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# Effects of Cosine Tapering Window on Quantum Phase Estimation 

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#### Abstract

We provide a modification to the quantum phase estimation algorithm (QPEA)[1-3] inspired on classical windowing methods for spectral density estimation. From this modification we obtain an upper bound in the cost that implies a cubic improvement with respect to the algorithm's error rate. Numerical evaluation of the costs also demonstrates an improvement. Moreover, with similar techniques, we detail an iterative projective measurement method for ground state preparation that gives an exponential improvement over previous bounds using QPEA. Numerical tests that confirm the expected scaling behavior are also obtained. For these numerical tests we have used a Lattice Thirring model as testing ground. Using well-known perturbation theory results, we also show how to more appropriately estimate the cost scaling with respect to state error instead of evolution operator error


## I. INTRODUCTION

In recent years, there has been increased interest in quantum computing techniques from the high-energy and nuclear physics community. Traditionally, physical systems that cannot be accessed through perturbative methods are studied through a combination of a Wick rotation and Monte Carlo methods [4] to perform observables estimation in the Lagrangian formalism. This approach amounts to estimating the path integral in Euclidean space time. Working in Euclidean space time ensures the integrand, which is the Boltzmann weight used to sample Field configurations, is strictly positive and thus can be assigned to the probabilistic weight in Monte Carlo methods. However, this excludes systems that have topological terms [5], non-zero chemical potentials [6], or limits the study of real-time dynamics in general [7, 8]. These particular cases spoil the positiveness of the sampling weight and can no longer use the probabilistic interpretation. Switching altogether to the Hamiltonian formalism avoids the requirement of probabilistic interpretation of Monte Carlo methods. The main drawback of working in the Hamiltonian formalism is the exponential growth of the Hilbert space dimension with respect to the system size. This renders most classical methods unfeasible even at relatively small system sizes. Quantum computing promises to bypass this problem [9] in a general sense. Thus, it will be of importance to further develop quantum algorithms so we are ready on the arrival of proper quantum hardware.

In quantum computing, state preparation is an essential part in the process of estimating observables. Here, we propose an iterative approach for state preparation which exponentially improves over previous bounds provided in [10] when using the quantum phase estimation algorithm (QPEA) [1]. We also introduce a minimal modification to the truncation window from the original QPEA [1] for which we also provide cost estimates
for state preparation and phase estimation.
This work is organized as follows: In Section II, we explain the original QPEA algorithm and the effects that time-domain truncation has on spectral estimation. In Section III we introduce the modified window function and its effects on the costs of QPEA itself. In Section IV, we detail the iterative approach involving projective measurements that exponentially improves the cost scaling with respect to the state error over previous estimates [10]. Moreover, we study the effects of using the QPEA modifications proposed in Section III for state preparation. This modification to the time window is also seen in Ref [11], and we present here a circuit to prepare such state at a cost scaling linearly with the number of ancillary qubits. Finally, in Section V, we show some state preparation tests using a lattice implementation of the Thirring model [12], and provide the tools to estimate excited state contamination.

## II. SPECTRAL DENSITY ESTIMATION AND THE EFFECTS OF WINDOWING

When calculating a spectral density numerically, we are bounded by the resources available, both classically or quantum mechanically. We cannot calculate the continuous-time Fourier transform, defined

$$
\begin{equation*}
\mathcal{F}(f)(q)=\int_{-\infty}^{\infty} f(x) e^{-2 \pi i x q} d q \tag{1}
\end{equation*}
$$

of a signal over an infinite time domain. Be it a sound recording, DC Voltage signal, or the Hamiltonian evolution of a quantum system, we must choose a finite sample rate and time domain.

Classically, one method to estimate the spectral density is to use instead the discrete Fourier transform (DFT) [13, 14]. It is defined as follows:

[^0]\[

$$
\begin{equation*}
F_{k}=\sum_{n=0}^{N-1} f_{n} \cdot e^{-\frac{i 2 \pi}{N} k n} \tag{2}
\end{equation*}
$$

\]

where $\left\{f_{n}\right\}_{n}$ is a set of discrete samples of a continuous signal. There is an implementation of the DFT as a quantum algorithm, called the quantum Fourier transform (QFT)[15]. In this section, we will demonstrate its usage for spectral estimation on a quantum computer.

There are two effects to consider when estimating the spectral density from the discrete and finite samples of a signal. The two effects are aliasing (due to frequency domain truncation), and spectral-leakage (due to time domain truncation). About aliasing we do not have to worry since the spectrum of our Hamiltonian is going to be bounded. However, spectral leakage has the effect of increasing costs in quantum computing as it reduces the certainty that each spectral estimation experiment provides. Now, we will exemplify this effect on the original quantum phase estimation algorithm [1-3, 16].

In the original algorithm (See Figure 3), we assume that we start with the state

$$
\begin{equation*}
|k=0\rangle_{a} \otimes\left|\psi_{j}\right\rangle \tag{3}
\end{equation*}
$$

where $\left|\psi_{j}\right\rangle$ is an eigenstate of $H$, i.e., $H\left|\psi_{j}\right\rangle=E_{j}\left|\psi_{j}\right\rangle$. The ancilla register, labeled by $a$, consists of $m$-qubits which is able to store $2^{m}$ elements (samples). The ancilla states are labelled through $|k\rangle_{a}:=\left|k_{m-1}\right\rangle \otimes\left|k_{m-2}\right\rangle \otimes \cdots \otimes$ $\left|k_{0}\right\rangle$ with $k_{i} \in\{0,1\}$ and $k=-k_{m-1} 2^{m-1}+k_{m-2} 2^{m-2}+$ $k_{m-3} 2^{m-3} \cdots+k_{0} 2^{0}$. One then applies the inverse quantum Fourier transform operator

$$
\begin{equation*}
Q F T^{-1}:=\frac{1}{\sqrt{2^{m}}} \sum_{x=-2^{m-1}}^{2^{m-1}-1} \sum_{k=-2^{m-1}}^{2^{m-1}-1} e^{\frac{2 \pi i x k}{2^{m}}}|x\rangle_{a}\langle k| \tag{4}
\end{equation*}
$$

after which we are left with the uniform superposition on the ancillary register

$$
\begin{equation*}
\left(\frac{1}{\sqrt{2^{m}}} \sum_{x}|x\rangle_{a}\right) \otimes\left|\psi_{j}\right\rangle \tag{5}
\end{equation*}
$$

Now, we want to store the phases of Hamiltonian evolution on the ancilla register such that we have

$$
\begin{equation*}
\left(\frac{1}{\sqrt{2^{m}}} \sum_{x} e^{i 2 \pi y \lambda E_{j}}|x\rangle_{a}\right) \otimes\left|\psi_{j}\right\rangle \tag{6}
\end{equation*}
$$

To achieve that, first consider the $\ell_{t h}$-qubit-controlled operation

$$
\begin{equation*}
C_{a, \ell}(W)=|0\rangle_{a, \ell}\langle 0| \otimes I+|1\rangle_{a, \ell}\langle 1| \otimes W \tag{7}
\end{equation*}
$$

where $U=e^{i 2 \pi \lambda H}$. Thus, we can store $2^{m}$ samples in the ancillary register by applying $m$ similar operations for each ancillar qubit

$$
\begin{array}{r}
\left(C_{a, m-1}\left(U^{-2^{m-1}}\right) \prod_{\ell=0}^{m-2} C_{a, \ell}\left(U^{2^{\ell}}\right)\right) \\
\cdot\left(\frac{1}{\sqrt{2^{m}}} \sum_{x}|x\rangle_{a}\right) \otimes\left|\psi_{j}\right\rangle
\end{array}
$$

$$
\begin{equation*}
=\left(\frac{1}{\sqrt{2^{m}}} \sum_{x} e^{i 2 \pi y \lambda E_{j}}|x\rangle_{a}\right) \otimes\left|\psi_{j}\right\rangle \tag{8}
\end{equation*}
$$

