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# First-Order Quantum Correction in Coherent State Expectation Value of Loop-Quantum-Gravity Hamiltonian

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Given the non-graph-changing Hamiltonian  $\widehat{H}[N]$  in Loop Quantum Gravity (LQG),  $\langle \widehat{H}[N] \rangle$ , the coherent state expectation value of  $\widehat{H}[N]$ , admits a semiclassical expansion in  $\ell_p^2$ . In this paper, we explicitly compute the expansion of  $\langle \widehat{H}[N] \rangle$  to the linear order in  $\ell_p^2$  on the cubic graph with respect to the coherent state peaked at the homogeneous and isotropic data of cosmology. In our computation, a powerful algorithm, supported by rigorous proofs and several theorems, is developed to overcome the complexity in the computation of  $\langle \widehat{H}[N] \rangle$ . Particularly, some key innovations in our algorithm substantially reduce the complexity in computing the Lorentzian part of  $\langle \widehat{H}[N] \rangle$ . Moreover, with the algorithm developed in the present work, we can compute the expectation value of arbitrary monomial of holonomies and fluxes on one edge up to arbitrary order of  $\ell_p^2$ . Finally, some quantum correction effects resulted from  $\langle \widehat{H}[N] \rangle$  in cosmology are discussed at the end of this paper.

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## I. Introduction

Loop Quantum Gravity (LQG) is an approach toward the background independent and nonperturbative quantum gravity theory in four and higher dimensions [1–5]. Several recent progresses have been made by the active research of the quantum dynamics of LQG [6–16]. Particularly, tremendous progresses have been made in both canonical and covariant LQG on the semiclassical limit and the consistency with respect to classical gravity e.g. [7, 8, 15, 17–27]. However, regarding the full theory of LQG dynamics, less progress has been made on its quantum corrections (see e.g. [28–31] for some results, and [32, 33] for some results in the covariant approach). As a candidate of quantum gravity theory, it is important that LQG should shed light on quantum corrections to the classical theory of gravity.

The present paper focuses on the canonical aspects of LQG. Due to the non-polynomial Hamiltonian constraint operator  $\widehat{H}[N] = \widehat{H}_E[N] + (1 + \beta^2)\widehat{H}_L[N]$ , there has been persistent confusion that the quantum dynamics of LQG might not be computable analytically [34]. A previous work [8] partially resolves this confusion, where it has been schematically shown that the coherent state expectation value of the Hamiltonian/master constraint are computable order-by-order by the semiclassical expansion in  $\hbar$ . It is remarkable that the proposed scheme in [8] can also be applied to a wide class of non-polynomial operators used in the study of LQG dynamics. Although this scheme was proposed as early as when [8] firstly published in 2006, the expectation value of  $\widehat{H}[N]$  has only been computed at its classical limit, i.e. the 0-th order (in  $\hbar$ ). However, due to the complexity of the operator, especially the Lorentzian part of  $\widehat{H}[N]$  (denoted as  $\widehat{H}_L[N]$ ), the  $O(\hbar)$  quantum correction has not been studied in the literature.

The goal of the present work is to fill this gap by providing an explicit computation of the  $O(\hbar)$  quantum correction in  $\langle \widehat{H}[N] \rangle$  with respect to a certain coherent state. In this paper, in order to compute the quantum correction in  $\langle \widehat{H}[N] \rangle$ , a powerful algorithm is developed to overcome the complexity of  $\widehat{H}[N]$  that is the non-graph-changing Hamiltonian on a cubic lattice  $\gamma$ . We explicitly expand  $\langle \widehat{H}[N] \rangle$  to linear order in  $\hbar$  by applying the algorithm, with respect to the coherent state that is peaked at the homogeneous and isotropic data of cosmology. Namely we explicitly compute  $H_0$  and  $H_1$  in

$$\langle \widehat{H}[1] \rangle = H_0 + \ell_p^2 H_1 + O(\ell_p^4), \quad \ell_p^2 = \hbar \kappa \quad (1.1)$$

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where  $\kappa = 8\pi G_{\text{Newton}}$  and the lapse function  $N = 1$ .  $H_0$ , representing the 0-th order, reproduces the cosmological effective Hamiltonian in the  $\mu_0$ -scheme [15, 16, 35, 36]. Whereas,  $H_1$  gives the first order quantum correction, which is presented by our result in this work. The explicit expression of  $H_1$  is given in Section VI. It is worth noting that the coherent state used for computing  $\langle \widehat{H}[N] \rangle$  is not  $\text{SU}(2)$  gauge invariant (the motivation is stated below).

This work is closely related to the reduced phase space formulation of LQG (see e.g. [9, 37, 38]). In this formulation, some matter fields that are known as the clock fields is coupled to gravity. These matter fields are regarded as material reference frames used to transform gravity variables to gauge invariant Dirac observables. This procedure resolves the Diffeomorphism constraint and Hamiltonian constraint at the classical level resulting in the reduced phase space  $\mathcal{P}_{red}$  of Dirac observables. The dynamics of the gravity-clock system is described by the material-time evolution generated by the physical Hamiltonian  $\mathbf{H}$  on the reduced phase space  $\mathcal{P}_{red}$ . As an interesting model, Gaussian dust is chosen to be our clock fields [38, 39]. Then the resulting reduced phase space,  $\mathcal{P}_{red}$ , is identical to the pure-gravity unconstrained phase space. This identification defines the pure-gravity Hamiltonian constraint with unit lapse (i.e.  $N = 1$ )  $H[1]$  on the resulting reduced phase space  $\mathcal{P}_{red}$ , which indicates that the physical Hamiltonian  $\mathbf{H}$  equals to  $H[1]$  for the case when gravity is coupled to Gaussian dust. In this model, the quantization of  $\mathcal{P}_{red}$  is the same as quantizing the pure-gravity unconstrained phase space, which leads to the physical Hilbert space  $\mathcal{H}$  that is identical to the kinematical Hilbert space in the usual LQG.  $\mathcal{H}$  is unconstrained because it is from the quantization of  $\mathcal{P}_{red}$ . The physical Hamiltonian operator is obtained by  $\widehat{\mathbf{H}} = \frac{1}{2}(\widehat{H}[1] + \widehat{H}[1]^\dagger)$  with the usual LQG quantization of  $\widehat{H}[N]$  [6, 40–42]. Therefore from the perspective of reduced-phase-space LQG, our work computes the  $\langle \widehat{\mathbf{H}} \rangle$  with respect to the coherent state peaked at cosmological data on the graph  $\gamma$ , which is given by the real part of  $\langle \widehat{H}[1] \rangle$  (1.1).

Recent works have been focused on building models of LQG on a single graph  $\gamma$  [15, 20, 21, 28, 43–46]. Particularly, the quantum dynamics in the reduced-phase-space LQG framework is formulated on a cubic lattice  $\gamma$  with a path integral [16, 22]

$$A_{[g],[g']} = \int dh[dg] \nu[g] e^{S[g,h]/\ell_p^2}, \quad (1.2)$$

which is the canonical-LQG analog of the spinfoam formulation.  $A_{[g],[g']}$  is regarded as the transition amplitude of  $\widehat{\mathbf{H}}$  between the initial and final  $\text{SU}(2)$  gauge invariant coherent states, denoted as  $|[g]\rangle, |[g']\rangle$  respectively. The integration variables contains both trajectories of  $g \in \mathcal{P}_{red,\gamma}$  and  $\text{SU}(2)$  gauge transformations  $h$  on  $\gamma$ , with  $\nu[g]$  being a measure factor. Due to a feature of the path integral (1.2), the  $\text{SU}(2)$  gauge invariant amplitude  $A_{[g],[g']}$  is expressed as an integral of  $\text{SU}(2)$  gauge non-invariant variables  $g$  and  $h$ <sup>1</sup>. The action  $S[g, h]$  is linear to  $\widehat{\mathbf{H}}$  at  $\text{SU}(2)$  gauge non-invariant coherent states when the trajectories of  $g$  are continuous in time. In contrast to the usual path integrals in quantum field theories,  $S[g, h]$  contains the  $O(\hbar)$  correction from  $\langle \widehat{\mathbf{H}} \rangle$ . Our work precisely computes this  $O(\hbar)$  correction in  $S[g, h]$  of cosmological dynamics.

A general study of the equation of motion provided by the semiclassical limit of  $A_{[g],[g']}$  is presented in [22]. The application of it in cosmology is presented in [16, 46]. The cosmological dynamics in the limit of  $\hbar \rightarrow 0$  gives the  $\mu_0$ -scheme effective cosmological dynamics which reduces to the classical FLRW cosmology at low energy density. Next, it is equally important to discover the  $O(\hbar)$  correction of the effective cosmological dynamics. The effective dynamics with  $O(\hbar)$  correction can be obtained by the *quantum effective action* [47], denoted as  $\Gamma$ , from the path integral defined in (1.2). Perturbatively, the  $O(\hbar)$  correction in  $\Gamma$  for cosmology contains 3 contributions: (1)  $O(\hbar)$  correction in  $S[g, h]$  which is computed in this work, (2)  $O(\hbar)$  correction in  $\log \nu[g]$  where  $\nu[g]$  has been given explicitly in [16], and (3)  $O(\hbar)$  correction in  $\frac{1}{2} \log \det(\mathfrak{H})$  where the “1-loop determinant”  $\det(\mathfrak{H})$  is the determinant of the Hessian matrix  $\mathfrak{H}$  of  $S[g, h]$ . The  $g$ - $g$  matrix elements in  $\mathfrak{H}$  has been computed in [46]. A continuous study of  $\frac{1}{2} \log \det(\mathfrak{H})$  is postponed for future work. Therefore, in terms of the quantum correction in the effective cosmological dynamics, the present work computes a significant part in the  $O(\hbar)$  correction of the quantum effective action  $\Gamma$ .

After introducing motivations above, let us summarize several key steps in the computation of the present work: First of all, an important complication in  $\widehat{H}[N]$  is the volume operator  $\widehat{V}_v = \sqrt{|\widehat{Q}_v|}$  which contains the square-root and absolute-value, indicating that the  $\widehat{H}[N]$  is non-polynomial. When  $\langle \widehat{H}[N] \rangle$  is studied with respect to the coherent

<sup>1</sup> Here is a brief explanation of the reason why the path integral is in terms of gauge non-invariant coherent states (see [16] for details): The transition amplitude between gauge invariant coherent states is  $A_{[g],[g']} = \langle \Psi_{[g]}^t | U(T) | \Psi_{[g']}^t \rangle$  where  $U(T) = \exp\left(-\frac{i}{\hbar} T \widehat{\mathbf{H}}\right)$ . The gauge invariant coherent state is the group average of the gauge non-invariant state:  $|\Psi_{[g]}^t\rangle = \int dh |\psi_{g,h}^t\rangle$  where  $h$  is the  $\text{SU}(2)$  gauge transformation. Since  $U(T)$  is gauge invariant, we have  $A_{[g],[g']} = \int dh \langle \psi_g^t | U(\tau) | \psi_{g,h}^t \rangle$  where the integrand can be written as a coherent state path integral with the standard method.

state, this issue is overcome by substituting  $\hat{V}_v$  with the semiclassical expansion [8]

$$\hat{V}_{GT}^{(v)} = \langle \hat{Q}_v \rangle^{2q} \left[ 1 + \sum_{n=1}^{2k+1} (-1)^{n+1} \frac{q(1-q) \cdots (n-1-q)}{n!} \left( \frac{\hat{Q}_v^2}{\langle \hat{Q}_v \rangle^2} - 1 \right)^n \right] + O(\hbar^{k+1}) \quad (1.3)$$

where  $\hat{Q}_v$  is formulated as a polynomial of flux operators and  $q = 1/4$ . Truncating  $\hat{V}_{GT}^{(v)}$  with a finite  $k$  and substituting it back into  $\widehat{H}[N]$  allows us to express  $\langle \widehat{H}[N] \rangle$  by a expectation value of a polynomial operator.

The resulting polynomial sums over a huge number of terms ( $\sim 10^{19}$ ), each of which is a monomial of holonomy and flux operators. Computing expectation values of all terms would lead to a large computational complexity. The major complexity is encoded in the Lorentzian part of  $\widehat{H}[N]$ , denoted as  $\widehat{H}_L[N]$ . Several key methods are used to reduce the number of computations:

- The expectation value of every monomial term can be factorized into expectation values of holonomy-flux monomials with respect to different edges. Only certain types of expectation values of monomials on a single edge shall be computed. We further reduce the number of types by using the commutation relations, and several general formulae are derived for the expectation values of the resulting types (see Section III).
- We develop a power-counting argument in order to specifically locate each power of  $\hbar$ , expression in  $O(\hbar)$  represents the leading order behavior of each expectation value of the monomial operator (see Section IV). Since we are only interested in expanding  $\langle \widehat{H}[N] \rangle$  to its linear order in  $\hbar$ , a substantial amount of expectation values of monomials can be neglected due to the fact that they are only contributing to higher order in  $\hbar$ .
- When the coherent states are peaked at homogeneous and isotropic data. A large amount of symmetries that identify different terms are realized, which can be used to reduce the computational complexity. (see Section V).

Our method exponentially reduces the computational complexity. In particular, it is useful in computing the expectation value of Lorentzian part in  $\widehat{H}[N]$ .

In Section VB, In order to present the reduction methodology more concretely, an example that contains  $3^{3m-1}$  ( $m$  can be large) monomials is demonstrated. By applying our method, only 5 monomials' expectation values need to be computed.

The purpose of the present paper is to give detailed derivations for the results presented in [48]. Computations in this paper are carried out by using Mathematica on the High Performance Computation server with two 48-Core Processors (AMD EPYC 7642). One can find the Mathematica codes at [49].

The explicit resulting expression of  $O(\hbar)$  quantum correction in  $\langle \widehat{H}[N] \rangle$  is summarized in Section VI. In order to demonstrate the physical significance of our results and effects from the  $O(\hbar)$  correction to the classical limit of  $\Re \langle \widehat{H}[1] \rangle$ , the proposal in [15] is adopted: We view  $\Re \langle \widehat{H}[1] \rangle$  in (1.1) as the effective Hamiltonian on the 2-dimensional phase space, denoted as  $\mathcal{P}_{cos}$ , of homogeneous and isotropic cosmology.  $\Re \langle \widehat{H}[1] \rangle$  generates the Hamiltonian time evolution on the 2-dimensional phase space  $\mathcal{P}_{cos}$ . Time evolution of the homogeneous spatial volume is plotted, and is compared with the evolution generated by  $\langle \widehat{H}[1] \rangle$  at the limit of  $\hbar \rightarrow 0$ . The comparison demonstrates the effects on  $\langle \widehat{H}[1] \rangle$ , which is generated from the  $O(\hbar)$  correction contribution (see Section VI for details). We emphasize that the proposal that we adopt for the cosmological evolution is not as rigorous as the path integral formula (1.2). Nevertheless, we have argued that the  $O(\hbar)$  correction in  $\langle \widehat{H}[1] \rangle$  only contributes partially to the quantum correction in  $\Gamma$  which ultimately determines the quantum effect in the dynamics. The cosmological dynamics studied in Section VI only aims for displaying the effect of the  $O(\hbar)$  correction in  $\langle \widehat{H}[1] \rangle$ , and is not a rigorous prediction from the principle of LQG.

The structure of the present paper follows. Section II reviews the theory of LQG on a cubic lattice, including the Hamiltonian and the coherent state. Section III, we demonstrate the computations of the expectation value of operators defined at a single edge. Section IV, we develop a power-counting argument in order to reduce the computational complexity. Section V discusses  $\langle \widehat{H}[1] \rangle$  with respect to the coherent states peaked at homogeneous and isotropic data, and the symmetries which reduce the computational complexity. Section VI presents the explicit results of the quantum correction in  $\langle \widehat{H}[1] \rangle$ . Section VII, we conclude and discuss a few outlooks of the present work.

## II. Preliminaries

### A. Quantization and Hamiltonian

Classically general relativity can be formulated with the Ashtekar-Barbero variables  $(A_a^i, E_i^a)$  consisting of  $SU(2)$  connection  $A_a^i$  and canonically conjugate densitized triad field  $E_i^a$  defined on the spatial manifold  $\Sigma$  [50]. We denote the coordinate on  $\Sigma$  by  $(x, y, z)$ . Let  $\gamma \subset \Sigma$  be a finite cubic lattice whose edges are parallel to the axes of the coordinates. The sets of edges and vertices in  $\gamma$  are denoted by  $E(\gamma)$  and  $V(\gamma)$  respectively. Taking advantage of  $\gamma$ , we define holonomies along the edges of  $\gamma$ ,

$$h_e(A) = \mathcal{P} \exp \int_e A = 1 + \sum_{n=1}^{\infty} \int_0^1 dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 A(t_1) \cdots A(t_n), \quad \forall e \in E(\gamma), \quad (2.1)$$

and gauge covariant fluxes [51] on the 2-faces  $S_e$  in the dual lattices  $\gamma^*$ ,

$$p_s^i(e) := -\frac{2}{\beta a^2} \text{tr} \left[ \tau^i \int_{S_e} \varepsilon_{abc} h(\rho_e^s(\sigma)) E^c(\sigma) h(\rho_e^s(\sigma)^{-1}) \right], \quad (2.2)$$

where  $S_e \in \gamma^*$  is the 2-face,  $\rho^s(\sigma) : [0, 1] \rightarrow \Sigma$  is a path connecting the source point  $s_e \in e$  to  $\sigma \in S_e$  such that  $\rho_e^s(\sigma) : [0, 1/2] \rightarrow e$  and  $\rho_e^s(\sigma) : [1/2, 1] \rightarrow S_e$ .  $a$  is a length unit (e.g.  $a = 1\text{mm}$ ) to make  $p_s(e)$  dimensionless. Alternatively, one can choose the target point  $t_e \in e$  rather than  $s_e$  to define

$$p_t^i(e) := \frac{2}{\beta a^2} \text{tr} \left[ \tau^i \int_{S_e} \varepsilon_{abc} h(\rho_e^t(\sigma)) E^c(\sigma) h(\rho_e^t(\sigma)^{-1}) \right]. \quad (2.3)$$

where  $\rho^t(\sigma) : [0, 1] \rightarrow \Sigma$  is a path connecting the target  $t_e \in e$  to  $\sigma \in S_e$  such that  $\rho_e^t(\sigma) : [0, 1/2] \rightarrow e$  and  $\rho_e^t(\sigma) : [1/2, 1] \rightarrow S_e$ . Given  $(A_a^i, E_i^a)$ , Eqs. (2.1) and (2.2) lead to a map from the  $E(\gamma)$  to  $SL(2, \mathbb{C})$ ,

$$\mathbf{g} : e \mapsto g_e = e^{ip_s^k(e)\tau_k} h_e. \quad (2.4)$$

Because of the relation between  $p_s$  and  $p_t$

$$p_s^k(e^{-1})\tau_k = p_t^k(e)\tau_k = -h_e^{-1}p_s^k(e)\tau_k h_e \quad (2.5)$$

we obtain that

$$g_{e^{-1}} = g_e^{-1}. \quad (2.6)$$

Thus the map  $\mathbf{g} : E(\gamma) \rightarrow SL(2, \mathbb{C})$  generate a homomorphism from the groupoid of the graph  $\gamma$  to  $SL(2, \mathbb{C})$ . The LQG phase space based on  $\gamma$  is  $SL(2, \mathbb{C})^{|E(\gamma)|}$  and consists of all such homomorphisms [51]. Given a  $SU(2)$ -valued scalar field  $\mathcal{G} : \Sigma \rightarrow SU(2)$  on  $\Sigma$ ,  $\mathcal{G}$  defines a gauge transformation on  $\mathbf{g}$ , taking  $\mathbf{g}$  to  $\mathcal{G} \triangleright \mathbf{g}$  with

$$(\mathcal{G} \triangleright \mathbf{g})(e) = \mathcal{G}(s_e)\mathbf{g}(e)\mathcal{G}(t_e)^{-1}, \quad \forall e \in E(\gamma). \quad (2.7)$$

The quantization of this classical lattice theory gives us LQG based on the graph  $\gamma$ . The Hilbert space  $\mathcal{H}_\gamma$  consists of the square integrable functions of the holonomies. Given two functions  $\psi_i : \{h_e\}_{e \in E(\gamma)} \rightarrow \mathbb{C}$ , the inner produce is

$$\langle \psi_1 | \psi_2 \rangle = \int_{SU(2)^{|E(\gamma)|}} d\mu_h \overline{\psi_1(\{h_e\}_{e \in E(\gamma)})} \psi_2(\{h_e\}_{e \in E(\gamma)}) \quad (2.8)$$

where  $|E(\gamma)|$  denote the number of elements (i.e. cardinality) of  $E(\gamma)$  and  $\mu_h$  is the Haar measure.  $\mathcal{H}_\gamma$  is the kinematical Hilbert space of the canonical LQG with the operator-constraint formalism. Moreover,  $\mathcal{H}_\gamma$  modulo gauge transformations represents the physical Hilbert space of the reduced-phase-space LQG, where any gauge invariant function of  $h_e(A)$  and  $p_{s,t}^i(e)$  are Dirac observables, realized from the deparametrization by coupling to clock fields [37].

On  $\mathcal{H}_\gamma$ ,  $p_s^i(e)$  and  $p_t^i(e)$  are quantized as the right- and left-invariant vector field, namely

$$\begin{aligned} (\hat{p}_s^i(e)\psi)(h_{e'}, \dots, h_e, \dots, h_{e''}) &= it \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \psi(h_{e'}, \dots, e^{\epsilon\tau^i} h_e, \dots, h_{e''}) \\ (\hat{p}_t^i(e)\psi)(h_{e'}, \dots, h_e, \dots, h_{e''}) &= -it \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \psi(h_{e'}, \dots, h_e e^{\epsilon\tau^i}, \dots, h_{e''}) \end{aligned} \quad (2.9)$$

where  $t = \kappa\hbar/a^2 =: \ell_p^2/a^2$  (if  $a = 1\text{mm}$ ,  $t \simeq 6.56 \times 10^{-63}$ ) and  $\tau^j = (-i/2)\sigma^j$  with  $\sigma^j$  being the Pauli matrix. Another kind of basic operators are the multiplication operators  $D_{ab}^t(h_e)$  which is defined as

$$(D_{ab}^t(h_e)\psi)(A) = D_{ab}^t(h_e(A))\psi(A)$$

where  $D^t(h_e(A))$  is the value of the Wigner-D matrix at  $h_e(A) \in \text{SU}(2)$ . In this paper,  $D^t(x)$  denotes the Wigner-D matrix only if  $x$  is some specific  $\text{SU}(2)$  element. Moreover,  $h_e$ , when it appears alone as an operator, denote the matrix-valued multiplication operator  $D^{1/2}(h_e)$  for simplicity. With this convention, the commutators between the basic operators read

$$\begin{aligned} [D^t(h_e), D^t(h_{e'})] &= 0 = [\hat{p}_s^i(e), \hat{p}_t^j(e')] \\ [\hat{p}_s^i(e), \hat{p}_s^j(e')] &= -it\delta_{ee'}\epsilon_{ijk}\hat{p}_s^k(e), \\ [\hat{p}_t^i(e), \hat{p}_t^j(e')] &= -it\delta_{ee'}\epsilon_{ijk}\hat{p}_t^k(e), \\ [\hat{p}_s^i(e), D^t(h_{e'})] &= it\delta_{ee'}D^{t'}(\tau^i)D^t(h_e), \\ [\hat{p}_t^i(e), D^t(h_{e'})] &= -it\delta_{ee'}D^t(h_e)D^{t'}(\tau^i). \end{aligned} \quad (2.10)$$

where  $D^{t'}(\tau^i)$  is the corresponding representation matrix of  $\tau^i$ .

It is useful to introduce the flux operators with respect to the spherical basis. We define

$$\hat{p}_v^{\pm 1}(e) := \mp \frac{1}{\sqrt{2}} (\hat{p}_v^x(e) \pm i\hat{p}_v^y(e)), \quad \hat{p}_v^0(e) = \hat{p}_v^z(e) \quad (2.11)$$

with  $v = s, t$ . In the following context,  $\alpha, \beta, \dots = 0, \pm 1$  is used to denote the indices in the spherical basis, and  $i, j, k, \dots = 1, 2, 3$ , the indices in the Cartesian basis.

Taking advantage of the basic operators, one can define operators representing geometric observables such as areas and volumes [52–54]. The volume operator plays an important role in the present work. Let  $\mathcal{R} \subset \Sigma$  be a region in  $\Sigma$ . The volume operator of  $\mathcal{R}$  is defined by

$$\hat{V}_{\mathcal{R}} := \sum_{v \in V(\gamma) \cap \mathcal{R}} \hat{V}_v = \sum_{v \in V(\gamma) \cap \mathcal{R}} \sqrt{|\hat{Q}_v|} \quad (2.12)$$

where

$$\hat{Q}_v = (\beta a^2)^3 \epsilon_{ijk} \frac{\hat{p}_s^i(e_x^+) - \hat{p}_t^i(e_x^-)}{2} \frac{\hat{p}_s^j(e_y^+) - \hat{p}_t^j(e_y^-)}{2} \frac{\hat{p}_s^k(e_z^+) - \hat{p}_t^k(e_z^-)}{2} \quad (2.13)$$

where  $e_i^{\pm}$  with  $i = x, y, z$  are the edges along the  $i$ th axis such that  $v$  is the source point of  $e_i^+$  and the target point of  $e_i^-$ . The total volume is denoted by  $\hat{V} = \sum_{v \in V(\gamma)} \hat{V}_v$ . In terms of the flux operators with respect to the spherical basis (2.11), the operator  $\hat{Q}_v$  defined in Eq. (2.13) becomes

$$\hat{Q}_v = -i(\beta a^2)^3 \epsilon_{\alpha\beta\gamma} \frac{\hat{p}_s^\alpha(e_x^+) - \hat{p}_t^\alpha(e_x^-)}{2} \frac{\hat{p}_s^\beta(e_y^+) - \hat{p}_t^\beta(e_y^-)}{2} \frac{\hat{p}_s^\gamma(e_z^+) - \hat{p}_t^\gamma(e_z^-)}{2} \quad (2.14)$$

where  $\epsilon_{\alpha\beta\gamma}$  is defined by  $\epsilon_{-1,0,1} = 1$ .

In the operator-constraint formalism, the dynamics of LQG is encoded in the Hamiltonian constraint, which can be written as

$$\widehat{H[N]} = \widehat{H_E[N]} + (1 + \beta^2)\widehat{H_L[N]} \quad (2.15)$$

where  $\widehat{H_E[N]}$  is called the Euclidean part and  $\widehat{H_L[N]}$  is the Lorentzian part.  $N$  is the smeared function.  $\widehat{H[N]}$  is constructed by using the Thiemann's trick [6, 41]. The operator corresponding to the Euclidean part is

$$\widehat{H_E[N]} = \frac{1}{i\beta a^2 t} \sum_{v \in V(\gamma)} N(v) \sum_{e_I, e_J, e_K \text{ at } v} \epsilon^{IJK} \text{tr}(h_{\alpha_{IJ}} h_{e_K} [\hat{V}_v, h_{e_K}^{-1}]) \quad (2.16)$$

where  $e_I, e_J$  and  $e_K$  are oriented to be outgoing from  $v$ ,  $\epsilon^{IJK} = \text{sgn}[\det(e_I \wedge e_J \wedge e_K)]$ ,  $\alpha_{IJ}$  is the minimal loop around a plaquette consisting of  $e_I$  and  $e_J$ , where it goes out via  $e_I$  and comes back through  $e_J$ , taking  $v$  as its end point. With the same notion, the Lorentzian part reads

$$\widehat{H_L[N]} = \frac{-1}{2i\beta^7 a^{10} t^5} \sum_v N(v) \sum_{e_I, e_J, e_K \text{ at } v} \epsilon^{IJK} \text{tr}([h_{e_I}, [\hat{V}, \hat{H}_E]] h_{e_I}^{-1} [h_{e_J}, [\hat{V}, \hat{H}_E]] h_{e_J}^{-1} [h_{e_K}, \hat{V}_v] h_{e_K}^{-1}). \quad (2.17)$$

In the reduced-phase-space LQG where the diffeomorphism and Hamiltonian constraints are solved classically, the quantum dynamics is governed by the physical Hamiltonian  $\widehat{\mathbf{H}}$ . When we deparametrize gravity by coupling to the Gaussian dust [38, 39], the classical physical Hamiltonian  $\mathbf{H}$  is formally the same as the Hamiltonian constraint with unit lapse, except all quantities in  $\mathbf{H}$  are understood as Dirac observables. The quantization gives the Hamiltonian operator

$$\widehat{\mathbf{H}} = \frac{1}{2} \left( \widehat{H[1]} + \widehat{H[1]}^\dagger \right) \quad (2.18)$$

$\widehat{\mathbf{H}}$  is defined on  $\mathcal{H}_\gamma$ , which can be understood from a similar perspective of quantizing Dirac observables. Note that here we consider the non-graph-changing version of the Hamiltonian (constraint). If  $H[N]$  is understood as constraint, the discretization and quantization on  $\gamma$  cause the constraint anomaly. However in the reduced-phase-space LQG, the constraint anomaly is absent, because  $H[N]$  is not a constraint anymore.<sup>2</sup> The self-adjoint extension of  $\widehat{\mathbf{H}}$  exists [40, 55], so we choose the extension and define the self-adjoint Hamiltonian which is still denoted by  $\widehat{\mathbf{H}}$ .

### B. Coherent states

Choosing a canonical orientation for each edge  $e \in E(\gamma)$ , the classical phase space based on the graph  $\gamma$  is

$$\Gamma_\gamma \cong [\mathrm{SL}(2, \mathbb{C})]^{|E(\gamma)|}. \quad (2.19)$$

The complexifier coherent state  $\Psi_{\mathbf{g}}$  is [17]

$$\Psi_{\mathbf{g}} = \bigotimes_{e \in E(\gamma)} \psi_{g_e}^t, \quad \psi_{g_e}^t(h_e) = \sum_j d_j e^{-\frac{i}{2}j(j+1)} \chi_j(g_e h_e^{-1}) \quad (2.20)$$

where  $\psi_{g_e}^t$  is the  $\mathrm{SU}(2)$  coherent state at the edge  $e$ . The character  $\chi_j(g_e h_e^{-1})$  is the trace of the  $j$ -representation of  $g_e h_e^{-1}$ . The property  $\chi_j(g_e h_e^{-1}) = \chi_j(g_e^{-1} h_e)$  leads to the useful relation

$$\psi_{g_e}(h_e) = \psi_{g_{e^{-1}}}(h_{e^{-1}}).$$

Given  $g \in \mathrm{SL}(2, \mathbb{C})$ , it can be decomposed as

$$g = e^{i p_k \tau^k} u = n^s e^{i(\eta + i\xi)\tau_3} (n^t)^{-1}, \quad (2.21)$$

where  $\eta = -\sqrt{\vec{p} \cdot \vec{p}}$  and  $n^s, n^t \in \mathrm{SU}(2)$ , as well as  $\xi \in \mathbb{R}$ , are given by

$$\begin{aligned} n^s \tau^3 (n^s)^{-1} &= -\frac{p_k}{\sqrt{\vec{p} \cdot \vec{p}}} \tau_k, \\ n^s e^{-\xi \tau_3} (n^t)^{-1} &= u. \end{aligned} \quad (2.22)$$

Although  $n^s$  and  $n^t$  are not uniquely defined by this equation, each of them relates to a unique vector through the equation, with  $v = s, t$ ,

$$n^v \tau_3 (n^v)^{-1} = \vec{n}^v \cdot \vec{\tau}. \quad (2.23)$$

It is shown in [19] and is revisited shortly that

$$\frac{\langle \psi_{g_e}^t | \vec{p}_s(e) | \psi_{g_e}^t \rangle}{\|\psi_{g_e}^t\|^2} = -\eta_e \vec{n}_e^s + O(t), \quad \frac{\langle \psi_{g_e}^t | \vec{p}_t(e) | \psi_{g_e}^t \rangle}{\|\psi_{g_e}^t\|^2} = \eta_e \vec{n}_e^t + O(t) \quad (2.24)$$

where  $\eta_e$ ,  $\vec{n}_e^s$  and  $\vec{n}_e^t$  are defined as the decomposition parameters of  $g_e$  as in Eq. (2.21). This equation indicates that  $\eta_e \vec{n}_e^t$  is the classical limit of the flux operator at  $e$ .

<sup>2</sup> Sometimes,  $H[N]$  relates to conserved charges, then  $\widehat{H[N]}$  on  $\gamma$  may break the classical symmetry.

The following properties of the  $\psi_g^t$  [17–19] are useful in our analysis. Firstly, the inner product of these states read

$$\langle \psi_{g_1}^t | \psi_{g_2}^t \rangle = \psi_{g_1^\dagger g_2}^{2t} (1) = \frac{2\sqrt{\pi}e^{t/4}}{t^{3/2}} \frac{\zeta e^{\frac{\zeta^2}{t}}}{\sinh(\zeta)} + O(t^\infty) \quad (2.25)$$

where  $\text{tr}(g_1^\dagger g_2) = 2 \cosh(\zeta)$  and  $\Im(\zeta) \in [0, \pi]$  with  $\Im(\zeta)$  the imaginary part of  $\zeta$ <sup>3</sup>. Consequently, the norm of the coherent state is

$$\langle 1 \rangle_g := \langle \psi_g^t | \psi_g^t \rangle = \frac{2\sqrt{\pi}e^{t/4}}{t^{3/2}} \frac{pe^{\frac{p^2}{t}}}{\sinh(p)} + O(t^\infty), \quad (2.26)$$

where  $p = \sqrt{\bar{p} \cdot p}$ . Secondly,  $\psi_g^t$  satisfy the completeness condition

$$\int d\nu_t(g) |\psi_g^t\rangle \langle \psi_g^t| = \mathbb{I}, \quad (2.27)$$

where the measure  $d\nu_t(g)$  is

$$d\nu_t(g) = \frac{2\sqrt{2}e^{-t/4}}{(2\pi t)^{3/2}} \frac{\sinh(p)}{p} e^{-\frac{p^2}{t}} d\mu_H(u) d^3p = \frac{2}{\langle 1 \rangle_g \pi t^3} d\mu_H(u) d^3p. \quad (2.28)$$

Let us complete this section with some discussions on the volume operator contained in the Hamiltonian operator  $\widehat{H}[N]$ . Because of the square root in the definition of the volume operator, matrix elements of these operators are difficult to compute analytically. However, as far as the coherent state expectation value is concerned, the volume operators  $\hat{V}_v$  in  $\widehat{H}[N]$  can be replaced by Giesel-Thiemann's volume [8]  $\hat{V}_{GT}^{(v)}$  which is a semiclassical expansion

$$\hat{V}_{GT}^{(v)} = \langle \hat{Q}_v \rangle^{2q} \left[ 1 + \sum_{n=1}^{2k+1} (-1)^{n+1} \frac{q(1-q) \cdots (n-1-q)}{n!} \left( \frac{\hat{Q}_v^2}{\langle \hat{Q}_v \rangle^2} - 1 \right)^n \right] + O(t^{k+1}) \quad (2.29)$$

where  $q = 1/4$ . By making use of  $\hat{V}_{GT}^{(v)}$ , firstly truncating  $\hat{V}_{GT}^{(v)}$  at finite  $k$  and replacing  $\hat{V}_v$  by  $\hat{V}_{GT}^{(v)}$ ,  $\widehat{H}[N]$  can be expressed by a polynomial of holonomies and fluxes. Up to higher order in  $t$ , it is now manageable to compute the expectation value of  $\widehat{H}[N]$ , through computing the expectation value of a polynomial of holonomies and fluxes.

