. J. **882**, 158 (2019) [arXiv:1801.00213].
- [54] C.-G. Park and B. Ratra, Phys. Rev. D **101**, 083508 (2020) [arXiv:1908.08477].
- [55] J. Lesgourgues and T. Tram, J. Cosmol. Astropart. Phys. **1409**, 032 (2014) [arXiv:1312.2697],
- [56] A. H. Guth, M. H. Namjoo, and B. Ratra, in preparation.
- [57] J. de Cruz Pérez, C.-G. Park, and B. Ratra, arXiv:2211.04268.
- [58] *Handbook of Mathematical Functions*, eds. M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
- [59] *Higher Transcendental Functions*, Vol. I, ed. A. Erdélyi (McGraw-Hill, New York, 1953).
- [60] *NIST Handbook of Mathematical Functions*, eds. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Cambridge University Press, Cambridge, 2010).
- [61] B. Ratra, Phys. Rev. D **43**, 3802 (1991).
- [62] M. H. Namjoo, private communication (2020).
- [63] T. Tram and J. Lesgourgues, J. Cosmol. Astropart. Phys. **1310**, 002 (2013) [arXiv:1305.3261],