Finally, we apply $Q F T=\left(Q F T^{-1}\right)^{\dagger}$ on the ancillary register which leaves us with

$$
\begin{equation*}
\sum_{k=-2^{m-1}}^{2^{m-1}-1} G\left(k-2^{m} \theta_{j}\right)|k\rangle_{a} \otimes\left|\psi_{j}\right\rangle \tag{9}
\end{equation*}
$$

where

$$
\begin{equation*}
G(q)=\frac{e^{\frac{i \pi q}{2^{m}}} \sin (\pi q)}{2^{m} \sin \left(\frac{\pi q}{2^{m}}\right)}=e^{\frac{i \pi q}{2^{m}}} D_{2^{m}}\left(\frac{q}{2^{m}}\right) \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{M}(x)=\frac{\sin (M \pi x)}{M \sin (\pi x)} \tag{11}
\end{equation*}
$$

Here, spectral leakage manifests itself as probability leakage displayed by $|G(k)|^{2}$. This means that when we measure the ancillary register at the end of Figure 3 we will not obtain the nearest integer to $2^{m} \lambda E_{j}$ with absolute certainty. Instead, there is a probability leak whose decay can be upper bounded by

$$
\begin{equation*}
|G(k)|^{2} \leq 1 /\left(2^{2 m+2} k^{2}\right) \tag{12}
\end{equation*}
$$

We have plotted $|G(k)|^{2}$ on the first row, second column of Figure 1 where the probability spread is evident. In Equation (5), by preparing a uniform superposition we have implicitly used a rectangular window to truncate the evolution samples. In the next section, we seek to improve the leakage through another truncation window using minimal modifications to the original circuit implementation.

## III. COSINE TAPERING WINDOW IN QUANTUM PHASE ESTIMATION

For the modified phase estimation with cosine tapering window, we again assume that we start with the state (See Figure 4)

$$
\begin{equation*}
|0\rangle_{a} \otimes\left|\psi_{i}\right\rangle \tag{13}
\end{equation*}
$$

where the subscript $a$ indicates that the first register is the ancillary register. Since the subsequent quantum operations does not alter the state on the target register, we suppress $\left|\psi_{i}\right\rangle$ in what follows. We first start by creating a super position of $|0\rangle_{a}$ and $|1\rangle_{a}$ on the ancillary register by applying a Hadamard gate on the least significant qubit

$$
\begin{equation*}
\frac{|0\rangle_{a}+|1\rangle_{a}}{\sqrt{2}} \tag{14}
\end{equation*}
$$



FIG. 1: Here we showcase the two different window functions and their corresponding filter functions, connected by DFT (QFT) and its inverse (QFT ${ }^{-1}$ ). On the left, the black dots represent the discrete node points at which the window function is sampled. For example, these black dots represent the distribution of the ancillary register at the black dashed line on Figure 3 for the rectangular window and on Figure 4 for the cosine window. On the right, the red marks represent the distributions that we use to generate the windows through the inverse DFT (QFT) (these ancillary register distributions are marked with a red dashed line in Figures 3 and 4 for the cosine and rectangular windows respectively).

After performing this $\mathrm{QFT}^{-1}$ operation we obtain

$$
\begin{equation*}
Q F T^{-1}\left(\frac{|0\rangle_{a}+|1\rangle_{a}}{\sqrt{2}}\right)=\sum_{x=-2^{m-1}}^{2^{m-1}-1} f(x)|x\rangle_{a} \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
f(x)=\frac{1+e^{\frac{2 i \pi x}{2^{m}}}}{\sqrt{2^{m+1}}} \tag{16}
\end{equation*}
$$

That is not the cosine window just yet. In order to obtain the cosine window, we must apply the phase $e^{-\frac{i \pi x}{2 m}}$. This is equivalent to centering the corresponding spectral filter seen on the lower right of Figure 1. This can be done
through applying

$$
\begin{equation*}
R_{\phi, m-1}\left(\frac{\pi 2^{m-1}}{2^{m}}\right) \otimes \bigotimes_{l=0}^{m-2} R_{\phi, l}\left(-\frac{\pi 2^{l}}{2^{m}}\right) \tag{17}
\end{equation*}
$$

We are left with

$$
\begin{equation*}
\sum_{x=-2^{m-1}}^{2^{m-1}-1} \tilde{f}(x)|x\rangle_{a} \tag{18}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{f}(x)=f(x) \exp \left(-\frac{\pi i x}{2^{m}}\right)=\frac{\sqrt{2} \cos \left(\frac{\pi x}{2^{m}}\right)}{\sqrt{2^{m}}} \tag{19}
\end{equation*}
$$

Thus, we have obtained the desired time-domain window.


FIG. 2: Comparison of the filter functions in Figure 1. We have shifted the cosine filter function in order for it to be centered at $q=0$. The filter corresponding to the cosine window has a wider main lobe, but it has a faster decaying behavior outside of that. As a consequence, when binning the probabilities, we get a higher peak for the cosine filter.


FIG. 3: Circuit to implement an $m$-qubit phase estimation algorithm. $U$ is $e^{2 \pi i \lambda H}$. The $Q F T^{-1}$ applied on $|0\rangle^{\otimes m}$ can be simplified to $H^{\otimes m}$ acting on $|0\rangle^{\otimes m}$ as is typically shown. The red dashed line marks the samples of the filter function stored in the ancillary register and the black dashed line the samples of the corresponding window function.

Performing the controlled evolution operations as dis-
played in Figure 4 we obtain

$$
\begin{equation*}
\sum_{k=-2^{m-1}}^{2^{m-1}-1} \tilde{f}(x) e^{\frac{i 2 \pi x \lambda E_{i}}{2^{m}}}|x\rangle_{a} \tag{20}
\end{equation*}
$$



FIG. 4: Circuit to implement improved $m$-qubit phase estimation algorithm. $U$ is $e^{2 \pi i \lambda H}$. The red dashed line marks the samples of the filter function stored in the ancillary register and the black dashed line the samples of the corresponding window function.

Note that the input state $|\psi\rangle$, which is suppressed here, is the target register of these controlled operations. Finally, we apply the QFT on the ancilla register to find,

$$
\begin{align*}
& Q F T \sum_{x=-2^{m-1}}^{2^{m-1}-1} \tilde{f}(x) e^{\frac{i 2 \pi x \lambda E_{i}}{2^{m}}}|x\rangle_{a}  \tag{21}\\
& =\sum_{k=-2^{m-1}}^{2^{m-1}-1} F\left(k-2^{m} \lambda E_{i}\right)|k\rangle_{a}
\end{align*}
$$

where the coefficient $F(k)$ takes the form,

$$
\begin{align*}
F(k) & =\frac{1}{\sqrt{2^{m}}} \sum_{x=-2^{m-1}}^{2^{m-1}-1} \tilde{f}[x] e^{\frac{-2 \pi i x k}{2^{m i n}}} \\
& =\frac{\sin \left(\frac{\pi}{2^{m}}\right) \cos (\pi k) \csc \left(\frac{\pi-2 \pi k}{2^{m+1}}\right) \csc \left(\frac{2 \pi k+\pi}{2^{m+1}}\right)}{\sqrt{2} 2^{m}} \tag{22}
\end{align*}
$$

which is the almost-centered DFT of $f(x)$. Thus, the probability of measuring each value of $k$ on the ancillary register is

$$
\begin{equation*}
P(k)=\left|F\left(k-2^{m} \theta_{i}\right)\right|^{2}, \tag{23}
\end{equation*}
$$

with $\theta_{i}:=\lambda E_{i}$. We now approximate $2^{m} \theta_{i}$ to the nearest integer. That is, $2^{m} \theta_{i}=z+2^{m} \delta$, where $z$ is the nearest integer to $2^{m} \theta_{i}$ and $\left|2^{m} \delta\right| \leq \frac{1}{2}$.

We find that the lowest probability of measuring $z$ on the ancillary register is when $\left|2^{m} \delta\right|=\frac{1}{2}$ and that corresponds to $\min _{\delta} \operatorname{Pr}(z)=\min _{\delta}\left|F\left(-2^{m} \delta\right)\right|^{2}=\frac{1}{2}$. Therefore,

$$
\begin{equation*}
\operatorname{Pr}(z) \geq \frac{1}{2} \tag{24}
\end{equation*}
$$

That is an improvement of worst-case probability from $\operatorname{Pr}(z) \geq \frac{4}{\pi^{2}}$ from the phase estimation in [1-3].