### III. Expectation values of operators on one edge

As becoming clear in a moment, computing the coherent state expectation value of  $\widehat{H}[N]$  can be reduced to computing expectation values of operator monomials on individual edges. In this section, let us firstly focus on the expectation value of operators on one edge.

Given a monomial of holonomies and fluxes on an edge  $e$ , its expectation value with respect to the coherent state  $\psi_{g_e}^t$  labelled by  $g_e = n_e^s e^{iz_e \tau_3} (n_e^t)^{-1}$  relates to its expectation value with respect to  $\psi_{e^{iz_e \tau_3}}^t \equiv \psi_{z_e}^t$ , by a gauge transformation generated by  $n_e^s$  and  $n_e^t$  [20]:

$$\begin{aligned} & \langle \psi_{g_e}^t | P(\{\hat{p}_s^{\alpha_i}(e)\}, \{\hat{p}_t^{\alpha_j}(e)\}, \{D_{a_k b_k}^{\iota_k}(h_e)\}) | \psi_{g_e}^t \rangle \\ &= \langle \psi_{z_e}^t | P(\{\hat{p}_s^{\beta_i}(e) D_{\beta_i \alpha_i}^1((n_e^s)^{-1})\}, \{\hat{p}_t^{\beta_j}(e) D_{\beta_j \alpha_j}^1((n_e^t)^{-1})\}, \{D_{a_k c_k}^{\iota_k}(n_e^s) D_{c_k d_k}^{\iota_k}(h_e) D_{d_k b_k}^{\iota_k}((n_e^t)^{-1})\}) | \psi_{z_e}^t \rangle, \end{aligned} \quad (3.1)$$

where  $P(x, y, z)$  represents any monomial of  $x = \{x_1, x_2, \dots, x_m\}$ ,  $y = \{y_1, \dots, y_n\}$  and  $z = \{z_1, z_2, \dots, z_k\}$ . This feature implies that one can always do the calculation with respect to  $\psi_{z_e}^t$ , then restore the information of  $n_e^s$  and  $n_e^t$  afterwards. In the following context, we denote

$$\langle \psi_{z_e}^t | \hat{F}_e | \psi_{z_e}^t \rangle =: \langle \hat{F}_e \rangle_{z_e}. \quad (3.2)$$

Now let us consider the algorithm to compute (3.2) for a general monomial  $\hat{F}_e$  of holonomies and fluxes step by step. Based on the algorithm described in the following subsections, our codes [49] are designed. By the codes, one can compute the expectation value of arbitrary monomial  $\hat{F}_e$  up to arbitrary order.

<sup>3</sup> Here we used the following result shown in [19]. For any complex number  $z = R + iI$ , there exist real numbers  $s \in \mathbb{R}$  and  $\phi \in [0, \pi]$  such that  $\cosh(s + i\phi) = z$ .  $s$  and  $\phi$  are uniquely determined except in the case  $I = 0$  and  $|R| > 1$  in which case the  $s$  is determined up to its sign.



## A. The algorithm

### 1. The first step

Given a monomial of holonomy and flux operators. To compute the expectation value of this monomial, we need at first to remove all the holonomies to the right with the basic commutation relations (2.10). We use the following proposition to implement the procedure.

**Proposition 1.** *Let  $\hat{O}$  be defined as*

$$\hat{O} = \hat{O}_1 \hat{O}_2 \cdots \hat{O}_m. \quad (3.3)$$

Denote  $\mathcal{I} := \{2, \dots, m\}$ . Let  $\mathcal{I}_k := \{i_1, i_2, \dots, i_k\}$  with  $i_1 < i_2 < \dots < i_k$  be a sublist of  $\mathcal{I}$  which contains  $k$  elements. Then, by the definition of commutator, it has

$$\hat{O} = \hat{O}_2 \cdots \hat{O}_m \hat{O}_1 + \sum_{k=1}^{m-1} \sum_{\mathcal{I}_k} \left( \prod_{l \in \mathcal{I} - \mathcal{I}_k} \hat{O}_l \right) [[\cdots [[\hat{O}_1, \hat{O}_{i_1}], \hat{O}_{i_2}] \cdots], \hat{O}_{i_k}]. \quad (3.4)$$

The proof is quite straightforward with using the relation  $\hat{A}\hat{B} = \hat{B}\hat{A} + [\hat{A}, \hat{B}]$  iteratively. In Eq. (3.4), the terms at  $k$  carry  $k$ -fold commutator. Due to the factor  $t$  in the right hand side of the commutation relation (2.10), the  $k$ -fold commutator produces a factor  $t^k$  in the final results, which implies that the contributions of these terms to the expectation value of  $\hat{O}$  are at least at  $t^k$ -order.

Now let us see how to use Proposition 1 to move the holonomies to the right precisely. Assume that  $\hat{O}_1 = D_{ab}^\iota(h_e)$  in Eq. (3.4) and that all of the other operators are fluxes. Then according to Eq. (3.4), we need to calculate

$$[\cdots [[\cdots [[D_{ab}^\iota(h_e), \hat{p}_s^{\alpha_1}(e)], \hat{p}_s^{\alpha_2}(e)] \cdots \hat{p}_s^{\alpha_m}(e)], \hat{p}_t^{\alpha_1}(e)] \cdots, \hat{p}_t^{\alpha_n}(e)].$$

The result can be derived by

$$[\cdots [[D_{ab}^\iota(h_e), \hat{p}_s^{\alpha_1}(e)], \hat{p}_s^{\alpha_2}(e)] \cdots \hat{p}_s^{\alpha_m}(e)] = (-it)^m D_{aa_1}^{\iota'}(\tau^{\alpha_1}) D_{a_1 a_2}^{\iota'}(\tau^{\alpha_2}) \cdots D_{a_{m-1} a_m}^{\iota'}(\tau^{\alpha_m}) D_{a_m b}^\iota(h_e) \quad (3.5)$$

with  $a_k = a - \sum_{i=1}^k \alpha_i$ , and

$$[\cdots [[D_{ab}^\iota(h_e), \hat{p}_t^{\alpha_1}(e)], \hat{p}_t^{\alpha_2}(e)] \cdots \hat{p}_t^{\alpha_m}(e)] = (it)^m D_{ab_m}^\iota(h_e) D_{b_m b_{m-1}}^{\iota'}(\tau^{\alpha_m}) D_{b_{m-1} b_{m-2}}^{\iota'}(\tau^{\alpha_{m-1}}) \cdots D_{b_1 b}^{\iota'}(\tau^{\alpha_1}) \quad (3.6)$$

with  $b_k = \sum_{i=1}^k b_i + b$ , where we used that  $D_{ab}^\iota(\tau^\alpha) \propto \delta_{a, b+\alpha}$ . Taking advantage of Eqs. (3.5) and (3.6), we have, for instance,

$$\begin{aligned} & D_{ab}^\iota(h_e) \left( \prod_{i=1}^m \hat{p}_s^{\alpha_i}(e) \right) \left( \prod_{j=1}^n \hat{p}_t^{\beta_j}(e) \right) \\ &= \left( \prod_{i=1}^m \hat{p}_s^{\alpha_i}(e) \right) \left( \prod_{j=1}^n \hat{p}_t^{\beta_j}(e) \right) D_{ab}^\iota(h_e) - it \sum_{k=1}^m \left( \prod_{i \neq k} \hat{p}_s^{\alpha_i}(e) \right) \left( \prod_{j=1}^n \hat{p}_t^{\beta_j}(e) \right) D_{ac}^{\iota'}(\tau^{\alpha_k}) D_{cb}^\iota(h_e) \\ &+ (-it)^2 \sum_{k < l} \left( \prod_{i \notin \{k, l\}} \hat{p}_s^{\alpha_i}(e) \right) \left( \prod_{j=1}^n \hat{p}_t^{\beta_j}(e) \right) D_{ac}^{\iota'}(\tau^{\alpha_k}) D_{cd}^{\iota'}(\tau^{\alpha_l}) D_{db}^\iota(h_e) \\ &+ it \sum_{k=1}^n \left( \prod_{i=1}^m \hat{p}_s^{\alpha_i}(e) \right) \left( \prod_{j \neq k} \hat{p}_t^{\beta_j}(e) \right) D_{ac}^\iota(h_e) D_{cb}^{\iota'}(\tau^{\beta_k}) \\ &+ (it)^2 \sum_{k < l} \left( \prod_{i=1}^m \hat{p}_s^{\alpha_i}(e) \right) \left( \prod_{j \notin \{k, l\}} \hat{p}_t^{\beta_j}(e) \right) D_{ac}^\iota(h_e) D_{cd}^{\iota'}(\tau^{\beta_l}) D_{db}^{\iota'}(\tau^{\beta_k}) \\ &- (it)^2 \sum_{k, l} \left( \prod_{i \neq k} \hat{p}_s^{\alpha_i}(e) \right) \left( \prod_{j \neq l} \hat{p}_t^{\beta_j}(e) \right) D_{ac}^{\iota'}(\tau^{\alpha_k}) D_{cd}^\iota(h_e) D_{db}^{\iota'}(\tau^{\beta_l}) + O(t^3). \end{aligned} \quad (3.7)$$

If there are more than one holonomies contained in  $\hat{O}$ , one can use this procedure to permute them one by one. Finally,  $\hat{O}$  is expressed as summation of terms taking the form

$$\prod_{k=1}^m \hat{p}_s^{\alpha_k}(e) \prod_{k=1}^n \hat{p}_t^{\alpha_k}(e) \prod_{i=1}^l D_{a_i b_i}^{\iota_i}(h_e). \quad (3.8)$$

Then, one can merge the holonomies by applying the formula (C4), we eventually simplify  $\hat{O}$  to be a sum of operators of the form

$$\left( \prod_{i=1}^m \hat{p}_s^{\alpha_i}(e) \right) \left( \prod_{j=1}^n \hat{p}_t^{\beta_j}(e) \right) D_{ab}^\iota(h_e) \quad (3.9)$$

### 2. The second step

The second step is to transform  $\hat{p}_t^\beta(e)$  in Eq. (3.9) to  $\hat{p}_s^\beta(e)$ . To do this, we employ the formula

$$\begin{aligned} & \langle \hat{p}_s^{\alpha_1}(e) \cdots \hat{p}_s^{\alpha_m}(e) \hat{p}_t^{\beta_1}(e) \cdots \hat{p}_t^{\beta_n}(e) D_{ab}^\iota(h_e) \rangle_{z_e} \\ &= (-1)^n e^{-(\beta_1 + \cdots + \beta_n) \bar{z}_e} \langle \hat{p}_s^{\beta_n}(e) \cdots \hat{p}_s^{\beta_1}(e) \hat{p}_s^{\alpha_1}(e) \cdots \hat{p}_s^{\alpha_m}(e) D_{ab}^\iota(h_e) \rangle_{z_e}. \end{aligned} \quad (3.10)$$

The proof of this formula is quite technical and is put in Appendix C. Because of this equation, we now only need to consider the expectation value of operators

$$\hat{F}_{lab}^{\alpha_1 \cdots \alpha_m} = \hat{p}_s^{\alpha_1}(e) \cdots \hat{p}_s^{\alpha_m}(e) D_{ab}^\iota(h_e). \quad (3.11)$$

### 3. The third step

To compute the expectation value of  $\hat{F}_{lab}^{\alpha_1 \cdots \alpha_m}$ , we need to consider the cases with  $\iota = 0$  and  $\iota \neq 0$  separately. As shown in Appendix D, the expectation value for  $\hat{F}_{000}^{\alpha_1 \cdots \alpha_m} \equiv \hat{F}^{\alpha_1 \cdots \alpha_m}$  reads

$$\begin{aligned} & \langle \hat{F}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} \\ &= \delta \left( \sum_i \alpha_i, 0 \right) t^m \prod_{i=1}^m \frac{1}{(1 + |\alpha_i|)^{1/2}} e^{t/4} \int_{-\infty}^{\infty} dx x \prod_{k=1}^m \left( \frac{\alpha_k - 1}{2} x - \frac{\partial_y}{2} + \sum_{i=1}^k \alpha_i - \frac{\alpha_k}{2} \right) \frac{e^{-\frac{t}{4}x^2 + x\eta}}{2 \sinh(y)} \Big|_{y \rightarrow \eta} + O(t^\infty). \end{aligned} \quad (3.12)$$

For the operator  $\hat{F}_{lab}^{\alpha_1 \cdots \alpha_m}$  with  $\iota \neq 0$ , the explicit results is presented by (D19) and (D24). Then, as discussed in Appendix D, at least for  $\iota \leq 20$ , the results can be simplified as

$$\langle \hat{F}_{lab}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} = t^m e^{bz_e} \sum_{\substack{0 \leq d \leq \iota \\ d + \iota \in \mathbb{Z}}} \frac{2 - \delta(d, 0)}{2} e^{-\frac{t}{4}(2d^2 - 1)} \int dx e^{-\frac{t}{4}t(x^2 - 2dx)} F_\iota \left( \frac{x-1}{2} - d, \frac{x-1}{2}, \frac{\partial_\eta}{2} \right) \frac{\sinh(x\eta)}{\sinh(\eta)} + O(t^{-\infty}). \quad (3.13)$$

where  $F_\iota \left( \frac{x-1}{2} - d, \frac{x-1}{2}, \frac{\partial_\eta}{2} \right)$ , also depending on the list  $\{\alpha_i\}_{i=1}^m$ , is given by

$$\begin{aligned} F_\iota \left( \frac{x-1}{2} - d, \frac{x-1}{2}, \frac{\partial_\eta}{2} \right) &= \delta \left( \sum_{i=1}^m \alpha_i - a + b, 0 \right) \prod_{i=1}^m \frac{1}{(1 + |\alpha_i|)^{1/2}} \left( \frac{1}{(\iota + a)! (\iota - a)! (\iota + b)! (\iota - b)!} \right)^{1/2} \\ & \frac{(-1)^{d-2a+b} x(x-2d)}{(x-d+\iota)_{2\iota+1}} \prod_{k=1}^m \left( \alpha_k \frac{x-1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \\ & \sum_{z=0}^{\iota+d} \frac{(-1)^z (\iota+d)_z \left( \frac{x-1}{2} + \frac{\partial_\eta}{2} + b - z + \iota \right)_{\iota+a} \left( \frac{x-1}{2} - \frac{\partial_\eta}{2} - d - b + z \right)_{\iota-a}}{z!} \\ & \sum_{z=0}^{\iota-d} \frac{(-1)^z (\iota-d)_z \left( \frac{x-1}{2} + \frac{\partial_\eta}{2} - d - z + \iota \right)_{\iota-b} \left( \frac{x-1}{2} - \frac{\partial_\eta}{2} + z \right)_{\iota+b}}{z!}. \end{aligned} \quad (3.14)$$

We would like to compare our results (3.13) with the known results in [56]. At first, our formula (3.12) and (3.13) generalize the known results in literature [56], in the sense that our formula gives the results for arbitrary lists  $\{\alpha_i\}_{i=1}^m$  of flux indices and triples  $(\iota, a, b)$  with at least  $\iota \leq 20$ . Moreover, with our formula, one can get the expectation values to arbitrary order of  $t$ . However, in [56], the authors give only the results for the special cases where the list  $\{\alpha_i\}_{i=1}^m$  contains at most either a single  $-1$ , or a single  $1$ , or a pair of  $(-1, 1)$ . They are all the cases such that

the expectation values have non-vanishing  $O(t^0)$  or  $O(t)$ -term. Other cases are also interesting when we study the higher-order correction, even though the higher-order correction is beyond the present work. Our formula reduce to these known results at the special cases. The current work only use these special cases, but our codes [49] are designed based on the generalization formulas (3.12), (3.13) and Theorem D.1, since the generalized formulas have the potential in the generalization for computing higher-order correction.

Finally, let us complete this subsection by sketching the algorithm based on Eqs. (3.12), (3.13) and Theorem D.1 to compute the expectation value of  $\hat{F}_{\iota ab}^{\alpha_1 \dots \alpha_m}$ . One can refer to [49] for more details. According to Theorem D.1, we can simplify (at least for  $\iota \leq 20$ ) the integrals in Eqs. (3.12) and (3.13) to a linear combination of integrals taking the forms

$$I_1 = \int_{-\infty}^{\infty} dx e^{-ax^2+bx} \frac{\sinh(x\eta)}{x} = \frac{\pi}{2} \left( \operatorname{erfi} \left( \frac{b+\eta}{2\sqrt{a}} \right) - \operatorname{erfi} \left( \frac{b-\eta}{2\sqrt{a}} \right) \right)$$

and

$$I_2 = \int_{-\infty}^{\infty} dx e^{-ax^2+bx} \operatorname{pol}(x, \partial_z) \frac{e^{\pm\eta x}}{f(z)} \Big|_{z=\eta}$$

where  $a > 0$ ,  $b \in \mathbb{R}$ ,  $f$  is some function and  $\operatorname{pol}(x, \partial_z)$  denotes a polynomial of  $x$  and  $\partial_z$ . This is the first step of our algorithm, without considering the realization of their concrete form for now. Because  $I_1$  as a function of  $a, b$  is known, the next step of the algorithm is to compute  $I_2$ . To do this, we first expand  $\operatorname{pol}(x, \partial_z)$  to write the integrand of  $I_2$  as a linear combination of  $\left( \partial_z^n \frac{1}{f(z)} \right) x^m e^{-ax^2+bx \pm \eta x}$ . Then by substituting the results  $\int dx x^n e^{-ax^2+bx \pm \eta x}$ ,  $I_2$  can be computed easily. By this discussion, the only remaining problem is how to realize the concrete linear combination form of  $I_1$  and  $I_2$ , which can be illustrated by the derivation of  $\langle \hat{F}_{\iota ab}^{\alpha_1 \dots \alpha_m} \rangle_{z_e}$  for  $\iota = 1$  in Appendix E. For this case, the crucial step to simplify  $F_1$  is to apply the formula (E5) and (E15) inspired by the proof of Theorem D.1. Taking advantage of Eq. (E5), Eq.(E15) and the trick (E7), one can finally get (E20) which is a linear combination of integrals taking the forms of  $I_1$  and  $I_2$ .

## B. The cases when all flux indices vanish

In our computation, we often use the operator

$$D_{a_1 b_1}^\iota(h_e) [\hat{p}_s^0(e)]^{m_1} [\hat{p}_t^0(e)]^{n_1} \dots D_{a_k b_k}^\iota(h_e) [\hat{p}_s^0(e)]^{m_k} [\hat{p}_t^0(e)]^{n_k}. \quad (3.15)$$

To deal with this kind of operators, let us consider the operator  $D_{ab}^\iota(h_e) (\hat{p}_s^0(e))^m (\hat{p}_t^0(e))^n$ . By applying Proposition 1, it can be simplified to

$$\begin{aligned} & D_{ab}^\iota(h_e) [\hat{p}_s^0(e)]^m [\hat{p}_t^0(e)]^n \\ &= [\hat{p}_s^0(e)]^m [\hat{p}_t^0(e)]^n D_{ab}^\iota(h_e) - atm [\hat{p}_s^0(e)]^{m-1} [\hat{p}_t^0(e)]^n D_{ab}^\iota(h_e) + btn [\hat{p}_s^0(e)]^m [\hat{p}_t^0(e)]^{n-1} D_{ab}^\iota(h_e) + O(t^2). \end{aligned} \quad (3.16)$$

Then, for the operator (3.15), it has

$$\begin{aligned} & D_{a_1 b_1}^\iota(h_e) [\hat{p}_s^0(e)]^{m_1} [\hat{p}_t^0(e)]^{n_1} \dots D_{a_k b_k}^\iota(h_e) [\hat{p}_s^0(e)]^{m_k} [\hat{p}_t^0(e)]^{n_k} \\ &= [\hat{p}_s^0(e)]^{\sum_{i=1}^k m_i} [\hat{p}_t^0(e)]^{\sum_{i=1}^k n_i} \prod_{i=1}^k D_{a_i b_i}^\iota(h_e) - t \left( \sum_{i=1}^k a_i \left[ \sum_{l=i}^k m_l \right] [\hat{p}_s^0(e)]^{(\sum_{i=1}^k m_i)-1} [\hat{p}_t^0(e)]^{\sum_{i=1}^k n_i} \prod_{i=1}^k D_{a_i b_i}^\iota(h_e) \right) \\ &+ t \left( \sum_{i=1}^k b_i \left[ \sum_{l=i}^k n_l \right] [\hat{p}_s^0(e)]^{\sum_{i=1}^k m_i} [\hat{p}_t^0(e)]^{(\sum_{i=1}^k n_i)-1} \prod_{i=1}^k D_{a_i b_i}^\iota(h_e) \right) + O(t^2) \end{aligned} \quad (3.17)$$

By (C13), we finally have

$$\begin{aligned} & D_{a_1 b_1}^\iota(h_e) [\hat{p}_s^0(e)]^{m_1} [\hat{p}_t^0(e)]^{n_1} \dots D_{a_k b_k}^\iota(h_e) [\hat{p}_s^0(e)]^{m_k} [\hat{p}_t^0(e)]^{n_k} \\ &= (-1)^{\sum_{i=1}^k n_i} [\hat{p}_s^0(e)]^{\sum_{i=1}^k (m_i+n_i)} \prod_{i=1}^k D_{a_i b_i}^\iota(h_e) - t (-1)^{\sum_{i=1}^k n_i} \left( \sum_{i=1}^k a_i \left[ \sum_{l=i}^k m_l \right] + \sum_{i=1}^k b_i \left[ \sum_{l=i}^k n_l \right] \right) \times \\ & [\hat{p}_s^0(e)]^{\sum_{i=1}^k (m_i+n_i)-1} \prod_{i=1}^k D_{a_i b_i}^\iota(h_e). \end{aligned} \quad (3.18)$$

Recalling the derivation of  $\langle \hat{F}_{ab}^{\alpha_1 \dots \alpha_m} \rangle_{z_e}$ , we get

$$\langle (\hat{p}_s^0(e))^m D_{ab}^\iota(h_e) \rangle_{z_e} = e^{b\eta} \left( -t \frac{\partial \eta}{2} \right)^m e^{-b\eta} \langle D_{ab}^\iota(h_e) \rangle_{z_e} \quad (3.19)$$

where the result of  $\langle (\hat{p}_s^0(e))^m \rangle_{z_e}$  is given by setting  $\iota = 0 = a = b$ . It can be verified that,  $\langle D_{ab}^\iota(h_e) \rangle_{z_e}$  takes the form that

$$\langle D_{ab}^\iota(h_e) \rangle_{z_e} = \langle 1 \rangle_{z_e} (g_0 + tg_1(\eta) + O(t^2)) = f(t) \frac{e^{\frac{\eta^2}{t}} \eta}{\sinh(\eta)} (g_0 + tg_1(\eta) + O(t^2)) \quad (3.20)$$

with some functions  $g_0$ ,  $g_1$  and  $f$ . Therefore, with Faà di Bruno's formula, we can have that

$$\begin{aligned} \langle (\hat{p}_s^0(e))^m D_{ab}^\iota(h_e) \rangle_{z_e} &= e^{b\eta} \left( -t \frac{\partial \eta}{2} \right)^m e^{-b\eta} \langle D_{ab}^\iota(h_e) \rangle_{z_e} \\ &= \langle 1 \rangle_{z_e} (-\eta)^m [g_0 + tg_1(\eta)] + \langle 1 \rangle_{z_e} \frac{m(m+1)}{4} (-\eta)^{m-2} g_0 t \\ &\quad + \langle 1 \rangle_{z_e} \frac{m}{2} (-\eta)^{m-1} (\coth(\eta) + b) g_0 t + O(t^2) \end{aligned} \quad (3.21)$$

Based on these formula, we can propose a faster algorithm to deal with these cases.

#### IV. Power counting

After introducing the derivations of expectation values of several characterized operators, we finally need to deal with a set of specific operators that takes the following form,  $\sum_{\vec{\alpha}} \mathcal{T}^{\alpha_1 \alpha_2 \dots \alpha_m} \hat{O}_{\alpha_1 \alpha_2 \dots \alpha_m}$ , where  $\mathcal{T}$  is some numerical factors and  $\hat{O}$  is some polynomial operators of holonomies and fluxes. In principle, we would need to compute the expectation values of  $\hat{O}_{\alpha_1 \dots \alpha_m}$  for all indices  $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ . This computation can be preformed thanks to previous sections. However, the computational complexity comes from the huge amount of terms in the sum over  $\vec{\alpha}$ . Since we are only interested in the expectation value up to  $O(t)$ , the complexity can be reduced by certain power-counting argument: we count the least power of  $t$  contains in each  $\langle \hat{O}_{\alpha_1 \dots \alpha_m} \rangle$  before explicit computation, then we omit those terms only contribute to higher order than  $O(t)$  in  $\langle \hat{H}[N] \rangle$ . It turns out that a large degree of complexity can be reduced in this manner. The following arguments in this section will be proven rigorously in Appendix F.

In this section, we will denote  $\Psi_{\vec{g}}$  defined in (2.20) by  $|\Psi_{\vec{g}}\rangle$  with  $\vec{g} = \{g_e\}_{e \in E(\gamma)}$ , namely

$$|\Psi_{\vec{g}}\rangle = \bigotimes_{e \in E(\gamma)} |\psi_{g_e}\rangle. \quad (4.1)$$

Similarly,  $|\Psi_{\vec{g}^{(i)}}\rangle$  denotes the coherent state that at the edge  $e$  is  $|\psi_{g_e^{(i)}}\rangle$ . Let  $\hat{O}$  take the form of

$$\hat{O} = \hat{O}_1 \hat{O}_2 \dots \hat{O}_k \quad (4.2)$$

with  $\hat{O}_i$  being arbitrary polynomial of fluxes and holonomies. Inserting the resolution of identity (2.27), we have

$$\langle \Psi_{\vec{g}} | \hat{O} | \Psi_{\vec{g}} \rangle = \int \prod_{m=1}^{k-1} d\nu(\vec{g}^{(m)}) \prod_{i=1}^k \langle \Psi_{\vec{g}^{(i-1)}} | \hat{O}_i | \Psi_{\vec{g}^{(i)}} \rangle \quad (4.3)$$

where  $|\Psi_{\vec{g}^{(0)}}\rangle = |\Psi_{\vec{g}^{(k)}}\rangle := |\Psi_{\vec{g}}\rangle$  and the measure  $d\nu(\vec{g}^{(m)})$  is

$$d\nu(\vec{g}^{(m)}) = \prod_{e \in E(\gamma)} d\nu(g_e^{(m)}) \quad (4.4)$$

with  $d\nu(g_e^{(m)})$  defined in (2.28). Eq. (4.3) relates the expectation value of  $\hat{O}$  to matrix elements of each individual  $\hat{O}_i$ . Thus we are motivated to study matrix elements of polynomial of holonomies and flux. One can refer to Appendix F for more details on this issue. According to the analysis therein, the matrix elements of the fluxes and holonomies are of a form described below

$$\langle \psi_{g_e} | \hat{O}_i | \psi_{g'_e} \rangle = \langle \psi_{g_e} | \psi_{g'_e} \rangle (E_0(g_e, g'_e) + tE_1(g_e, g'_e) + O(t^\infty)). \quad (4.5)$$

Assigning to each edge  $e$  a complex number  $w_e = p_e - i\theta_e$ , we have the coherent state

$$|\Psi_{\vec{w}}\rangle := \bigotimes_{e \in E(\gamma)} |\psi_{w_e}\rangle. \quad (4.6)$$

For the operator  $\hat{O}$  in Eq. (4.2), we state the following result obtained firstly in [8]

**Theorem IV.1.** *Consider an operator  $\hat{O} = \prod_{i=1}^k \hat{O}_i$ . Assume that, for each operator  $\hat{O}_i$ , its matrix elements  $\langle \Psi_{\vec{g}^{(1)}} | \hat{O}_i | \Psi_{\vec{g}^{(2)}} \rangle$  take the following form*

$$\langle \Psi_{\vec{g}^{(1)}} | \hat{O}_i | \Psi_{\vec{g}^{(2)}} \rangle = \langle \Psi_{\vec{g}^{(1)}} | \Psi_{\vec{g}^{(2)}} \rangle \left( E_0^{(i)}(\vec{g}^{(1)}, \vec{g}^{(2)}) + t E_1^{(i)}(\vec{g}^{(1)}, \vec{g}^{(2)}) + O(t^\infty) \right). \quad (4.7)$$

Let  $N_0$  be the number of operators  $\hat{O}_m \in \{\hat{O}_i\}_{i=1}^k$  such that

$$\frac{\langle \Psi_{\vec{w}} | \hat{O}_m | \Psi_{\vec{w}} \rangle}{\langle \Psi_{\vec{w}} | \Psi_{\vec{w}} \rangle} = O(t), \quad (4.8)$$

where the  $O(t^0)$  term vanishes on the RHS. Then the expectation value of  $\hat{O}$  with respect to the coherent state  $|\Psi_{\vec{w}}\rangle$  satisfies

$$\frac{\langle \Psi_{\vec{w}} | \hat{O} | \Psi_{\vec{w}} \rangle}{\langle \Psi_{\vec{w}} | \Psi_{\vec{w}} \rangle} = O(t^n), \quad \text{with } n \geq \lfloor \frac{N_0 + 1}{2} \rfloor \quad (4.9)$$

where  $\lfloor x \rfloor$  is the largest integer no larger than  $x$ .

A detailed proof of the above result is provided in Appendix F, including a careful stationary phase analysis, the computation of nondegenerate Hessian matrix, and power-counting.

Because of the vanishing leading-order term of  $\langle \frac{\hat{Q}}{\langle \hat{Q} \rangle} - 1 \rangle$ , it can be regarded as operator  $\hat{O}_m$  satisfying (4.8). Thus, Theorem IV.1 is applied to count the power of  $t$  for the term including  $\left( \frac{\hat{Q}^2}{\langle \hat{Q} \rangle^2} - 1 \right)^k$  in  $\langle \widehat{H[N]} \rangle$ . Moreover, in order to apply Theorem IV.1, matrix elements of  $\hat{O}_i$  have to be computable. We have to factorize  $\frac{\hat{Q}^2}{\langle \hat{Q} \rangle^2} - 1 = \left( \frac{\hat{Q}}{\langle \hat{Q} \rangle} + 1 \right) \left( \frac{\hat{Q}}{\langle \hat{Q} \rangle} - 1 \right)$  because every matrix element of  $\hat{Q}$  is a polynomial of matrix elements of the flux operators, while that of  $\hat{Q}^2$  is not.

Because the expectation values of  $\hat{p}_s^{\pm 1}(e)$ ,  $\hat{p}_t^{\pm 1}(e)$  and  $D_{ab}^{\nu}(h_e)$  ( $a \neq b$ ) with respect to  $\Psi_{\vec{z}}$  vanish, each of them can also be considered as operator  $\hat{O}_m$  in (4.8). Therefore, this theorem can be applied to study the leading order of monomial of holonomies and fluxes. Let us use  $\hat{p}^\beta(e)$  to denote either  $\hat{p}_s^\beta(e)$  or  $\hat{p}_t^\beta(e)$ , and use  $\mathcal{M}$  to denote the monomial of holonomies and fluxes. Let  $N_\pm$  be the number of  $\hat{p}^{\pm 1}(e)$  respectively and,  $M_+$  (respectively  $M_-$ ) be the number of  $D_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}(h_e)$  (respectively  $D_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}(h_e)$ ) in  $\mathcal{M}$ . According to our analysis above, the expectation value of  $\mathcal{M}$  with respect to the coherent state  $|\psi_{z_e}\rangle$  with  $z_e \in \mathbb{C}$  is non-vanishing if

$$\sum_{i=1}^m \beta_i + \sum_{j=1}^k (b_j - a_j) = 0. \quad (4.10)$$

Hence, we have

$$N_+ + M_+ = N_- + M_-. \quad (4.11)$$

Therefore, this theorem gives us that the leading order the expectation value  $\langle \mathcal{M} \rangle_{z_e}$  is  $O(t^{N_+ + M_+})$  or higher. We have more discussions on this case. Since the matrix elements of  $\hat{p}^\alpha(e)$  and  $D_{ab}^{\frac{1}{2}}(h_e)$  are computable, the results on the leading order of  $\mathcal{M}$  can be calculated more concretely. The result is summarized as the following theorem.

**Theorem IV.2.** *Given  $\mathcal{M}$  an arbitrary monomial of holonomies and fluxes. Let  $\mathcal{M}'$  be the operator resulting from  $\mathcal{M}$  by deleting all factors  $\hat{p}^0(e)$  and  $D_{aa}^{\frac{1}{2}}(h_e)$ . Denote the number of  $\hat{p}_s^0(e)$  and  $\hat{p}_t^0(e)$  in  $\mathcal{M}$  as  $N_{0,s}$  and  $N_{0,t}$  respectively, and the number of  $D_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}(h_e)$  and  $D_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}(h_e)$  as  $M_{0+}$  and  $M_{0-}$  respectively. Then the leading order of  $\langle \mathcal{M} \rangle_{z_e}$  is exactly*

$O(t^{M_++N_+})$  if and only if the leading order of  $\langle \mathcal{M}' \rangle_{z_e}$  is exactly  $O(t^{M_++N_+})$ , where  $N_\pm$  be the number of  $\hat{p}^{\pm 1}(e)$  respectively and,  $M_+$  (respectively  $M_-$ ) be the number of  $D_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}(h_e)$  (respectively  $D_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}(h_e)$ ) in  $\mathcal{M}$ . Moreover, it has

$$\langle \mathcal{M} \rangle_{z_e} \cong (\langle \hat{p}_s^0(e) \rangle_{z_e})^{N_{0,s}} (\langle \hat{p}_t^0(e) \rangle_{z_e})^{N_{0,t}} (\langle D_{\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}(h_e) \rangle_{z_e})^{M_{0,+}} (\langle D_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}}(h_e) \rangle_{z_e})^{M_{0,-}} \langle \mathcal{M}' \rangle_{z_e} \quad (4.12)$$

where  $\cong$  means the  $O(t^{M_++N_+})$  terms of the left and right hand sides are equal to each other.

The proof of this theorem is quite technical and, thus, presented in Appendix F 3

## V. Cosmological expectation value

We apply our computation of expectation values to coherent states labelled by homogeneous and isotropic data. The symmetry group of the homogeneous and isotropic cosmology is  $\mathbb{T} \rtimes F$  where  $F$  is the isotropic subgroup and  $\mathbb{T}$  is the translation subgroup. Denote the subgroup of  $\mathbb{T} \rtimes F$  preserving  $\gamma$  by  $S_\gamma$ . A classical state  $\mathbf{g}$  is said to be symmetric with respect to  $S_\gamma$  if  $s^* \mathbf{g} := \mathbf{g} \circ s$  is identical with  $\mathbf{g}$  up to a gauge transformation  $s$  ( $\forall s \in S_\gamma$ ). According to this definition, classically symmetric states  $\mathbf{g}$  are of the form [57]

$$\mathbf{g} : e \mapsto g_e = n_e e^{iz\tau_3} n_e^{-1} \quad (5.1)$$

with  $n_e \in \text{SU}(2)$  satisfying

$$n_e \tau_3 n_e^{-1} = \vec{n}_e \cdot \vec{\tau}. \quad (5.2)$$

In the last equation,  $\vec{n}_e$  is the unit vector pointing to direction of edge  $e$ . Then, for each  $s = (t, f) \in \mathbb{T} \rtimes F$ , it can be verified that

$$\mathbf{g} \circ s = \text{Ad}_f \circ \mathbf{g} \quad (5.3)$$

where  $\text{Ad}_f \circ \mathbf{g}(e) = f \mathbf{g}(e) f^{-1}$  for all  $e \in E(\gamma)$ .

