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Diffusion-limited Relic Particle Production

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We examine the thermal evolution of particle number densities in the early universe when the particles have a finite diffusion length. Assuming that annihilations are impossible when the mean separation of the particles is larger than their diffusion length, we derive a version of the Boltzmann equation for freeze out in this scenario and an approximate solution, accurate to better than 2%. The effect of a finite diffusion length is to increase the final relic freeze-out abundance over its corresponding value when diffusion effects are ignored. When diffusion is limited only by scattering off of the thermal background, and the annihilation cross section is bounded by unitarity, a significant effect on the freeze-out abundance requires a scattering cross section much larger than the annihilation cross section. A similar effect is demonstrated when the relic particles are produced via the freeze-in mechanism, but in this case the finite diffusion length is due to the scattering of particles that annihilate into the relic particle of interest. For freeze in, the effect of a finite diffusion length is to reduce the final relic particle abundance. The effects of a finite diffusion length are most important when the scattering cross section or the relic mass are very large. While we have not found a particularly compelling example where this would affect previous results, with the current interest in new dark matter candidates it could become an important consideration.

I. INTRODUCTION

The evolution of relic particle densities is one of the central topics in early-universe cosmology. In the standard treatment, the relic particles are taken to be initially relativistic and in chemical and kinetic equilibrium with the thermal background. As the temperature T drops below the mass m of these particles, they become nonrelativistic, and their number density is Boltzmann suppressed as $e^{-m/T}$. Finally, the particles drop out of thermal equilibrium with a fixed number density per comoving volume. This freeze-out scenario has been developed and refined over the past 50 years [1–12]. It remains one of the favored models for the evolution of dark matter, and it can also apply to the evolution of other particles as well.

Here we consider an effect that has not been previously discussed in these treatments of thermal evolution: a finite diffusion length for the evolving massive particles. In addition to undergoing pair annihilations, these particles will scatter off of the thermal background particles, resulting in a finite diffusion length. A finite diffusion length could also result from more exotic scattering processes, such as interactions with relic magnetic fields, domain walls, or other early-universe phenomena. If this diffusion length is much larger than the mean particle separation, then it has little effect on the particle evolution. However, if the diffusion length drops below the mean particle separation during the freeze-out process, then the effect on the evolution of the particle number density can be profound.

Although the effect discussed here has not been previously examined, there are other discussions of annihilating, diffusing particles in the literature. Zeldovich and Khlopov investigated the diffusion and annihilation of monopoles in the early universe [13]. Several authors (see, e.g. Refs. [14, 15]) examined systems of particles and antiparticles in a non-expanding background, with initial inhomogeneities in the particle-antiparticle distribution, diffusion, and annihilation (but no particle-antiparticle creation from the background). These systems tend to evolve into domains of particles and antiparticles within which further annihilation is impossible.

In the next section, we show how diffusion effects can be incorporated into the Boltzmann equation, and we derive an approximate solution to the diffusion-limited evolution of the relic particle number density that is accurate to within 2%. In Sec. 3, we examine the particular case where scattering off of thermal background particles limits the diffusion length of the relic particles. In Sec. 4, we extend our discussion to the freeze-in mechanism for relic particles. Our conclusions are summarized in Sec. 5.

II. DIFFUSION AND FREEZE OUT

Recall first the standard picture of thermal freeze out. Consider a fermion χ with g_χ spin degrees of freedom. For simplicity, we will take the particle to be its own antiparticle, but our results generalize easily to the case of

distiguishable particles and antiparticles, as well as to the case where χ is bosonic. The number density evolves as

$$\frac{dn_\chi}{dt} + 3Hn_\chi = \langle \sigma_A v \rangle [n_{eq}^2 - n_\chi^2], \quad (1)$$

where n_χ is the number density of the χ particles, H is the Hubble parameter ($\equiv a^{-1}da/dt$, with a being the scale factor), and $\langle \sigma_A v \rangle$ is the thermally-averaged cross section times relative velocity. When $T \gg m$, the χ particles are highly relativistic, and the equilibrium number density n_{eq} is given by

$$n_{eq} = (3/4)(g_\chi/\pi^2)\zeta(3)T^3, \quad (2)$$

where $\zeta(3) \approx 1.202$. (We take $\hbar = c = k = 1$ throughout). In the opposite limit, when $T \ll m$, we have instead

$$n_{eq} = g_\chi \left(\frac{m_\chi T}{2\pi} \right)^{3/2} \exp(-m_\chi/T). \quad (3)$$