In order to amplify the probability of success of obtaining an estimate of $\theta_{i}$, we must sacrifice in precision or cost [2]. This can be understood as coarsening the data by summing the probabilities of $k=2^{p-1}$ results to the left and to the right (including $z$ ) of the nearest integer, $z$. That is,

$$
\begin{equation*}
P(-k \leq l<k)=\sum_{z-k \leq l<z+k}\left|\alpha_{l z}\right|^{2} \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha_{l z}=F\left(l-\delta 2^{m}\right) \tag{26}
\end{equation*}
$$

Equivalently, we can define the complementary error rate $e=1-P(-k \leq l<k)$, which is

$$
\begin{equation*}
e=\sum_{k \leq l<2^{m-1}}\left|\alpha_{l z}\right|^{2}+\sum_{-2^{m-1} \leq l<-k}\left|\alpha_{l z}\right|^{2} . \tag{27}
\end{equation*}
$$

Summing up the probabilities (coarsening the data) has the effect of raising the tolerance in precision from $1 / 2^{m+1}$ to $k / 2^{m}=1 / 2^{m-p+1}$. Therefore, it is convenient to define a new variable $t$ through the following

$$
\begin{equation*}
m=t+p \tag{28}
\end{equation*}
$$

such that the target precision to which we estimate the phase $\theta_{i}$ is $1 / 2^{t+1}$. This way the number of qubits $t$ determines the target precision and the number of extra qubits $p$ determines the probability of failure of the method, $e$.

We can obtain an upper bound on $e$ as done in Refs. [2, 3]. In these references, authors obtain this upper bound
on the error rate, but for the circuit in Figure 3 ( with $\alpha_{l z}$ corresponding to rectangular window). With this, they then solve for the number of summed-up qubits $p_{\text {rect }}$, and obtain that the minimum requirement for a certain target rate $e$ is

$$
\begin{equation*}
p_{\text {rect }}=\left\lceil\log _{2} \frac{1}{2 e}+\frac{1}{2}\right\rceil \tag{29}
\end{equation*}
$$

In Appendix A 5, we derive an analogous upper bound on $e$ for the cosine-window filter from which we obtain the minimum number of extra qubits

$$
\begin{equation*}
p=\left\lceil\log _{2}\left(\frac{\pi^{2 / 3}}{48^{1 / 3} e^{1 / 3}}+2\right)\right\rceil \tag{30}
\end{equation*}
$$

for a target error rate $e$.
The cost in the gate complexity increases multiplicatively by $2^{p}$. This means a constant improvement in the minimum bound for $p$, but a cubic improvement in gate complexity compared to Refs. [1-3] with respect to $e$. This can be also seen from the shape of both filters superposed in Figure 2, the amplitude appears to be more concentrated for the cosine-window filter than for the rectangular-window one.

Since we have only obtained a bound, we also perform numerical tests and plot $e$ against $p$ for $m=10$ on Figure 5 for different values of $2^{m} \delta$. The cases $2^{m} \delta=0$ and -0.5 are exceptional because we expect the error rate to be zero for the rectangular-window filter for $2^{m} \delta=0$ for any value of $p$, and zero for the cosine-window filter for $\left|2^{m} \delta\right|=0.5$ and for $p \geq 1$. In the corresponding plots in Figure 5 we see that those cases expected to be zero are of the order $\sim 10^{-30}$ due to numerical precision. We can see that the cosine-window filter outperforms the rectangular-window filter as we increase $p$, except for the particular case $2^{m} \delta=0$.

In the following section, we will estimate the circuit depth for state preparation using the rectangular-window filter as well as the cosine-window filter in order to compare.

## IV. ITERATIVE STATE PREPARATION AND THE EFFECTS OF COSINE TAPERING WINDOW

The phase estimation algorithm can also be used for state preparation. Here, we will detail how that is accomplished. First, using the rectangular-window variation (Figure 6), and finally comparing with our cosinewindow version (Figure 7). First, we assume the target register starts with the state

$$
\begin{equation*}
|\phi\rangle=\sum_{i} \phi_{i}\left|\psi_{i}\right\rangle, \tag{31}
\end{equation*}
$$

where $\left|\psi_{i}\right\rangle$ are the eigenstates of the Hamiltonian in question and $\phi_{i}$ are the overlap factors of the initial guess with those eigenstates.

In the broad sense, the state preparation method detailed here consists of applying the phase estimation circuit in Figure 6 on the state $|0\rangle_{a}|\phi\rangle$. This is done multiple times until the resulting state, $|\psi\rangle$, is $\epsilon$-close to the ground state $\left|\psi_{0}\right\rangle$,

$$
\begin{equation*}
\||\psi\rangle-\left|\psi_{0}\right\rangle \| \leq \epsilon \tag{32}
\end{equation*}
$$

It is only through using this iterative approach that we achieve a logarithmic cost scaling with respect to $1 / \epsilon$ as explained below.

The phase estimation algorithm in Figure 6 is an approximate implementation of the projection operator on the ground state. We assume here that we can simulate the Hamiltonian evolution exactly. Therefore, applying the phase estimation circuit Figure 6, just before the measurement on $|0\rangle_{a}|\phi\rangle$, leaves us with the state

$$
\begin{equation*}
\sum_{i} \sum_{q=-2^{m-1}}^{2^{m-1}-1} G\left(q-\left(2^{m} \theta_{i}-2^{m} \theta_{0}^{(\xi)}\right)\right) \phi_{i}|q\rangle_{a} \otimes\left|\psi_{i}\right\rangle \tag{33}
\end{equation*}
$$

which up to this point is a unitary operation. In the last expression, $\theta_{i}$ represents $\lambda E_{i}$ and $\theta_{0}^{(\xi)}$ is our best estimate of $\theta_{0}=\lambda E_{0}$, where $\xi$ is the precision to which we know $\theta_{0}$. Here, we have used $G(q)$ corresponding to the rectangular window instead of $F(q)$ for the cosine window. We can check that the rectangular-window filter takes the form,

$$
\begin{equation*}
G(q)=\frac{e^{\frac{i \pi q}{2^{m}}} \sin (\pi q)}{2^{m} \sin \left(\frac{\pi q}{2^{m}}\right)}=e^{\frac{i \pi q}{2^{m}}} D_{2^{m}}\left(\frac{q}{2^{m}}\right) \tag{34}
\end{equation*}
$$

The $R_{\phi}$ gates in Figure 6 provide the shift $\theta_{0}^{(\xi)}$ to the filter function in Equation (33).

Measuring the ancillary register and post-selecting $|0\rangle_{a}$ results in the state,

$$
\begin{equation*}
\sum_{i} G\left(2^{m} \theta_{0}^{(\xi)}-2^{m} \theta_{i}\right) \phi_{i}|0\rangle_{a} \otimes\left|\psi_{i}\right\rangle \tag{35}
\end{equation*}
$$

The whole circuit (with measurement included) is equivalent to applying the following filter or approximate projector on the target register

$$
\begin{equation*}
\tilde{P}_{\psi_{0}}=\sum_{i} \gamma_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right| \tag{36}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma_{i}=G\left(2^{m} \theta_{0}^{(\xi)}-2^{m} \theta_{i}\right) \tag{37}
\end{equation*}
$$

We can also recast the approximate projector to the form

$$
\begin{equation*}
\tilde{P}_{\psi_{0}}=\sqrt{1-\rho_{0}}\left|\psi_{0}\right\rangle\left\langle\psi_{0}\right|+\hat{R} \tag{38}
\end{equation*}
$$

where $\|\hat{R}\|=O\left(\epsilon^{\prime}\right)(\|\cdot\|$ stands for the spectral norm) for some value of $\epsilon^{\prime}$ and we also have that $\hat{R}\left|\psi_{0}\right\rangle=0$, thus,

$$
\begin{equation*}
\left(\tilde{P}_{\psi_{0}}\right)^{r}=\left(1-\rho_{0}\right)^{r / 2}\left|\psi_{0}\right\rangle\left\langle\psi_{0}\right|+\hat{R}^{r} . \tag{39}
\end{equation*}
$$



FIG. 5: Here we plot the error rate $e$ defined in Equation (27) against the extra number of qubits $p$ in Equation (28). We have chosen $t=10$ and $2^{m} \delta \in\{0.0,-0.1,-0.2,-0.3,-0.4,-0.5\}$. It is clear that, except for the exceptional cases $2^{m} \delta=0$, the filter function coming from the cosine window outperforms the one from the rectangular window as we increase $p$.