### A. Symmetries of the expectation value

Given  $\hat{F}_e$  as a polynomial of fluxes and holonomies on  $e$ . For  $s = (t, f) \in \mathbb{T} \rtimes F$ , Eq. (5.3) results in

$$\langle \psi_{g_{s(e)}} | \hat{F}_{s(e)} | \psi_{g_{s(e)}} \rangle = \langle \psi_{f g_e f^{-1}} | \hat{F}_e | \psi_{f g_e f^{-1}} \rangle = \langle \psi_{g_e} | (f \triangleright \hat{F}_e) | \psi_{g_e} \rangle. \quad (5.4)$$

where  $f \triangleright \hat{F}_e$  denote the gauge transformed operator of  $\hat{F}_e$  by  $f$  and the last equality can be derived by using the similar procedure as to derive Eq. (3.1).

To expand the expectation value of  $\hat{H}_E$  and  $\hat{H}_L$  to order  $O(t)$ , one needs to replace the operator  $\hat{V}_v$  by  $\hat{V}_{GT}^{(v)}$  defined in (2.29). Then the Euclidean part  $\widehat{H_E[N]}$  is rewritten in terms of (there is no summation over  $I, J, K$  here)

$$\hat{H}_E^{(n)}(v; e_I, e_J, e_K) = \frac{1}{i\beta a^2 t} \epsilon^{IJK} \text{tr}(h_{\alpha IJ} [h_{e_K}, \hat{Q}_v^{2n}] h_{e_K}^{-1}), \quad (5.5)$$

and the Lorentzian part, in terms of

$$\begin{aligned} & \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_I, e_J, e_K) \\ &= \frac{-1}{2i\beta^7 a^{10} t^5} \epsilon^{IJK} \text{tr}([h_{e_I}, [\hat{Q}_{v_1}^{2k_1}, \hat{H}_E^{(k_2)}(v_2)]] h_{e_I}^{-1} [h_{e_J}, [\hat{Q}_{v_3}^{2k_3}, \hat{H}_E^{(k_4)}(v_4)]] h_{e_J}^{-1} [h_{e_K}, \hat{Q}_v^{2k_5}] h_{e_K}^{-1}) \end{aligned} \quad (5.6)$$

with  $\vec{k} = (k_1, k_2, k_3, k_4, k_5)$ . Define

$$\hat{H}_E^{(n)}(v) = \sum_{e_I, e_J, e_K} \hat{H}_E^{(n)}(v; e_I, e_J, e_K) \quad (5.7)$$

and

$$\hat{H}_L^{(\vec{k})}(v) = \sum_{v_1, v_2, v_3, v_4, e_I, e_J, e_K} \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_I, e_J, e_K). \quad (5.8)$$

The Euclidean and Lorentzian parts, with the replacement  $\hat{V}_v \rightarrow \hat{V}_{GT}^{(n)}$  truncated at a finite  $n$ , are linear combinations of  $\hat{H}_E^{(n)}(v)$  and  $\hat{H}_L^{(\vec{k})}(v)$  with various  $n$  and  $\vec{k}$  respectively.

### 1. Symmetries of the Euclidean part

According to Eq. (5.4) and the gauge invariance of  $\hat{H}_E^{(n)}(v; e_I, e_J, e_K)$ , one realizes the following symmetry

$$\langle \hat{H}_E^{(n)}(v; e_I, e_J, e_K) \rangle = \langle \hat{H}_E^{(n)}(s(v); s(e_I), s(e_J), s(e_K)) \rangle, \quad (5.9)$$

where  $\langle \cdot \rangle$  denotes the expectation value with respect to the cosmological coherent state given by (5.1), and  $s = (t, f)$  is a symmetry of the graph.

By this relation, Eq. (5.7) is simplified as

$$\hat{H}_E^{(n)}(v) = 24(\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+) + \hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^-)) \quad (5.10)$$

where the prefactor 24 is deduced by the fact that there are totally 48 terms in the RHS of (5.7).

Moreover,  $[h_{e_z^\pm}, \hat{Q}_v^{2n}]h_{e_z^\pm}^{-1}$  appearing in  $\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^\pm)$  potentially relates  $\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^\pm)$  with  $\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^\mp)$ . Concisely,

$$[h_{e_z^\pm}, \hat{Q}_v^{2n}]h_{e_z^\pm}^{-1} = \sum_{l=1}^{2n} \sum_{\mathcal{P}_l} \left( \mp it \frac{(\beta a^2)^3}{8} \right)^l \hat{Q}_v^{p_1} \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}^{\alpha_1} \hat{Y}^{\beta_1} \tau^{\gamma_1} \hat{Q}_v^{p_2} \epsilon_{\alpha_2 \beta_2 \gamma_2} \hat{X}^{\alpha_2} \hat{Y}^{\beta_2} \tau^{\gamma_2} \dots \hat{Q}_v^{p_l} \epsilon_{\alpha_l \beta_l \gamma_l} \hat{X}^{\alpha_l} \hat{Y}^{\beta_l} \tau^{\gamma_l} \hat{Q}_v^{p_{l+1}} \quad (5.11)$$

where the edges  $e_z^\pm$  are oriented so that  $s(e_z^+) = v = s(e_z^-)$ ,  $\hat{X}^\alpha = \hat{p}_s^\alpha(e') - \hat{p}_t^\alpha(e')$  and  $\hat{Y}^\alpha = \hat{p}_s^\alpha(e'') - \hat{p}_t^\alpha(e'')$ , and  $\mathcal{P} = \{p_1, p_2, \dots, p_{l+1}\}$  with  $p_i \in \mathbb{Z}$ ,  $p_i \geq 0$  and  $\sum_{i=1}^{l+1} p_i = 2n - l$ .<sup>4</sup> Substituting the last equation into the expression of  $\hat{H}_E^{(n)}(v)$ , one has that

$$\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+) + \hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^-) = 2\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+) \quad (5.12)$$

where  $\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+)$  is the operator  $\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+)$  with applying the following replacement

$$[h_{e_z^+}, \hat{Q}_v^{2n}]h_{e_z^+}^{-1} \rightarrow \sum_{l \text{ is odd}} \sum_{\mathcal{P}_l} \left( -it \frac{(\beta a^2)^3}{8} \right)^l \hat{Q}_v^{p_1} \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}^{\alpha_1} \hat{Y}^{\beta_1} \tau^{\gamma_1} \hat{Q}_v^{p_2} \epsilon_{\alpha_2 \beta_2 \gamma_2} \hat{X}^{\alpha_2} \hat{Y}^{\beta_2} \tau^{\gamma_2} \dots \hat{Q}_v^{p_l} \epsilon_{\alpha_l \beta_l \gamma_l} \hat{X}^{\alpha_l} \hat{Y}^{\beta_l} \tau^{\gamma_l} \hat{Q}_v^{p_{l+1}}. \quad (5.13)$$

By Eq. (5.10),  $\hat{H}_E^{(n)}(v)$  becomes  $\hat{H}_E^{(n)}(v) = 48\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+)$ . Thus, when we calculate the expectation value of the Euclidean part, it is only necessary to consider  $\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+)$  rather than  $\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^\pm)$ .

Further, according to Eq. (5.11), the Euclidean Hamiltonian is of the form

$$\hat{H}_E^{(n)}(v; e_I, e_J, e_K) = \epsilon_{IJK} \text{tr}(h_{\alpha_{IJ}} \tau^\alpha) \hat{O}_\alpha,$$

where  $\hat{O}_\alpha$  is a polynomial of fluxes. Then the fact  $\text{tr}(h\tau^\alpha) = -\text{tr}(h^{-1}\tau^\alpha)$  gives

$$\hat{H}_E^{(n)}(v; e_I, e_J, e_K) = \hat{H}_E^{(n)}(v; e_J, e_I, e_K). \quad (5.14)$$

In summary, originally there are totally 48 terms for every  $\hat{H}_E^{(n)}(v)$  in Eq. (5.7). However, thanks to the symmetries discussed in this section, we have  $\hat{H}_E^{(n)}(v) = 48\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+)$ , which means that only the expectation value of  $\hat{H}_E^{(n)}(v; e_x^+, e_y^+, e_z^+)$  is necessary to be computed.

### 2. Symmetries of the Lorentzian part

Considering a list of vertices and edges  $(v; v_1, v_2, v_3, v_4; e_I, e_J, e_K)$  with  $e_I, e_J$  and  $e_K$  being outgoing from  $v$ , we have that  $(e_I, e_J, e_K)$  is either left-handed or right-handed. Thus, there exists a rotation  $f$  which leaves  $v$  invariant such that  $(v; f(v_1), f(v_2), f(v_3), f(v_4); f(e_I), f(e_J), f(e_K))$  is either

$$(v; f(v_1), f(v_2), f(v_3), f(v_4); e_x^+, e_y^+, e_z^+)$$

<sup>4</sup> A general equation can be obtained analogously if the holonomy  $h_{e_z^\pm}$  is replaced by a holonomy along other edges. In the following context, Eq. (5.11) will be usually referred as this general equation.

or

$$(v; f(v_1), f(v_2), f(v_3), f(v_4); e_x^+, e_y^+, e_z^-).$$

Therefore, (5.8) is simplified to

$$\hat{H}_L^{(\vec{k})}(v) = 24 \sum_{v_1, v_2, v_3, v_4} \left( \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^+) + \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^-) \right) \quad (5.15)$$

Moreover, since the term  $[h_{e_z^+}, \hat{Q}_v^{2n}]h_{e_z^+}^{-1}$  appears in  $\hat{H}_L^{(\vec{k})}$  too, Eq. (5.11) can be applied again to simplify Eq. (5.15) to obtain the following expression

$$\hat{H}_L^{(\vec{k})}(v) = 48 \sum_{v_1, v_2, v_3, v_4} \hat{\hat{H}}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^+), \quad (5.16)$$

where this  $\hat{\hat{H}}_L^{(\vec{k})}$  operator is given by  $\hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^+)$  with the replacement (5.13). As a consequence, it is only necessary to compute the expectation value of

$$\hat{\hat{H}}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^+)$$

for different vertices  $v_1, v_2, v_3$  and  $v_4$ .

The above discussion simplifies the computation of the Lorentzian part. However, more symmetries are required in order to reduce the computation time to an acceptable level. For this purpose, let us firstly look at the term  $[h_e[\hat{Q}_{v_1}^m, h_e^{-1}], \hat{Q}_{v_2}]$  which is from the commutator between the volume and the Euclidean part. We obtain the following proposition which can be proven by Eq. (5.11) directly.

**Proposition 2.** *Given an edge  $e$  with the source  $s(e)$  and the target  $t(e)$ ,  $[h_e[\hat{Q}_{s(e)}^m, h_e^{-1}], \hat{Q}_v] = 0$  for all  $v \neq s(e)$ .*

With this proposition, we consider the commutator  $[\hat{Q}_{v_1}^{2k}, \hat{H}_E^{(n)}(v_2; e_I, e_J, e_K)]$  which defines the operator  $\hat{K}$  as

$$\hat{K} = \frac{1}{it}[\hat{V}, \hat{H}_E]. \quad (5.17)$$

By definition, we have

$$[\hat{Q}_{v_1}^{2k}, \hat{H}_E^{(n)}(v_2; e_I, e_J, e_K)] = \frac{2}{i\beta a^2 t}(\hat{K}_1 + \hat{K}_2) \quad (5.18)$$

with

$$\begin{aligned} \hat{K}_1 &:= \epsilon_{IJK} \text{tr}([\hat{Q}_{v_1}^{2k}, h_{\alpha_{IJ}}][h_{e_K}, \hat{Q}_{v_2}^{2n}]h_{e_K}^{-1}) \\ \hat{K}_2 &:= \epsilon_{IJK} \text{tr}(h_{\alpha_{IJ}}[\hat{Q}_{v_1}^{2k}, [h_{e_K}, \hat{Q}_{v_2}^{2n}]h_{e_K}^{-1}]). \end{aligned} \quad (5.19)$$

The classical analogy of Eq. (5.17) is

$$K = \{V, H_E\}. \quad (5.20)$$

Substituting the expression of  $H_E$ , one has

$$K = \{V, H_E\} = \int d^3x \{V, F_{ab}^i(x)\} \frac{\epsilon_{ijk} E_i^a E_j^b}{\sqrt{\det(E)}}. \quad (5.21)$$

According to Eq. (5.21), only the Poisson bracket between volume  $V$  and the curvature  $F_{ab}^i$  is involved in the classical expression of  $K$ . In the quantum theory,  $F_{ab}^i$  is quantized to a holonomy along some loop  $\alpha_{IJ}$ . Thus, comparing to Eq. (5.18), the operator  $\hat{K}_1$  corresponds to the RHS of Eq. (5.21), while  $\hat{K}_2$  gives an extra term in  $\hat{K}$ . According to Proposition 2, this extra term  $\hat{K}_2$  vanishes unless  $v_1 = v_2 = s(e_K)$  at which Eq. (5.11) can be applied to cancel the holonomies inside the commutators of  $\hat{K}_2$ . Then  $\hat{K}_2$  is simplified to the following form

$$\text{tr}(h_{\alpha_{IJ}}[\hat{Q}_{v_1}^{2k}, \text{polynomial of only fluxes}]).$$



Therefore, it is because of the non-commutativity between the flux operators that the operator  $\hat{K}_2$  appears in  $\hat{K}$ . Note that the existence of  $\hat{K}_2$  does not affect the continuum limit of  $\lim_{t \rightarrow 0} \langle \widehat{H}[N] \rangle$  (the classical limit of  $\langle \widehat{H}[N] \rangle$  reduces to the classical continuum expression of  $H[N]$  when the sizes of lattice edges are neglected [22]).

By Eq. (5.13),  $\hat{K}_2$  can be simplified as

$$\begin{aligned} \hat{K}_2 = & \epsilon_{IJK} \sum_{p_1+p_2=2n-1} (2k) \frac{-it(\beta a^2)^3}{8} \text{tr}(h_{\alpha_{IJ}} \tau^\gamma) \hat{Q}_v^{p_1} [\hat{Q}_v, \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}_I^{\alpha_1} \hat{X}_J^{\beta_1}] \hat{Q}_v^{2k-1+p_2} \\ & + \epsilon_{IJK} \sum_{p_1+p_2=2n-1} \frac{2k(2k-1)}{2} \frac{-it(\beta a^2)^3}{8} \text{tr}(h_{\alpha_{IJ}} \tau^\gamma) \hat{Q}_v^{p_1} [\hat{Q}_v, [\hat{Q}_v, \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}_I^{\alpha_1} \hat{X}_J^{\beta_1}]] \hat{Q}_v^{2k-2+p_2} \\ & + O(t^4) \end{aligned} \quad (5.22)$$

where  $\hat{X}_I^\alpha = \hat{p}_s^\alpha(e_I) - \hat{p}_t^\alpha(e_I)$  and the conclusion that  $\hat{K}_2 \neq 0$  if  $v_1 = v_2 \equiv v$  is used. For the first term, we have up to  $O(t^4)$

$$\begin{aligned} \text{first term} = & \epsilon_{IJK} (2n) (2k) \frac{-it(\beta a^2)^3}{8} \text{tr}(h_{\alpha_{IJ}} \tau^\gamma) \hat{Q}_v^{2n} [\hat{Q}_v, \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}_I^{\alpha_1} \hat{X}_J^{\beta_1}] \hat{Q}_v^{2k-2} \\ & - (2k) \frac{2n(2n+1)}{2} \frac{-it(\beta a^2)^3}{8} \text{tr}(h_{\alpha_{IJ}} \tau^\gamma) \hat{Q}_v^{2n-1} [\hat{Q}_v, [\hat{Q}_v, \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}_I^{\alpha_1} \hat{X}_J^{\beta_1}]] \hat{Q}_v^{2k-2} \end{aligned} \quad (5.23)$$

For the second term, up to  $O(t^4)$  we have

$$\text{second term} = \epsilon_{IJK} (2n) \frac{2k(2k-1)}{2} \frac{-it(\beta a^2)^3}{8} \text{tr}(h_{\alpha_{IJ}} \tau^\gamma) \hat{Q}_v^{2n-1} [\hat{Q}_v, [\hat{Q}_v, \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}_I^{\alpha_1} \hat{X}_J^{\beta_1}]] \hat{Q}_v^{2k-2} \quad (5.24)$$

Finally,  $\hat{K}_2$  is

$$\begin{aligned} \hat{K}_2 = & \epsilon_{IJK} (2n) (2k) \frac{-it(\beta a^2)^3}{8} \text{tr}(h_{\alpha_{IJ}} \tau^\gamma) \hat{Q}_v^{2n} [\hat{Q}_v, \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}_I^{\alpha_1} \hat{X}_J^{\beta_1}] \hat{Q}_v^{2k-2} \\ & + \epsilon_{IJK} \frac{(2k)(2n)(2k-2n-2)}{2} \frac{-it(\beta a^2)^3}{8} \text{tr}(h_{\alpha_{IJ}} \tau^\gamma) \hat{Q}_v^{2n-1} [\hat{Q}_v, [\hat{Q}_v, \epsilon_{\alpha_1 \beta_1 \gamma_1} \hat{X}_I^{\alpha_1} \hat{X}_J^{\beta_1}]] \hat{Q}_v^{2k-2} \\ & + O(t^4) \end{aligned} \quad (5.25)$$

Because of the commutators between fluxes operators,

$$[\hat{p}^\alpha(e), \hat{p}^\beta(e)] = t(-1)^\gamma \varepsilon_{-\gamma \alpha \beta} \hat{p}^\gamma(e) =: tC_{\alpha\beta\gamma} \hat{p}^\gamma(e) \quad (5.26)$$

with  $\varepsilon_{-1,0,1} = 1$  and  $p^\alpha(e)$  denoting  $p_t^\alpha(e)$  or  $p_s^\alpha(e)$ , one obtains the following

$$[\hat{p}_s^\alpha(e^+) + s_1 \hat{p}_t^\alpha(e^-), \hat{p}_s^\beta(e^+) + s_2 \hat{p}_t^\beta(e^-)] = tC_{\alpha\beta\gamma} (\hat{p}_s^\gamma(e^+) + s_1 s_2 \hat{p}_t^\gamma(e^-)), \quad (5.27)$$

with  $s_1, s_2 = \pm 1$ . Substituting Eq. (5.27) into Eq. (5.25), we express  $\hat{K}_2$ , as well as  $\hat{K}$ , as a polynomial of  $h_e$  and  $p_s^\alpha(e^+) \pm p_t^\alpha(e^-)$ .

Moreover, thanks to the above results,  $\hat{H}_L^{(\vec{k})}(v)$  in Eq. (5.16) can finally be simplified to be in terms of

$$\frac{C}{t^2} \text{tr}(h_{e_x^+} F_1 h_{e_x^+}^{-1} h_{e_y^+} F_2 h_{e_y^+}^{-1} G_1) \quad (5.28)$$

where  $C$  is some constant of order  $t^0$  or higher,  $F_i$  with  $i = 1, 2$  are some monomials of holonomies and  $(p_s^\alpha(e^+) \pm p_t^\alpha(e^-))$  and  $G_1$  is a monomial of  $(p_s^\alpha(e^+) - p_t^\alpha(e^-))$ .

The results in Sec. IV can be used to reduce the computational complexity too. To use these results, one needs to apply the basic commutation relations (2.10) to simplify the Hamiltonian operator such that the operators after the simplification are written in terms of  $C\hat{P}$  with  $C$  being some constant of order  $t^0$  or higher, and  $\hat{P}$  being some monomial of holonomies and fluxes.

In order to achieve so, one needs to permute  $h_{e_x^+}$  and  $\hat{F}_1$ , as well as  $h_{e_y^+}$  and  $\hat{F}_2$ , in Eq. (5.28) with applying Eq. (3.4). Take the permutation of  $h_{e_x^+}$  and  $\hat{F}_1$  as an example: Implementing the results of Eq. (3.4), one substitutes  $\hat{O}_1$  by  $h_{e_x^+}$ , and  $\hat{O}_{i_k}$  by  $\hat{p}_s^\alpha(e_x^+)$  and/or  $\hat{p}_t^\alpha(e_x^+)$ . One of many these substitutions inevitably generates some special

terms in which the commutators only contain  $\hat{p}_s^\alpha(e_x^+)$ . The computation of these commutators with (2.10) will lead to results that are proportional to  $h_{e_x^+}$ . After substituting these permuted results into Eq. (5.28), this  $h_{e_x^+}$  eventually cancels with  $h_{e_x^+}^{-1}$ . (similar to Eq. (5.11)). One can apply the same mechanism to permute  $h_{e_y^+}$  and  $\hat{F}_2$ .

Let us collect these special terms coming from permuting  $h_{e_x^+}$  and  $\hat{F}_1$  as well as permuting  $h_{e_y^+}$  and  $\hat{F}_2$ . Denote the partial sum of these special terms in  $\hat{H}_L^{(k)}$  by  $\text{alt} \hat{H}_L^{(\vec{k})}$ . Because of the cancellation between holonomies and their inverses,  $\text{alt} \hat{H}_L^{(\vec{k})}$  no longer depends on  $h_{e_x^+}$  and  $h_{e_y^+}$ .

It turns out that  $\text{alt} \hat{H}_L^{(\vec{k})}$  possesses more symmetries which will be discussed shortly below. These special terms can be equivalently selected by considering only the non-commutativity between  $h_{e_x^+}$  and  $p_s^\alpha(e_x^+)$  but ignoring the non-commutativity between  $h_{e_x^+}$  and  $p_t^\alpha(e_x^+)$ . That is

$$\text{the special terms of } h_{e_x^+} \hat{F}_1 h_{e_x^+}^{-1} = h_{s_x^+} \hat{F}_1 h_{s_x^+}^{-1} \quad (5.29)$$

where  $s_x^+$  is the segment within  $e_x^+$  and does not contain the target  $t(e_x^+)$ . Because of the aforementioned cancellation between the holonomies and their inverses, it is remarkable that the length of the segment does not cause any ambiguity and the operator in Eq. (5.29) does not change graph ever segment of edges in the holonomy is chosen. Concretely, Eq. (5.29) results in

$$\begin{aligned} & \text{alt} \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_I, e_J, e_K) \\ &= \frac{-1}{2i\beta^7 a^{10} t^5} \epsilon^{IJK} \text{tr} \left( [h_{s_I}, [\hat{Q}_{v_1}^{2k_1}, \hat{H}_E^{(k_2)}(v_2)]] h_{s_I}^{-1} [h_{s_J}, [\hat{Q}_{v_3}^{2k_3}, \hat{H}_E^{(k_4)}(v_4)]] h_{s_J}^{-1} [h_{s_K}, \hat{Q}_v^{2k_5}] h_{s_K}^{-1} \right). \end{aligned} \quad (5.30)$$

It is interesting that the RHS could be understood as that from an alternative definition of the Lorentzian part,

$$\text{alt} \widehat{H}_L[N] = \frac{-1}{2i\beta^7 a^{10} t^5} \sum_v N(v) \sum_{s_I, s_J, s_K \text{ at } v} \epsilon^{IJK} \text{tr} \left( [h_{s_I}, [\hat{V}, \hat{H}_E]] h_{s_I}^{-1} [h_{s_J}, [\hat{V}, \hat{H}_E]] h_{s_J}^{-1} [h_{s_K}, \hat{V}_v] h_{s_K}^{-1} \right). \quad (5.31)$$

in which all edges  $e_I, e_J, e_K$  are replaced by their corresponding segments  $s_I, s_J, s_K$  with  $s_I \subset e_I$ . Indeed,  $\text{alt} \widehat{H}_L[N]$  is obtained by an alternative regularization/quantization of the Hamiltonian, i.e. via the following replacement

$$\{K, \dot{e}^a A_a(x)\} \rightarrow \frac{-1}{2\kappa(i\hbar\beta)^2} [h_{s_e}, [\hat{V}, \hat{H}_E]] h_{s_e}^{-1},$$

where the holonomy along the segment  $s_e \subset e$  instead of the entire edge  $e$  is used. Here,  $\dot{e}^a$  denote the vector tangent to  $e$ .

Collect the terms in  $\hat{H}_L^{(k)}$  other than the special terms discussed above, and denote their sum by  $\text{extra} \hat{H}_L^{(\vec{k})}$ , namely

$$\begin{aligned} & \text{extra} \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_I, e_J, e_K) \\ &= \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_I, e_J, e_K) - \text{alt} \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_I, e_J, e_K). \end{aligned} \quad (5.32)$$

The operators  $\text{alt} \hat{H}_L^{(\vec{k})}$  and  $\text{extra} \hat{H}_L^{(\vec{k})}$  are dealt with separately in our algorithm.

For  $\text{alt} \hat{H}_L^{(\vec{k})}$ , the simplification procedures discussed above result in

$$\frac{C}{t^2} h_{s_x^+} F_1 h_{s_x^+}^{-1} h_{s_y^+} F_2 h_{s_y^+}^{-1} G_1, \quad (5.33)$$

instead of Eq. (5.28). Since  $[h_{s_I}, \hat{p}_t^\alpha(e_I)] = 0$ , we can simplify these terms with

$$h_{s^\pm} \prod_{i=1}^m (\sigma_i^+ \hat{p}_s^{\alpha_i}(e^+) + \sigma_i^- \hat{p}_t^{\alpha_i}(e^-)) h_{s^\pm}^{-1} = \sum_{\mathcal{I}} (it)^{|\mathcal{I}|} \prod_{i \notin \mathcal{I}} (\sigma_i^+ \hat{p}_s^{\alpha_i}(e^+) + \sigma_i^- \hat{p}_t^{\alpha_i}(e^-)) \prod_{j \in \mathcal{I}} \sigma_j^\pm \tau^{\alpha_j} \quad (5.34)$$

where  $\mathcal{I}$  is a subsets of  $\{1, 2, \dots, m\}$  with its length denoted by  $|\mathcal{I}|$ ,  $s^+$  and  $s^-$  are segments of  $e^+$  and  $(e^-)^{-1}$  respectively with  $e^\pm$  oriented such that  $e^+$  and  $(e^-)^{-1}$  are both outgoing,  $\sigma_i^+ = 1$  and  $\sigma_i^- = \pm 1$ . The summation over  $e_I, e_J$  and  $e_K$  in Eq. (5.8) motivates us to compute the following

$$h_{s^+} \prod_{i=1}^m (\sigma_i^+ \hat{p}_s^{\alpha_i}(e^+) + \sigma_i^- \hat{p}_t^{\alpha_i}(e^-)) h_{s^+}^{-1} - h_{s^-} \prod_{i=1}^m (\sigma_i^+ \hat{p}_s^{\alpha_i}(e^+) + \sigma_i^- \hat{p}_t^{\alpha_i}(e^-)) h_{s^-}^{-1}. \quad (5.35)$$

Then, one can apply the aforementioned replacement

$$h_{s^+} \prod_{i=1}^m (\sigma_i^+ \hat{p}_s^{\alpha_i}(e^+) + \sigma_i^- \hat{p}_t^{\alpha_i}(e^-)) h_{s^+}^{-1} \rightarrow \sum_{\mathcal{J}} (it)^{|\mathcal{J}|} \prod_{i \notin \mathcal{J}} (\sigma_i^+ \hat{p}_s^{\alpha_i}(e^+) + \sigma_i^- \hat{p}_t^{\alpha_i}(e^-)) \prod_{j \in \mathcal{J}} \tau^{\alpha_j} \quad (5.36)$$

with  $\mathcal{J} \subset \{1, 2, \dots, m\}$  such that

$$\prod_{j \in \mathcal{J}} \sigma_j^- = -1. \quad (5.37)$$

Substitute Eq. (5.36) into (5.33), one cancels the prefactor  $1/t^2$ , simplifying (5.33) to be in the form of  $C\hat{P}$  with some constant  $C$  of order  $t^0$  or higher. Hence the results in Sec. IV can be applied.

Further,  $\text{alt } \hat{H}_L^{(\vec{k})}$  brings the following symmetries. Consider a  $\pi$ -rotation which transforms either  $e_x^+$  to  $e_x^-$  or  $e_y^+$  to  $e_y^-$ . Denote it by  $\mathfrak{s}$ . Moreover, to indicate the dependence of  $F_i$  on vertices and edges, we will rewrite  $F_i$  in Eq. (5.33) as  $F_i(v, e)$ . Then, since  $\mathfrak{s}(s_k^+)$  with  $k = x, y$  is either  $s_k^+$  or  $s_k^-$  by definition of  $\mathfrak{s}$ , Eq. (5.34) tells us

$$\begin{aligned} & h_{\mathfrak{s}(s_x^+)} F_1(\mathfrak{s}(v), \mathfrak{s}(e)) h_{\mathfrak{s}(s_x^+)}^{-1} h_{\mathfrak{s}(s_y^+)} F_2(\mathfrak{s}(v), \mathfrak{s}(e)) h_{\mathfrak{s}(s_y^+)}^{-1} G_1(\mathfrak{s}(v), \mathfrak{s}(e)) \\ &= h_{s_x^+} F_1(\mathfrak{s}(v), \mathfrak{s}(e)) h_{s_x^+}^{-1} h_{s_y^+} F_2(\mathfrak{s}(v), \mathfrak{s}(e)) h_{s_y^+}^{-1} G_1(\mathfrak{s}(v), \mathfrak{s}(e)). \end{aligned} \quad (5.38)$$

Furthermore, with recalling Eq. (5.4), we obtain the following equation

$$\begin{aligned} & \langle h_{s_x^+} F_1(v, e) h_{s_x^+}^{-1} h_{s_y^+} F_2(v, e) h_{s_y^+}^{-1} G_1(v, e) \rangle \\ &= \langle h_{s_x^+} F_1(\mathfrak{s}(v), \mathfrak{s}(e)) h_{s_x^+}^{-1} h_{s_y^+} F_2(\mathfrak{s}(v), \mathfrak{s}(e)) h_{s_y^+}^{-1} G_1(\mathfrak{s}(v), \mathfrak{s}(e)) \rangle, \end{aligned} \quad (5.39)$$

which reduces the number of contributing vertices and edges in the computation of  $\text{alt } \hat{H}_L^{(\vec{k})}$ .

For the operator  $\text{extr } \hat{H}_L^{(\vec{k})}$ , Eq. (5.34) can no longer be applied. Therefore, the symmetry implied by (5.39) is not manifested. To reduce the complexity of the computation, the following strategy is proposed. Consider a rotation, denoted by  $\mathfrak{r}$ , about the axis  $(1/\sqrt{2}, 1/\sqrt{2}, 0)$  for  $\pi$  radians which exchanges the  $x$ - and  $y$ -axes, and flips the  $z$ -axis. We obtain

$$\begin{aligned} & \langle \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^+) \rangle = \langle \hat{H}_L^{(\vec{k})}(\mathfrak{r}(v); \mathfrak{r}(v_1), \mathfrak{r}(v_2), \mathfrak{r}(v_3), \mathfrak{r}(v_4); \mathfrak{r}(e_x^+), \mathfrak{r}(e_y^+), \mathfrak{r}(e_z^+)) \rangle \\ &= \langle \hat{H}_L^{(\vec{k})}(v; \mathfrak{r}(v_1), \mathfrak{r}(v_2), \mathfrak{r}(v_3), \mathfrak{r}(v_4); e_y^+, e_x^+, e_z^-) \rangle = \langle \hat{H}_L^{(\vec{k})}(v; \mathfrak{r}(v_1), \mathfrak{r}(v_2), \mathfrak{r}(v_3), \mathfrak{r}(v_4); e_y^+, e_x^+, e_z^+) \rangle \end{aligned} \quad (5.40)$$

in which we use the definition of  $\hat{H}_L^{(\vec{k})}$  in (5.16), assume  $\mathfrak{r}(v) = v$  without loss of generality. Moreover, the first equality of the last equation is obtained by using Eq. (5.4), the second one, by the definition of  $\mathfrak{r}$  and, the last one, by Eq. (5.34). Then consider the operator

$$\begin{aligned} & \hat{F}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^+) \\ &:= \hat{H}_L^{(\vec{k})}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^+) + \hat{H}_L^{(\vec{k})}(v; \mathfrak{r}(v_3), \mathfrak{r}(v_4), \mathfrak{r}(v_1), \mathfrak{r}(v_2); e_x^+, e_y^+, e_z^+) \end{aligned} \quad (5.41)$$

According to (5.40),  $\hat{F}$  is

$$\begin{aligned} & \hat{F}(v; v_1, v_2, v_3, v_4; e_x^+, e_y^+, e_z^+) \\ &= -\text{tr}([h_{e_y^+}, [\hat{Q}_{\mathfrak{r}(v_1)}^{2k_1}, \hat{H}_E^{(k_2)}(\mathfrak{r}(v_2))]] h_{e_y^+}^{-1} [h_{e_x^+}, [\hat{Q}_{\mathfrak{r}(v_3)}^{2k_3}, \hat{H}_E^{(k_4)}(\mathfrak{r}(v_4))]] h_{e_x^+}^{-1} [h_{e_z^+}, \hat{Q}_v^{2k_5}] h_{e_z^+}^{-1}) \\ & \quad + \text{tr}([h_{e_x^+}, [\hat{Q}_{\mathfrak{r}(v_3)}^{2k_1}, \hat{H}_E^{(k_2)}(\mathfrak{r}(v_4))]] h_{e_x^+}^{-1} [h_{e_y^+}, [\hat{Q}_{\mathfrak{r}(v_1)}^{2k_3}, \hat{H}_E^{(k_4)}(\mathfrak{r}(v_2))]] h_{e_y^+}^{-1} [h_{e_z^+}, \hat{Q}_v^{2k_5}] h_{e_z^+}^{-1}) \end{aligned} \quad (5.42)$$

up to an overall factor. For clarity, we will denote

$$\begin{aligned} \hat{X} &\equiv [h_{e_x^+}, [\hat{Q}_{\mathfrak{r}(v_3)}^{2k_3}, \hat{H}_E^{(k_4)}(\mathfrak{r}(v_4))]] h_{e_x^+}^{-1} \\ \hat{Y} &\equiv [h_{e_y^+}, [\hat{Q}_{\mathfrak{r}(v_1)}^{2k_1}, \hat{H}_E^{(k_2)}(\mathfrak{r}(v_2))]] h_{e_y^+}^{-1} \\ \hat{Z} &\equiv [h_{e_z^+}, \hat{Q}_v^{2k_5}] h_{e_z^+}^{-1}. \end{aligned} \quad (5.43)$$

Because of the holonomies therein, they are all operator-valued matrices whose entries, thus, will be denoted as  $\hat{X}_{\bar{a}\bar{b}}$ ,  $\hat{Y}_{\bar{a}\bar{b}}$  and  $\hat{Z}_{\bar{a}\bar{b}}$  respectively. Then, one has

$$\hat{F} = \hat{X}_{\bar{a}\bar{b}}\hat{Y}_{\bar{b}\bar{c}}\hat{Z}_{\bar{c}\bar{a}} - \hat{Y}_{\bar{a}\bar{b}}\hat{X}_{\bar{b}\bar{c}}\hat{Z}_{\bar{c}\bar{a}} = \hat{Y}_{\bar{b}\bar{c}}\hat{X}_{\bar{a}\bar{b}}\hat{Z}_{\bar{c}\bar{a}} - \hat{Y}_{\bar{a}\bar{b}}\hat{X}_{\bar{b}\bar{c}}\hat{Z}_{\bar{c}\bar{a}} + [\hat{X}_{\bar{a}\bar{b}}, \hat{Y}_{\bar{b}\bar{c}}]\hat{Z}_{\bar{c}\bar{a}}. \quad (5.44)$$

According to Eq. (3.1), we first compute the expectation values of the operators with respect to coherent states labeled by  $e^{iz_e\tau_3}$ , and then gauge transform the results correspondingly. Hence, applying this procedure to the the first two terms of Eq. (5.44), one finally simplifies the subtraction of their expectation values as

$$\langle \hat{Y}_{\bar{b}\bar{c}}\hat{X}_{\bar{a}\bar{b}}\hat{Z}_{\bar{c}\bar{a}} - \hat{Y}_{\bar{a}\bar{b}}\hat{X}_{\bar{b}\bar{c}}\hat{Z}_{\bar{c}\bar{a}} \rangle_{\{g_e\}} = (\mathcal{C}_{abcdef} - \mathcal{C}'_{abcdef}) \langle \hat{Y}_{ab}\hat{X}_{cd}\hat{Z}_{ef} \rangle_{\{z_e\}} \quad (5.45)$$

where  $\mathcal{C}_{abcdef}$  and  $\mathcal{C}'_{abcdef}$  are two sets of constant coefficients produced by the gauge-transformation. Thanks to the the last equation, one has

$$\langle \hat{F} \rangle_{\{g_e\}} = (\mathcal{C}_{abcdef} - \mathcal{C}'_{abcdef}) \langle \hat{Y}_{ab}\hat{X}_{cd}\hat{Z}_{ef} \rangle_{\{z_e\}} + \langle [\hat{X}_{\bar{a}\bar{b}}, \hat{Y}_{\bar{b}\bar{c}}]\hat{Z}_{\bar{c}\bar{a}} \rangle_{\{g_e\}}. \quad (5.46)$$

Regardless of the factors  $\mathcal{C}_{abcdef} - \mathcal{C}'_{abcdef}$  which is easy to compute, according to Eq. (5.45), the expectation values of  $\langle \hat{Y}_{ab}\hat{X}_{cd}\hat{Z}_{ef} \rangle_{\{z_e\}}$  and  $\langle [\hat{X}_{\bar{a}\bar{b}}, \hat{Y}_{\bar{b}\bar{c}}]\hat{Z}_{\bar{c}\bar{a}} \rangle_{\{g_e\}}$  are needed to compute to get  $\langle \hat{F} \rangle_{\{g_e\}}$ . While for the last term,  $[\hat{X}_{\bar{a}\bar{b}}, \hat{Y}_{\bar{b}\bar{c}}]\hat{Z}_{\bar{c}\bar{a}}$  itself is of  $O(t)$ . Thus we only need to compute the leading-order term of  $\langle [\hat{X}_{\bar{a}\bar{b}}, \hat{Y}_{\bar{b}\bar{c}}]\hat{Z}_{\bar{c}\bar{a}} \rangle$  by applying Theorem IV.2. By definition of  $\hat{F}$ , without applying Eq. (5.46), we need to compute  $\langle \hat{Y}_{\bar{a}\bar{b}}\hat{X}_{\bar{b}\bar{c}}\hat{Z}_{\bar{c}\bar{a}} \rangle$  and  $\langle \hat{Y}_{\bar{a}\bar{b}}\hat{X}_{\bar{b}\bar{c}}\hat{Z}_{\bar{c}\bar{a}} \rangle$  to get  $\langle \hat{F} \rangle$ . Comparing the computed terms before and after applying Eq. (5.46), it is concluded that the computational complexity is reduced due to Eq. (5.46), because it is much easier to compute  $\langle [\hat{X}_{\bar{a}\bar{b}}, \hat{Y}_{\bar{b}\bar{c}}]\hat{Z}_{\bar{c}\bar{a}} \rangle$  by applying Theorem IV.2.

To explicitly compute  $\Delta\mathcal{C} := \mathcal{C}_{abcdef} - \mathcal{C}'_{abcdef}$ , let us use  $O_i$  to denote expectation values of polynomials of fluxes and holonomies with respect to coherent states labeled by  $e^{iz_e\tau_3}$ . Then  $\hat{H}_L^{(\vec{k})}$  and the corresponding results of  $\Delta\mathcal{C}$  take the following forms, which are discussed case by case.

(1) If  $\hat{H}_L^{(\vec{k})}$  is expressed in terms of the form

$$\begin{aligned} (1.1) \quad & (n_x O_1 \tau^\alpha n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^\beta n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}}; \\ (1.2) \quad & (n_x O_1 \tau^\alpha n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^{\beta_1} \tau^{\beta_2} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}}; \\ (1.3) \quad & (n_x O_1 \tau^{\alpha_1} \tau^{\alpha_2} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^\beta n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}}, \end{aligned}$$

then  $\Delta\mathcal{C}$  is defined as the following,

$$\begin{aligned} \Delta\mathcal{C} = & \text{tr} \left[ (n_x^{-1} n_y) \left( \prod_{\beta=1}^{m_y} \tau^\beta \right) (n_y^{-1} n_z) \tau^\gamma (n_z^{-1} n_x) \left( \prod_{\alpha=1}^{m_x} \tau^\alpha \right) \right] \\ & - \text{tr} \left[ (n_y^{-1} n_x) \left( \prod_{\alpha=1}^{m_x} \tau^\alpha \right) (n_x^{-1} n_z) \tau^\gamma (n_z^{-1} n_y) \left( \prod_{\beta=1}^{m_y} \tau^\beta \right) \right] \end{aligned}$$

where  $(m_y, m_x) = (1, 1)$ ,  $(2, 1)$  and  $(1, 2)$  for the cases (1.1), (1.2) and (1.3) respectively.

(2) If  $\hat{H}_L^{(\vec{k})}$  is expressed in terms of the form

$$\begin{aligned} (2.1) \quad & (n_x O_1 h_{e_x^\pm} \tau^\alpha h_{e_x^\pm}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 h_{e_y^\pm} \tau^\beta h_{e_y^\pm}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}}; \\ (2.2) \quad & (n_x O_1 h_{e_x^\pm} \tau^\alpha h_{e_x^\pm}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 h_{e_y^\pm} \tau^{\beta_1} \tau^{\beta_2} h_{e_y^\pm}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}}; \\ (2.3) \quad & (n_x O_1 h_{e_x^\pm} \tau^{\alpha_1} \tau^{\alpha_2} h_{e_x^\pm}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 h_{e_y^\pm} \tau^\beta h_{e_y^\pm}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}}, \end{aligned}$$

then  $\Delta\mathcal{C}$  is defined as the following,

$$\Delta\mathcal{C} = \left( \prod_{\beta=1}^{m_y} \tau^\beta \right)_{bc} \left( \prod_{\alpha=1}^{m_x} \tau^\alpha \right)_{da} \left\{ (n_x^{-1} n_y)_{ab} [(n_y^{-1} n_z) \tau^\gamma (n_z^{-1} n_x)]_{cd} - (n_y^{-1} n_x)_{cd} [(n_x^{-1} n_z) \tau^\gamma (n_z^{-1} n_y)_{ab}] \right\}$$

(3) If  $\hat{H}_L^{(\vec{k})}$  is expressed in terms of the form

$$(3.1) \quad (n_x O_1 h_{e_x^+} \tau^\alpha h_{e_x^+}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^{\beta_1} h_{e_y^+} \tau^{\beta_2} h_{e_y^+}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}},$$

then  $\Delta\mathcal{C}$  is defined as the following,

$$\Delta\mathcal{C} = (\tau^{\beta_2})_{bc} (\tau^\alpha)_{da} \left\{ [n_x^{-1} n_y \tau^{\beta_1}]_{ab} [(n_y^{-1} n_z) \tau^\gamma (n_z^{-1} n_x)]_{cd} - [(n_x^{-1} n_z) \tau^\gamma n_z^{-1} n_y \tau^{\beta_1}]_{ab} [(n_y^{-1} n_x)]_{cd} \right\}$$

(4) If  $\hat{H}_L^{(\vec{k})}$  is expressed in terms of the form

$$(4.1) \quad (n_x O_1 \tau^{\alpha_1} h_{e_x^+} \tau^{\alpha_2} h_{e_x^+}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 h_{e_y^+} \tau^\beta h_{e_y^+}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}},$$

then  $\Delta\mathcal{C}$  is defined as the following,

$$\Delta\mathcal{C} = (\tau^{\alpha_2})_{da} (\tau^\beta)_{bc} \left\{ [(n_x^{-1} n_y)_{ab}] [(n_y^{-1} n_z \tau^\gamma n_z^{-1} n_x \tau^{\alpha_1})_{cd}] - [(n_x^{-1} n_z) \tau^\gamma (n_z^{-1} n_y)]_{ab} [(n_y^{-1} n_x) \tau^{\alpha_1}]_{cd} \right\}$$

(5) If  $\hat{H}_L^{(\vec{k})}$  is expressed in terms of the form

$$(5.1) \quad (n_x O_1 h_{e_x^+} \tau^\alpha h_{e_x^+}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^\beta n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}};$$

$$(5.2) \quad (n_x O_1 h_{e_x^+} \tau^\alpha h_{e_x^+}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^{\beta_1} \tau^{\beta_2} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}};$$

$$(5.3) \quad (n_x O_1 h_{e_x^+} \tau^{\alpha_1} \tau^{\alpha_2} h_{e_x^+}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^\beta n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}},$$

then  $\Delta\mathcal{C}$  is defined as the following,

$$\Delta\mathcal{C} = \left( \prod_{\alpha=1}^{m_x} \tau^\alpha \right)_{ba} \left\{ [(n_x^{-1} n_y)_{ab}] \left( \prod_{\beta=1}^{m_y} \tau^\beta \right) (n_y^{-1} n_z) \tau^\gamma (n_z^{-1} n_x) \right\} - [(n_x^{-1} n_z) \tau^\gamma (n_z^{-1} n_y)]_{ab} \left( \prod_{\beta=1}^{m_y} \tau^\beta \right) (n_y^{-1} n_x) \right\}$$

(6) If  $\hat{H}_L^{(\vec{k})}$  is expressed in terms of the form

$$(6.1) \quad (n_x O_1 \tau^\alpha n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 h_{e_y^+} \tau^\beta h_{e_y^+}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}};$$

$$(6.2) \quad (n_x O_1 \tau^\alpha n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 h_{e_y^+} \tau^{\beta_1} \tau^{\beta_2} h_{e_y^+}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}};$$

$$(6.3) \quad (n_x O_1 \tau^{\alpha_1} \tau^{\alpha_2} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 h_{e_y^+} \tau^\beta h_{e_y^+}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}},$$

then  $\Delta\mathcal{C}$  is defined as the following,

$$\Delta\mathcal{C} = \left( \prod_{\beta=1}^{m_y} \tau^\beta \right)_{ba} \left\{ [(n_y^{-1} n_z) \tau^\gamma (n_z^{-1} n_x)]_{ab} \left( \prod_{\alpha=1}^{m_x} \tau^\alpha \right) (n_x^{-1} n_y) \right\} - [(n_y^{-1} n_x) \left( \prod_{\alpha=1}^{m_x} \tau^\alpha \right) (n_x^{-1} n_z) \tau^\gamma (n_z^{-1} n_y)]_{ab} \right\}$$

(7) If  $\hat{H}_L^{(\vec{k})}$  is expressed in terms of the form

$$(7.1) \quad (n_x O_1 \tau^\alpha n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^{\beta_1} h_{e_y^+} \tau^{\beta_2} h_{e_y^+}^{-1} n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}},$$

then  $\Delta\mathcal{C}$  is defined as the following,

$$\Delta\mathcal{C} = (\tau^{\beta_2})_{ba} \left\{ [(n_y^{-1} n_z) \tau^\gamma (n_z^{-1} n_x) \tau^\alpha (n_x^{-1} n_y) \tau^{\beta_1}]_{ab} - [(n_y^{-1} n_x) \tau^\alpha (n_x^{-1} n_z) \tau^\gamma (n_z^{-1} n_y) \tau^{\beta_1}]_{ab} \right\}$$

(8) If  $\hat{H}_L^{(\vec{k})}$  is expressed in terms of the form

$$(8.1) \quad (n_x O_1 \tau^{\alpha_1} h_{e_x^+} \tau^{\alpha_2} h_{e_x^+}^{-1} n_x^{-1})_{\bar{a}\bar{b}} (n_y O_2 \tau^\beta n_y^{-1})_{\bar{b}\bar{c}} (n_z O_3 \tau^\gamma n_z^{-1})_{\bar{c}\bar{a}}$$

the subtraction of the coefficients are of the form

$$\Delta\mathcal{C} = (\tau^{\alpha_2})_{ba} \left\{ [(n_x^{-1} n_y) \tau^\beta (n_y^{-1} n_z) \tau^\gamma (n_z^{-1} n_x) \tau^{\alpha_1}]_{ab} - [(n_x^{-1} n_z) \tau^\gamma (n_z^{-1} n_y) \tau^\beta n_y^{-1} n_x \tau^{\alpha_1}]_{ab} \right\}.$$

In summary, let  $N_1$  be the original number of terms in  $\text{alt} \hat{H}_L^{(k)}$ . The symmetries implied by Eqs. (5.12) and (5.14) reduces this number to  $N_1/4^2$ , where the power 2 is from the fact that  $\text{alt} \hat{H}_L^{(k)}$  consists of two  $\hat{H}_E$ . Then Eq. (5.16) reduces this number further to  $N_1/(4^2 \times 48)$ . Finally the symmetry (5.39) reduces this number to

$$\frac{N_1}{4^2 \times 48 \times 4} = \frac{N_1}{3072}.$$

Let  $N_2$  be the original number of terms in  $\text{extr} \hat{H}_L^{(k)}$ . The symmetries implied by Eqs. (5.12) and (5.14) are used at first to reduce this number to  $N_2/4^2$ . Then Eq. (5.16) reduces it to  $N_2/(4^2 \times 48)$ . Finally, Eq. (5.46) reduce this number further to be about<sup>5</sup>

$$\frac{N_2}{4^2 \times 48 \times 2} = \frac{N_2}{1536}.$$

## B. Exhibit an explicit computation

In order to demonstrate the idea of our algorithm, some simple examples are used in this section. All of the cases illustrated by these examples finally occur in our computation. Particularly, some functions like  $P(v, i, \mathcal{I}^{(i,m)})$ ,  $\text{WDt}(\mathcal{I}^{(i,k)})$  and  $\text{WD}(\mathcal{I}^{(i,k)})$  indeed exist in our codes [49] as the same manner.

Roughly speaking, the computation is divided into two steps. The first is to simplify the operator with applying the commutation relations (2.10) and the second is to compute the expectation values of the simplified operator.

In the first step, the non-commutative multiplications between operator make a non-trivial simplification. Because of the operator  $\hat{Q}$ , we actually need to deal with

$$\hat{P}^\alpha(v, i) = \hat{p}_s^\alpha(e_v^+) - \hat{p}_t^\alpha(e_v^-)$$

where  $e_v^\pm$  are the edges along the  $i$ th direction satisfying  $s(e_v^+) = v = t(e_v^-)$ . In the computation, we mainly need to deal with the commutators between  $\hat{P}^\alpha(v, i)$  and holonomy. Thus let us consider the example

$$\left[ \prod_{j=1}^m \hat{P}^{\alpha_j}(v, i), h_{e_v^+} \right] = it \sum_{k=1}^m \tau^{\alpha_k} h_{e_v^+} \prod_{j \neq k} \hat{P}^{\alpha_j}(v, i) + (it)^2 \sum_{k < l} \tau^{\alpha_l} \tau^{\alpha_k} h_{e_v^+} \prod_{j \neq k, l} \hat{P}^{\alpha_j}(v, i) + O(t^3). \quad (5.47)$$

Because these  $\alpha_j$  appear in the computed operator as dummy indices, their specific values do not matter in the operator-simplification procedure. In other words,  $\prod_{j=1}^m \hat{P}^{\alpha_j}(v, i)$  is treated like a tensor for which only the type does matter. Therefore, based on our algorithm, we define a function

$$P(v, i, \mathcal{I}^{(i,m)}) := \prod_{\alpha \in \mathcal{I}^{(i,m)}} \hat{P}^\alpha(v, i), \quad (5.48)$$

where  $\mathcal{I}^{(i,m)}$  denote an index set of length  $m$ . In the superscript of  $\mathcal{I}^{(i,m)}$ ,  $i$  is used again for convenience so that the multiplication of  $(\hat{P}^\alpha(v, 1)\hat{P}^\beta(v, 2)\hat{P}^\gamma(v, 3))^m$  occuring in  $\hat{Q}_v^m$  can be denoted as

$$(\hat{P}^\alpha(v, 1)\hat{P}^\beta(v, 2)\hat{P}^\gamma(v, 3))^m = P(v, 1, \mathcal{I}^{(1,m)})P(v, 2, \mathcal{I}^{(2,m)})P(v, 3, \mathcal{I}^{(3,m)}).$$

Similarly, let us define

$$\text{WDt}(\mathcal{I}^{(i,k)}) := \tau^{\alpha_k} \tau^{\alpha_{k-1}} \dots \tau^{\alpha_1} \quad (5.49)$$

where  $\mathcal{I}^{(i,k)} = \{\alpha_1, \dots, \alpha_k\}$ . With these notions, Eq. (5.47) can be written as

$$\begin{aligned} [P(v, i, \mathcal{I}^{(i,m)}), h_{e_v^+}] &= (it) \sum_{k=1}^m \text{WDt}(\tilde{\mathcal{I}}_k^{(i,1)}) h_{e_v^+} P(v, i, \mathcal{I}_k^{(i,m-1)}) \\ &\quad + (it)^2 \sum_{k < l} \text{WDt}(\tilde{\mathcal{I}}_{k,l}^{(i,2)}) h_{e_v^+} P(v, i, \mathcal{I}_{k,l}^{(i,m-2)}) + O(t^3) \end{aligned} \quad (5.50)$$

<sup>5</sup> The word ‘‘about’’ is because there exists cases with  $\hat{X} = \hat{Y}$  in Eq. (5.44).

where  $\mathcal{I}_{k_1, k_2, \dots, k_q}^{(i, m)}$  is the index set of  $\mathcal{I}^{(i, m)}$  without the  $k_1$ th,  $k_2$ th,  $\dots$ ,  $k_q$ th indices and  $\tilde{\mathcal{I}}_{k_1, k_2, \dots, k_q}^{(i, m)}$  is the complement of  $\mathcal{I}_{k_1, k_2, \dots, k_q}^{(i, m)}$ .

Further, let us implement an non-operator factor  $\mathcal{F}(\mathcal{I}^{(i, m)})$  contracted with  $P(v, i, \mathcal{I}^{(i, m)})$  to consider  $\sum_{\mathcal{I}^{(i, m)}} \mathcal{F}(\mathcal{I}^{(i, m)})[P(v, i, \mathcal{I}^{(i, m)}), h_{e_v^+}]$  which is finally simplified as

$$\begin{aligned} & \sum_{\mathcal{I}^{(i, m)}} \mathcal{F}(\mathcal{I}^{(i, m)})[P(v, i, \mathcal{I}^{(i, m)}), h_{e_v^+}] \\ &= (it) \sum_{\mathcal{I}^{(i, m-1)}, \mathcal{I}^{(i, 1)}} \text{WDt}(\mathcal{I}^{(i, 1)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-1)}) \sum_{k=1}^m \mathcal{F}(\{\mathcal{I}^{(i, m-1)}, \mathcal{I}^{(i, 1)}\}_{(k)}) \\ & \quad + (it)^2 \sum_{\mathcal{I}^{(i, m-2)}, \mathcal{I}^{(i, 2)}} \text{WDt}(\mathcal{I}^{(i, 2)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-2)}) \sum_{k < l} \mathcal{F}(\{\mathcal{I}^{(i, m-2)}, \mathcal{I}^{(i, 2)}\}_{(k, l)}) + O(t^3). \end{aligned} \quad (5.51)$$

Here  $\{\mathcal{I}^{(i, m-n)}, \mathcal{I}^{(i, n)}\}_{(k_1, k_2, \dots, k_n)}$  is a joint set obtained by merging  $\mathcal{I}^{(i, m-n)}, \mathcal{I}^{(i, n)}$  in such a way that the elements in  $\mathcal{I}^{(i, n)}$  (respecting their original order) are distributed in  $k_1$ th,  $\dots$ ,  $k_n$ th position in  $\{\mathcal{I}^{(i, m-n)}, \mathcal{I}^{(i, n)}\}_{(k_1, k_2, \dots, k_n)}$ . Instead of deleting  $n$  indices in all possible  $\mathcal{I}^{(i, m)}$ , one replace this deleting procedure by an inserting procedure, namely merging all possible sets  $\mathcal{I}^{(i, m-n)}$  with length  $m-n$  and  $\mathcal{I}^{(i, n)}$  with length  $n$  in the above way for all values of  $k_1 < k_2 < \dots < k_n$ . Therefore, Eq. (5.50) can be simplified further as

$$\begin{aligned} & [h_{e_v^+}, P(v, i, \mathcal{I}^{(i, m)})] \rightarrow \\ & it \text{WDt}(\mathcal{I}^{(i, 1)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-1)}) + (it)^2 \text{WDt}(\mathcal{I}^{(i, 2)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-2)}) + O(t^3). \end{aligned} \quad (5.52)$$

Then, for each evaluation of the index sets  $\mathcal{I}^{(i, m-n)}$  and  $\mathcal{I}^{(i, n)}$ , we joint them in the aforementioned way to consider  $\{\mathcal{I}^{(i, m-n)}, \mathcal{I}^{(i, n)}\}_{(k_1, k_2, \dots, k_n)}$  for all values of  $k_1 < k_2 < \dots < k_n$ .

Similarly for the commutator  $[h_{(e_v^-)^{-1}}, P(v, i, \mathcal{I}^{(i, m)})]$ , we have

$$\begin{aligned} & [h_{(e_v^-)^{-1}}, P(v, i, \mathcal{I}^{(i, m)})] \rightarrow \\ & -it \text{WDt}(\mathcal{I}^{(i, 1)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-1)}) + (-it)^2 \text{WDt}(\mathcal{I}^{(i, 2)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-2)}) + O(t^3). \end{aligned} \quad (5.53)$$

For  $[h_{e_v^-}, P(v, i, \mathcal{I}^{(i, m)})]$  and  $[h_{(e_v^+)^{-1}}, P(v, i, \mathcal{I}^{(i, m)})]$ , we need to define

$$\text{WD}(\mathcal{I}^{(i, k)}) := \tau^{\alpha_1} \tau^{\alpha_1} \dots \tau^{\alpha_k} \quad (5.54)$$

where  $\mathcal{I}^{(i, k)} = \{\alpha_1, \dots, \alpha_k\}$ .

$$\begin{aligned} & [h_{e_v^-}, P(v, i, \mathcal{I}^{(i, m)})] \rightarrow \\ & it \text{WD}(\mathcal{I}^{(i, 1)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-1)}) + (it)^2 \text{WD}(\mathcal{I}^{(i, 2)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-2)}) + O(t^3), \\ & [h_{(e_v^+)^{-1}}, P(v, i, \mathcal{I}^{(i, m)})] \rightarrow \\ & -it \text{WD}(\mathcal{I}^{(i, 1)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-1)}) + (-it)^2 \text{WD}(\mathcal{I}^{(i, 2)}) h_{e_v^+} P(v, i, \mathcal{I}^{(i, m-2)}) + O(t^3). \end{aligned} \quad (5.55)$$

The second step is to compute the expectation value of the simplified operator. Let us still take  $\sum_{\mathcal{I}^{(i, m)}} \mathcal{F}(\mathcal{I}^{(i, m)})[P(v, i, \mathcal{I}^{(i, m)}), h_{e_v^+}]$  as an example. A subtlety here is that the operator  $\sum_{\mathcal{I}^{(i, m)}} \mathcal{F}(\mathcal{I}^{(i, m)})[P(v, i, \mathcal{I}^{(i, m)}), h_{e_v^+}]$  involves two edges  $e_v^\pm$  because of the operator  $\hat{P}^\alpha(v, i)$ . We consider its expectation value with respect to coherent state  $|\psi_{g_{e_v^+}}\rangle \otimes |\psi_{g_{e_v^-}}\rangle$ , where, without loss of generality, we set

$$g_{e_v^+} = g_{e_v^-} = g = n e^{iz\tau_3} n^{-1}.$$

According to the above discussion, one can apply the result of Eq. (5.51) to compute the expectation value of its LHS. Let us take the first term for instance. We need to compute the expectation value of

$$\sum_{\mathcal{I}^{(i, m-1)}, \mathcal{I}^{(i, 1)}, b} \text{WDt}(\mathcal{I}^{(i, 1)}, a, b) \left\langle D_{bc}^{1/2}(h_{e_v^+}) P(v, i, \mathcal{I}^{(i, m-1)}) \right\rangle_g \sum_{k=1}^m \mathcal{F}(\{\mathcal{I}^{(i, m-2)}, \mathcal{I}^{(i, 1)}\}_{(k)}) \quad (5.56)$$

Applying Eq. (3.1), we finally obtain the following

$$\begin{aligned}
& \sum_{\mathcal{I}^{(i,m-1)}, \mathcal{I}^{(i,1)}, b} \text{WDt}(\mathcal{I}^{(i,1)}, a, b) \left\langle D_{bc}^{1/2}(h_{e_v^+}) P(v, i, \mathcal{I}^{(i,m-1)}) \right\rangle_g \sum_{k=1}^m \mathcal{F}(\{\mathcal{I}^{(i,m-2)}, \mathcal{I}^{(i,1)}\}_{(k)}) \\
&= \sum_{\mathcal{I}^{(i,m-1)}, \mathcal{I}^{(i,1)}, b} \text{WDt}(\mathcal{I}^{(i,1)}, a, b) D_{bb'}^{1/2}(n) \left\langle D_{b'c'}^{1/2}(h_{e_v^+}) P(v, i, \mathcal{I}^{(i,m-1)}) \right\rangle_z \\
& D_{c'c}^{1/2}(n^{-1}) \sum_{k=1}^m \tilde{\mathcal{F}}(\{\mathcal{I}^{(i,m-2)}, \mathcal{I}^{(i,1)}\}_{(k)})
\end{aligned} \tag{5.57}$$

where  $\tilde{\mathcal{F}}$  is given by

$$\tilde{\mathcal{F}}(\{\alpha_1, \dots, \alpha_k\}) := \sum_{\beta_1, \dots, \beta_k} \mathcal{F}(\{\beta_1, \dots, \beta_k\}) D_{\alpha_1 \beta_1}^1(n^{-1}) \cdots D_{\alpha_k \beta_k}^1(n^{-1}).$$

To compute Eq. (5.57), the results of Theorem IV.2 can be applied. Thus, one obtains the possible  $\mathcal{I}^{(i,m-1)}$  are:

- (i) If one only computes the leading-order term of the expectation value, then  $\mathcal{I}^{(i,m-1)}$  is evaluated at  $\underbrace{\{0, 0, \dots, 0\}}_{m-1}$
- (ii) If one only computes the expectation value up to  $O(t)$ , then  $\mathcal{I}^{(i,m-1)}$  is evaluated at  $\underbrace{\{0, 0, \dots, 0\}}_{m-1}$ ,  $\underbrace{\{0, 0, \dots, 0, \pm 1\}}_{m-1}$ ,  $\underbrace{\{0, 0, \dots, 0, 1, -1\}}_{m-1}$  and  $\underbrace{\{0, 0, \dots, 0, -1, 1\}}_{m-1}$ .

With this discussion, Eq. (5.57) can be computed by considering both the expectation-value part

$$\left\langle D_{b'c'}^{1/2}(h_{e_v^+}) P(v, i, \mathcal{I}^{(i,m-1)}) \right\rangle_z$$

and the non-operator-factor part

$$\sum_b \text{WDt}(\mathcal{I}^{(i,1)}, a, b) D_{bb'}^{1/2}(n) D_{c'c}^{1/2}(n^{-1}) \sum_{k=1}^m \tilde{\mathcal{F}}(\{\mathcal{I}^{(i,m-2)}, \mathcal{I}^{(i,1)}\}_{(k)})$$

for each possibility of  $\mathcal{I}^{(i,m-1)}$ .

For the expectation-value part, the values of  $\left\langle D_{b'c'}^{1/2}(h_{e_v^+}) P(v, i, \mathcal{I}^{(i,m-1)}) \right\rangle_z$  for Case (i) and the cases where  $\mathcal{I}^{(i,m-1)}$  contains  $\pm 1$  in Case (ii) are computed by Eq. (4.12). Furthermore, for the cases where  $\mathcal{I}^{(i,m-1)}$  contains  $\pm 1$  in Case (ii), the results are independent of the position of  $\pm 1$  in  $\mathcal{I}^{(i,m-1)}$ . Therefore, only  $1 + 2 + 2 = 5$  cases are considered in Case (ii) finally. Comparing with the original number of cases  $3^{3m-1}$ , one can obtain an advantage of our algorithm.

For the non-operator-factor part, given a possible  $\mathcal{I}^{(i,m-2)}$  and consider all possible  $\mathcal{I}^{(i,1)}$ s. Then all possible  $\{\mathcal{I}^{(i,m-2)}, \mathcal{I}^{(i,1)}\}_{(k)}$ s can be reconstructed. It is noted that, for Case (ii), the positions of  $\pm 1$  in  $\mathcal{I}^{(i,m-2)}$  does matter to the value of  $\tilde{\mathcal{F}}$ . Thus we need to consider the permutations of indices in (with keeping the relative order between  $\pm 1$ )  $\mathcal{I}^{(i,m-2)}$  in Case (ii) when reconstructing  $\{\mathcal{I}^{(i,m-2)}, \mathcal{I}^{(i,1)}\}_{(k)}$ . With the results of these two parts, the results of Eq.(5.56) is obtained correspondingly.

Finally, we complete this section with the discussion on the values of  $n$  and  $\vec{k}$  in  $\hat{H}_E^{(n)}$  and  $\hat{H}_L^{(\vec{k})}$ , which is summarized as the following lemma

**Lemma V.1.** *To get the expectation values of  $\hat{H}_E^{(n)}$  and  $\hat{H}_L^{(\vec{k})}$  up to order  $O(t)$ , it is sufficient to set  $n$  and  $\vec{k} = (k_1, k_2, k_3, k_4, k_5)$  respectively such that*

$$n \leq 3 \tag{5.58}$$

and

$$\frac{|k_1 + k_2 - 3| + (k_1 + k_2 - 3)}{2} + \frac{|k_3 + k_4 - 3| + (k_3 + k_4 - 3)}{2} + k_5 \leq 3. \tag{5.59}$$



The proof of this lemma is sketched as follows. When one replaces  $\hat{V}_v$  in  $\widehat{H}_E$  by  $\hat{V}_{GT}^{(v)}$ , the term  $(\hat{Q}^2/\langle\hat{Q}\rangle^2 - 1)^k$  in  $\hat{V}_{GT}^{(v)}$  contributes as the following

$$\begin{aligned} & \frac{1}{t} \text{tr}(h_{\alpha_{IJ}} h_{s_K} [(\hat{Q}^2/\langle\hat{Q}\rangle^2 - 1)^k, h_{s_K}^{-1}]) \\ &= \sum_{m=0}^{k-1} \text{tr}(h_{\alpha_{IJ}} h_{s_K} (\hat{Q}^2/\langle\hat{Q}\rangle^2 - 1)^m \frac{[\hat{Q}^2/\langle\hat{Q}\rangle^2, h_{s_K}^{-1}]}{t} (\hat{Q}^2/\langle\hat{Q}\rangle^2 - 1)^{k-1-m}) \end{aligned} \quad (5.60)$$

where the overall coefficient is neglected. Since  $\hat{Q}^2/\langle\hat{Q}\rangle^2 - 1 = (\hat{Q}/\langle\hat{Q}\rangle + 1)(\hat{Q}/\langle\hat{Q}\rangle - 1)$ , and  $\langle\hat{Q}/\langle\hat{Q}\rangle - 1\rangle = O(t)$ , Theorem IV.1 can be applied, which indicates that the leading-order expectation value of the operator (5.60) is at least  $O(t^{\lfloor k/2 \rfloor})$ . Thus, as far as expanding the expectation value to  $O(t)$  is concerning, one can sufficiently choose  $k \leq 3$ , which leads to  $n \leq 3$  in  $\hat{H}_E^{(n)}$ . A very similar discussion for  $\hat{H}_L^{(\bar{k})}$  can be done, which gives us Eq. (5.59).

## VI. Quantum correction in the expectation value

The resulting expectation value of the Hamiltonian with unit lapse  $\widehat{H}[1] = \widehat{H}_E + (1 + \beta^2) \widehat{H}_L$  at coherent states with cosmological data ( $\eta < 0$  in our convention) is shown as follows

$$\begin{aligned} \langle \widehat{H}_E \rangle &= 6a\sqrt{-\beta\eta} \sin^2(\xi) - \frac{3}{4}at\sqrt{-\frac{\beta}{\eta^3}} \sin^2\left(\frac{\xi}{2}\right) \cos\left(\frac{\xi}{2}\right) \left\{ \cos\left(\frac{\xi}{2}\right) [8\eta^2 + 8\eta(4 \cosh(\eta) - 3)\text{csch}(\eta) - 9] \right. \\ &\quad \left. - 12i\eta \sin\left(\frac{\xi}{2}\right) \right\} + O(t^2), \end{aligned} \quad (6.1)$$

$$\begin{aligned} \langle \widehat{H}_L \rangle &= -\frac{6a\sqrt{-\beta\eta} \sin^2(\xi) \cos^2(\xi)}{\beta^2} - \frac{3at}{262144(-\beta\eta)^{3/2}} \left\{ 2(3 - 220\eta^2) \cos(6\xi) \right. \\ &\quad + 4i\eta(4838 \sin(\xi) - 6284 \sin(2\xi) + 4685 \sin(3\xi) - 5222 \sin(4\xi) - 105 \sin(5\xi)) \\ &\quad + 2(-3611 + 8\eta(492\eta + 11i)) \cos(\xi) - 2(-789 + 4\eta(305\eta + 18i)) \cos(2\xi) \\ &\quad + (4413 - 4\eta(928\eta + 49i)) \cos(3\xi) + 8(-1978 + \eta(4192\eta - 7i)) \cos(4\xi) \\ &\quad + (-7 + 4(-272\eta + 5i)\eta) \cos(5\xi) - 4\eta \coth(\eta) [536 \cos(\xi) + 1731 \cos(2\xi) \\ &\quad + 1524 \cos(3\xi) - 40548 \cos(4\xi) + 116 \cos(5\xi) + 117 \cos(6\xi) + 37292] \\ &\quad + 8\eta \text{csch}(\eta) [130 \cos(\xi) + 918 \cos(2\xi) + 801 \cos(3\xi) - 18618 \cos(4\xi) \\ &\quad \left. + 125 \cos(5\xi) + 58 \cos(6\xi) + 16362] + 8(1436 + \eta(-4056\eta + 25i)) \right\} + O(t^2). \end{aligned} \quad (6.2)$$

According to (5.32),  $\langle \widehat{H}_L \rangle = \langle^{\text{extr}} \widehat{H}_L \rangle + \langle^{\text{alt}} \widehat{H}_L \rangle$ , in which  $\langle^{\text{alt}} \widehat{H}_L \rangle$  is,

$$\begin{aligned} \langle^{\text{alt}} \widehat{H}_L \rangle &= -\frac{3a\sqrt{-\beta\eta} \sin^2(2\xi)}{8\beta^2} - \frac{3at}{32768(-\beta\eta)^{3/2}} \left\{ 4(104\eta^2 - 79) \cos(\xi) + (68 - 160\eta^2) \cos(2\xi) \right. \\ &\quad + 4(55 - 104\eta^2) \cos(3\xi) + (944\eta^2 - 501) \cos(4\xi) - 848\eta^2 \\ &\quad - 2\eta \left[ \coth(\eta)(-262 \cos(\xi) - 6(46 \cos(2\xi) + 65 \cos(3\xi) - 380 \cos(4\xi) + 366)) \right. \\ &\quad - \text{csch}(\eta)(-292 \cos(\xi) + 268 \cos(2\xi) + 420 \cos(3\xi) - 2035 \cos(4\xi) + 1799) \\ &\quad \left. \left. - 64i(6 \sin(\xi) - 10 \sin(2\xi) + 6 \sin(3\xi) - 7 \sin(4\xi)) \right] + 241 \right\} + O(t^2), \end{aligned} \quad (6.3)$$

and  $\langle \widehat{H}_L \rangle^{\text{extr}}$  is

$$\begin{aligned}
\langle \widehat{H}_L \rangle^{\text{extr}} = & -\frac{9a\sqrt{-\beta\eta}\sin^2(2\xi)}{8\beta^2} + \frac{3at}{262144(-\beta\eta)^{3/2}} \left\{ -2(580\eta^2 + 72i\eta - 517)\cos(2\xi) \right. \\
& + 2(3 - 220\eta^2)\cos(6\xi) + 4i\eta[3302\sin(\xi) - 3724\sin(2\xi) + 3149\sin(3\xi) \\
& - 35(98\sin(4\xi) + 3\sin(5\xi))] + 2(-2347 + 8\eta(284\eta + 11i))\cos(\xi) \\
& + (2653 - 4\eta(96\eta + 49i))\cos(3\xi) + 56(-211 + \eta(464\eta - i))\cos(4\xi) \\
& + (-7 + 4(-272\eta + 5i)\eta)\cos(5\xi) - 4\eta\coth(\eta)[1584\cos(\xi) + 627\cos(2\xi) \\
& - 36\cos(3\xi) - 31428\cos(4\xi) + 116\cos(5\xi) + 117\cos(6\xi) + 28508] \\
& + 8\eta\text{csch}(\eta)[714\cos(\xi) + 382\cos(2\xi) - 39\cos(3\xi) \\
& - 14548\cos(4\xi) + 125\cos(5\xi) + 58\cos(6\xi) + 12764] + 8\eta(-3208\eta + 25i) + 9560 \left. \right\} + O(t^2).
\end{aligned} \tag{6.4}$$

When expressing  $\langle \widehat{H}[1] \rangle$  to be  $\langle \widehat{H}[1] \rangle = H_0 + tH_1 + O(t^2)$ , we notice that the  $O(t)$ -term  $H_1$  contains  $\eta$  in its denominator and, thus, is divergent if  $\eta \rightarrow 0$ . This feature is implied by the fact that if  $\langle \widehat{Q}_v \rangle \rightarrow 0$ ,  $\widehat{V}_{GT}^{(v)}$  is divergent. This is because when  $\eta \rightarrow 0$ ,  $\langle \widehat{Q}_v \rangle \rightarrow 0$  since  $\langle \widehat{Q}_v \rangle \sim |\eta|^3$ .