Therefore,

$$
\begin{equation*}
\left\|\hat{R}^{r}\right\|=O\left(\epsilon^{\prime r}\right) \tag{40}
\end{equation*}
$$

Here, we have illustrated that by applying this projector $r$ times we obtain a smaller projector residue that decreases exponentially with $r$.

Now, we will estimate an upper bound on $r$. The state on the target register after $r$ iterations of the filtering operation from Equation (36) is

$$
\begin{equation*}
|\psi\rangle=\frac{\sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle}{\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|} \tag{41}
\end{equation*}
$$

In Appendix A 4 we derive

$$
\begin{equation*}
\epsilon=\Theta\left(\frac{\| \sum_{i \neq 0} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|}{\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|}\right) \tag{42}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(\frac{\| \sum_{i \neq 0} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|}{\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|}\right)^{2}=\frac{\sum_{i \neq 0}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r}}{\sum_{i}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r}} \tag{43}
\end{equation*}
$$

For $i \neq 0$, we have (see Appendix A 2 for bounds)

$$
\begin{equation*}
\left|\gamma_{i}\right| \leq \frac{1}{2^{m+1} \Delta} \tag{44}
\end{equation*}
$$



FIG. 6: Circuit to implement projection with peak centered at the estimate of the ground state energy. The resulting filter is the DFT of a rectangular window or a constant signal.
where $\Delta$ is the lower bound on the spectral gap,

$$
\begin{equation*}
\Delta_{0}=\theta_{1}-\theta_{0}=\lambda E_{1}-\lambda E_{0} \tag{45}
\end{equation*}
$$

Now, we derive an inequality for the quantity in Equation (43):

$$
\begin{equation*}
\frac{\sum_{i \neq 0}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r}}{\sum_{i}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r}} \leq \frac{1}{\left|\phi_{0}\right|^{2}\left|\gamma_{0}\right|^{2 r}}\left(\frac{1}{2^{m+1} \Delta}\right)^{2 r} \tag{46}
\end{equation*}
$$

In deriving this we have used Equation (44), $\sum_{i \neq 0}\left|\phi_{i}\right|^{2} \leq$ 1 , and $\left|\phi_{i}\right|^{2} \geq 0$. We would like to replace the term $\left|\gamma_{0}\right|^{2 r}$ in Equation (46) with some more meaningful parameters like the probability of success,

$$
\begin{equation*}
P_{r}=\| \tilde{P}_{\psi_{0}}^{r}|\phi\rangle \|^{2} \tag{47}
\end{equation*}
$$

As explained in Appendix A 3, $P_{r}$ approaches $\left|\phi_{0}\right|^{2}$ from below as $\epsilon \rightarrow 0$. Thus, it is more convenient to parametrize $P_{r}$ the following way:

$$
\begin{equation*}
P_{r}=(1-\rho)\left|\phi_{0}\right|^{2} \tag{48}
\end{equation*}
$$

where $\rho \geq 0$. Also, in Appendix A 3 we derive

$$
\begin{equation*}
\left|\gamma_{0}\right|^{2 r}=\left(1-\rho_{0}\right)^{r}=\Theta(1-\rho) \tag{49}
\end{equation*}
$$

Thus, Equation (46) can be rewritten in terms of $\rho$ the following way

$$
\begin{equation*}
\frac{\sum_{i \neq 0}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r}}{\sum_{i}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r}} \leq \frac{1}{\left|\phi_{0}\right|^{2} \Omega(1-\rho)}\left(\frac{1}{2^{m+1} \Delta}\right)^{2 r} \tag{50}
\end{equation*}
$$

We can now relate $\epsilon$ to other relevant quantities through Equation (42) or more specifically

$$
\begin{equation*}
\epsilon^{2}=O\left(\frac{\sum_{i \neq 0}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r}}{\sum_{i}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r}}\right) \tag{51}
\end{equation*}
$$

Solving for $r$, we get

$$
\begin{equation*}
r \leq \frac{\log O\left(\frac{1}{\epsilon\left|\phi_{0}\right| \sqrt{1-\rho}}\right)}{\log 2^{m+1} \Delta} \tag{52}
\end{equation*}
$$

Now, in order to ensure that the relative (to the overlap) error rate $\rho$ introduced by the $r$ filter operations is constant, we have to calculate the precision to which the ground energy has to be known. Using Equation (49) and the bound

$$
\begin{equation*}
\left(1-\rho_{0}\right)^{r} \leq e^{-\rho_{0} r} \tag{53}
\end{equation*}
$$

we can write

$$
\begin{equation*}
\rho_{0} \leq \frac{1}{r} \ln \Omega\left(\frac{1}{1-\rho}\right) \tag{54}
\end{equation*}
$$

Provided that $2 \xi$ is less than the width of the main lobe from the base, we can bound $\left|\gamma_{0}\right|$ from above with a parabola. That is,

$$
\begin{equation*}
\left|\gamma_{0}\right|^{2}=\left(1-\rho_{0}\right) \leq 1-a 2^{2 m} \xi^{2} \tag{55}
\end{equation*}
$$

Finally, this means that

$$
\begin{align*}
\xi & \leq \frac{1}{2^{m}} \sqrt{\frac{1}{a r} \ln \Omega\left(\frac{1}{1-\rho}\right)} \\
& \leq \frac{1}{2^{m}} \sqrt{\frac{1}{a} \frac{\log 2^{m+1} \Delta}{\log O\left(\frac{1}{\epsilon\left|\phi_{0}\right| \sqrt{1-\rho}}\right)} \ln \Omega\left(\frac{1}{1-\rho}\right)} \tag{56}
\end{align*}
$$

where $a$ has to be chosen such that the parabola has the same zero crossings as filter's main lobe. For the rectangular-window filter, $a=1$.


FIG. 7: Circuit to implement projection with peak centered at the estimate of the ground state energy. The resulting filter is the DFT of a cosine tapering window. The last Hadamard gate is there to create a coherent binning of odd and even $x$ states.

If we use the same method but applying the cosinewindow filter in Equation (22), we run into a problem: $\rho_{0}$ is prohibitively large. To see this, first consider that instead of Equation (37) we have

$$
\begin{equation*}
\gamma_{i}=F\left(2^{m} \theta_{0}^{(\xi)}-2^{m} \theta_{0}\right) \tag{57}
\end{equation*}
$$

The maximum possible $\gamma_{0}$ is when $\theta_{0}^{(\xi)}=\theta_{0}$ at which

$$
\begin{equation*}
\left|\gamma_{0}\right|^{2} \sim 0.8 \tag{58}
\end{equation*}
$$

which in turn gives us a minimum $\rho_{0}$ of $\sim 0.2$. Therefore, according to Equation (53), the exponential suppression becomes important at $r \sim 5$. This severely limits the number of iterations that one can practically do. Thus, we propose the modified circuit projection in Figure 7 in order to solve this. From the projection circuit in

Figure 7 we have that

$$
\begin{equation*}
\gamma_{i}=F_{+}\left(2^{m} \theta_{0}^{(\xi)}-2^{m} \theta_{i}\right) \tag{59}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{+}(q)=\frac{F(q-1 / 2)+F(q+1 / 2)}{\sqrt{2}} \tag{60}
\end{equation*}
$$

An upper bound was derived for this function in Appendix A 1, similar to the one in Equation (44) for Equation (34). With this bound, we repeated the same procedure as above. Through this, we obtained analogous inequalities for the number of iterations:

$$
\begin{equation*}
r_{\cos } \leq \frac{\log O\left(\frac{1}{\epsilon\left|\phi_{0}\right| \sqrt{1-\rho}}\right)}{\log \frac{2^{3 m+3} \Delta\left(\Delta+2^{-m}\right)\left(\Delta-2^{-m}\right)}{\pi^{2}}} \tag{61}
\end{equation*}
$$

and required precision:

$$
\begin{equation*}
\xi_{\mathrm{cos}} \leq \frac{1}{2^{m}} \sqrt{\frac{1}{a_{\mathrm{cos}}} \frac{\log \frac{2^{3 m+3} \Delta\left(\Delta+2^{-m}\right)\left(\Delta-2^{-m}\right)}{\pi^{2}}}{\log O\left(\frac{1}{\epsilon\left|\phi_{0}\right| \sqrt{1-\rho}}\right)} \ln \Omega\left(\frac{1}{1-\rho}\right)} \tag{62}
\end{equation*}
$$

Here, a similar bounding parabola was used for Equation (59), but with $a_{\text {cos }}=1 / 4$.

It would appear that the required precision for the ground state energy value scales polylogarithmically with respect to all relevant quantities; however, we should no-
tice the $1 / 2^{m}$ factor always appearing in front. For the case of the rectangular window, we have the condition $2^{m+1} \Delta>1$, which means that the required precision scales $\tilde{O}(\Delta)$ (We adopt the $\tilde{O}$ notation from [10]). We have pointed out this fact in Table I, where the other
methods show a similar scaling for the required ground state energy precision.

One way of estimating the ground state energy with the required precision is to perform the state preparation algorithm with an initial guess of the ground state energy $\theta_{0}^{(\xi)}=-0.5$. If not successful, we increase $\theta_{0}^{(\xi)}$ by the target precision $\xi=\tilde{O}(\Delta)$ and repeat the algorithm. The search can stop when one succeeds the state preparation algorithm at a rate that is not exponentially suppressed. This classical search method is something similar to what is proposed in Ref [10] when the ground state energy is not known a priori.

So far in this section, we have shown that the iterative approach of state preparation through QPE has an exponential speed up over the single-iteration approach. As shown in Table I, the scaling with respect to the required precision, $\epsilon$, and the overlap, $\left|\phi_{0}\right|^{-1}$, has been improved exponentially.

Now, we will account for the error incurred by approximating the Hamiltonian evolution. Consider a Hamiltonian with $\Gamma$ non-commuting terms

$$
\begin{equation*}
H=\sum_{\gamma=1}^{\Gamma} H_{\gamma} \tag{63}
\end{equation*}
$$

for which we will approximate the Hamiltonian evolution through $\tilde{U}$, where

$$
\begin{equation*}
\|\tilde{U}-U\|=\epsilon^{\prime} \tag{64}
\end{equation*}
$$

Most methods of approximation for Hamiltonian simulation give their cost estimates through the operator error $\epsilon^{\prime}$; however, what we care about is the vector error, i.e., the error on the state. To evaluate the source of vector error coming from using $\tilde{U}$, first consider the effective Hamiltonian

$$
\begin{equation*}
\tilde{H}=-i \frac{1}{2 \pi \lambda} \log \tilde{U} \tag{65}
\end{equation*}
$$

for which, through Taylor expansion, the operator error is

$$
\begin{equation*}
\|H-\tilde{H}\|=O\left(\frac{\epsilon^{\prime}}{2 \pi \lambda}\right) \tag{66}
\end{equation*}
$$

Finally, we can relate this Hamiltonian error to the vector state error through Matrix perturbation theory [17]:

$$
\begin{align*}
\|\left|\tilde{\psi}_{0}\right\rangle-\left|\psi_{0}\right\rangle \| & \leq \frac{\| P_{1}(\tilde{H}-H)\left|\psi_{0}\right\rangle \|}{E_{1}-E_{0}} \\
& \leq \frac{\lambda\|\tilde{H}-H\|}{\Delta} \tag{67}
\end{align*}
$$

where $P_{1}$ is $\sum_{i \neq 0}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$ and $\left|\tilde{\psi}_{0}\right\rangle$ is the ground state of the effective Hamiltonian $\tilde{H}$. Thus,

$$
\begin{equation*}
\|\left|\tilde{\psi}_{0}\right\rangle-\left|\psi_{0}\right\rangle \|=O\left(\frac{\epsilon^{\prime}}{\Delta}\right) \tag{68}
\end{equation*}
$$

With this, we can now correctly estimate the cost of preparing the ground state. We have made a more precise treatment of the cost by looking how the ground state of the effective Hamiltonian perturbs the ground state. This introduces a modified cost factor $1 /(\Delta \epsilon)$ as opposed to just $1 / \epsilon$. This is an aspect that only until recently [18] was taken into account for estimating ground state energy using QPEA and preparing states using digital adiabatic simulation.

All of this means that an algorithm which can simulate a Hamiltonian evolution $e^{i 2 \pi \lambda H}$ with an error $\epsilon^{\prime}$ that costs $O\left(\operatorname{polylog}\left(2^{n}, 1 / \epsilon^{\prime}\right)\right)$ will be sufficient to obtain a polylogarithmic cost in $1 /(\Delta \epsilon)$ with our state preparation method. Any algorithm in Refs [19-22] fulfills these requirements. The implied costs of using these algorithms for state preparation using the algorithms presented here are shown on the second row of Table I.

Generally speaking, methods based on product formulas alone do not meet the requirements. For example, in Ref [23] authors used the $k t h$-order product formulas provided by Suzuki in Ref [24]. The authors find that, in general, the number of product terms in the product expansion is $N_{\text {exp }}=O\left(5^{2 k} /\left(\epsilon^{\prime}\right)^{1 / 2 k}\right)$. From this formula, it is evident that we cannot increase $k$ arbitrarily as the cost increases exponentially with respect to $k$. The scaling of the cost with respect to $\epsilon$ and $\Delta$, using the Suzuki formulas, is reflected on the first row of Table I.

## V. NUMERICAL TESTS OF STATE PREPARATION

The theory that will serve as a test ground will be the $(1+1)$-dimensional massive Thirring model

$$
\begin{align*}
S_{\mathrm{Th}}[\psi, \bar{\psi}]=\int d^{2} x & {\left[\bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi-m \bar{\psi} \psi\right.} \\
& \left.-\frac{g}{2}\left(\bar{\psi} \gamma_{\mu} \psi\right)\left(\bar{\psi} \gamma^{\mu} \psi\right)\right] \tag{69}
\end{align*}
$$

For our simulations, we use the lattice Hamiltonian derived in [12],

$$
\begin{align*}
\bar{H}_{\mathrm{sim}} & =-\frac{1}{2} \sum_{n}^{N-2}\left(S_{n}^{+} S_{n+1}^{-}+S_{n+1}^{+} S_{n}^{-}\right) \\
& +a \tilde{m}_{0} \sum_{n}^{N-1}(-1)^{n}\left(S_{n}^{z}+\frac{1}{2}\right) \\
& +\Delta(g) \sum_{n}^{N-1}\left(S_{n}^{z}+\frac{1}{2}\right)\left(S_{n+1}^{z}+\frac{1}{2}\right) . \tag{70}
\end{align*}
$$

In order to boost the overlap of our initial guess with the ground state we use a variational approach. The ansatz for the ground state consists of alternating noncommuting operators that comprise the Hamiltonian [2630]:

$$
|\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})\rangle=e^{-i \gamma_{p} H_{\gamma}} e^{-i \beta_{p} H_{\beta}} e^{-i \alpha_{p} H_{\alpha}} \ldots
$$

| Preparation (ground energy known) | Gates | Qubits | Required precision |
| :---: | :---: | :---: | :---: |
| This paper + Product formulas [23, 24] | $\tilde{O}\left(\frac{1}{\left\|\phi_{0}\right\|^{2} \Delta^{1+1 / 2 k} \epsilon^{1 / 2 k}}\right)$ | $O\left(\log N+\log \frac{1}{\Delta}\right)$ | $\tilde{O}(\Delta)$ |
| $\begin{aligned} & \text { This } \\ & \text { LCU/Qubitization/Interp. } \\ & {[19-22, ~ 25]}\end{aligned}+$ | $\tilde{O}\left(\frac{1}{\left\|\phi_{0}\right\|^{2} \Delta}\right)$ | $\begin{array}{r} O\left(\log N+\log \log \frac{1}{\Delta \epsilon}\right. \\ \left.+\log \frac{1}{\Delta}\right) \end{array}$ | $\tilde{O}(\Delta)$ |
| Ge et al. [10] | $\tilde{O}\left(\frac{1}{\left\|\phi_{0}\right\|^{2} \Delta}\right)$ | $\begin{array}{r} O\left(\log N+\log \log \frac{1}{\Delta \epsilon}\right. \\ \left.+\log \frac{1}{\Delta}\right) \end{array}$ | $\tilde{O}(\Delta)$ |
| Single-round Phase estimation + amp. amplif. [10] | $\tilde{O}\left(\frac{1}{\left\|\phi_{0}\right\|^{3} \Delta \epsilon}\right)$ | $\begin{aligned} O(\log N & +\log \frac{1}{\epsilon} \\ & \left.+\log \frac{1}{\Delta}\right) \end{aligned}$ | $O\left(\left\|\phi_{0}\right\| \epsilon \Delta\right)$ |