Hence, the expansion of  $\langle \widehat{H}[1] \rangle$  becomes invalid when  $\eta$  is too small. More precisely, expressing  $\langle \widehat{H}[1] \rangle$  as  $\langle \widehat{H}[1] \rangle = \sqrt{|\eta|} [\mathfrak{f}_0 + (t/\eta^2)\mathfrak{f}_1 + O(t^2)]$ , we get that  $\mathfrak{f}_0$  is independent of  $\eta$ , and  $\mathfrak{f}_1$  is regular at  $\eta \rightarrow 0$ . Thus, it is concluded that our expansion is valid when  $\eta^2 \gg t$ . This aspect is important for a new improvement of cosmological effective dynamics derived from the full LQG [58]. The expansion is valid for large  $|\eta|$ , because  $\mathfrak{f}_1$  is regular at  $|\eta| \rightarrow \infty$ .

Consider the reduced-phase-space LQG of gravity coupled to Gaussian dust. Then, the relational evolution with respect to the dust time  $T$  will be generated by the physical Hamiltonian  $\widehat{\mathbf{H}} = \frac{1}{2} (\widehat{H}[1] + \widehat{H}[1]^\dagger)$ . Its coherent state expectation value reads

$$\langle \widehat{\mathbf{H}} \rangle = \Re(\langle \widehat{H}[1] \rangle) \equiv H_0 + t\widetilde{H}_1 + O(t^2), \quad \widetilde{H}_1 = \Re(H_1). \tag{6.5}$$

In order to demonstrate the physical application and effects from the  $O(\hbar)$  correction we adopt the proposal in [15] as follows. Firstly, we view  $\langle \widehat{\mathbf{H}} \rangle$  as the effective Hamiltonian on the 2-dimensional phase space  $\mathcal{P}_{cos}$  of homogeneous and isotropic cosmology. Then, one can verify that  $\eta = \frac{\mu^2 P_0}{\beta a^2}$  and  $\xi = \mu\beta K_0$  where  $\mu$  is the coordinate length of  $e \in E(\gamma)$ ,  $P_0$  is the square of the scale factor and  $K_0$  is the extrinsic curvature. Thus, the Poisson bracket between  $\xi$  and  $\eta$  reads  $\{\eta, \xi\} = \frac{\kappa}{3a^2}$ . With this Poisson bracket, the Hamiltonian time evolution on  $\mathcal{P}_{cos}$  generated by  $\langle \widehat{\mathbf{H}} \rangle$  is computable. The numerical result is shown in Fig. 1, which respectively depicts the dynamics of the spatial volume governed by  $H_0$ ,  $\langle \widehat{\mathbf{H}} \rangle = H_0 + t\widetilde{H}_1$  and the classical FLRW Hamiltonian  $H_{cl} = -6a\beta^{-3/2}\sqrt{-\eta}\xi^2$ .

In the example shown in Fig. 1, a relatively small  $t = 10^{-5}$  is chosen to display the effects of the next-to-leading-order term. The coincided initial data of  $\eta, d\eta/dT$  are chosen to be at  $T = 0$  for all of the three cases. Since  $\eta|_{T=0}$  gives a large spatial volume,  $T = 0$  is in the low-energy-density regime. As shown in Fig.1, toward  $T < 0$ , the evolution with respect to  $H_0 + t\widetilde{H}_1$  is similar as that corresponding to  $H_0$ , where the latter one gives the  $\mu_0$ -scheme effective dynamics. Both of the effective Hamiltonian  $H_0 + t\widetilde{H}_1$  and  $H_0$  resolve the big-bang singularity by ‘‘a bounce’’. Moreover, the dynamics of  $H_0$  and  $H_{cl}$  at late time  $T > 0$  are compatible. Further more, it is shown in Fig. 1 that  $t\widetilde{H}_1$  in  $H_0 + t\widetilde{H}_1$  behaves like an additional matter distribution with negative energy density of  $O(t)$ , which causes the universe to re-collapse and have another bounce at very late time. It should be noted that the time of re-collapse is extremely late because of the tiny value of  $t$ . It can be verified numerically that the time when the recollapse happens is correlated to  $t$ , namely, the recollapse happens earlier for a larger  $t$  but happens later for a smaller  $t$ . For vanishing  $t$ , it needs infinitely long time before the recollapse happens, i.e. there is no recollapse. Finally, the collapse occurs in the FLRW phase with  $\xi \ll 1$ . Thus expanding  $\widetilde{H}_1$  at  $\xi = 0$ , we have

$$\widetilde{H}_1 \cong \frac{3a(-\gamma\eta)^{3/2}(1280\eta^2 - 3072\eta\coth(\eta) - 1792\eta\text{csch}(\eta) - 5568)}{262144\gamma^3\eta^3} + O(\xi^2). \tag{6.6}$$

That is, when  $\xi$  is vanishing,  $\widetilde{H}_1$  contains terms which do not vanish. It is these terms that lead to the recollapse. These terms are all contributed by the Lorentzian part, and is the consequence from the Thiemann’s regularization of the Lorentzian Hamiltonian.

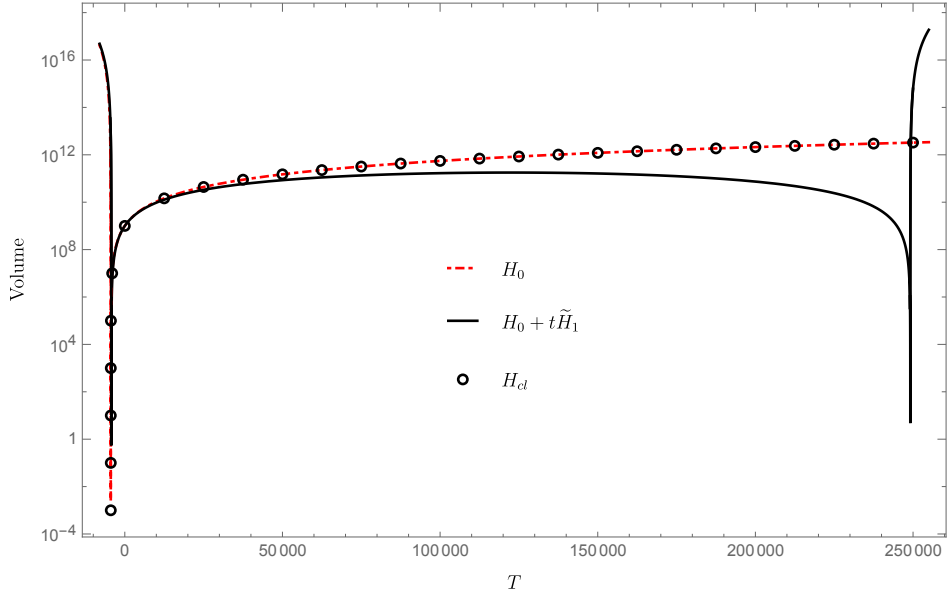


FIG. 1: Evolution of the spatial volume generated by  $\langle \hat{\mathbf{H}} \rangle = H_0 + t\tilde{H}_1$  in (6.5) (black curve),  $H_0$  (red dashed curve), and  $H_{cl}$  (black circles). The coincided initial data of  $\eta, d\eta/dT$  are chosen to be at  $T = 0$  for these 3 cases. The parameters are set to be  $a = 1$ ,  $t = 10^{-5}$ , and  $\beta = 0.2375$ .

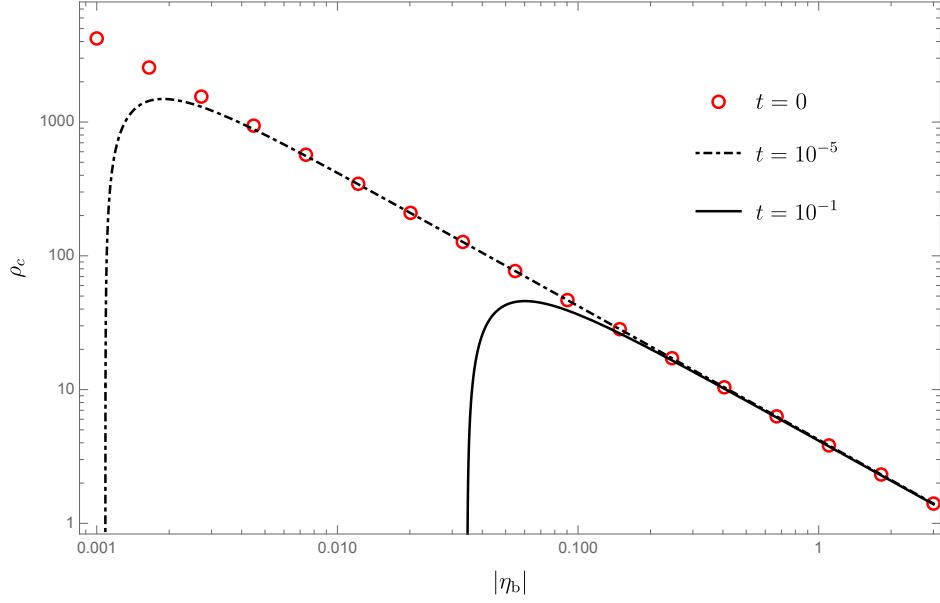


FIG. 2: Plots of the critical density  $\rho_c$  of the dynamics governed by  $H_0$  (red circle) and,  $H_0 + t\tilde{H}_1$  with  $t = 10^{-5}$  (dash-dotted curve)  $t = 0.01$  (solid curve). The parameters are set as  $a = 1$  and  $\beta = 0.2375$ .

For the critical density  $\rho_c$  plotted in Fig.2, it also receives correction from  $t\tilde{H}_1$  at the  $T < 0$  bounce. As is known, the critical density from the  $\mu_0$ -scheme dynamics governed by  $H_0$  reads  $\rho_c = \frac{3}{2a^2\kappa\beta^3(\beta^2+1)|\eta_b|}$  with  $|\eta_b|$  being the value of  $|\eta|$  at the bounce. When  $\tilde{H}_1$  term is considered, the dependence of  $\rho_c$  on  $\eta_b$  is only known numerically. Some numerical results are presented in Fig. 2. As shown there, instead of blowing up for small  $|\eta_b|$ , the corrected  $\rho_c$  from  $\langle \hat{\mathbf{H}} \rangle = H_0 + t\tilde{H}_1$  is bounded from above for small values of  $|\eta_b|$ . In an optimistic viewpoint, this correction of  $\rho_c$  might hint that the correction from higher-order term in  $t$  could potentially flatten the dependence of  $\eta_b$  in  $\rho_c$ . This flattened behavior of  $\rho_c$  is also supported by the current model of  $\bar{\mu}$ -scheme effective dynamics with complete quantum corrections, which is considered as an important feature of  $\bar{\mu}$ -scheme Loop Quantum Cosmology. However, by recalling

the fact that our expansion in  $t$  requires  $\eta^2 \gg t$ , one realizes that the small  $|\eta_b|$  regime, where the correction of  $\rho_c$  becomes significant, mostly violates this requirement (see fig.2). Hence the quantum dynamics near the bounce is still an open problem from this point of view.

It should be emphasized that the proposal [15] adopted here for studying the effective dynamics is not as rigorous as the path integral formula (1.2). As argued in Section I, the  $O(t)$  correction in  $\langle \widehat{H}[1] \rangle$  is only a partial contribution in the quantum effective action that ultimately determining the quantum effect in the dynamics. Hence, the cosmological dynamics plotted in Fig.1 only shows  $O(t)$  correction in  $\langle \widehat{H}[1] \rangle$  from one of the three  $O(t)$  contributions in  $\Gamma$ , and is not yet a rigorous prediction from the principle of LQG. We have only focused on non-gauge-invariant coherent state and neglected the group averaging that impose the gauge invariance in the path integral (1.2). Moreover, this paper only focus on the effects of the quantum correction of linear order in  $t$ , and the effects from higher order corrections are still unclear and beyond the scope of this paper.

## VII. Conclusion and outlook

In this paper, we developed an algorithm to overcome the complexity of computing the expectation value of LQG Hamiltonian operator  $\widehat{H}[N]$ . With this algorithm, the  $O(\hbar)$  correction in the expectation value  $\langle \widehat{H}[N] \rangle$  at the coherent state peaked at the homogeneous and isotropic data of cosmology is computed. In the current work, there are several perspectives which should be addressed in the future:

The first one is to complete the computation of the quantum effective action  $\Gamma$  mentioned in Section I. After completing of this current work, the only missing ingredient in the  $O(\hbar)$  terms of  $\Gamma$  is the “1-loop determinant”  $\det(\mathfrak{H})$ . Therefore, a research to be carried out immediately is to compute this correction of  $\det(\mathfrak{H})$  at the homogeneous and isotropic background. Once we obtain all of the  $O(\hbar)$  contribution to  $\Gamma$ , the variation of  $\Gamma$  should give the quantum corrected effective equations which will demonstrate the quantum correction to the cosmological model implied by LQG.

The next step of generalizing our computation is to study the expectation values of  $\langle \widehat{H}[N] \rangle$  with respect to the coherent states peaked at cosmological perturbations. The semiclassical limits of the expectation value and the cosmological perturbation theory from the path integral (1.2) have been studied in [46]. Thus, it is interesting to study the  $O(\ell_p)$  correction to the cosmological perturbation theory.

Finally, the computation of the quantum correction in  $\langle \widehat{H}[N] \rangle$  should also be extended to the model of gravity coupled to standard matter fields. The contributions of matter fields to  $\widehat{H}[N]$  have been studied in [59, 60]. Since the matter parts in  $\widehat{H}[N]$  is much simpler than the Lorentzian part in  $\widehat{H}[N]$ , the computation of their expectation values should not be hard. Study of the matter contributions and their quantum corrections is a project currently undergoing [61].

Our work expands  $\langle \widehat{H}[N] \rangle$  to the order  $t^1$  and neglects the higher order terms. It is important to understand higher order contributions, or namely the reminder of the expansion, in order to rigorously estimate the regime in the phase space where the expansion is good (see e.g. the paragraph above (6.5) for the discussion of the phase space regime such that  $t\tilde{H}_1$  is small).

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### A. $SL(2, \mathbb{C})$ and $SU(2)$ groups

Let  $\sigma_i$  with  $i = 1, 2, 3$  be the Pauli matrices and  $\tau_k := -i\sigma_k/2$ . Define

$$\vec{\theta} = (\theta \sin(\psi) \cos(\phi), \theta \sin(\psi) \sin(\phi), \theta \cos(\psi)). \quad (\text{A1})$$

$h \in \text{SU}(2)$  can be coordinatized as

$$h = e^{\vec{\theta} \cdot \vec{\tau}} = \begin{pmatrix} \cos(\frac{\theta}{2}) - i \sin(\frac{\theta}{2}) \cos(\psi) & -i \sin(\frac{\theta}{2}) \sin(\psi) e^{-i\phi} \\ -i \sin(\frac{\theta}{2}) \sin(\psi) e^{i\phi} & \cos(\frac{\theta}{2}) + i \sin(\frac{\theta}{2}) \cos(\psi) \end{pmatrix} \quad (\text{A2})$$

with  $\theta, \phi \in (0, 2\pi)$  and  $\psi \in (0, \pi)$ . The Haar measure then is

$$d\mu_H(\vec{\theta}) = \frac{1}{4\pi^2} \sin^2(\frac{\theta}{2}) \sin(\psi) d\theta d\psi d\phi. \quad (\text{A3})$$

Moreover, by defining

$$\vec{p} = (p \sin(\alpha) \cos(\beta), p \sin(\alpha) \sin(\beta), p \cos(\alpha)) \quad (\text{A4})$$

with  $p > 0$ ,  $\alpha \in (0, \pi)$  and  $\beta \in (0, 2\pi)$ ,  $g \in \text{SL}(2, \mathbb{C})$  can be parameterized as

$$g(\vec{p}, \vec{\theta}) = e^{i\vec{p} \cdot \vec{\tau}} e^{\vec{\theta} \cdot \vec{\tau}} =: e^{i\vec{p} \cdot \vec{\tau}} h(\vec{\theta}). \quad (\text{A5})$$

For each  $p > 0$ , there exists  $u_{\vec{p}}^{\pm} \in \text{SU}(2)$  such that

$$(u_{\vec{p}}^{\pm})^{-1} \vec{p} \cdot \vec{\tau} (u_{\vec{p}}^{\pm}) = \pm p \tau_3. \quad (\text{A6})$$

Note that  $u_{\vec{p}}^{\pm}$  is determined by Eq. (A6) up to a right transformation by  $e^{\alpha \tau_3}$ . Namely  $u_{\vec{p}}^{\pm} e^{\alpha \tau_3}$  for all  $\alpha \in \mathbb{R}$  are solution to Eq. (A6) provided  $u_{\vec{p}}^{\pm}$  does. Moreover,  $u_{\vec{p}}^{\pm}$  has the relation

$$u_{\vec{p}}^{\pm} = u_{-\vec{p}}^{\mp}. \quad (\text{A7})$$

Let us denote  $\eta \equiv \pm p$  and  $u \equiv u_{\vec{p}}^{\pm}$  for convenience. Then

$$g(\vec{p}, \vec{\theta}) = u e^{i\eta \tau_3} u^{-1} h(\vec{\theta}). \quad (\text{A8})$$

For  $u^{-1} h(\vec{\theta}) \in \text{SU}(2)$ , decomposing it as

$$u^{-1} h(\vec{\theta}) = e^{-\xi \tau_3} n^{-1}, \quad (\text{A9})$$

one get

$$g(\vec{p}, \vec{\theta}) = u e^{i(\eta + i\xi) \tau_3} n^{-1}. \quad (\text{A10})$$

Note that Eq. (A9) determines  $n$  up to a right transformation by  $e^{\alpha \tau_3}$  as that for  $u_{\vec{p}}^{\pm}$ .  $n$  satisfies

$$n(\eta \tau_3) n^{-1} = h(\vec{\theta}) (\vec{p} \cdot \vec{\tau}) h(\vec{\theta})^{-1}. \quad (\text{A11})$$

The Wigner 3- $j$  symbol  $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$  is an  $\text{SU}(2)$ -invariant tensor, namely

$$D_{n_1 m_1}^{j_1}(h) D_{n_2 m_2}^{j_2}(h) D_{n_3 m_3}^{j_3}(h) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \quad \forall h \in \text{SU}(2). \quad (\text{A12})$$

Define  $\tau_{\alpha}$  with  $\alpha = -1, 0, 1$  as

$$\tau_{\pm 1} = \mp \frac{\tau_1 \pm i\tau_2}{\sqrt{2}}, \quad \tau_0 = \tau_3. \quad (\text{A13})$$

We obtain the  $j$ -representation of  $\tau_{\alpha}$  in terms of the 3 $j$  symbol, according to the WignerEckart theorem, as

$$D_{mm'}^j(\tau_{\alpha}) = i w_j \epsilon_{nm}^j \begin{pmatrix} j & j & 1 \\ n & m' & \alpha \end{pmatrix} \quad (\text{A14})$$

where  $w_j = \sqrt{j(j+1)(2j+1)}$  and  $\epsilon_{nm}^j = (-1)^{j+m} \delta(n, -m)$  is the 2- $j$  symbol. The 2- $j$  symbol is also  $\text{SU}(2)$  invariant. By 3- $j$  symbol and  $\epsilon_{nm}^j$ , any  $\text{SU}(2)$ -intertwiner can be constructed as

$$\iota(k_1 k_2 \cdots k_{n-1})_{m_1 m_2 m_3 \cdots m_n} = \begin{pmatrix} j_1 & j_2 & k_1 \\ m_1 & m_2 & l_1 \end{pmatrix} \epsilon_{l_1 l'_1}^{k_1} \begin{pmatrix} k_1 & j_3 & k_2 \\ l'_1 & m_3 & l_2 \end{pmatrix} \epsilon_{l_2 l'_2}^{k_2} \cdots \epsilon_{l_{n-2} l'_{n-2}}^{k_{n-2}} \begin{pmatrix} k_{n-2} & j_{n-1} & j_n \\ l'_{n-2} & m_{n-1} & m_n \end{pmatrix}. \quad (\text{A15})$$

Moreover, the Clebsch-Gordan coefficients relates to 3 $j$ -symbol as

$$\begin{aligned} \langle j_1 m_1 j_2 m_2 | J M \rangle &= (-1)^{2j_2} \sqrt{2J+1} \epsilon_{MN}^J \begin{pmatrix} J & j_2 & j_1 \\ N & m_2 & m_1 \end{pmatrix} \\ &= (-1)^{j_1 - j_2 - J} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & N \end{pmatrix} \epsilon_{NM}^J \end{aligned} \quad (\text{A16})$$

### B. the Clebsch-Gordan coefficients with negative parameters

Given  $j_1, m_1, j_2, m_2$  and  $m = m_1 + m_2$ , the Clebsch-Gordan coefficients  $\langle j_1 m_1 j_2 m_2 | p m \rangle \equiv \begin{bmatrix} j_1 & j_2 & p \\ m_1 & m_2 & m \end{bmatrix}$  for various  $p$  satisfy the difference equation [62]

$$A(p+1) \begin{bmatrix} j_1 & j_2 & p+1 \\ m_1 & m_2 & m \end{bmatrix} + A(p) \begin{bmatrix} j_1 & j_2 & p-1 \\ m_1 & m_2 & m \end{bmatrix} + (A_0(p) - m_1 + m_2) \begin{bmatrix} j_1 & j_2 & p \\ m_1 & m_2 & m \end{bmatrix} = 0 \quad (\text{B1})$$

where  $\max(|m|, |j_1 - j_2|) \leq p \leq j_1 + j_2$  and

$$A(p) = \frac{1}{p} \sqrt{-\frac{(p^2 - \xi_1^2)(p^2 - \xi_2^2)(p^2 - \xi_3^2)}{4p^2 - 1}}, \quad (\text{B2})$$

$$A_0(p) = \frac{\xi_1 \xi_2 \xi_3}{p(p+1)}$$

with

$$\xi_1 = j_1 - j_2, \quad \xi_2 = j_1 + j_2 + 1, \quad \xi_3 = m.$$

With the initial data

$$\begin{bmatrix} j_1 & j_2 & j_1 + j_2 \\ m_1 & m_2 & m \end{bmatrix} = \sqrt{\frac{(2j_1)!(2j_2)!(j_1 + j_2 - m)!(j_1 + j_2 + m)!}{(2j_1 + 2j_2)!(j_1 - m_1)!(j_1 + m_1)!(j_2 - m_2)!(j_2 + m_2)!}} \quad (\text{B3})$$

the Clebsch-Gordan coefficients for other values of  $p$  are computable with Eq. (B1).

In order to extend the Clebsch-Gordan coefficients to negative parameters, we define a function

$$\mathcal{C}_0(x) := \sqrt{\frac{(2j_2)!\Gamma(2x+1)\Gamma(x+j_2-m+1)\Gamma(x+j_2+m+1)}{(j_2-m_2)!(j_2+m_2)!\Gamma(2x+2j_2+1)\Gamma(x-m_1+1)\Gamma(x+m_1+1)}}, \quad (\text{B4})$$

with which

$$\begin{bmatrix} j_1 & j_2 & j_1 + j_2 \\ m_1 & m_2 & m \end{bmatrix} = \mathcal{C}_0(j_1). \quad (\text{B5})$$

It is remarkable that  $\mathcal{C}_0(x)$  is well-defined not only for positive  $x$  such that

$$x - m_1 \in \mathbb{Z}, \quad x \geq |m_1| \quad (\text{B6})$$

but also for negative  $x$  satisfying

$$x - m_1 \in \mathbb{N}, \quad x \leq \min(-|m_1|, -j_2 - |m| - 1) \quad (\text{B7})$$

where those gamma functions with negative integers is understood as

$$\frac{\Gamma(-m_1) \cdots \Gamma(-m_k)}{\Gamma(-n_1) \cdots \Gamma(-n_k)} = \lim_{z \rightarrow 0} \frac{\Gamma(z - m_1) \cdots \Gamma(z - m_k)}{\Gamma(z - n_1) \cdots \Gamma(z - n_k)} = (-1)^{n_1 + \cdots + n_k - m_1 - \cdots - m_k} \frac{n_1! \cdots n_k!}{m_1! \cdots m_k!}. \quad (\text{B8})$$

By definition, the Clebsch-Gordan coefficients  $\begin{bmatrix} j_1 & j_2 & j_1 + j_2 - \iota \\ m_1 & m_2 & m \end{bmatrix}$  is obtained by applying the recurrence relation (B1) successively for  $\iota$  steps with the initial data  $\mathcal{C}_0(j_1, j_1 + j_2)$ . Then, we defined  $\begin{bmatrix} -j_1 & j_2 & -j_1 + j_2 - \iota \\ m_1 & m_2 & m \end{bmatrix}$ , the Clebsch-Gordan coefficients with negative parameters, as the result by applying the recurrence relation

$$\begin{bmatrix} -j_1 & j_2 & -q-1 \\ m_1 & m_2 & m \end{bmatrix} = -\frac{1}{\tilde{A}(-q)} \left( \tilde{A}(-q+1) \begin{bmatrix} -j_1 & j_2 & -q+1 \\ m_1 & m_2 & m \end{bmatrix} + (\tilde{A}_0(-q) - m_1 + m_2) \begin{bmatrix} -j_1 & j_2 & -q \\ m_1 & m_2 & m \end{bmatrix} \right) \quad (\text{B9})$$

with the initial data  $\mathcal{C}_0(-j_1)$ , where

$$\tilde{A}(q) = A(q)|_{j_1 \rightarrow -j_1}, \quad \tilde{A}_0(q) = A_0(q)|_{j_1 \rightarrow -j_1}. \quad (\text{B10})$$

This definition extended the Clebsch-Gordan coefficients to negative parameters. It guarantees that,

$$\begin{bmatrix} -j_1 & j_2 & -j_1 + j_2 - \iota \\ m_1 & m_2 & m \end{bmatrix} = \begin{bmatrix} j_1 & j_2 & j_1 + j_2 - \iota \\ m_1 & m_2 & m \end{bmatrix} \Big|_{j_1 \rightarrow -j_1}. \quad (\text{B11})$$

By definition, it has

$$\begin{aligned} \mathcal{C}_0(-j_1) &= \sqrt{\frac{(2j_1 - 2j_2 - 1)!(2j_2)!(j_1 - m_1 - 1)!(j_1 + m_1 - 1)!}{(2j_1 - 1)!(j_1 - j_2 - m - 1)!(j_1 - j_2 + m - 1)!(j_2 - m_2)!(j_2 + m_2)!}} \\ &= (-1)^{j_2 + m_2} \begin{bmatrix} j_1 - 1 & j_2 & j_1 - j_2 - 1 \\ m_1 & m_2 & m \end{bmatrix}. \end{aligned} \quad (\text{B12})$$

Moreover,  $\begin{bmatrix} j_1 - 1 & j_2 & j_1 - j_2 + \iota - 1 \\ m_1 & m_2 & m \end{bmatrix}$  can also be obtained by applying successively the recurrence relation

$$\begin{bmatrix} j_1 & j_2 & q \\ m_1 & m_2 & m \end{bmatrix} = -\frac{1}{B(q)} \left( B(q-1) \begin{bmatrix} j_1 & j_2 & q-2 \\ m_1 & m_2 & m \end{bmatrix} + (B_0(q-1) - m_1 + m_2) \begin{bmatrix} j_1 & j_2 & q-1 \\ m_1 & m_2 & m \end{bmatrix} \right) \quad (\text{B13})$$

with the initial data  $\begin{bmatrix} j_1 & j_2 & j_1 - j_2 - 1 \\ m_1 & m_2 & m \end{bmatrix}$ , where

$$B(q) = A(q) \Big|_{j_1 \rightarrow j_1 - 1}, \quad B_0(q) := A_0(q) \Big|_{j_1 \rightarrow j_1 - 1}. \quad (\text{B14})$$

Furthermore, it can be verified that

$$B(q) = -\tilde{A}(-q), \quad B_0(q-1) = \tilde{A}_0(-q). \quad (\text{B15})$$

Therefore, according to Eqs. (B12), (B9) and (B13), we finally have

$$\begin{bmatrix} -j_1 & j_2 & -j_1 + j_2 - \iota \\ m_1 & m_2 & m \end{bmatrix} = (-1)^{j_2 + m_2 - \iota} \begin{bmatrix} j_1 - 1 & j_2 & j_1 - j_2 + \iota - 1 \\ m_1 & m_2 & m \end{bmatrix}, \quad (\text{B16})$$

namely

$$\begin{bmatrix} -j_1 & j_2 & -j_1 + \Delta \\ m_1 & m_2 & m \end{bmatrix} = (-1)^{\Delta + m_2} \begin{bmatrix} j_1 - 1 & j_2 & j_1 - 1 - \Delta \\ m_1 & m_2 & m \end{bmatrix}. \quad (\text{B17})$$

### C. Proof of (3.10)

One can refer to [63–65] for more details on this method. The 2- $j$  symbol is graphically represented as

$$\epsilon_{mn}^j = (-1)^{j+n} \delta(m, -n) = \begin{array}{c} m \\ \downarrow \\ \xrightarrow{j} \\ \downarrow \\ n \end{array}. \quad (\text{C1})$$

The 3- $j$ -symbols is graphically represented as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{array}{c} m_1 \\ | \\ j_1 \\ \swarrow \quad \searrow \\ j_2 \quad j_3 \\ \swarrow \quad \searrow \\ m_2 \quad m_3 \\ + \end{array}. \quad (\text{C2})$$

The Wigner-D matrix  $D_{mn}^j(h)$ , as a tensor  $h \in \mathcal{H}_j \otimes \mathcal{H}_j^*$ , is

$$D_{mn}^j(h) = \langle jm | h | jn \rangle = \begin{array}{c} j \\ \triangleleft \\ m \quad \quad \quad n \end{array}. \quad (\text{C3})$$

For the multiplication operator  $D_{ab}^\iota(h_e)$ , its action on  $D_{mn}^j(h_e)$  reads

$$D_{ab}^\iota(h_e)D_{mn}^j(h_e) = \sum_{J=j-\iota}^{j+\iota} d_J (-1)^{M-N} \begin{pmatrix} \iota & j & J \\ a & m & -M \end{pmatrix} \begin{pmatrix} \iota & j & J \\ b & n & -N \end{pmatrix} D_{MN}^J(h), \quad (\text{C4})$$

where  $\begin{pmatrix} \iota & j & J \\ a & m & -M \end{pmatrix}$  denotes the Wigner  $3j$ -symbol. This graphically corresponds to

$$\begin{array}{c} j_1 \\ \diagdown \\ \text{---} \triangleleft h \\ \diagup \\ j_2 \end{array} = \sum_{J=|j_1-j_2|}^{j_1+j_2} d_J \begin{array}{c} j_1 \\ \diagdown \\ \text{---} \triangleleft h \\ \diagup \\ j_2 \end{array} \begin{array}{c} J \\ \diagdown \\ \text{---} \triangleleft h \\ \diagup \\ J \end{array} \begin{array}{c} j_1 \\ \diagdown \\ \text{---} \triangleleft h \\ \diagup \\ j_2 \end{array}. \quad (\text{C5})$$

For the operators  $\hat{p}_s^i(e)$  and  $\hat{p}_t^j(e)$ , by (2.9) we have

$$\begin{aligned} \hat{p}_s^\alpha(e)D_{mn}^j(h_e) &= -tw_j e_{\tilde{m}m}^j \begin{pmatrix} j & j & 1 \\ \tilde{m} & m' & \alpha \end{pmatrix} D_{m'n}^j(h_e), \\ \hat{p}_t^\alpha(e)D_{mn}^j(h_e) &= tw_j e_{\tilde{n}n'}^j \begin{pmatrix} j & j & 1 \\ \tilde{n} & n & \alpha \end{pmatrix} D_{mn'}^j(h_e), \end{aligned} \quad (\text{C6})$$

where we used (A14). Thus, one has

$$p_s^\alpha(e) \begin{array}{c} j \\ \diagdown \\ \text{---} \triangleleft h_e \\ \diagup \\ m \end{array} \begin{array}{c} n \end{array} = -tw_j \begin{array}{c} m \\ \leftarrow \\ \text{---} \triangleleft h_e \\ \downarrow \alpha \\ 1 \end{array} \begin{array}{c} j \\ \diagdown \\ \text{---} \triangleleft h_e \\ \diagup \\ n \end{array} \quad (\text{C7})$$

$$p_t^\alpha(e) \begin{array}{c} j \\ \diagdown \\ \text{---} \triangleleft h_e \\ \diagup \\ m \end{array} \begin{array}{c} n \end{array} = tw_j \begin{array}{c} m \\ \text{---} \triangleleft h_e \\ \leftarrow \\ \downarrow \alpha \\ 1 \end{array} \begin{array}{c} j \\ \diagdown \\ \text{---} \triangleleft h_e \\ \diagup \\ n \end{array} \quad (\text{C8})$$