TABLE I: Algorithms for ground state preparation for the case when the ground energy is known beforehand to the required precision. For simplicity of comparison, we have omitted speed-ups in $\left|\phi_{0}\right|$ through amplitude amplification. We have also omitted some overhead costs, like for example the base oracle gate cost in the oracle based approach of Hamiltonian simulation [19, 20].

$$
\begin{equation*}
e^{-i \gamma_{1} H_{\gamma}} e^{-i \beta_{1} H_{\beta}} e^{-i \alpha_{1} H_{\alpha}}\left|\phi_{0}\right\rangle \tag{71}
\end{equation*}
$$

where

$$
\begin{align*}
H_{\alpha} & =-\frac{1}{2} \sum_{n=0}\left(S_{2 n}^{+} S_{2 n+1}^{-}+S_{2 n+1}^{+} S_{2 n}^{-}\right)  \tag{72}\\
H_{\beta} & =-\frac{1}{2} \sum_{n=1}\left(S_{2 n-1}^{+} S_{2 n}^{-}+S_{2 n}^{+} S_{2 n-1}^{-}\right) \\
H_{\gamma} & =a \tilde{m}_{0} \sum_{n}^{N-1}(-1)^{n}\left(S_{n}^{z}+\frac{1}{2}\right) \\
& +\Delta(g) \sum_{n}^{N-1}\left(S_{n}^{z}+\frac{1}{2}\right)\left(S_{n+1}^{z}+\frac{1}{2}\right) . \tag{73}
\end{align*}
$$

We proceed to minimize

$$
\begin{equation*}
\langle\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})| \bar{H}_{\operatorname{sim}}|\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})\rangle \tag{74}
\end{equation*}
$$

With these new $|\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})\rangle$ states we then follow the state preparation procedure in the last section for both the cosine window and rectangular window.

We would like to corroborate the bounds from the previous section. Two things are important to us: an exponential decay in the error with respect to the number of iterations, and an improvement by using the cosine window.

We also wish to separate the error from the implementation of the Hamiltonian evolution and the error from the projection procedure. As mentioned in the last section, we perform the projection procedure with respect to the effective Hamiltonian resulting from approximating the Hamiltonian evolution. In order to perform a demonstration, we choose Suzuki's second-order formula [24] to approximate Hamiltonian evolution:

$$
\begin{equation*}
\tilde{U}_{\text {step }}=U^{i \frac{\Delta t}{2} H_{\alpha}} U^{i \frac{\Delta t}{2} H_{\beta}} U^{i \Delta t H_{\gamma}} U^{i \frac{\Delta t}{2} H_{\beta}} U^{i \frac{\Delta t}{2} H_{\alpha}} \tag{75}
\end{equation*}
$$

where $\tilde{U}=\tilde{U}_{\text {step }}^{d}$. Here, $\Delta t=2 \pi \lambda / d$ and $d$ is an integer equal or greater to 1 .

In order to estimate the error $\epsilon$ we evolve the final state $|\psi\rangle$ and measure the expectation value of an operator of interest $\hat{O}$. That is,

$$
\begin{equation*}
\mathbf{O}=\left\{\langle\lambda|\left(U_{\text {step }}^{\dagger}\right)^{n} \hat{O} U_{\text {step }}^{n}|\lambda\rangle: n=\{1,2,3 \ldots, N\}\right\} \tag{76}
\end{equation*}
$$

We then estimate a central value defined as the mean:

$$
\begin{equation*}
\bar{O}=\mathbf{E}[\mathbf{O}] \tag{77}
\end{equation*}
$$

and a standard deviation

$$
\begin{equation*}
\sigma_{O}=\sqrt{\mathbf{E}\left[(\mathbf{O}-\bar{O})^{2}\right]} \tag{78}
\end{equation*}
$$

For this numerical test we have chosen the chiral condensate as our observable.

$$
\begin{equation*}
\hat{\chi}=\frac{1}{n} \sum_{i=0}(-1)^{i+1} Z_{i} \tag{79}
\end{equation*}
$$

In Figure 8 we show a comparisons of $\sigma_{\chi}$ versus the number of iterations, $r$, for both the rectangular and the cosine window. This comparison is done for $d \in\{1,2,3\}$ and system sizes $N \in\{4,6,8\}$. We would also like to stress that $\sigma_{\chi}$ estimates the excited state contamination with respect to the effective Hamiltonian in Equation (65). The two main takeaways of these numerical tests are that: $\sigma_{\chi}$ appears to decay exponentially with respect to the number of iterations, and that the cosine window outperforms the rectangular window.

## VI. CONCLUSION

In this work, we have presented the effects of using a cosine tapering window on the quantum phase estimation


FIG. 8: Here we show comparisons of the excited state contamination, $\sigma_{\chi}$, in the chiral condensate for the values $d \in 1,2,3$ and $N \in 4,6,8$. The excited state contamination is plotted against the number of iterations, $r$, showing that the cosine-window filter achieves an improvement over the rectangular-window one.
algorithm. It was demonstrated that one obtains a cubic improvement in gate complexity scaling with respect to the error rate, $e$. It is left to future research the exploration of other windows, their optimization with respect to other metrics, as well as the exploration of possible hybrid quantum-classical approaches.

We also showed the effects of this window when the phase estimation algorithm is re-purposed for quantum state preparation. Simultaneously, we showed that using repeated blunted filter operations was more efficient than performing a single sharper filter operation. The improvements were exponential in $\epsilon^{-1}$, the state error, over previous estimates. Nevertheless, the linear scaling with respect to $\Delta^{-1}$ remains.

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## Appendix A: Detailed derivations of some formulas

## 1. $\quad F_{+}(x)$ and its upper bound

We convert the filter function, $F_{+}(q)(60)$, to upperbound its magnitude.

$$
\begin{aligned}
& F_{+}(q)=\frac{F(q-1 / 2)+f(q+1 / 2)}{\sqrt{2}} \\
& =\frac{\sin \left(\frac{\pi}{2^{m}}\right) \cos \left(\pi q-\frac{\pi}{2}\right)}{2^{m+1} \sin \left(\frac{2 \pi-2 \pi q}{2^{m+1}}\right) \sin \left(\frac{2 \pi q}{2^{m+1}}\right)} \\
& +\frac{\sin \left(\frac{\pi}{2^{m}}\right) \cos \left(\pi q+\frac{\pi}{2}\right)}{2^{m+1} \sin \left(\frac{-2 \pi q}{2^{m+1}}\right) \sin \left(\frac{2 \pi+2 \pi q}{2^{m+1}}\right)}
\end{aligned}
$$

$$
\begin{align*}
& =\frac{\sin \left(\frac{\pi}{2^{m}}\right) \sin (\pi q)\left(\sin \left(\frac{\pi+\pi q}{2^{m}}\right)+\sin \left(\frac{\pi-\pi q}{2^{m}}\right)\right)}{2^{m+1} \sin \left(\frac{\pi q}{2^{m}}\right) \sin \left(\frac{\pi-\pi q}{2^{m}}\right) \sin \left(\frac{\pi+\pi q}{2^{m}}\right)} \\
& =\frac{\sin \left(\frac{\pi}{2^{m}}\right) \sin (\pi q) \sin \left(\frac{\pi}{2^{m}}\right) \cos \left(\frac{\pi q}{2^{m}}\right)}{2^{m} \sin \left(\frac{\pi q}{2^{m}}\right) \sin \left(\frac{\pi-\pi q}{2^{m}}\right) \sin \left(\frac{\pi+\pi q}{2^{m}}\right)} \\
& =\frac{\sin ^{2}\left(\frac{\pi}{2^{m}}\right) \sin (\pi q) \cot \left(\frac{\pi q}{2^{m}}\right)}{2^{m} \sin \left(\frac{\pi-\pi q}{2^{m}}\right) \sin \left(\frac{\pi+\pi q}{2^{m}}\right)} . \tag{A1}
\end{align*}
$$

Hence, the absolute value of $F_{+}(q)$ is upper-bounded as,

$$
\begin{align*}
& \left|F_{+}(q)\right| \\
& \leq \frac{\sin ^{2}\left(\frac{\pi}{2^{m}}\right)}{2^{m}\left|\sin \left(\frac{\pi q}{2^{m}}\right)\right|\left|\sin \left(\frac{\pi-\pi q}{2^{m}}\right)\right|\left|\sin \left(\frac{\pi+\pi q}{2^{m}}\right)\right|} \\
& \leq \frac{\left(2^{m-1}\right)^{3} \sin ^{2}\left(\frac{\pi}{2^{m}}\right)}{2^{m}|q \||q-1|| q+1 \mid}, \text { for } 0 \leq 1+|q| \leq 2^{m-1} \\
& \leq \frac{\pi^{2}}{8|q||q-1||q+1|}, \text { for } 0 \leq 1+|q| \leq 2^{m-1} \tag{A2}
\end{align*}
$$

## 2. Some useful bounds

Using the inequality,

$$
\begin{equation*}
\left|\sin \frac{x}{2}\right| \geq \frac{|x|}{\pi} \text { for } 0 \leq|x| \leq \pi \tag{A3}
\end{equation*}
$$

we provide an upper bound on the Dirichlet kernel from Equation (11).