Given a monomial of holonomies and fluxes  $\hat{F}_e$ . According to Eqs. (C5), (C7) and (C8), we draw  $\hat{F}_e$  graphically as

$$\int d\mu_h \overline{D_{m'n'}^{j'}(h)} \hat{F}_e D_{mn}^j(h) = \frac{1}{d_{j'}} \begin{array}{c} j \\ \downarrow n \\ \text{---} \boxed{F_l} \text{---} \\ \uparrow j' \\ j' \downarrow n' \end{array} \begin{array}{c} m \\ \uparrow j \\ \text{---} \boxed{F_r} \text{---} \\ \downarrow j' \\ m' \downarrow j' \end{array}, \quad (\text{C9})$$

where the dashed lines represents the indices possessed by  $\hat{F}_e$ . Some dashed lines may carry arrows depending on the form of  $\hat{F}_e$ . With this formula, the expectation value of  $\langle \hat{F}_e \rangle_{z_e}$  is

$$\langle \hat{F}_e \rangle_{z_e} = \sum_{jj'} d_j e^{-\frac{t}{2}(j(j+1)+j'(j'+1))} \begin{array}{c} j \\ \downarrow z_e \\ \text{---} \boxed{F_l} \text{---} \\ \uparrow z_e \\ j' \downarrow z_e \end{array} \begin{array}{c} m \\ \uparrow z_e \\ \text{---} \boxed{F_r} \text{---} \\ \downarrow z_e \\ m' \downarrow z_e \end{array}. \quad (\text{C10})$$





Therefore

$$\langle \hat{F}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} = t^m \sum_{n \in \mathbb{Z}} e^{-\frac{t}{4}n^2 + \frac{t}{4}n} F_0\left(\frac{n-1}{2}, \frac{\partial_\eta}{2}\right) \frac{e^{n\eta}}{2 \sinh(\eta)} \quad (\text{D6})$$

where  $n \equiv 2j + 1$ . Substituting the values of the 3- $j$  and 2- $j$  symbols, we have that

$$F_0(j, \frac{\partial_\eta}{2}) = \delta\left(\sum_{i=1}^m \alpha_i, 0\right) \left(\prod_{i=1}^m \frac{1}{(1 + |\alpha_i|)^{1/2}}\right) \prod_{n=1}^m \left(\alpha_n j - \frac{\partial_\eta}{2} + \sum_{k=1}^n \alpha_k\right) \quad (\text{D7})$$

Applying the Poisson summation formula to Eq. (D6), we get

$$\begin{aligned} & \langle \hat{F}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} \\ &= \delta\left(\sum_i \alpha_i, 0\right) t^m \prod_{i=1}^m \frac{1}{(1 + |\alpha_i|)^{1/2}} e^{t/4} \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}x^2} x \prod_{n=1}^m \left(\alpha_n \frac{x-1}{2} - \frac{\partial_\eta}{2} + \sum_{k=1}^n \alpha_k\right) \frac{e^{x\eta}}{2 \sinh(\eta)} + O(t^\infty). \end{aligned} \quad (\text{D8})$$

By the trick

$$\partial_\eta^n \frac{e^{x\eta}}{\sinh(\eta)} = \left. \left( (x + \partial_y)^n \frac{e^{x\eta}}{\sinh(y)} \right) \right|_{y \rightarrow \eta}, \quad (\text{D9})$$

Eq. (D8) can be finally simplified as

$$\begin{aligned} & \langle \hat{F}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} \\ &= \delta\left(\sum_i \alpha_i, 0\right) t^m \prod_{i=1}^m \frac{1}{(1 + |\alpha_i|)^{1/2}} e^{t/4} \int_{-\infty}^{\infty} dx x \prod_{k=1}^m \left( \frac{\alpha_k - 1}{2} x - \frac{\partial_y}{2} + \sum_{i=1}^k \alpha_i - \frac{\alpha_k}{2} \right) \frac{e^{-\frac{t}{4}x^2 + x\eta}}{2 \sinh(y)} \Big|_{y \rightarrow \eta} + O(t^\infty) \end{aligned} \quad (\text{D10})$$

The integrand is a polynomial of  $x$  multiplied by a Gaussian function  $e^{-tx^2/4 + x\eta}$ . Thus let us integrate  $x^n e^{-tx^2/4 + x\eta}$

$$\int_{-\infty}^{\infty} dx x^n e^{-\frac{t}{4}x^2 + \eta x} = \frac{t}{e^{t/4}} \frac{\sinh(\eta)}{\eta} \langle 1 \rangle \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \eta^{n-2k} \left(\frac{t}{2}\right)^{k-n} (2k-1)!! \quad (\text{D11})$$

where  $\lfloor y \rfloor$  denotes the greatest integer less than or equal to  $y$  and Eq. (2.26) is substituted because we are concerned on the expectation value with respect to the normalized coherent states, i.e.  $\langle \hat{F}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} / \langle 1 \rangle$ . The leading-order term of the RHS of Eq. (D11) is  $O(t^{-n+1})$  with  $n$  the power of  $x$  in the integrand. Thus it is the highest-power term of  $x$  in the integrand of (D10) that leads to the leading-order term of  $\langle \hat{F}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e}$ , and the highest-power and second-highest-power terms of  $x$  lead to the next-to-leading order term of  $\langle \hat{F}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e}$ , and so on. Therefore, we finally get

$$\begin{aligned} & \langle \hat{F}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} \\ &= \delta\left(\sum_i \alpha_i, 0\right) \langle 1 \rangle \prod_{i=1}^m \frac{1}{(1 + |\alpha_i|)^{1/2}} \left\{ \left( \prod_{k=1}^m (\alpha_k - 1) \right) \eta^m + \left( \prod_{k=1}^m (\alpha_k - 1) \right) \frac{(m+1)m}{2} \eta^{m-2} \frac{t}{2} + \right. \\ & \quad \left. + \sum_{k=1}^m \left( \prod_{i \neq k} (\alpha_i - 1) \right) \left( \frac{\coth(\eta) + \alpha_k}{2} + \alpha_1 + \cdots + \alpha_{k-1} \right) \eta^{m-1} t + \cdots \right\}. \end{aligned} \quad (\text{D12})$$

## 2. for the case with $\iota \neq 0$

Now let us consider the operator

$$\hat{F}_{lab}^{\alpha_1 \cdots \alpha_n} = \hat{p}_s^{\alpha_1}(e) \cdots \hat{p}_s^{\alpha_m}(e) D_{ab}^\iota(h_e), \quad \iota \neq 0. \quad (\text{D13})$$

By using (C6) and (C4), it can be obtained that

$$\begin{aligned} \langle \hat{F}_{lab}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} &= \sum_{jj'} d_j e^{-\frac{t}{2}(j(j+1)+j'(j'+1))} d_{j'} (-tw_{j'})^m \sum_{kk'} e^{kz_e+k'\bar{z}_e} \epsilon_{k'k_1}^{j'} \begin{pmatrix} j' & 1 & j' \\ k_1 & \alpha_1 & n_1 \end{pmatrix} \epsilon_{n_1 k_2}^{j'} \begin{pmatrix} j' & 1 & j' \\ k_2 & \alpha_2 & n_2 \end{pmatrix} \cdots \\ &\begin{pmatrix} j' & 1 & j' \\ k_{m-1} & \alpha_{m-1} & n_{m-1} \end{pmatrix} \epsilon_{n_{m-1} k_m}^{j'} \begin{pmatrix} j' & 1 & j' \\ k_m & \alpha_m & n_m \end{pmatrix} \epsilon_{n_m k_{m+1}}^{j'} \begin{pmatrix} j' & \iota & j \\ k_{m+1} & a' & k \end{pmatrix} \epsilon_{aa'}^\iota \begin{pmatrix} j' & \iota & j \\ k' & b & k'' \end{pmatrix} \epsilon_{k''k}^j \end{aligned} \quad (D14)$$

Because the products of these  $3j$ -symbols vanish unless  $k = k' + b$ , the last equation can be simplified as

$$\langle \hat{F}_{lab}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} = t^m e^{bz_e} \sum_{j,j'} e^{-\frac{t}{2}(j(j+1)+j'(j'+1))} F_\iota(j, j', \frac{\partial \eta}{2}) \frac{\sinh((2j' + 1)\eta)}{\sinh(\eta)} \quad (D15)$$

where  $F_\iota$  is given by

$$\begin{aligned} F_\iota(j, j', k') &:= d_j d_{j'} (-w_{j'})^m \epsilon_{k'k_1}^{j'} \begin{pmatrix} j' & 1 & j' \\ k_1 & \alpha_1 & n_1 \end{pmatrix} \epsilon_{n_1 k_2}^{j'} \begin{pmatrix} j' & 1 & j' \\ k_2 & \alpha_2 & n_2 \end{pmatrix} \cdots \begin{pmatrix} j' & 1 & j' \\ k_{m-1} & \alpha_{m-1} & n_{m-1} \end{pmatrix} \epsilon_{n_{m-1} k_m}^{j'} \\ &\begin{pmatrix} j' & 1 & j' \\ k_m & \alpha_m & n_m \end{pmatrix} \epsilon_{n_m k_{m+1}}^{j'} \begin{pmatrix} j' & \iota & j \\ k_{m+1} & a' & k' + b \end{pmatrix} \epsilon_{aa'}^\iota \begin{pmatrix} j' & \iota & j \\ k' & b & k'' \end{pmatrix} \epsilon_{k''k'+b}^j. \end{aligned} \quad (D16)$$

The possible values of  $j$  are  $|j' - \iota|$ ,  $|j' - \iota| + 1$ ,  $\dots$ ,  $j' + \iota$ . By applying Eqs. (A16) and (B17) again, we have

$$\begin{aligned} &\epsilon_{n_m k_{m+1}}^{j'} \begin{pmatrix} j' & \iota & j \\ k_{m+1} & a' & k' + b \end{pmatrix} \begin{pmatrix} j' & \iota & j \\ k' & b & k'' \end{pmatrix} \epsilon_{k''k'+b}^j \Big|_{j=j'-\Delta} \\ &= (-1)^{b-a+1} \left[ \epsilon_{n_m k_{m+1}}^{j'} \begin{pmatrix} j' & \iota & j \\ k_{m+1} & a' & k' + b \end{pmatrix} \begin{pmatrix} j' & \iota & j \\ k' & b & k'' \end{pmatrix} \epsilon_{k''k'+b}^j \Big|_{j=j'+\Delta} \right]_{j' \rightarrow -j'-1}. \end{aligned} \quad (D17)$$

Together with Eq. (D4), it leads to

$$F_\iota(j' + \Delta, j', k') = -F_\iota(j' - \Delta, j', k') \Big|_{j' \rightarrow -j'-1} \quad (D18)$$

where we use  $F_\iota(j, j', k) \propto \delta(\sum_{i=1}^m \alpha_i - a + b, 0)$ . Thus for  $j' = j \pm d$  with  $d > 0$ , one has

$$\begin{aligned} &\sum_{2j'+d \geq \iota} f(d_{j'}^2) F_\iota(j' + d, j', \frac{\partial \eta}{2}) \frac{\sinh(d_{j'} \eta)}{\sinh(\eta)} + \sum_{2j'-d \geq \iota} f(d_{j'}^2) F_\iota(j' - d, j', \frac{\partial \eta}{2}) \frac{\sinh(d_{j'} \eta)}{\sinh(\eta)} \\ &= \sum_{-2j'-2+d \geq \iota} f(d_{j'}^2) F_\iota(-j' - 1 + d, -j' - 1, \frac{\partial \eta}{2}) \frac{\sinh(-d_{j'} \eta)}{\sinh(\eta)} + \sum_{2j'-d \geq \iota} f(d_{j'}^2) F_\iota(j' - d, j', \frac{\partial \eta}{2}) \frac{\sinh(d_{j'} \eta)}{\sinh(\eta)} \\ &= \sum_{\substack{n \in \mathbb{Z} \\ n \notin [d-\iota, d+\iota]}} f(d_{j'}^2) F_\iota(\frac{n-1}{2} - d, \frac{n-1}{2}, \frac{\partial \eta}{2}) \frac{\sinh(n\eta)}{\sinh(\eta)}. \end{aligned}$$

Here the conditions  $2j' \pm d \geq \iota$  are from the the triangle condition i.e.  $j' + j \geq \iota$ , of the  $3j$ -symbol. The second equality is because of the replacement  $j' \rightarrow -j' - 1$  in the first summation,  $d_{j'} = 2j' + 1 \equiv n$  and  $f$  is an arbitrary function. Therefore,

$$\langle \hat{F}_{lab}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} = t^m e^{bz_e} \sum_{\substack{0 \leq d \leq \iota \\ d+\iota \in \mathbb{Z}}} \frac{2 - \delta(d, 0)}{2} e^{-\frac{t}{4}(2d^2-1)} \sum_{\substack{n \in \mathbb{Z} \\ n \notin [d-\iota, d+\iota]}} e^{-\frac{1}{4}t(n^2-2dn)} F_\iota(\frac{n-1}{2} - d, \frac{n-1}{2}, \frac{\partial \eta}{2}) \frac{\sinh(n\eta)}{\sinh(\eta)} \quad (D19)$$

To calculate  $F_\iota(j, j', k')$ , we notice that

$$-w_{j'} \epsilon_{n_{i-1} k_i}^{j'} \begin{pmatrix} j' & 1 & j' \\ k_i & \alpha_i & n_i \end{pmatrix} = \delta(k_i + \alpha_i + n_i, 0) (\alpha_i j' - n_i) \sqrt{\frac{(j' - n_{i-1})! (n_{i-1} + j')!}{(1 - \alpha_i)! (\alpha_i + 1)! (j' - n_i)! (n_i + j')!}} \quad (D20)$$

Moreover, by applying Eq. (A16) and the expression of the Clebsch-Gordan coefficients (Eq. (5), Section 8.2.1 in [66]), we have

$$\begin{aligned} \epsilon_{n_m k_{m+1}}^{j'} \begin{pmatrix} j' & \iota & j \\ k_{m+1} & a' & k \end{pmatrix} \begin{pmatrix} j' & \iota & j \\ k' & b & k'' \end{pmatrix} \epsilon_{k'' k}^j = (-1)^{j'-\iota-j+a'+b} \delta_{a'+k, n_m} \delta_{k'+b, k} \times \\ \frac{(j+\iota-j')!(j-\iota+j')!(-j+\iota+j')!}{(j+\iota+j'+1)!} \left( \frac{(j'+n_m)!(j'-n_m)!}{(j'+k')!(j'-k')!(\iota+a')!(\iota-a')!(\iota+b)!!(\iota-b)!} \right)^{1/2} \\ \sum_z \frac{(-1)^z (j'+\iota+k-z)!(j-k+z)!}{z!(j'-j+\iota-z)!(j'+k+a'-z)!(j-\iota-k-a'+z)!} \sum_z \frac{(-1)^z (j+\iota+k'-z)!(j'-k'+z)!}{z!(j-j'+\iota-z)!(j+k-z)!(j'-\iota-k+z)!}. \end{aligned} \quad (\text{D21})$$

Eqs. (D20) and (D21) lead to

$$\begin{aligned} F_\iota(j'-d, j', k') = \delta\left(\sum_{i=1}^m \alpha_i - a + b, 0\right) \prod_{i=1}^m \frac{1}{(1+|\alpha_i|)^{1/2}} \left( \frac{1}{(\iota+a)! (\iota-a)! (\iota+b)! (\iota-b)!} \right)^{1/2} \\ \frac{(-1)^{d-2a+b} (2j'+1)(2j'-2d+1)}{(2j'-d+\iota+1)_{2\iota+1}} \prod_{n=1}^m (\alpha_i j' - k' + \sum_{i=1}^n \alpha_i) \\ \sum_z \frac{(-1)^z (d+\iota)_z (j'+k'+b-z+\iota)_{\iota+a} (j'-d-k'-b+z)_{\iota-a}}{z!} \\ \sum_z \frac{(-1)^z (\iota-d)_z (j'-d+k'-z+\iota)_{\iota-b} (j'-k'+z)_{\iota+b}}{z!} \end{aligned} \quad (\text{D22})$$

where  $(x)_n := x(x-1)\cdots(x-n+1)$  is the falling factorial.

There is subtle issue on the summation over  $k$  and  $k'$  in Eq. (D14). In order to obtain the factor  $\sinh((2j'+1)\eta)/\sinh(\eta)$  in (D15), one has to consider a summation of  $e^{2k'\eta}$  over all values of  $k'$  in  $[-j', j']$ . However, except the constraint of  $|k'| \leq j'$  for  $k'$ , there exists another condition of  $|k'+b| = |k| \leq j$  implied by  $k = k'+b$ , which seemingly narrows the range of  $k'$ . In order to preserve the range of  $k'$ , one can always “artificially engineer”  $F_\iota(j, j', k')$  such that it vanishes for  $|k'+b| > j$ , which has actually been done by the triangle inequality of  $3j$ -symbols in the RHS Eq. (D16). Thus the subtlety is now encoded in the condition that  $F_\iota(j, j', k')$  vanishes for  $|k'+b| > j$ . We need to verify this for the algebraic expression of  $F_\iota(j, j', k')$  in the RHS of Eq. (D22). To check it, assume  $k'+b = j+\delta = j'-d+\delta$  with  $0 < \delta \leq d+b$ . Then  $(j'-k'+z)_{\iota+b}$ , in the second summand over  $z$ , vanishes because

$$j'-k'+z \geq d+b-\delta \geq 0, \quad j'-k'+z-(\iota+b)+1 \leq \delta < 0 \quad (\text{D23})$$

where  $0 \leq z \leq \iota-d$  is applied. Therefore, we conclude that the RHS of Eq.(D22) vanishes for  $j-b < k' \leq j'$ . Similarly, it can be checked that  $(j'-d+k'-z+\iota)_{\iota-b}$  vanishes, indicating the vanishing of the RHS of Eq.(D22) for  $-j' \leq k'+b < -j-b$ . We thus claim that the RHS of Eq.(D22) vanishes for  $|k'+b| > j$ , which indicates that the expression of the RHS of Eq.(D22) can be analytically extended to give  $F_\iota(j, j', k')$  for the full range of  $k'$ , i.e.  $|k'| \leq j$ .

Replacing  $j'$  and  $k'$  in Eq. (D22) by  $\frac{n-1}{2}$  and  $\frac{\partial_\eta}{2}$  respectively, we have

$$\begin{aligned} F_\iota\left(\frac{n-1}{2} - d, \frac{n-1}{2}, \frac{\partial_\eta}{2}\right) = \delta\left(\sum_{i=1}^m \alpha_i - a + b, 0\right) \prod_{i=1}^m \frac{1}{(1+|\alpha_i|)^{1/2}} \left( \frac{1}{(\iota+a)! (\iota-a)! (\iota+b)! (\iota-b)!} \right)^{1/2} \\ \frac{(-1)^{d-2a+b} n(n-2d)}{(n-d+\iota)_{2\iota+1}} \prod_{k=1}^m \left( \alpha_i \frac{n-1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \\ \sum_{z=0}^{\iota+d} \frac{(-1)^z (\iota+d)_z \left(\frac{n-1}{2} + \frac{\partial_\eta}{2} + b - z + \iota\right)_{\iota+a} \left(\frac{n-1}{2} - \frac{\partial_\eta}{2} - d - b + z\right)_{\iota-a}}{z!} \\ \sum_{z=0}^{\iota-d} \frac{(-1)^z (\iota-d)_z \left(\frac{n-1}{2} + \frac{\partial_\eta}{2} - d - z + \iota\right)_{\iota-b} \left(\frac{n-1}{2} - \frac{\partial_\eta}{2} + z\right)_{\iota+b}}{z!}. \end{aligned} \quad (\text{D24})$$

In order to apply the Poisson summation formula to (D19) to calculate the summation over  $n$ , we need to extend the summation to entire  $\mathbb{Z}$ , while Eq. (D19) sums  $n$  over  $\mathbb{Z} \setminus [d-\iota, d+\iota]$ . However, because of the term  $(n-d+\iota)_{2\iota+1}$  in

the denominator of  $F_\iota$ , this extension is not trivial. We need to prove that  $F_\iota(\frac{n-1}{2} - d, \frac{n-1}{2}, \frac{\partial_\eta}{2}) \frac{\sinh(n\eta)}{\sinh(\eta)}$  is well-defined at the points where  $F_\iota$  itself does not. Once it is proven, we have

$$\begin{aligned} \langle \hat{F}_{\iota ab}^{\alpha_1 \dots \alpha_m} \rangle_{z_e} &= t^m e^{bz_e} \sum_{\substack{0 \leq d \leq \iota \\ d+\iota \in \mathbb{Z}}} \frac{2 - \delta(d, 0)}{2} e^{-\frac{t}{4}(2d^2-1)} \int dx e^{-\frac{1}{4}t(x^2-2dx)} F_\iota\left(\frac{x-1}{2} - d, \frac{x-1}{2}, \frac{\partial_\eta}{2}\right) \frac{\sinh(x\eta)}{\sinh(\eta)} \\ &- \sum_{n \in [d-\iota, d+\iota] \cap \mathbb{Z}} e^{-\frac{1}{4}t(n^2-2dn)} F_\iota\left(\frac{n-1}{2} - d, \frac{n-1}{2}, \frac{\partial_\eta}{2}\right) \frac{\sinh(n\eta)}{\sinh(\eta)} + O(t^\infty). \end{aligned} \quad (\text{D25})$$

Moreover, the integrals in the last equation usually produce a factor  $e^{\eta^2/t}$ . Thus the second term given by the summation over  $n \in [d-\iota, d+\iota] \cap \mathbb{Z}$  decays exponentially as  $t \rightarrow 0$  after normalization. Therefore, we finally obtain

$$\langle \hat{F}_{\iota ab}^{\alpha_1 \dots \alpha_m} \rangle_{z_e} = t^m e^{bz_e} \sum_{\substack{0 \leq d \leq \iota \\ d+\iota \in \mathbb{Z}}} \frac{2 - \delta(d, 0)}{2} e^{-\frac{t}{4}(2d^2-1)} \int dx e^{-\frac{1}{4}t(x^2-2dx)} F_\iota\left(\frac{x-1}{2} - d, \frac{x-1}{2}, \frac{\partial_\eta}{2}\right) \frac{\sinh(x\eta)}{\sinh(\eta)} + O(t^\infty). \quad (\text{D26})$$

In order to show that  $F_\iota(\frac{n-1}{2} - d, \frac{n-1}{2}, \frac{\partial_\eta}{2}) \frac{\sinh(n\eta)}{\sinh(\eta)}$  is well-defined at the poles of  $F_\iota$ , we have checked that  $F_\iota(\frac{n-1}{2} - d, \frac{n-1}{2}, \frac{\partial_\eta}{2})$  for at least  $\iota \leq 20$  can be simplified as summation of terms taking the form<sup>6</sup>

$$\frac{1}{x-n} f(x, \partial_\eta) \frac{\sinh(x\eta)}{\sinh(\eta)}$$

where  $f(x, \partial_\eta)$ , a polynomial of  $x$  and  $\partial_\eta$ , satisfies that  $f(n, \cdot)$  takes  $\partial_\eta = 2k$  for all  $k \in \mathbb{Z} \cap [-\frac{n-1}{2}, \frac{n-1}{2}]$  as its zeros. Then, the following theorem will be helpful.

**Theorem D.1.** *Let  $f(x, k)$  be a polynomials of  $x$  and  $k$  and  $n \in \mathbb{Z}$  be some integer. Let*

$$h(x) := \frac{1}{x-n} f(x, \partial_\eta) \frac{\sinh(x\eta)}{\sinh(\eta)}. \quad (\text{D27})$$

Then  $h(n) := \lim_{x \rightarrow n} h(x)$  is well-defined, i.e.,  $n$  is a removable singularity of  $h(x)$ , provided that for  $x = n$   $f(x, \partial_\eta)$  satisfies

$$f(n, \partial_\eta) = g(n, \partial_\eta) \left( \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} (\partial_\eta + 2k) \right), \quad (\text{D28})$$

with some polynomial  $g(n, \partial_\eta)$  of  $n$  and  $\partial_\eta$ . Moreover, if (D28) holds,  $h(x)$  will take the form

$$h(x) = g(x, \partial_\eta) \left( \tilde{g}_0(x, \eta) \frac{\sinh((x-n)\eta)}{x-n} + \sum_{l \geq 1} g_l(x, \eta) (x-n)^l \right). \quad (\text{D29})$$

*Proof.* We will prove that

$$\left( \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} (\partial_\eta + 2k) \right) \frac{\sinh(x\eta)}{\sinh(\eta)} = \sum_l g_l(x, \eta) (x-n)^l \quad (\text{D30})$$

with  $g_l$  some function of  $x$  and  $\eta$ , and  $g_0$  taking the form

$$g_0(x, \eta) = \tilde{g}_0(x, \eta) \sinh((x-n)\eta). \quad (\text{D31})$$

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<sup>6</sup> Here, we conjecture that this statement is true for all  $\iota$ .

Let  $y = x - n$ . Then we have

$$\frac{\sinh(x\eta)}{\sinh(\eta)} = \frac{1}{\sinh(\eta)} (\cosh(y\eta) \sinh(n\eta) + \sinh(y\eta) \cosh(n\eta)). \quad (\text{D32})$$

Substitute the above expression into the LHS of Eq. (D30) and expand the result. One then expresses the LHS of Eq. (D30) as a linear combination of terms taking the forms of  $\partial_\eta^l (h_1(\eta) \sinh(y\eta))$  and  $\partial_\eta^l (h_2(\eta) \cosh(y\eta))$ , where  $h_1$  and  $h_2$  are some arbitrary functions. Expanding the action of the differential operator by Leibniz's rule, we obtain linear combinations of  $q_1(\eta) \cosh(y\eta)y^m$  and  $q_2(\eta) \sinh(y\eta)y^m$ , where  $q_1$  and  $q_2$  are some arbitrary functions. Thus, (D30) is obtained.

To get  $g_0$ , one just act  $\partial_\eta$  on neither  $\cosh(y\eta)$  nor  $\sinh(y\eta)$ . Thus

$$g_0 = \left( \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} (\partial_\eta + 2k) \frac{\sinh(n\eta)}{\sinh(\eta)} \right) \cosh(y\eta) + \left( \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} (\partial_\eta + 2k) \frac{\cosh(n\eta)}{\sinh(\eta)} \right) \sinh(y\eta) \quad (\text{D33})$$

Since  $\frac{\sinh(n\eta)}{\sinh(\eta)} = \sum_{k=-j}^j e^{-2k\eta}$  with  $2j+1 = n$  and  $(\partial_\eta + k)e^{-k\eta} = 0$ , we have

$$\prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} (\partial_\eta + 2k) \frac{\sinh(n\eta)}{\sinh(\eta)} = 0. \quad (\text{D34})$$

Therefore

$$g_0 = \left( \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} (\partial_\eta + 2k) \frac{\cosh(n\eta)}{\sinh(\eta)} \right) \sinh(y\eta) \quad (\text{D35})$$

□

### E. $\langle \hat{F}_{lab}^{\alpha_1 \dots \alpha_m} \rangle_{z_e}$ for the case of $\iota = 1$

By setting  $\iota = 1$  in (D19), we can obtain

$$\begin{aligned} & \langle \hat{F}_{lab}^{\alpha_1 \dots \alpha_m} \rangle_{z_e} \\ &= -\delta \left( \sum_{i=1}^m \alpha_i - a + b \right) \frac{(-1)^a}{2} t^m e^{bz_e} \left( \prod_{i=1}^m \frac{1}{\sqrt{1+|\alpha_i|}} \right) \frac{1}{\sqrt{(1+|a|)(1+|b|)}} \sum_{\substack{n \in \mathbb{Z} \\ n \notin [-1,1]}} e^{-\frac{t}{4}(n^2-1)} \\ & \quad \prod_{k=1}^m \left( \alpha_k \frac{n-1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \frac{n \left( a \frac{n-1}{2} + b + \frac{\partial_\eta}{2} \right) \left( b \frac{n-1}{2} - \frac{\partial_\eta}{2} \right) \sinh(n\eta)}{\frac{n^2-1}{4} \sinh(\eta)} \\ & + \delta \left( \sum_{i=1}^m \alpha_i - a + b \right) (-1)^{b-a} t^m e^{bz_e} \left( \prod_{i=1}^m \frac{1}{\sqrt{1+|\alpha_i|}} \right) \frac{1}{\sqrt{(1+|a|)(1+|b|)}} \sum_{\substack{n \in \mathbb{Z} \\ n \notin [-2,0]}} e^{-\frac{t}{4}(n+1)^2} \\ & \quad \prod_{k=1}^m \left( -\alpha_k \frac{n+1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \frac{\left( \frac{n+1}{2} - (1-|b|-b) \frac{\partial_\eta}{2} \right) \left( \frac{n+1}{2} - (|b|-1-b) \frac{\partial_\eta}{2} + |b| \right) \sinh(n\eta)}{\frac{n+1}{2} \sinh(\eta)}. \end{aligned} \quad (\text{E1})$$

where in the second term we replaced  $n$  by  $-n$  for the further convenience. It is easy to check that Theorem D.1 can be applied to extend the summation in the last equation to all  $n \in \mathbb{Z}$ . Therefore, we obtain that

$$\begin{aligned}
& \langle \hat{F}_{1ab}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} \\
&= -\frac{(-1)^a}{2} \delta \left( \sum_{i=1}^m \alpha_i - a + b \right) t^m e^{bz_e} \left( \prod_{i=1}^m \frac{1}{\sqrt{1+|\alpha_i|}} \frac{1}{\sqrt{1+|a|\sqrt{1+|b|}}} \right) \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x^2-1)} \\
& \quad \prod_{k=1}^m \left( \alpha_k \frac{x-1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( a \frac{x-1}{2} + \frac{\partial_\eta}{2} + b \right) \frac{2x(b(x-1) - \partial_\eta) \sinh(x\eta)}{x^2-1} \frac{\sinh(x\eta)}{\sinh(\eta)} \\
& + (-1)^{b-a} \delta \left( \sum_{i=1}^m \alpha_i - a + b \right) t^m e^{bz_e} \left( \prod_{i=1}^m \frac{1}{\sqrt{1+|\alpha_i|}} \frac{1}{\sqrt{1+|a|\sqrt{1+|b|}}} \right) \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x+1)^2} \\
& \quad \prod_{k=1}^m \left( -\alpha_k \frac{n+1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{n+1}{2} - (|b|-1-b) \frac{\partial_\eta}{2} + |b| \right) \frac{(x+1 - (1-|b|-b)\partial_\eta) \sinh(n\eta)}{x+1} \frac{\sinh(n\eta)}{\sinh(\eta)} \\
& + O(t^\infty).
\end{aligned} \tag{E2}$$

To show the algorithms to compute Eq. (E2),  $m \neq 0$  will be assumed without the loss of generality. For the case of  $m = 0$ , the algorithm can be applied very similarly.

To begin with, we will rewrite Eq. (E2) as

$$\begin{aligned}
& \langle \hat{F}_{1ab}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} \\
&= -\frac{(-1)^a}{2} \delta \left( \sum_{i=1}^m \alpha_i - a + b \right) t^m e^{bz_e} \left( \prod_{i=1}^m \frac{1}{\sqrt{1+|\alpha_i|}} \frac{1}{\sqrt{1+|a|\sqrt{1+|b|}}} \right) \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x^2-1)} \\
& \quad \prod_{k=2}^m \left( \alpha_k \frac{x-1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( a \frac{x-1}{2} + \frac{\partial_\eta}{2} + b \right) x \left( b - \frac{\partial_\eta}{x-1} \right) \left( \alpha_1 - \frac{\partial_\eta}{x+1} \right) \frac{\sinh(n\eta)}{\sinh(\eta)} \\
& + (-1)^{b-a} \delta \left( \sum_{i=1}^m \alpha_i - a + b \right) t^m e^{bz_e} \left( \prod_{i=1}^m \frac{1}{\sqrt{1+|\alpha_i|}} \frac{1}{\sqrt{1+|a|\sqrt{1+|b|}}} \right) \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x+1)^2} \\
& \quad \prod_{k=1}^m \left( -\alpha_k \frac{x+1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{x+1}{2} - (|b|-1-b) \frac{\partial_\eta}{2} + |b| \right) \left( 1 - (1-|b|-b) \frac{\partial_\eta}{x+1} \right) \frac{\sinh(x\eta)}{\sinh(\eta)}.
\end{aligned} \tag{E3}$$

Then for the first integral, it is

$$I_1 = \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x^2-1)} \prod_{k=2}^m \left( \alpha_k \frac{x-1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( a \frac{x-1}{2} + \frac{\partial_\eta}{2} + b \right) x \left( b - \frac{\partial_\eta}{x-1} \right) \left( \alpha_1 - \frac{\partial_\eta}{x+1} \right) \frac{\sinh(x\eta)}{\sinh(\eta)}. \tag{E4}$$

Because

$$\begin{aligned}
& \left( b - \frac{\partial_\eta}{x-1} \right) \left( \alpha_1 - \frac{\partial_\eta}{x+1} \right) \frac{\sinh(x\eta)}{\sinh(\eta)} \\
&= \left( b \alpha_1 \frac{\sinh(x\eta)}{\sinh(\eta)} - \alpha_1 \left( \frac{\cosh(\eta x)}{\sinh(\eta)} - \frac{\sinh(\eta(x-1))}{\sinh^2(\eta)(x-1)} \right) - b \left( \frac{\cosh(\eta x)}{\sinh(\eta)} - \frac{\sinh(\eta(x+1))}{\sinh^2(\eta)(x+1)} \right) + \right. \\
& \quad \left. \left( \frac{\sinh(\eta x)}{\sinh(\eta)} + \frac{\cosh(\eta)}{\sinh(\eta)^3} \frac{\sinh(\eta(x-1))}{x-1} - \frac{\cosh(\eta)}{\sinh(\eta)^3} \frac{\sinh((x+1)\eta)}{x+1} \right) \right),
\end{aligned} \tag{E5}$$

$I_1$  can be deduced further as the following

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x^2-1)} \prod_{k=2}^m \left( \alpha_k \frac{x-1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( a \frac{x-1}{2} + \frac{\partial_\eta}{2} + b \right) x \frac{(b\alpha_1 + 1) \sinh(x\eta) - (\alpha_1 + b) \cosh(\eta x)}{\sinh(\eta)} \\
&+ \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}(y^2+2y)} \prod_{k=2}^m \left( \alpha_k \frac{y}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( a \frac{y}{2} + \frac{\partial_\eta}{2} + b \right) (y+1) \frac{\alpha_1 \sinh(\eta) + \cosh(\eta)}{\sinh(\eta)^3} \frac{\sinh(\eta y)}{y} \\
&- \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}(y^2-2y)} \prod_{k=2}^m \left( \alpha_k \frac{y}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^{k-1} \alpha_i \right) \left( a \frac{y}{2} + \frac{\partial_\eta}{2} + b - a \right) (y-1) \frac{\cosh(\eta) - b \sinh(\eta)}{\sinh(\eta)^3} \frac{\sinh(\eta y)}{y}.
\end{aligned} \tag{E6}$$

Then by using the trick

$$\partial_\eta^n e^{\pm x\eta} f(\eta) = (\pm x + \partial_z)^n e^{\pm x\eta} f(z) \Big|_{z \rightarrow \eta}, \tag{E7}$$

we have

$$\begin{aligned}
I_1 &= \sum_{s=\pm 1} \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x^2-1)} \prod_{k=2}^m \left( \frac{\alpha_k - s}{2} x - \frac{\alpha_k + \partial_z}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{a+s}{2} x + \frac{-a + \partial_z}{2} + b \right) \\
&\quad x \frac{(-(\alpha_1 + b) + s(b\alpha_1 + 1))e^{sx}}{2 \sinh(z)} \Big|_{z \rightarrow \eta} \\
&+ \sum_{s=\pm 1} \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}(y^2+2y)} \prod_{k=2}^m \left( \frac{\alpha_k - s}{2} y - \frac{\partial_z}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{a+s}{2} y + \frac{\partial_z}{2} + b \right) \\
&\quad (y+1) \frac{se^{s\eta y}}{2y} \frac{\alpha_1 \sinh(z) + \cosh(z)}{\sinh(z)^3} \Big|_{z \rightarrow \eta} \\
&- \sum_{s=\pm 1} \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}(y^2-2y)} \prod_{k=2}^m \left( \frac{\alpha_k - s}{2} y - \frac{\partial_z}{2} + \sum_{i=1}^{k-1} \alpha_i \right) \left( \frac{a+s}{2} y + \frac{\partial_z}{2} + b - a \right) \\
&\quad (y-1) \frac{se^{s\eta y}}{y} \frac{\cosh(z) - b \sinh(z)}{\sinh(z)^3} \Big|_{z \rightarrow \eta}
\end{aligned} \tag{E8}$$

By defining

$$\begin{aligned}
C_1 &= \prod_{k=2}^m \left( -\frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{\partial_z}{2} + b \right) \frac{\alpha_1 \sinh(\eta) + \cosh(\eta)}{\sinh(\eta)^3} \\
C_2 &= - \prod_{k=2}^m \left( -\frac{\partial_\eta}{2} + \sum_{i=1}^{k-1} \alpha_i \right) \left( \frac{\partial_\eta}{2} + b - a \right) \frac{\cosh(\eta) - b \sinh(\eta)}{\sinh(\eta)^3}
\end{aligned} \tag{E9}$$



$I_1$  can be finally simplified as

$$\begin{aligned}
I_1 = & \sum_{s=\pm 1} \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x^2-1)} \prod_{k=2}^m \left( \frac{\alpha_k - s}{2} x - \frac{\alpha_k + \partial_z}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{a+s}{2} x + \frac{-a + \partial_z}{2} + b \right) \\
& x \frac{(-(\alpha_1 + b) + s(b\alpha_1 + 1))e^{sx}}{2 \sinh(z)} \Big|_{z \rightarrow \eta} \\
& + \sum_{s=\pm 1} \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}(y^2+2y)} \left( \prod_{k=2}^m \left( \frac{\alpha_k - s}{2} y - \frac{\partial_z}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{a+s}{2} y + \frac{\partial_z}{2} + b \right) \right. \\
& \left. (y+1) \frac{\alpha_1 \sinh(z) + \cosh(z)}{\sinh(z)^3} \Big|_{z \rightarrow \eta} - C_1 \right) \frac{se^{s\eta y}}{2y} \\
& - \sum_{s=\pm 1} \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}(y^2-2y)} \left( \prod_{k=2}^m \left( \frac{\alpha_k - s}{2} y - \frac{\partial_z}{2} + \sum_{i=1}^{k-1} \alpha_i \right) \left( \frac{a+s}{2} y + \frac{\partial_z}{2} + b - a \right) \right. \\
& \left. (y-1) \frac{\cosh(z) - b \sinh(z)}{\sinh(z)^3} \Big|_{z \rightarrow \eta} - C_2 \right) \frac{se^{s\eta y}}{y} \\
& + C_1 \int_{-\infty}^{\infty} e^{-\frac{t}{4}(y^2+2y)} \frac{\sinh(\eta y)}{y} dy - C_2 \int_{-\infty}^{\infty} e^{-\frac{t}{4}(y^2-2y)} \frac{\sinh(\eta y)}{y} dy.
\end{aligned} \tag{E10}$$

In this expression, the integrands of first three terms involve only polynomials of the integral variables, which can be computed by using the same procedure as that to compute  $\langle \hat{F}_{\frac{1}{2}ab}^{\alpha_1 \dots \alpha_m} \rangle$ . For the last two terms, by using the formula

$$\int_{-\infty}^{\infty} dy e^{-\frac{t}{4}(y^2+by)} \frac{\sinh(y\eta)}{y} = \frac{1}{2} \pi \left( \operatorname{erfi} \left( \frac{1 + \frac{tb}{4\eta}}{2\sqrt{\frac{t}{4\eta^2}}} \right) + \operatorname{erfi} \left( \frac{1 - \frac{tb}{4\eta}}{2\sqrt{\frac{t}{4\eta^2}}} \right) \right), \tag{E11}$$

and

$$\operatorname{erfi} \left( \frac{1}{x} \right) = e^{\frac{1}{x^2}} \frac{x}{\pi} \sum_{n=0}^{\infty} \Gamma\left(\frac{1}{2} + n\right) x^{2n}, \tag{E12}$$

a straightforward calculation can be performed as the following

$$\int_{-\infty}^{\infty} dy e^{-\frac{t}{4}(y^2+2y)} \frac{\sinh(y\eta)}{y} = \langle 1 \rangle \frac{1}{\sqrt{\pi}} \frac{\sinh(\eta)}{\eta^2} \sum_{m=0}^{\infty} \sum_{n=0}^m \Gamma\left(\frac{1}{2} + n\right) \frac{2^n}{\eta^{n+m}} \binom{n+m}{m-n} \left\{ e^{-\eta} + e^{\eta} (-1)^{m-n} \right\} \left( \frac{t}{2} \right)^{m+2}, \tag{E13}$$

where Eq. (2.26) is substitute because we are concerned about the expectation value with respect to the normalized coherent state, i.e.  $\langle \hat{F}_{\nu ab}^{\alpha_1 \dots \alpha_m} \rangle / \langle 1 \rangle$ . This complete our computation for  $I_1$ .

Secondly,  $I_2$  can be simplified as the following

$$\begin{aligned}
I_2 = & \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x+1)^2} \prod_{k=1}^m \left( -\alpha_k \frac{x+1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{x+1}{2} - (|b| - 1 - b) \frac{\partial_\eta}{2} + |b| \right) \frac{\sinh(x\eta)}{\sinh(\eta)} \\
& - \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x+1)^2} \prod_{k=1}^m \left( -\alpha_k \frac{x+1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right) \left( \frac{x+1}{2} + (|b| - 1 - b) \frac{\partial_\eta}{2} + |b| \right) \frac{(1 - |b| - b) \partial_\eta}{x+1} \frac{\sinh(x\eta)}{\sinh(\eta)}.
\end{aligned} \tag{E14}$$

Because

$$\frac{\partial_\eta}{x+1} \frac{\sinh(x\eta)}{\sinh(\eta)} = \frac{1}{\sinh(\eta)^2} \left( \sinh(\eta) \cosh(x\eta) - \frac{\sinh((x+1)\eta)}{x+1} \right) \tag{E15}$$

we have

$$\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x+1)^2} \prod_{k=1}^m \left[ -\alpha_k \frac{x+1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right] \left[ \frac{x+1}{2} - (|b|-1-b) \frac{\partial_\eta}{2} + |b| \right] \times \\
&\quad \frac{\sinh(x\eta) - (1-|b|-b) \cosh(x\eta)}{\sinh(\eta)} \\
&\quad + \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}y^2} \prod_{k=1}^m \left[ -\alpha_k \frac{y}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right] \left[ \frac{y}{2} - (|b|-1-b) \frac{\partial_\eta}{2} + |b| \right] \frac{(1-|b|-b) \sinh(y\eta)}{\sinh(\eta)^2 y}.
\end{aligned} \tag{E16}$$

By defining

$$C = \left( \alpha_1 - \frac{\partial_\eta}{2} \right) \left( \alpha_1 + \alpha_2 - \frac{\partial_\eta}{2} \right) \cdots \left( \alpha_1 + \cdots + \alpha_m - \frac{\partial_\eta}{2} \right) \left( |b| - (|b|-1-b) \frac{\partial_\eta}{2} \right) \frac{1-|b|-b}{\sinh(\eta)^2}, \tag{E17}$$

we obtain

$$\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} dx e^{-\frac{t}{4}(x+1)^2} \prod_{k=1}^m \left[ -\alpha_k \frac{x+1}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right] \times \\
&\quad \left[ \frac{x+1}{2} - (|b|-1-b) \frac{\partial_\eta}{2} + |b| \right] \frac{\sinh(x\eta) - (1-|b|-b) \cosh(x\eta)}{\sinh(\eta)} \\
&\quad + \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}y^2} \left\{ \prod_{k=1}^m \left[ -\alpha_k \frac{y}{2} - \frac{\partial_\eta}{2} + \sum_{i=1}^k \alpha_i \right] \left[ \frac{y}{2} - (|b|-1-b) \frac{\partial_\eta}{2} + |b| \right] - C \right\} \frac{(1-|b|-b) \sinh(y\eta)}{\sinh(\eta)^2 y} \\
&\quad + C \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}y^2} \frac{\sinh(y\eta)}{y}.
\end{aligned} \tag{E18}$$

Similar as  $I_1$ , the first two integrals are computable because the integrands therein are just polynomials multiplied by the Gaussian functions. For the last term, we have

$$\begin{aligned}
C \int_{-\infty}^{\infty} dy e^{-\frac{t}{4}y^2} \frac{\sinh(y\eta)}{y} &= C \operatorname{sgn}(\eta) \pi \operatorname{erfi} \left( \frac{1}{2\sqrt{\frac{t}{4\eta^2}}} \right) \\
&= \langle 1 \rangle C \frac{t^2}{2\sqrt{\pi} e^{t/4}} \frac{\sinh(\eta)}{\eta^2} \sum_{n=0}^{\infty} \Gamma\left(\frac{1}{2} + n\right) \left(\frac{t}{\eta^2}\right)^n.
\end{aligned} \tag{E19}$$

where Eq. (2.26) is substitute because we are concerned about the expectation value with respect to the normalized coherent state, i.e.  $\langle \hat{F}_{lab}^{\alpha_1 \cdots \alpha_m} \rangle / \langle 1 \rangle$ . Therefore,  $I_2$  is computable.

In summary, we obtain

$$\begin{aligned}
\langle \hat{F}_{lab}^{\alpha_1 \cdots \alpha_m} \rangle_{z_e} &= -\frac{(-1)^a}{2} \delta \left( \sum_{i=1}^m \alpha_i - a + b \right) t^m e^{bz_e} \left( \prod_{i=1}^m \frac{1}{\sqrt{1+|\alpha_i|}} \frac{1}{\sqrt{1+|a|}\sqrt{1+|b|}} \right) I_1 \\
&\quad + (-1)^{b-a} \delta \left( \sum_{i=1}^m \alpha_i - a + b \right) t^m e^{bz_e} \left( \prod_{i=1}^m \frac{1}{\sqrt{1+|\alpha_i|}} \frac{1}{\sqrt{1+|a|}\sqrt{1+|b|}} \right) I_2 + O(t^\infty)
\end{aligned} \tag{E20}$$

where  $I_1$  and  $I_2$  can be computed by using the algorithm introduced above. Taking the operator  $\hat{F}_{lab} = D_{ab}^1(h_e)$  as an example, we can have

$$\langle \hat{F}_{lab} \rangle_{z_e} / \langle 1 \rangle = \begin{cases} e^{i\xi} - \left( \frac{e^{i\xi} \tanh(\frac{\eta}{2})}{2\eta} + \frac{1}{4} e^{i\xi} \right) t + O(t^2), & a = b = 1, \\ 1 - \frac{\tanh(\frac{\eta}{2})}{\eta} t + O(t^2), & a = b = 0, \\ e^{-i\xi} - \left( \frac{e^{-i\xi} \tanh(\frac{\eta}{2})}{2\eta} + \frac{1}{4} e^{-i\xi} \right) t + O(t^2), & a = b = -1, \end{cases} \tag{E21}$$

which is compatible with the corresponding result given in [20].

### F. Mathematical supports for Sec. IV

To begin with, we need to study the matrix element of the holonomy and flux. According to the results in [19], we have

$$\langle \psi_{g_e} | \hat{p}_s^\alpha | \psi_{g'_e} \rangle = i \langle \psi_{g_e} | \psi_{g'_e} \rangle \frac{\text{tr}(\tau^\alpha g'_e g_e^\dagger)}{\sinh(\zeta_e^{(0)})} \left( -\zeta_e^{(0)} + (\coth(\zeta_e^{(0)}) - \frac{1}{\zeta_e^{(0)}}) \frac{t}{2} \right) + O(t^\infty). \quad (\text{F1})$$

where  $\zeta_e^{(0)}$  is given by  $2 \cosh(\zeta_e^{(0)}) = \text{tr}(g'_e g_e^\dagger)$ . For  $\hat{p}_t^\alpha(e)$ , one has

$$\langle \psi_{g_e} | \hat{p}_t^\alpha | \psi_{g'_e} \rangle = i \langle \psi_{g_e} | \psi_{g'_e} \rangle \frac{\text{tr}(g_e^\dagger g'_e \tau^\alpha)}{\sinh(\zeta_e^{(0)})} \left( \zeta_e^{(0)} - (\coth(\zeta_e^{(0)}) - \frac{1}{\zeta_e^{(0)}}) \frac{t}{2} \right) + O(t^\infty). \quad (\text{F2})$$

For the holonomy operator  $D_{ab}^l(h_e)$ , we have [19]

$$\begin{aligned} & \langle \psi_{g_e} | D_{ab}^{\frac{1}{2}}(h_e) | \psi_{g'_e} \rangle \\ &= \langle \psi_{g_e} | \psi_{g'_e} \rangle \left( \frac{2 \text{tr}(\tau^k g'_e g_e^\dagger) D_{ab}^{\frac{1}{2}}(\tau^k g'_e)}{(e^{\zeta_e^{(0)}} + 1)^2 \zeta_e^{(0)}} e^{\frac{\zeta_e^{(0)}}{2}} + \frac{D_{ab}^{\frac{1}{2}}(g'_e)}{2 \zeta_e^{(0)}} e^{-\frac{\zeta_e^{(0)}}{2}} \right) e^{-\frac{t}{16}} \left( (e^{\zeta_e^{(0)}} + 1) \zeta_e^{(0)} + (-e^{\zeta_e^{(0)}} + 1) \frac{t}{4} \right) + O(t^\infty). \end{aligned} \quad (\text{F3})$$

According to Eqs. (F1), (F2) and (F3), the matrix elements of the fluxes and holonomies are of a form described below

$$\langle \psi_{g_e} | \hat{O}_i | \psi_{g'_e} \rangle = \langle \psi_{g_e} | \psi_{g'_e} \rangle (E_0(g_e, g'_e) + t E_1(g_e, g'_e) + O(t^\infty)). \quad (\text{F4})$$

Recalling Eq. (4.3), we are going to consider integrals containing  $\frac{\langle \psi_{g_e} | \psi_{g'_e} \rangle}{\|\psi_{g_e}\| \|\psi_{g'_e}\|}$ . These integrals can be analyzed with the generalized stationary phase approximation guided by Hörmanders theorem 7.7.5 in [67].

**Theorem F.1.** *Let  $K$  be a compact subset in  $\mathbb{R}^n$ ,  $X$  be an open neighborhood of  $K$ , and  $k$  be a positive integer. If (1) the complex functions  $u \in C_0^{2k}(K)$ ,  $f \in C^{3k+1}(X)$  and  $\Im(f) \geq 0$  in  $X$ , with  $\Im(f)$  being the imaginary part of  $f$ ; (2) there is a unique point  $x_0 \in K$  satisfying  $\Im(f(x_0)) = 0$ ,  $f'(x_0) = 0$ , and  $\det(f''(x_0)) \neq 0$  ( $f''$  denotes the Hessian matrix),  $f' \neq 0$  in  $K \setminus \{x_0\}$  then we have the following estimation:*

$$\left| \int_K u(x) e^{i\lambda f(x)} dx - e^{i\lambda f(x_0)} \left[ \det \left( \frac{\lambda f''(x_0)}{2\pi i} \right) \right]^{-\frac{1}{2}} \sum_{s=0}^{k-1} \left( \frac{1}{\lambda} \right)^s L_s u(x_0) \right| \leq C \left( \frac{1}{\lambda} \right)^k \sum_{|\alpha| \leq 2k} \sup |D^\alpha u|. \quad (\text{F5})$$

Here the constant  $C$  is bounded when  $f$  stays in a bounded set in  $C^{3k+1}(X)$ . We have used the standard multi-index notation  $\alpha = \langle \alpha_1, \dots, \alpha_n \rangle$  and

$$D^\alpha = (-i)^\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \text{where } |\alpha| = \sum_{i=1}^n \alpha_i \quad (\text{F6})$$

$L_s u(x_0)$  denotes the following operation on  $u$ :

$$L_s u(x_0) = i^{-s} \sum_{l-m=s} \sum_{2l \geq 3m} \frac{(-1)^l 2^{-l}}{l! m!} \left[ \sum_{a,b=1}^n H_{ab}^{-1}(x_0) \frac{\partial^2}{\partial x_a \partial x_b} \right]^l (g_{x_0}^m u)(x_0), \quad (\text{F7})$$

where  $H(x) = f''(x)$  denotes the Hessian matrix and the function  $g_{x_0}(x)$  is given by

$$g_{x_0}(x) = f(x) - f(x_0) - \frac{1}{2} H^{ab}(x_0) (x - x_0)_a (x - x_0)_b \quad (\text{F8})$$

such that  $g_{x_0}(x_0) = g'_{x_0}(x_0) = g''_{x_0}(x_0) = 0$ . For each  $s$ ,  $L_s$  is a differential operator of order  $2s$  acting on  $u(x)$ .

1. Analysis of integrals containing  $\frac{\langle \psi_{g_e} | \psi_{g'_e} \rangle}{\|\psi_{g_e}\| \|\psi_{g'_e}\|}$

Define

$$G(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}) := \frac{\langle \psi_{g^{(1)}} | \psi_{g^{(2)}} \rangle}{\|\psi_{g^{(1)}}\| \|\psi_{g^{(2)}}\|} \quad (\text{F9})$$

with  $g^{(k)}$  parameterized by

$$g^{(k)} = e^{i\vec{p}^{(k)} \cdot \vec{\tau}} e^{\vec{\theta}^{(k)} \cdot \vec{\tau}}, \quad k = 1, 2. \quad (\text{F10})$$

According to Eq. (2.25) and Eq. (2.26),  $G(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)})$  reads

$$G(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}) = \frac{\zeta \sqrt{\sinh(p^{(1)}) \sinh(p^{(2)})}}{\sqrt{p^{(1)} p^{(2)}} \sinh(\zeta)} e^{-\frac{-2\zeta^2 + (p^{(1)})^2 + (p^{(2)})^2}{2\zeta}} \quad (\text{F11})$$

where  $p^{(i)} = \sqrt{\vec{p}^{(i)} \cdot \vec{p}^{(i)}}$  and  $\zeta$  is given by

$$2 \cosh(\zeta) = \text{tr}(g_1^\dagger g_2). \quad (\text{F12})$$

Denote

$$S(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}) := -2\zeta^2 + (p^{(1)})^2 + (p^{(2)})^2.$$

We first claim that

**Lemma F.1.**  $\Re(S(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}))$ , the real part of  $S(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)})$ , is non-negative and vanishes iff  $\vec{p}^{(2)} = \vec{p}^{(1)}$  and  $\vec{\theta}^{(2)} = \vec{\theta}^{(1)}$ .

To prove this lemma, let us introduce the following proposition which is given in [18].

**Proposition 3.** Let  $g = e^{i\vec{p} \cdot \vec{\tau}} e^{\vec{\theta} \cdot \vec{\tau}}$ . Considering  $\zeta = s + i\phi \in \mathbb{C}$  with  $s \in \mathbb{R}$  and  $\phi \in [0, \pi]$  determined by  $\cosh(\zeta) = \text{tr}(g)$ , we have, with denoting  $p := \sqrt{\vec{p} \cdot \vec{p}}$ ,

$$\delta = \frac{p^2}{4} - s^2 + \phi^2 \geq 0 \quad (\text{F13})$$

where the equality occurs iff  $\theta := \sqrt{\vec{\theta} \cdot \vec{\theta}} = 0$ .

Thanks to this proposition, we prove Lemma F.1 as follows.

*Proof of Lemma F.1.*  $g_1^\dagger g_2$  can be decomposed as

$$g_1^\dagger g_2 = e^{i\vec{x} \cdot \vec{\tau}} e^{\vec{y} \cdot \vec{\tau}}.$$

Denote  $x = \sqrt{\vec{x} \cdot \vec{x}}$  and  $y = \sqrt{\vec{y} \cdot \vec{y}}$ . Then

$$\Re(S(\vec{p}, \vec{\theta}, \vec{p}, \vec{\theta})) = 2\delta - \frac{x^2}{2} + (p^{(1)})^2 + (p^{(2)})^2. \quad (\text{F14})$$

where  $\delta := x^2/4 - \Re(\zeta)^2$  is non-negative according to the proposition 3. Thus we only need to prove that  $-x^2/2 + (p^{(1)})^2 + (p^{(2)})^2 \geq 0$ .

By definition, we have  $2 \cosh(x) = \text{tr}(g_1^\dagger g_2 g_2^\dagger g_1)$ , which leads to

$$2 \cosh(x) = \text{tr}(e^{2i\vec{p}^{(1)} \cdot \vec{\tau}} e^{2i\vec{p}^{(2)} \cdot \vec{\tau}}). \quad (\text{F15})$$

Since  $e^{i\vec{\mu} \cdot \vec{\tau}} = \cosh(\frac{\mu}{2})\mathbb{I} + 2i\frac{\vec{\mu} \cdot \vec{\tau}}{\mu} \sinh(\frac{\mu}{2})$  with  $\mu = \sqrt{\vec{\mu} \cdot \vec{\mu}}$ , we have

$$\cosh(x) = \frac{1-\beta}{2} \cosh(p^{(1)} - p^{(2)}) + \frac{1+\beta}{2} \cosh(p^{(1)} + p^{(2)}) \leq \cosh(p^{(1)} + p^{(2)}) \quad (\text{F16})$$

where  $\beta = \frac{\vec{p}^{(1)} \cdot \vec{p}^{(2)}}{p^{(1)} p^{(2)}} \in [-1, 1]$  and  $\cosh(p^{(1)} + p^{(2)}) \geq \cosh(p^{(1)} - p^{(2)})$  is used. Moreover, because of  $\sqrt{2(p^{(1)})^2 + 2(p^{(2)})^2} \geq p^{(1)} + p^{(2)} \geq 0$ , it has

$$\cosh(\sqrt{2(p^{(1)})^2 + 2(p^{(2)})^2}) \geq \cosh(p^{(1)} + p^{(2)}). \quad (\text{F17})$$

Combining the results of (F16) and (F17), one finally have

$$-\frac{x^2}{2} + (p^{(1)})^2 + (p^{(2)})^2 \geq 0 \quad (\text{F18})$$

where the equality occurs only if  $\vec{p}^{(1)} = \vec{p}^{(2)}$ .

In summary we have

$$\Re(S(\vec{p}, \vec{\theta}, \vec{p}, \vec{\theta})) \geq 0 \quad (\text{F19})$$

and  $\Re(S(\vec{p}, \vec{\theta}, \vec{p}, \vec{\theta})) = 0$  only if  $\vec{p}^{(1)} = \vec{p}^{(2)}$  and  $\delta = 0$  which means  $\vec{\theta}^{(1)} = \vec{\theta}^{(2)}$ .  $\square$

It turns out below that the integrand of Eq. (4.3) consists of a Gaussian-like function

$$e^{-\frac{1}{2i}(S(\vec{p}, \vec{\theta}, \vec{p}^{(1)}, \vec{\theta}^{(1)}) + S(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}) + \dots + S(\vec{p}^{(k)}, \vec{\theta}^{(k)}, \vec{p}, \vec{\theta}))}. \quad (\text{F20})$$

Lemma F.1 suggests us to do the stationary phase approximation analysis at  $\vec{p}^{(i)} = \vec{p}$  and  $\vec{\theta}^{(i)} = \vec{\theta}$ . Notice that  $\vec{p}$  and  $\vec{\theta}$  are given to parameterize  $g$  as  $g := e^{i\vec{p} \cdot \vec{\tau}} e^{\vec{\theta} \cdot \vec{\tau}}$ , and  $g$  labels the coherent state  $|\psi_g\rangle$  with respect to which the expectation value of  $\hat{O}$  is computed. Thus, rather than considering all values of  $\vec{p}$  and  $\vec{\theta}$ , it is sufficient to set

$$\vec{p}_o = (0, 0, p), \quad \vec{\theta}_o = (0, 0, \theta) \quad (\text{F21})$$

according to (3.1). Denote

$$f_{k; \vec{p}, \vec{\theta}}(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \dots, \vec{p}^{(k)}, \vec{\theta}^{(k)}) \equiv S(\vec{p}, \vec{\theta}, \vec{p}^{(1)}, \vec{\theta}^{(1)}) + S(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}) + \dots + S(\vec{p}^{(k)}, \vec{\theta}^{(k)}, \vec{p}, \vec{\theta}), \quad (\text{F22})$$

we have the following result:

**Theorem F.2.** (i)  $\Re(f_{k; \vec{p}, \vec{\theta}}) \geq 0$  and the equality occurs only when all  $g^{(i)}$  coincide, namely  $\vec{p}^{(i)} = \vec{p}$  and  $\vec{\theta}^{(i)} = \vec{\theta}$ .

(ii) At  $\vec{p}^{(i)} = \vec{p}_o = (0, 0, p)$  and  $\vec{\theta}^{(i)} = \vec{\theta}_o = (0, 0, \theta)$ , it has

$$\nabla_{\vec{p}^{(i)}} f_{k; \vec{p}_o, \vec{\theta}_o} = 0 = \nabla_{\vec{\theta}^{(i)}} f_{k; \vec{p}_o, \vec{\theta}_o}, \quad \forall i = 1, \dots, k. \quad (\text{F23})$$

(iii) The Hessian matrix  $f''_{k; \vec{p}_o, \vec{\theta}_o}$  of  $f_{k; \vec{p}_o, \vec{\theta}_o}$  at  $\vec{p}^{(i)} = \vec{p}_o = (0, 0, p)$  and  $\vec{\theta}^{(i)} = \vec{\theta}_o = (0, 0, \theta)$  is non-degenerate with the determinant

$$\det \left( f''_{k; \vec{p}_o, \vec{\theta}_o} \right) \Big|_{\vec{p}^{(i)} = \vec{p}_o, \vec{\theta}^{(i)} = \vec{\theta}_o} = \left( \frac{1024 \sin^4 \left( \frac{\theta}{2} \right)}{\theta^4} \right)^k \quad (\text{F24})$$

*Proof.* The first statement is true by using Lemma F.1. For the second statement, let us consider

$$S(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}) = -2\zeta^2 + (\vec{p}^{(1)})^2 + (\vec{p}^{(2)})^2 \quad (\text{F25})$$

where  $\zeta$  is given by

$$\cosh(\zeta) = \frac{1}{2} \text{tr}(e^{\vec{\theta}^{(2)} \cdot \vec{\tau}} e^{-\vec{\theta}^{(1)} \cdot \vec{\tau}} e^{i\vec{p}^{(1)} \cdot \vec{\tau}} e^{i\vec{p}^{(2)} \cdot \vec{\tau}}). \quad (\text{F26})$$

Then it has that

$$\begin{aligned} \frac{\partial \cosh(\zeta)}{\partial p_j^{(1)}} \Big|_{\vec{p}_o, \vec{\theta}_o} &= \frac{\partial \cosh(\zeta)}{\partial p_j^{(2)}} \Big|_{\vec{p}_o, \vec{\theta}_o} = \frac{\delta_{j,3}}{2} \sinh(p) \\ \frac{\partial \cosh(\zeta)}{\partial \theta_j^{(1)}} \Big|_{\vec{p}_o, \vec{\theta}_o} &= -\frac{\partial \cosh(\zeta)}{\partial \theta_j^{(2)}} \Big|_{\vec{p}_o, \vec{\theta}_o} = \frac{i\delta_{j,3}}{2} \sinh(p) \end{aligned} \quad (\text{F27})$$

where the subscript  $\vec{p}_o, \vec{\theta}_o$  indicates to take values at  $\vec{p}^{(1)} = \vec{p}_o = \vec{p}^{(2)}$  and  $\vec{\theta}^{(1)} = \vec{\theta}_o = \vec{\theta}^{(2)}$ . According to Eq. (F27), we have

$$\nabla_{\vec{p}^{(1)}} S(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}) \Big|_{\vec{p}_o, \vec{\theta}_o} = 0 = \nabla_{\vec{p}^{(2)}} S(\vec{p}^{(1)}, \vec{\theta}^{(1)}, \vec{p}^{(2)}, \vec{\theta}^{(2)}) \Big|_{\vec{p}_o, \vec{\theta}_o} \quad (\text{F28})$$

and, thus,  $\nabla_{\vec{p}^{(i)}} f_{k; \vec{p}_o, \vec{\theta}_o} \Big|_{\vec{p}_o, \vec{\theta}_o} = 0$  for all  $i = 1, 2, \dots, k$ . For  $\nabla_{\vec{\theta}^{(i)}} f_{k; \vec{p}_o, \vec{\theta}_o} \Big|_{\vec{p}_o, \vec{\theta}_o}$ , the second equation in Eq. (F27) gives us

$$\nabla_{\vec{\theta}^{(i)}} f_{k; \vec{p}_o, \vec{\theta}_o} \Big|_{\vec{p}_o, \vec{\theta}_o} = \nabla_{\vec{\theta}^{(i)}} S(\vec{p}^{(i-1)}, \vec{\theta}^{(i-1)}, \vec{p}^{(i)}, \vec{\theta}^{(i)}) \Big|_{\vec{p}_o, \vec{\theta}_o} + \nabla_{\vec{\theta}^{(i)}} S(\vec{p}^{(i)}, \vec{\theta}^{(i)}, \vec{p}^{(i+1)}, \vec{\theta}^{(i+1)}) \Big|_{\vec{p}_o, \vec{\theta}_o} = 0, \quad (\text{F29})$$

which completes the proof of the second statement.

For the last statement, using the conclusion of the second statement, we can immediately get that

$$\frac{\partial f_{k; \vec{p}_o, \vec{\theta}_o}}{\partial x_m^{(i)} \partial y_n^{(j)}} \Big|_{\vec{p}_o, \vec{\theta}_o} = 0, \quad |i - j| > 1 \quad (\text{F30})$$

where  $x_m^{(i)}$  and  $y_n^{(j)}$  represent  $\vec{\theta}_m^{(i)}$  or  $\vec{p}_m^{(i)}$ . Therefore, if we order the arguments  $\vec{p}^{(i)}, \vec{\theta}^{(i)}$  as

$$p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, \theta_1^{(1)}, \theta_2^{(1)}, \theta_3^{(1)}, p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, \theta_1^{(2)}, \theta_2^{(2)}, \theta_3^{(2)}, \dots \quad (\text{F31})$$

to arrange the matrix elements of  $f''_{k; \vec{p}_o, \vec{\theta}_o}(\vec{p}_o, \vec{\theta}_o) \equiv f''_{k; \vec{p}_o, \vec{\theta}_o} \Big|_{\vec{p}_o, \vec{\theta}_o}$ , the resulting matrix is block-tridiagonal matrix. Moreover, since all of the  $p$ -arguments, as well as the  $\theta$ -arguments, in  $f_{k; \vec{p}_o, \vec{\theta}_o}$  are symmetric, we conclude that

$$\frac{\partial f_{k; \vec{p}_o, \vec{\theta}_o}}{\partial x_m^{(i)} \partial y_l^{(j)}} \Big|_{\vec{p}_o, \vec{\theta}_o} = \frac{\partial f_{k; \vec{p}_o, \vec{\theta}_o}}{\partial x_m^{(i')} \partial y_l^{(j')}} \Big|_{\vec{p}_o, \vec{\theta}_o}. \quad (\text{F32})$$

Consequently  $f''_{k; \vec{p}_o, \vec{\theta}_o}(\vec{p}_o, \vec{\theta}_o)$  takes the form

$$f''_{k; \vec{p}_o, \vec{\theta}_o}(\vec{p}_o, \vec{\theta}_o) = \begin{pmatrix} A & B & 0 & 0 & \cdots & 0 \\ B^T & A & B & 0 & \cdots & 0 \\ 0 & B^T & A & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B^T & A & B \\ 0 & 0 & \cdots & 0 & B^T & A \end{pmatrix} \quad (\text{F33})$$

where  $A$  and  $B$  are  $6 \times 6$  matrix. The matrix  $A$  and  $B$  can be calculated by considering the case with  $k = 2$ , which gives us

$$A = \begin{pmatrix} \frac{4 \tanh(\frac{p}{2})}{p} & 0 & 0 & -\frac{4 \sin^2(\frac{\theta}{2}) \tanh(\frac{p}{2})}{\theta} & -\frac{2 \sin(\theta) \tanh(\frac{p}{2})}{\theta} & 0 \\ 0 & \frac{4 \tanh(\frac{p}{2})}{p} & 0 & \frac{2 \sin(\theta) \tanh(\frac{p}{2})}{\theta} & -\frac{4 \sin^2(\frac{\theta}{2}) \tanh(\frac{p}{2})}{\theta} & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -\frac{4 \sin^2(\frac{\theta}{2}) \tanh(\frac{p}{2})}{\theta} & \frac{2 \sin(\theta) \tanh(\frac{p}{2})}{\theta} & 0 & -\frac{4p(\cos(\theta)-1) \coth(p)}{\theta^2} & 0 & 0 \\ -\frac{2 \sin(\theta) \tanh(\frac{p}{2})}{\theta} & -\frac{4 \sin^2(\frac{\theta}{2}) \tanh(\frac{p}{2})}{\theta} & 0 & 0 & -\frac{4p(\cos(\theta)-1) \coth(p)}{\theta^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -\frac{2 \tanh(\frac{p}{2})}{p} & 0 & 0 & \frac{2 \sin^2(\frac{\theta}{2}) \tanh(\frac{p}{2}) + i \sin(\theta)}{\theta} & \frac{\sin(\theta) \tanh(\frac{p}{2}) + i(\cos(\theta)-1)}{\theta} & 0 \\ 0 & -\frac{2 \tanh(\frac{p}{2})}{p} & 0 & \frac{-i \cos(\theta) - \sin(\theta) \tanh(\frac{p}{2}) + i}{\theta} & \frac{2 \sin^2(\frac{\theta}{2}) \tanh(\frac{p}{2}) + i \sin(\theta)}{\theta} & 0 \\ 0 & 0 & -1 & 0 & 0 & i \\ \frac{2 \sin^2(\frac{\theta}{2}) \tanh(\frac{p}{2}) - i \sin(\theta)}{\theta} & \frac{i(\cos(\theta) + i \sin(\theta) \tanh(\frac{p}{2}) - 1)}{\theta} & 0 & \frac{2p(\cos(\theta)-1) \coth(p)}{\theta^2} & -\frac{2ip(\cos(\theta)-1)}{\theta^2} & 0 \\ \frac{-i \cos(\theta) + \sin(\theta) \tanh(\frac{p}{2}) + i}{\theta} & \frac{2 \sin^2(\frac{\theta}{2}) \tanh(\frac{p}{2}) - i \sin(\theta)}{\theta} & 0 & \frac{2ip(\cos(\theta)-1)}{\theta^2} & \frac{2p(\cos(\theta)-1) \coth(p)}{\theta^2} & 0 \\ 0 & 0 & -i & 0 & 0 & -1 \end{pmatrix}.$$

With the expression of  $A$  and  $B$ , it can be verified that

$$BA^{-1}B^T = B^T A^{-1}B = 0. \quad (\text{F34})$$

To calculate  $\det(f''_{k;\vec{p}_o,\vec{\theta}_o}(\vec{p}_o,\vec{\theta}_o))$ , we define matrices of  $\tilde{B}$ ,  $\tilde{C}$  and  $D$  of dimensions  $6 \times 6(k-1)$ ,  $6(k-1) \times 6$  and  $6(k-1) \times 6(k-1)$  respectively such that

$$f''_{k;\vec{p}_o,\vec{\theta}_o}(\vec{p}_o,\vec{\theta}_o) = \begin{pmatrix} A & \tilde{B} \\ \tilde{C} & D \end{pmatrix}. \quad (\text{F35})$$

Then by using the property of the Schur complement, it has

$$\det(f''_{k;\vec{p}_o,\vec{\theta}_o}(\vec{p}_o,\vec{\theta}_o)) = \det(A) \det(D - \tilde{C}A^{-1}\tilde{B}) = \det(A) \det(D) \quad (\text{F36})$$

where we used  $\tilde{C}A^{-1}\tilde{B} = 0$  because of Eq. (F34) and that  $f''_{k;\vec{p}_o,\vec{\theta}_o}$  is block-tridiagonal matrix. Because  $D$  is the Hessian matrix  $f''_{k-1;\vec{p}_o,\vec{\theta}_o}(\vec{p}_o,\vec{\theta}_o)$ , we finally have

$$\det(f''_{k;\vec{p}_o,\vec{\theta}_o}(\vec{p}_o,\vec{\theta}_o)) = \det(A)^k = \left( \frac{1024 \sin^4(\frac{\theta}{2})}{\theta^4} \right)^k. \quad (\text{F37})$$

□

By these results, the stationary phase approximation method introduced in Theorem F.1 can be applied to calculate the integral (4.3).