Its magnitude bounded as

$$
\begin{align*}
\left|D_{M}(x)\right|= & \frac{1}{M} \frac{|\sin M \pi x|}{|\sin \pi x|} \\
& \leq \frac{1}{M} \frac{1}{|\sin \pi x|} \leq \frac{1}{2 M|x|} \tag{A4}
\end{align*}
$$

for $0 \leq|x| \leq 1 / 2$.

## 3. Useful relations of success rate

The probability of successfully applying the filter in Equation (36) $r$ times is

$$
\begin{align*}
P_{r} & =\| P_{\psi_{0}}^{r}|\phi\rangle \|^{2} \\
& =\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|^{2}=\sum_{i}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r} \\
& =\left|\gamma_{0}\right|^{2 r}\left|\phi_{0}\right|^{2}+\sum_{i \neq 0}\left|\phi_{i}\right|^{2}\left|\gamma_{i}\right|^{2 r} \\
& =\left|\gamma_{0}\right|^{2 r}\left|\phi_{0}\right|^{2}+P_{r} \frac{\| \sum_{i \neq 0} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|^{2}}{\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|^{2}} \\
& =\left|\gamma_{0}\right|^{2 r}\left|\phi_{0}\right|^{2}+P_{r} O\left(\epsilon^{2}\right) \tag{A5}
\end{align*}
$$

To get to the last line we have used Equation (A11). Solving for $P_{r}$, we obtain that

$$
\begin{equation*}
P_{r}=\frac{\left|\gamma_{0}\right|^{2 r}}{1-O\left(\epsilon^{2}\right)}\left|\phi_{0}\right|^{2} \tag{A6}
\end{equation*}
$$

It is clear that $P_{r}$ approaches $\left|\phi_{0}\right|^{2}$ from below as $\epsilon \rightarrow 0$. Therefore, we find it convenient to parameterize $P_{r}$ the following way

$$
\begin{equation*}
P_{r}=(1-\rho)\left|\phi_{0}\right|^{2} \tag{A7}
\end{equation*}
$$

where $\rho \geq 0$. From this definition and Equation (A6), it immediately follows that

$$
\begin{equation*}
\left|\gamma_{0}\right|^{2 r} \leq(1-\rho) \tag{A8}
\end{equation*}
$$

More generally,

$$
\begin{equation*}
\left|\gamma_{0}\right|^{2 r} \rightarrow \frac{P_{r}}{\left|\phi_{0}\right|^{2}}, \text { as } \epsilon \rightarrow 0 \tag{A9}
\end{equation*}
$$

Equivalently, using definition in Equation (A6),

$$
\begin{equation*}
\left|\gamma_{0}\right|^{2 r}=\Theta(1-\rho) \tag{A10}
\end{equation*}
$$

## 4. Asymptotic expression for $\epsilon$

First, we assume that we can choose $\left|\psi_{0}\right\rangle$ such that the product $\gamma_{0}^{r} \phi_{0}$ is always real and positive. Thus, we can obtain the following bound for $\epsilon$

$$
\begin{align*}
\epsilon & \equiv \||\psi\rangle-\left|\psi_{0}\right\rangle \| \\
& =\| \frac{\sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle}{\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|}-\left|\psi_{0}\right\rangle \| \\
& =\frac{\|\left(\gamma_{0}^{r} \phi_{0}-\sqrt{\left.P_{r}\right)}\left|\psi_{0}\right\rangle+\sum_{i \neq 0} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|\right.}{\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|} \\
& \geq \frac{\| \sum_{i \neq 0} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|}{\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|} \tag{A11}
\end{align*}
$$

Now, if we use Equation (A10) we also obtain that

$$
\begin{equation*}
\||\psi\rangle-\left|\psi_{0}\right\rangle \|=\Theta\left(\frac{\| \sum_{i \neq 0} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|}{\| \sum_{i} \phi_{i} \gamma_{i}^{r}\left|\psi_{i}\right\rangle \|}\right) \tag{A12}
\end{equation*}
$$

## 5. Inequalities for Phase Estimation

Here, we find an upper bound on the amplitude $F(q)$ Equation (22):

$$
\begin{align*}
|F(q ; m)| & \leq \frac{\sin \left(\frac{\pi}{2^{m}}\right)}{\sqrt{2}\left|\sin \left(\frac{\pi+2 \pi q}{2^{m+1}}\right)\right|} \frac{1}{2|1 / 2-q|} \text { for } 0 \leq|1 / 2-q| \leq 2^{m-1} \\
& \leq \frac{\sin \left(\frac{\pi}{2^{m}}\right)}{\sqrt{2}\left|\frac{1+2 q}{2^{m}}\right|} \frac{1}{2|1 / 2-q|} \text { for } 0 \leq|1 / 2-q| \leq 2^{m-1}, \text { and for } 0 \leq|1 / 2+q| \leq 2^{m-1} \\
& \leq \frac{\sin \left(\frac{\pi}{2^{m}}\right) 2^{m}}{\sqrt{2} 4} \frac{1}{|1 / 2+q||1 / 2-q|} \text { for } 0 \leq|q|+1 / 2 \leq 2^{m-1} \tag{A13}
\end{align*}
$$

where we have used the fact that $|\sin (x)|$ is $\pi$-periodic in order to get a similar inequalities when $|x|+1 / 2>2^{m-1}$.

Therefore, the probability of getting an error greater than $\frac{k}{2^{m}}$ is

$$
\begin{align*}
e & =\sum_{k \leq l<2^{m-1}}\left|\alpha_{l z}\right|^{2}+\sum_{-2^{m-1} \leq l<k}\left|\alpha_{l z}\right|^{2} \leq c^{2} \sum_{k \leq l<2^{m-1}}\left(\frac{1}{\left(l-\delta 2^{m}+1 / 2\right)\left(l-\delta 2^{t}-1 / 2\right)}\right)^{2} \\
& +c^{2} \sum_{-2^{m-1}<l<-k}\left(\frac{1}{\left(l-\delta 2^{m}+1 / 2\right)\left(l-\delta 2^{m}-1 / 2\right)}\right)^{2}+c^{2}\left(\frac{1}{\left.\left|-2^{m-1}-\delta 2^{m}+1 / 2\right| \mid 2^{m-1}+\delta 2^{m}+1 / 2\right) \mid}\right)^{2} \\
& =c^{2} \sum_{l=k-1}^{2^{m-1}-2}\left(\frac{1}{\left(l-\delta 2^{m}+1 / 2+1\right)\left(l-\delta 2^{m}-1 / 2+1\right)}\right)^{2} \\
& +c^{2} \sum_{l=k-2}^{2^{m-1}-2}\left(\frac{1}{\left(l+\delta 2^{m}-1 / 2+1\right)\left(l+\delta 2^{m}+1 / 2+1\right)}\right)^{2}+c^{2}\left(\frac{\left.2^{m-1}+\delta 2^{m}-1 / 2| | 2^{m-1}+\delta 2^{m}+1 / 2\right) \mid}{l}\right)^{2} \\
& \leq c^{2^{2}} \sum_{l=k-1}^{2^{m-1}-2} \frac{1}{l^{4}}+c^{2} \sum_{l=k-2}^{2^{m-1}-1} \frac{1}{l^{4}} \leq 2 c^{2^{m-1}-1} \sum_{l=k-1} \frac{1}{l^{4}} \leq 2 c^{2} \int_{l=k-2}^{2^{m-1}-1} \frac{d l}{l^{4}} \\
& =\frac{2 c^{2}}{3(k-2)^{3}}<\frac{\pi^{2}}{48(k-2)^{3}} . \tag{A14}
\end{align*}
$$

For the term in the fourth line, we have used the fact that $|\sin (x)|$ is $\pi$-periodic to find a bound when $l=-2^{m}$. Therefore, to get an estimate that is within $k / 2^{m}=2^{p-1} / 2^{m}=1 / 2^{t+1}$ of the value of $\theta_{i}$ with an error rate of at most $e$, we need the total number qubits $m$ to be

$$
\begin{equation*}
m=t+p=t+\left\lceil\log _{2}\left(\frac{\pi^{2 / 3}}{48^{1 / 3} e^{1 / 3}}+2\right)\right\rceil \tag{A15}
\end{equation*}
$$

## Appendix B: Ancillary Circuits

Figures 9 and 10 provide ancillary circuits for the preparation of the filter distributions. Figure 11 just establishes the convention for $Q F T$.