Taking advantage of the above results, we now can come to the proof of Theorem IV.1.

## 2. proof of Theorem IV.1

As in Eq. (4.3), it has

$$\frac{\langle \Psi_{\vec{w}} | \hat{O} | \Psi_{\vec{w}} \rangle}{\langle \Psi_{\vec{w}} | \Psi_{\vec{w}} \rangle} = \int \prod_{j=1}^{k-1} \prod_{e \in E(\gamma)} \frac{2}{\pi t^3} d\mu_H(u_e^{(j)}) d^3 \vec{p}_e^{(j)} \prod_{i=1}^k \frac{\langle \Psi_{\vec{g}^{(i-1)}} | \hat{O}_i | \Psi_{\vec{g}^{(i)}} \rangle}{\| \Psi_{\vec{g}^{(i-1)}} \| \| \Psi_{\vec{g}^{(i)}} \|} \quad (\text{F38})$$

where we denoted  $|\Psi_{\vec{g}^{(0)}}\rangle = |\Psi_{\vec{g}^{(k)}}\rangle := |\Psi_{\vec{w}}\rangle$ , applied Eq. (2.28) and used the decomposition

$$g_e^{(i)} = e^{i\vec{p}_e^{(i)} \cdot \vec{\tau}} e^{\vec{\theta}_e^{(i)} \cdot \vec{\tau}} = e^{i\vec{p}_e^{(i)} \cdot \vec{\tau}} u_e^{(i)}. \quad (\text{F39})$$

By the assumption, we have

$$\frac{\langle \Psi_{\vec{g}^{(i-1)}} | \hat{O}_i | \Psi_{\vec{g}^{(i)}} \rangle}{\| \Psi_{\vec{g}^{(i-1)}} \| \| \Psi_{\vec{g}^{(i)}} \|} = \frac{\langle \Psi_{\vec{g}^{(i-1)}} | \Psi_{\vec{g}^{(i)}} \rangle}{\| \Psi_{\vec{g}^{(i-1)}} \| \| \Psi_{\vec{g}^{(i)}} \|} \left( E_0^{(i)}(\vec{g}^{(i-1)}, \vec{g}^{(i)}) + t E_1^{(i)}(\vec{g}^{(i-1)}, \vec{g}^{(i)}) + O(t^\infty) \right). \quad (\text{F40})$$

Thus Eq. (F38) takes the form

$$\frac{\langle \Psi_{\vec{w}} | \hat{O} | \Psi_{\vec{w}} \rangle}{\langle \Psi_{\vec{w}} | \Psi_{\vec{w}} \rangle} = \int \prod_{j=1}^{k-1} \prod_{e \in E(\gamma)} \frac{2}{\pi t^3} d\mu_H(u_e^{(j)}) d^3 \vec{p}_e^{(j)} \prod_{i=1}^k \frac{\langle \Psi_{\vec{g}^{(i-1)}} | \Psi_{\vec{g}^{(i)}} \rangle}{\| \Psi_{\vec{g}^{(i-1)}} \| \| \Psi_{\vec{g}^{(i)}} \|} \prod_{l=1}^k E^{(l)}(\vec{g}^{(l-1)}, \vec{g}^{(l)}) \quad (\text{F41})$$

with  $P(\{\vec{g}^{(i)}\}_{i=1}^k)$  denoting the function

$$E^{(l)}(\vec{g}^{(l-1)}, \vec{g}^{(l)}) := \left( E_0^{(l)}(\vec{g}^{(l-1)}, \vec{g}^{(l)}) + t E_1^{(l)}(\vec{g}^{(l-1)}, \vec{g}^{(l)}) + O(t^\infty) \right). \quad (\text{F42})$$

Eq. (F41) can be analyzed with the stationary phase approximation according to Theorem F.2. It should be noticed that the Haar measure  $d\mu_H$  can be expressed as

$$d\mu_H(u) = \frac{\sin^2\left(\frac{1}{2}\sqrt{\vec{\theta} \cdot \vec{\theta}}\right)}{4\pi^2(\vec{\theta} \cdot \vec{\theta})} d^3 \vec{\theta} \quad (\text{F43})$$

where  $u \in \text{SU}(2)$  is coordinatized as  $u = e^{\vec{\theta} \cdot \vec{\tau}}$  and  $d^3\vec{\theta}$  is the Lebesgue measure on  $\mathbb{R}^3$ . Substituting the last equation into Eq. (F41) and applying Eq. (F5) as well as Eq. (F24), we finally obtain

$$\frac{\langle \Psi_{\vec{w}} | \hat{O} | \Psi_{\vec{w}} \rangle}{\langle \Psi_{\vec{w}} | \Psi_{\vec{w}} \rangle} = \prod_{e \in E(\gamma)} \left( \frac{\theta_e^2}{\sin^2(\theta_e/2)} \right)^{k-1} \sum_{s=0}^{l-1} (2t)^s \mathfrak{D}^{(s)} \Big|_{g_e^{(k)} = e^{i\omega_e \tau_3}, \forall k, e} + O(t^l) \quad (\text{F44})$$

where  $\mathfrak{D}^{(s)}$  takes the form

$$\mathfrak{D}^{(s)} = (-1)^s \sum_{l-j=s} \sum_{2l \geq 3j} \frac{(-1)^{l-2l}}{l!j!} \left[ \sum_{a,b=1}^n H_{ab}^{-1}(x_0) \frac{\partial^2}{\partial x_a \partial x_b} \right]^l \left( G^{(j)} \prod_{i=1}^k E^{(i)} \right) \Big|_{g_e^{(l)} = e^{i\omega_e \tau_3}, \forall l, e} \quad (\text{F45})$$

with  $G^{(j)}$  being defined as the following,

$$G^{(j)} \left( \{\vec{g}^{(i)}\}_{i=k}^{k-1} \right) = \left( \prod_{e \in E(\gamma)} \prod_{n=1}^{k-1} \frac{\sin^2(|\vec{\theta}_e^{(n)}|/2)}{|\vec{\theta}_e^{(n)}|^2} \right) \left( \prod_{e \in E(\gamma)} \prod_{n=0}^{k-1} \frac{\zeta_e^{(n,n+1)} \sqrt{\sinh(p_e^{(n)}) \sinh(p_e^{(n+1)})}}{\sqrt{p_e^{(n)} p_e^{(n+1)} \sinh(\zeta_e^{(n,n+1)})}} \right) g_{x_0}(\{\vec{g}^{(i)}\}_{i=1}^{k-1})^j. \quad (\text{F46})$$

Here  $g_{x_0}$  is some function defined by applying (F8) to the current case.

If the leading order term of  $\mathfrak{D}^{(s)}$  is claimed to be  $O(t^{d_s})$ , then each term in the summation over  $s$  of Eq. (F44) is  $O(t^{s+d_s})$ . Moreover, for each  $l$  in Eq. (F45), the derivative acting on  $G^{(j)} \prod E^{(i)}$  is of order  $2l$  in total. Because of the properties given by Eq.(F8), the non-vanishing result appears when there are at least  $3j$  derivatives acting on  $G^{(j)}$ , which indicates that the order of derivative that acts on the term  $\prod E^{(i)}$  in Eq. (F45) is  $2l - 3j$ . Because  $2l - 3j = s$  and  $l = s + j$ , there are at most  $2s - j$  derivatives acting on  $\prod E^{(i)}$ . Further, since  $j \geq 0$ , the maximum order of derivative acting on the term  $\prod E^{(i)}$  is  $2s$ , which only occurs for  $l = s$ . Then, let us count the leading order of  $\mathfrak{D}^{(s)}$  for a given  $s$ . According to the expression of  $\mathfrak{D}^{(s)}$ , once it is evaluated at the critical point given by  $g_e^{(k)} = e^{i\omega_e \tau_3}$ , those  $E^{(m)}$  contributed by operators  $\hat{O}_m$  satisfying (4.8) will inevitably increase the power of its leading order term if they are not acted by any derivative operators. For a fixed  $s$  there are at least  $n_s$  of these ‘‘non-acted’’ terms with

$$n_s = \frac{(N_0 - 2s) + |N_0 - 2s|}{2}. \quad (\text{F47})$$

Therefore,  $\frac{\langle \Psi_{\vec{w}} | \hat{O} | \Psi_{\vec{w}} \rangle}{\langle \Psi_{\vec{w}} | \Psi_{\vec{w}} \rangle}$  is of order of  $t^n$  with

$$n \geq \min_{s \in \mathbb{Z}^+} (s + n_s) = \lfloor \frac{N_0 + 1}{2} \rfloor. \quad (\text{F48})$$

### 3. proof of Theorem IV.2

At first, denote  $\hat{p}^0(e)$ ,  $D_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}(h_e)$  and  $D_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}}(h_e)$  by  $\hat{O}_a^{\text{d}}$  with  $a = 1, 2, 3$  respectively, and  $p^{\pm 1}(e)$ ,  $D_{\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}}(h_e)$  or  $D_{-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}(h_e)$  by  $\hat{O}_a^{\text{nd}}$  with  $a = 1, 2, 3, 4$  respectively. According to (F1), (F2) and (F3), the matrix elements of  $\hat{O}_a^i$  take the form

$$\langle \psi_{g^{(1)}} | \hat{O}_a^i | \psi_{g^{(2)}} \rangle = \langle \psi_{g^{(1)}} | \psi_{g^{(2)}} \rangle E_{a;0}^i(g^{(1)}, g^{(2)}) + O(t), \quad \forall i = \text{d, nd}. \quad (\text{F49})$$

Then by formulae listed in Sec. F4, we obtain:

**Lemma F.2.** *Given  $g^{(i)} = e^{i\vec{p}^{(i)} \cdot \vec{\tau}} e^{i\vec{\theta}^{(i)} \cdot \vec{\tau}}$ , we have*

$$\begin{aligned} (\partial_{x_k^{(j)}} E_{a;0}^{\text{d}})(e^{i\omega \tau_3}, e^{i\omega \tau_3}) &= 0, \quad j = 1, 2, \quad \text{and } k = 1, 2 \\ (\partial_{x_3^{(j)}} E_{a;0}^{\text{nd}})(e^{i\omega \tau_3}, e^{i\omega \tau_3}) &= 0, \quad j = 1, 2 \end{aligned} \quad (\text{F50})$$



for all  $w = p - i\theta \in \mathbb{C}$ , where  $x_j^{(i)}$  denotes  $p_j^{(i)}$  or  $\theta_j^{(i)}$ . Moreover, consider the matrix  $A$  and  $B$  in Eq. (F33).  $E_{a;0}^{\text{nd}}$  satisfies that

$$\begin{aligned}
& \nabla_{\bar{x}(1)}^T E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3})A^{-1}B = 0 = \nabla_{\bar{x}(2)}^T E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3})A^{-1}B^T, \\
& \nabla_{\bar{x}(1)}^T E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3})A^{-1}B^T = -\nabla_{\bar{x}(1)}^T E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3}), \\
& \nabla_{\bar{x}(2)}^T E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3})A^{-1}B = -\nabla_{\bar{x}(2)}^T E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3}), \\
& B^T A^{-1} \nabla_{\bar{x}(1)} E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3}) = 0 = BA^{-1} \nabla_{\bar{x}(2)} E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3}), \\
& BA^{-1} \nabla_{\bar{x}(1)} E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3}) = -\nabla_{\bar{x}(1)} E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3}), \\
& B^T A^{-1} \nabla_{\bar{x}(2)} E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3}) = -\nabla_{\bar{x}(2)} E_{a;0}^{\text{nd}}(e^{iw\tau_k}, e^{iw\tau_3}),
\end{aligned} \tag{F51}$$

where  $\nabla_{\bar{x}^{(i)}} = (\partial_{\bar{p}_1^{(i)}}, \partial_{\bar{p}_2^{(i)}}, \partial_{\bar{p}_3^{(i)}}, \partial_{\bar{\theta}_1^{(i)}}, \partial_{\bar{\theta}_2^{(i)}}, \partial_{\bar{\theta}_3^{(i)}})^T$  and  $\nabla_{\bar{x}^{(i)}}^T$  is its transpose.

The second lemma is on the inverse of the Hessian matrix (F33). Recalling Eq. (F35), one has that the inverse of  $f''_{k;\bar{p}_o, \bar{\theta}_o}$ , denoted as  $H_{k;\bar{p}_o, \bar{\theta}_o}^{-1}$ , is

$$H_{k;\bar{p}_o, \bar{\theta}_o}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}\tilde{B}(D - \tilde{C}A^{-1}\tilde{B})^{-1}\tilde{C}A^{-1} & -A^{-1}\tilde{B}(D - \tilde{C}A^{-1}\tilde{B})^{-1} \\ -(D - \tilde{C}A^{-1}\tilde{B})^{-1}\tilde{C}A^{-1} & (D - \tilde{C}A^{-1}\tilde{B})^{-1} \end{pmatrix}. \tag{F52}$$

Since  $\tilde{C}A^{-1}\tilde{B} = 0$  and  $D = f''_{k-1;\bar{p}_o, \bar{\theta}_o}$ , we have

$$H_{k;\bar{p}_o, \bar{\theta}_o}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}\tilde{B}H_{k-1;\bar{p}_o, \bar{\theta}_o}^{-1}\tilde{C}A^{-1} & -A^{-1}\tilde{B}H_{k-1;\bar{p}_o, \bar{\theta}_o}^{-1} \\ -H_{k-1;\bar{p}_o, \bar{\theta}_o}^{-1}\tilde{C}A^{-1} & H_{k-1;\bar{p}_o, \bar{\theta}_o}^{-1} \end{pmatrix}. \tag{F53}$$

For  $k = 1$ ,  $H_{1;\bar{p}_o, \bar{\theta}_o}^{-1} = A^{-1}$ . Thus one has that  $\tilde{B}H_{1;\bar{p}_o, \bar{\theta}_o}^{-1}\tilde{C} = 0$  and

$$H_{2;\bar{p}_o, \bar{\theta}_o}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\tilde{B}H_{1;\bar{p}_o, \bar{\theta}_o}^{-1} \\ -H_{1;\bar{p}_o, \bar{\theta}_o}^{-1}\tilde{C}A^{-1} & H_{1;\bar{p}_o, \bar{\theta}_o}^{-1} \end{pmatrix}.$$

Doing this successively, one has  $\tilde{B}H_{k-1;\bar{p}_o, \bar{\theta}_o}^{-1}\tilde{C} = 0$ , and

$$H_{k;\bar{p}_o, \bar{\theta}_o}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\tilde{B}H_{k-1;\bar{p}_o, \bar{\theta}_o}^{-1} \\ -H_{k-1;\bar{p}_o, \bar{\theta}_o}^{-1}\tilde{C}A^{-1} & H_{k-1;\bar{p}_o, \bar{\theta}_o}^{-1} \end{pmatrix}. \tag{F54}$$

Finally,  $H_{k;\bar{p}_o, \bar{\theta}_o}^{-1}$  can be obtained with this recurrence relation and the initial data  $H_{1;\bar{p}_o, \bar{\theta}_o}^{-1} = A^{-1}$ . The result is as follows:

**Lemma F.3.**  $H_{k;\bar{p}_o, \bar{\theta}_o}^{-1}$  satisfies that

$$(H_{k;\bar{p}_o, \bar{\theta}_o}^{-1})_{mn} = \begin{cases} (-1)^{|m-n|}A^{-1}(BA^{-1})^{|m-n|}, & m < n \\ A^{-1}, & m = n \\ (-1)^{|m-n|}A^{-1}(B^T A^{-1})^{|m-n|}, & m > n \end{cases} \tag{F55}$$

where  $H_{k;\bar{p}_o, \bar{\theta}_o}^{-1}$  is arranged as a block matrix as  $f''_{k;\bar{p}_o, \bar{\theta}_o}$  in (F33), with  $(H_{k;\bar{p}_o, \bar{\theta}_o}^{-1})_{mn}$  as a block.

Now the theorem (IV.2) can be proven.

*Proof of Theorem IV.2.* For convenience, we define  $s_o = M_+ + N_+$ . By Theorem IV.1,  $\langle \mathcal{M} \rangle_{z_c}$  is of order  $t^{s_o}$  or higher. Adopting the result from equation (F44),  $O(t^{s_o})$  only occurs when  $s = s_o$  and  $\mathfrak{D}^{(s_o)}$  is of  $O(t^0)$ .

Set  $s = s_o$  in the definition (F45) of  $\mathfrak{D}^{(s)}$ . For given  $l$  and  $j$ , if there is one  $E^{(m)}$  taking the form of  $E_{a;0}^{\text{nd}} + O(t)$  not being acted by derivatives, then the eventual evaluation at the critical point will vanish. Moreover, in  $\mathfrak{D}^{(s_o)}$ , on one hand it contains  $2s_o$  of  $E_{a;0}^{\text{nd}} + O(t)$ , and on the other hand the maximum order of derivative that can act on  $\prod E^{(i)}$  is

$2s_o$ , which only occurs when  $l = s_o$ . Therefore, only when  $l = s_o$  and  $m = 0$ , all  $E_{a;0}^{\text{nd}} + O(t)$  are acted by derivatives. Finally,

$$\mathfrak{D}^{(s_o)} = \left( \frac{\sin^2(\frac{\theta}{2})}{\theta^2} \right)^{|\mathcal{M}|-1} \frac{2^{-s_o}}{s_o!} \left[ \sum_{a,b=1}^{6(|\mathcal{M}|-1)} H_{ab}^{-1} \frac{\partial^2}{\partial x_a \partial x_b} \right]^{s_o} \prod_{i=1}^{|\mathcal{M}|-1} E^{(i)} \Big|_{g^{(l)} = e^{i\omega\tau_3}, \forall l}, \quad (\text{F56})$$

where  $|\mathcal{M}|$  denotes the number of factors in the monomial  $\mathcal{M}$ .

To calculate (F56), we employ the notion introduced in Lemma F.2 and F.3 to treat  $H^{-1}$  as a block matrix. Then  $\sum_{a,b=1}^{6(|\mathcal{M}|-1)} H_{ab}^{-1} \frac{\partial^2}{\partial x_a \partial x_b}$  is rewritten as

$$\sum_{a,b=1}^{6(|\mathcal{M}|-1)} H_{ab}^{-1} \frac{\partial^2}{\partial x_a \partial x_b} = \sum_{m,n=1}^{|\mathcal{M}|-1} \nabla_{\vec{x}^{(m)}}^T \left( \hat{H}_{|\mathcal{M}|-1; \vec{p}_o, \vec{\theta}_o}^{-1} \right)_{mn} \nabla_{\vec{x}^{(n)}}. \quad (\text{F57})$$

We then expand  $\left[ \sum_{m,n=1}^{|\mathcal{M}|-1} \nabla_{\vec{x}^{(m)}}^T \left( \hat{H}_{|\mathcal{M}|-1; \vec{p}_o, \vec{\theta}_o}^{-1} \right)_{mn} \nabla_{\vec{x}^{(n)}} \right]^{s_o}$  and let each individual term of the expansion act on  $\prod_{i=1}^{|\mathcal{M}|-1} E^{(i)}$ . In each individual term of the expansion, it contains derivative with respect to certain  $\vec{x}^{(q)}$ . Because each  $E_{a;0}^{\text{nd}} + O(t)$  only depends on certain  $\vec{x}^{(q)}$ , we only consider the case when all derivatives are paired with all  $E_{a;0}^{\text{nd}} + O(t)$  with the same argument. For the other terms of the expansion, they give vanishing results because of the evaluation at the critical point.

The procedure mentioned above is equivalent to the follows. We first partition these  $E_{a;0}^{\text{nd}} + O(t)$  into ordering pairs. Denote all possibilities of the partition as  $\mathcal{P}$ . Given a pair  $(E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) + O(t), E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) + O(t))$  in a partition  $p \in \mathcal{P}$ . It can be acted by

$$\nabla_{\vec{x}^{(m)}}^T (H_{\mathcal{M}-1; \vec{p}_o, \vec{\theta}_o}^{-1})_{mn} \nabla_{\vec{x}^{(n)}}, \quad (\text{F58})$$

$$\nabla_{\vec{x}^{(m-1)}}^T (H_{\mathcal{M}-1; \vec{p}_o, \vec{\theta}_o}^{-1})_{m-1,n} \nabla_{\vec{x}^{(n)}}, \quad (\text{F59})$$

$$\nabla_{\vec{x}^{(m)}}^T (H_{\mathcal{M}-1; \vec{p}_o, \vec{\theta}_o}^{-1})_{m,n-1} \nabla_{\vec{x}^{(n-1)}}, \quad (\text{F60})$$

$$\nabla_{\vec{x}^{(m-1)}}^T (H_{\mathcal{M}-1; \vec{p}_o, \vec{\theta}_o}^{-1})_{m-1,n-1} \nabla_{\vec{x}^{(n-1)}} \quad (\text{F61})$$

According to Lemmas F.3 and F.2,

(1) If  $m < n$ , only the operator (F60) gives non-vanishing results which reads

$$\begin{aligned} & \nabla_{\vec{x}^{(m)}}^T E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) (H_{\mathcal{M}-1; \vec{p}_o, \vec{\theta}_o}^{-1})_{m,n-1} \nabla_{\vec{x}^{(n-1)}} E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) \\ &= \nabla_{\vec{x}^{(m)}}^T E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) A^{-1} \nabla_{\vec{x}^{(n-1)}} E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) \end{aligned} \quad (\text{F62})$$

(2) If  $m > n$ , only the operator (F59) gives non-vanishing results which reads

$$\begin{aligned} & \nabla_{\vec{x}^{(m-1)}}^T E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) (H_{\mathcal{M}-1; \vec{p}_o, \vec{\theta}_o}^{-1})_{m-1,n} \nabla_{\vec{x}^{(n)}} E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) \\ &= \nabla_{\vec{x}^{(m-1)}}^T E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) A^{-1} \nabla_{\vec{x}^{(n)}} E_{a_n}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) \\ &= \nabla_{\vec{x}^{(n)}}^T E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) A^{-1} \nabla_{\vec{x}^{(m-1)}} E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}), \end{aligned} \quad (\text{F63})$$

where in the last step we used  $A = A^T$ .

It should be reminded that an evaluation at the critical point has been done in Eqs.(F62) and (F63).

According to the results in Eqs. (F62) and (F63), rather than partitioning the  $E_{a;0}^{\text{nd}} + O(t)$  into *ordering* pairs, we can identify the partitions  $p_1$  and  $p_2$  if  $p_1$  can be the same as  $p_2$  by reordering each of its pairs. The set with this identification will be denoted by  $\tilde{\mathcal{P}}$ . Then we finally have

$$\begin{aligned} & \left[ \sum_{m,n=1}^{|\mathcal{M}|-1} \nabla_{\vec{x}^{(m)}}^T \left( \hat{H}_{|\mathcal{M}|-1; \vec{p}_o, \vec{\theta}_o}^{-1} \right)_{mn} \nabla_{\vec{x}^{(n)}} \right]^{s_o} \prod_{i=1}^{|\mathcal{M}|-1} E^{(i)} \Big|_{g^{(l)} = e^{i\omega\tau_3}, \forall l} \\ &= 2^{s_o} \sum_{p \in \tilde{\mathcal{P}}} \prod_{\substack{(m,n) \in p \\ \text{with } m < n}} \nabla_{\vec{x}^{(m)}}^T E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) A^{-1} \nabla_{\vec{x}^{(n-1)}} E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) \prod_{l \in \mathcal{P}^c} E_{a_l;0}^{\text{nd}}(e^{i\omega\tau_3}, e^{i\omega\tau_3}) + O(t) \end{aligned} \quad (\text{F64})$$

where  $\mathcal{P}^{\mathbb{C}}$  denotes the set of those  $E^{(i)}$  of the form  $E_{a;0}^{\text{d}}$ . Substituting the results, we finally have

$$\frac{\langle \psi_{z_e} | \mathcal{M} | \psi_{z_e} \rangle}{\langle \psi_{z_e} | \psi_{z_e} \rangle} = \left( (2t)^{s_o} \frac{1}{s_o!} \sum_{p \in \bar{\mathcal{P}}} \prod_{\substack{(m,n) \in p \\ \text{with } m < n}} \nabla_{\vec{x}^{(m)}}^T E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) A^{-1} \nabla_{\vec{x}^{(n-1)}} E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) \right) \times \prod_{l \in \mathcal{P}^{\mathbb{C}}} E_{a_l;0}^{\text{d}}(e^{i\omega\tau_3}, e^{i\omega\tau_3}) + O(t^{s_o+1}) \quad (\text{F65})$$

Note that in

$$\prod_{\substack{(m,n) \in p \\ \text{with } m < n}} \nabla_{\vec{x}^{(m)}}^T E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) A^{-1} \nabla_{\vec{x}^{(n-1)}} E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}),$$

the first argument in the most left factor and the second argument in the most right factor cannot be acted by the derivative operator. Therefore, by the similar discussion as above, one verifies that

$$\frac{\langle \psi_{z_e} | \mathcal{M}' | \psi_{z_e} \rangle}{\langle \psi_{z_e} | \psi_{z_e} \rangle} = \left( (2t)^{s_o} \frac{1}{s_o!} \sum_{p \in \bar{\mathcal{P}}} \prod_{\substack{(m,n) \in p \\ \text{with } m < n}} \nabla_{\vec{x}^{(m)}}^T E_{a_m;0}^{\text{nd}}(\vec{x}^{(m-1)}, \vec{x}^{(m)}) A^{-1} \nabla_{\vec{x}^{(n-1)}} E_{a_n;0}^{\text{nd}}(\vec{x}^{(n-1)}, \vec{x}^{(n)}) \right) + O(t^{s_o+1}). \quad (\text{F66})$$

Moreover,  $E_{a_l;0}^{\text{d}}(e^{i\omega\tau_3}, e^{i\omega\tau_3})$  represents the leading order of the expectation value of its corresponding operator. Since these  $E_{a_l;0}^{\text{d}}(e^{i\omega\tau_3}, e^{i\omega\tau_3})$  do not vanish, we conclude that  $\langle \mathcal{M} \rangle_{z_e}$  is of order  $t^{s_o}$  if and only if  $\langle \mathcal{M}' \rangle_{z_e}$  is. Thus Eq. (4.12) is true for the case when both sides are of order  $t^{s_o}$ . Moreover, both sides of Eq. (4.12) are of  $O(t^{M_++N_+})$  or higher, and if the leading order terms of both sides are not  $O(t^{M_++N_+})$ , then they both vanish at  $O(t^{M_++N_+})$ .  $\square$

#### 4. derivative of the matrix element of $p^i(e)$ and $D_{ab}^{\frac{1}{2}}(h_e)$

Note that the results show below ignore the terms of order  $t$  and higher. We denote  $\nabla_{\vec{x}^{(i)}} := (\nabla_{\vec{p}^{(i)}}, \nabla_{\vec{\theta}^{(i)}})$  and  $x^*$  is the complex conjugate of  $x$ .

$$\begin{aligned} \nabla_{\vec{x}^{(1)}} \frac{\langle \psi_{g(1)} | \hat{p}_s^1(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( -\frac{1}{2}, \frac{1}{2} i \tanh\left(\frac{p}{2}\right), 0, -\frac{ip \sin(\theta) \text{csch}(p)}{2\theta}, \frac{ip \sin^2\left(\frac{\theta}{2}\right) \text{csch}(p)}{\theta}, 0 \right)^T \\ &= \left( \nabla_{\vec{x}^{(2)}} \frac{\langle \psi_{g(1)} | \hat{p}_s^1(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} \right)^*, \\ \nabla_{\vec{x}^{(1)}} \frac{\langle \psi_{g(1)} | \hat{p}_s^2(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( -\frac{1}{2} i \tanh\left(\frac{p}{2}\right), -\frac{1}{2}, 0, -\frac{ip \sin^2\left(\frac{\theta}{2}\right) \text{csch}(p)}{\theta}, -\frac{ip \sin(\theta) \text{csch}(p)}{2\theta}, 0 \right)^T \\ &= \left( \nabla_{\vec{x}^{(2)}} \frac{\langle \psi_{g(1)} | \hat{p}_s^2(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} \right)^*, \\ \nabla_{\vec{x}^{(1)}} \frac{\langle \psi_{g(1)} | \hat{p}_s^3(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( 0, 0, -\frac{1}{2}, 0, 0, -\frac{i}{2} \right)^T = \left( \nabla_{\vec{x}^{(2)}} \frac{\langle \psi_{g(1)} | \hat{p}_s^3(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} \right)^*. \end{aligned} \quad (\text{F67})$$

$$\begin{aligned}
\nabla_{\bar{x}^{(1)}} \frac{\langle \psi_{g(1)} | \hat{p}_t^1(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( \frac{e^{i\theta} + e^{p-i\theta}}{2e^p + 2}, \frac{ie^{-i\theta} (e^p - e^{2i\theta})}{2(e^p + 1)}, 0, \frac{e^{-i\theta} (-1 + e^{i\theta}) p (e^{2p} + e^{i\theta})}{2\theta (e^{2p} - 1)}, \right. \\
&\left. \frac{ie^{-i\theta} (-1 + e^{i\theta}) p (e^{2p} - e^{i\theta})}{2\theta (e^{2p} - 1)}, 0 \right)^T = \left( \nabla_{\bar{x}^{(2)}} \frac{\langle \psi_{g(1)} | \hat{p}_t^1(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} \right)^*, \\
\nabla_{\bar{x}^{(1)}} \frac{\langle \psi_{g(1)} | \hat{p}_t^2(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( \frac{ie^{-i\theta} (-e^p + e^{2i\theta})}{2(e^p + 1)}, \frac{e^{i\theta} + e^{p-i\theta}}{2e^p + 2}, 0, \frac{ie^{-i\theta} (-1 + e^{i\theta}) p (-e^{2p} + e^{i\theta})}{2\theta (e^{2p} - 1)}, \right. \\
&\left. \frac{e^{-i\theta} (-1 + e^{i\theta}) p (e^{2p} + e^{i\theta})}{2\theta (e^{2p} - 1)}, 0 \right)^T = \left( \nabla_{\bar{x}^{(2)}} \frac{\langle \psi_{g(1)} | \hat{p}_t^2(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} \right)^*, \\
\nabla_{\bar{x}^{(1)}} \frac{\langle \psi_{g(1)} | \hat{p}_t^3(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( 0, 0, \frac{1}{2}, 0, 0, \frac{i}{2} \right)^T = \left( \nabla_{\bar{x}^{(2)}} \frac{\langle \psi_{g(1)} | \hat{p}_t^3(e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} \right)^*.
\end{aligned} \tag{F68}$$

$$\begin{aligned}
\nabla_{\bar{x}^{(1)}} \frac{\langle \psi_{g(1)} | D_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}(h_e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( 0, 0, -\frac{1}{4}e^{-\frac{i\theta}{2}}, 0, 0, -\frac{1}{4}ie^{-\frac{i\theta}{2}} \right)^T, \\
\nabla_{\bar{x}^{(1)}} \frac{\langle \psi_{g(1)} | D_{\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}}(h_e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( -\frac{e^{\frac{i\theta}{2}} \tanh(\frac{p}{2})}{2p}, \frac{ie^{\frac{i\theta}{2}} \tanh(\frac{p}{2})}{2p}, 0, -\frac{i \sin(\frac{\theta}{2})}{\theta + \theta e^p}, -\frac{\sin(\frac{\theta}{2})}{\theta + \theta e^p}, 0 \right)^T, \\
\nabla_{\bar{x}^{(1)}} \frac{\langle \psi_{g(1)} | D_{-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}(h_e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( -\frac{e^{-\frac{i\theta}{2}} \tanh(\frac{p}{2})}{2p}, -\frac{ie^{-\frac{i\theta}{2}} \tanh(\frac{p}{2})}{2p}, 0, -\frac{ie^p \sin(\frac{\theta}{2})}{\theta + \theta e^p}, \frac{e^p \sin(\frac{\theta}{2})}{\theta + \theta e^p}, 0 \right)^T, \\
\nabla_{\bar{x}^{(1)}} \frac{\langle \psi_{g(1)} | D_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}(h_e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( 0, 0, \frac{1}{4}e^{\frac{i\theta}{2}}, 0, 0, \frac{1}{4}ie^{\frac{i\theta}{2}} \right)^T, \\
\nabla_{\bar{x}^{(2)}} \frac{\langle \psi_{g(1)} | D_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}(h_e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( 0, 0, \frac{1}{4}e^{-\frac{i\theta}{2}}, 0, 0, -\frac{1}{4}ie^{-\frac{i\theta}{2}} \right)^T, \\
\nabla_{\bar{x}^{(2)}} \frac{\langle \psi_{g(1)} | D_{\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}}(h_e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( \frac{e^{\frac{i\theta}{2}} \tanh(\frac{p}{2})}{2p}, -\frac{ie^{\frac{i\theta}{2}} \tanh(\frac{p}{2})}{2p}, 0, -\frac{ie^p \sin(\frac{\theta}{2})}{\theta + \theta e^p}, -\frac{e^p \sin(\frac{\theta}{2})}{\theta + \theta e^p}, 0 \right)^T, \\
\nabla_{\bar{x}^{(2)}} \frac{\langle \psi_{g(1)} | D_{-\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}(h_e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( \frac{e^{-\frac{i\theta}{2}} \tanh(\frac{p}{2})}{2p}, \frac{ie^{-\frac{i\theta}{2}} \tanh(\frac{p}{2})}{2p}, 0, -\frac{i \sin(\frac{\theta}{2})}{\theta + \theta e^p}, \frac{\sin(\frac{\theta}{2})}{\theta + \theta e^p}, 0 \right)^T, \\
\nabla_{\bar{x}^{(2)}} \frac{\langle \psi_{g(1)} | D_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}}(h_e) | \psi_{g(2)} \rangle}{\langle \psi_{g(1)} | \psi_{g(2)} \rangle} &= \left( 0, 0, -\frac{1}{4}e^{\frac{i\theta}{2}}, 0, 0, \frac{1}{4}ie^{\frac{i\theta}{2}} \right)^T.
\end{aligned} \tag{F69}$$

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