FIG. 9: Here, we have taken the circuit up to the black dashed line in Figure 3 and simplified it given the initial state; same goes for Figure 6.

(a) Unsimplified circuit

(b) Simplified circuit

FIG. 10: Here, we have taken the circuit up to the black dashed line in Figure 4 and simplified it given the initial state; same goes for Figure 7.


FIG. 11: Our convention for $Q F T^{-1}$. The corresponding $Q F T$ can be obtained by simply inverting the circuit. This circuit is most widely labeled as $Q F T$ due to conventions in quantum computing literature [2, 3].
[1] D. S. Abrams and S. Lloyd, "Quantum Algorithm Providing Exponential Speed Increase for Finding Eigenvalues and Eigenvectors," Phys. Rev. Lett. 83 no. 24, (Dec., 1999) 5162-5165, arXiv:quant-ph/9807070 [quant-ph].
[2] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, "Quantum algorithms revisited," Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 454 no. 1969, (Jan, 1998) 339-354. http://dx.doi.org/10.1098/rspa.1998.0164.
[3] M. A. Nielsen and I. Chuang, "Quantum computation and quantum information," 2002.
[4] C. Gattringer and C. B. Lang, Quantum chromodynamics on the lattice, vol. 788. Springer, Berlin, 2010.
[5] T. Izubuchi, S. Aoki, K. Hashimoto, Y. Nakamura, T. Sekido, and G. Schierholz, "Dynamical QCD simulation with theta terms," PoS LATTICE2007 (2007) 106, arXiv:0802.1470 [hep-lat].
[6] G. Aarts, "Introductory lectures on lattice QCD at nonzero baryon number," J. Phys. Conf. Ser. 706 no. 2, (2016) 022004, arXiv:1512.05145 [hep-lat].
[7] A. Alexandru, G. Basar, P. F. Bedaque, S. Vartak, and N. C. Warrington, "Monte Carlo Study of Real Time Dynamics on the Lattice," Phys. Rev. Lett. 117 no. 8, (2016) 081602, arXiv:1605.08040 [hep-lat].
[8] S. Takeda, "Tensor network approach to real-time path integral," PoS LATTICE2019 (2019) 033, arXiv:1908.00126 [hep-lat].
[9] R. P. Feynman, "Simulating physics with computers," Int. J. Theor. Phys 21 no. 6/7, (1982) .
[10] Y. Ge, J. Tura Brugués, and J. Cirac, "Faster ground state preparation and high-precision ground energy estimation with fewer qubits," Journal of Mathematical Physics $60(02,2019) 022202$, arXiv:1712. 03193 [quant-ph].
[11] W. van Dam, G. M. D'Ariano, A. Ekert, C. Macchiavello, and M. Mosca, "Optimal Quantum Circuits for General Phase Estimation," Phys. Rev. Lett. 98 no. 9, (Mar., 2007) 090501, arXiv:quant-ph/0609160 [quant-ph].
[12] M. C. Bañuls, K. Cichy, Y.-J. Kao, C.-J. D. Lin, Y.-P. Lin, and D. T.-L. Tan, "Phase structure of the ( $1+1$ )-dimensional massive thirring model from matrix product states," Physical Review D 100 no. 9, (Nov, 2019) . http://dx.doi.org/10.1103/PhysRevD.100.094504.
[13] F. J. Harris, "On the use of windows for harmonic analysis with the discrete fourier transform," in Proc. IEEE, pp. 51-83. 1978.
[14] R. B. Blackman and J. W. Tukey, "The measurement of power spectra from the point of view of communications engineering-part i," Bell System Technical Journal 37 no. 1, (1958) 185-282.
[15] D. Coppersmith, "An approximate Fourier transform useful in quantum factoring," arXiv e-prints (Jan., 2002) quant-ph/0201067, arXiv:quant-ph/0201067 [quant-ph].
[16] A. Y. Kitaev, "Quantum measurements and the abelian stabilizer problem," arXiv preprint quant-ph/9511026 (1995) .
[17] J. H. Wilkinson, The Algebraic Eigenvalue Problem. Oxford University Press, Inc., USA, 1988.
[18] C. Yi and E. Crosson, "Spectral Analysis of Product Formulas for Quantum Simulation," arXiv e-prints (Feb., 2021) arXiv:2102.12655, arXiv:2102.12655 [quant-ph].
[19] D. W. Berry, A. M. Childs, and R. Kothari, "Hamiltonian simulation with nearly optimal dependence on all parameters," 2015 IEEE 56th Annual Symposium on Foundations of Computer Science (Oct, 2015) . http://dx.doi.org/10.1109/FOCS.2015.54.
[20] D. W. Berry, A. M. Childs, R. Cleve, R. Kothari, and R. D. Somma, "Simulating hamiltonian dynamics with a truncated taylor series," Physical Review Letters 114 no. 9, (Mar, 2015) . http: //dx.doi.org/10.1103/PhysRevLett.114.090502.
[21] G. H. Low and I. L. Chuang, "Optimal hamiltonian simulation by quantum signal processing," Physical Review Letters 118 no. 1, (Jan, 2017). http: //dx.doi.org/10.1103/PhysRevLett.118.010501.
[22] G. H. Low and I. L. Chuang, "Hamiltonian Simulation by Qubitization," Quantum 3 (July, 2019) 163. https://doi.org/10.22331/q-2019-07-12-163.
[23] D. W. Berry, G. Ahokas, R. Cleve, and B. C. Sanders, "Efficient quantum algorithms for simulating sparse hamiltonians," Communications in Mathematical Physics 270 no. 2, (Dec, 2006) 359-371. http://dx.doi.org/10.1007/s00220-006-0150-x.
[24] M. Suzuki, "General theory of fractal path integrals with applications to many-body theories and statistical physics," Journal of Mathematical Physics 32 (1991) 400-407.
[25] G. Hao Low, V. Kliuchnikov, and N. Wiebe, "Well-conditioned multiproduct Hamiltonian simulation," arXiv e-prints (July, 2019) arXiv:1907.11679, arXiv:1907.11679 [quant-ph].
[26] E. Farhi, J. Goldstone, and S. Gutmann, "A quantum approximate optimization algorithm," arXiv:1411.4028 [quant-ph].
[27] D. Wecker, M. B. Hastings, and M. Troyer, "Progress towards practical quantum variational algorithms," Phys. Rev. A 92 (Oct, 2015) 042303. https: //link.aps.org/doi/10.1103/PhysRevA.92.042303.
[28] E. Farhi and A. W. Harrow, "Quantum supremacy through the quantum approximate optimization algorithm," arXiv:1602.07674 [quant-ph].
[29] L. Zhou, S.-T. Wang, S. Choi, H. Pichler, and M. D. Lukin, "Quantum approximate optimization algorithm: Performance, mechanism, and implementation on near-term devices," Phys. Rev. X 10 (Jun, 2020) 021067.
[30] W. W. Ho and T. H. Hsieh, "Efficient variational simulation of non-trivial quantum states," SciPost Physics 6 no. 3, (Mar, 2019) .
http://dx.doi.org/10.21468/SciPostPhys.6.3.029.


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