Following Refs. [5, 6], we parametrize the annihilation cross section as

$$\langle \sigma_A v \rangle = \sigma_0 \left(\frac{m_\chi}{T} \right)^{-n}, \quad (4)$$

so that $n = 0$ corresponds to s -wave annihilation, $n = 1$ gives p -wave annihilation, and so on. We make the standard change of variables $x = m_\chi/T$ and $Y = n_\chi/s$, where s is the entropy density, given by

$$s = \frac{2\pi^2}{45} g_* T^3, \quad (5)$$

and g_* is the effective number of degrees of freedom in thermal equilibrium, all assumed to be at the same temperature. We obtain [5, 6]

$$\frac{dY}{dx} = -\lambda x^{-n-2} (Y^2 - Y_{eq}^2), \quad (6)$$

where the constant λ is given by

$$\lambda = 0.264 g_*^{1/2} m_{Pl} m_\chi \sigma_0, \quad (7)$$

and m_{Pl} is the Planck mass.

In the standard scenario, n_χ tracks n_{eq} , and as T drops below m_χ ($x > 1$), the χ number density becomes exponentially suppressed as in Eq. (3). When x reaches a value of $x_f \sim 10 - 30$, the rates for the reactions that produce χ become negligible compared to the annihilation rate, and the abundance freezes out. However, annihilations continue, reducing the value of Y relative to its value at $x = x_f$. The final abundance is well-approximated by [5, 6]

$$Y_\infty = \frac{n+1}{\lambda} x_f^{n+1}. \quad (8)$$

This derivation assumes an effectively infinite mean free path for the particles. However, in addition to annihilations, the χ particles scatter off of the thermal background, resulting in a finite diffusion length d , which will be a function of T . (Here we are taking d to be the physical, not the comoving diffusion length). Although scattering is the only unavoidable source of a finite diffusion length, it is not the only possibility. Magnetic fields, domain walls, or other early-universe phenomena might also reduce the χ diffusion length. Hence, we will take $d(T)$ for now to be a free parameter and determine under what conditions a finite diffusion length affects the freeze-out process. In the next section, we will examine the specific case of scattering interactions with thermal background particles.

Now consider how a finite value of d alters the freeze-out process. It has no effect at all on the creation of χ particles, since the creation rate is determined by the rate of annihilation of thermal background particles into χ particles. However, a finite diffusion length *does* affect the $\chi\chi$ annihilation rate. A given χ particle can only travel a distance $d(T)$ at a temperature T to annihilate with another χ particle. Hence, when $n_\chi \ll 1/d^3$, the annihilations effectively cease. Conversely, when $n_\chi \gg 1/d^3$, diffusion has no effect on $\chi\chi$ annihilations. We will make the approximation that annihilation is completely unaffected for $n_\chi > 1/d^3$ and that annihilations cease completely for $n_\chi < 1/d^3$. In reality, this behavior will not be a step function but will vary more gradually with n_χ and d . However, our approximation, while admittedly crude, will be sufficient for a first calculation of the effect considered here.

With this approximation, we can multiply n_χ^2 in Eq. (1) by the appropriate step function $\theta(n - 1/d^3)$, where the Heaviside step function $\theta(x)$ is defined by $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$. This gives

$$\frac{dn_\chi}{dt} + 3Hn_\chi = \langle\sigma_A v\rangle[n_{eq}^2 - n_\chi^2\theta(n_\chi - 1/d^3)]. \quad (9)$$

Consider how this change alters the evolution of the particle abundances. As long as $n_\chi > 1/d^3$, the evolution is unaffected, but when n_χ drops below $1/d^3$, the annihilations shut off. However, creation of χ continues, ultimately pushing n_χ back above $1/d^3$. The net affect is that in thermal equilibrium, n_χ does not necessarily track n_{eq} ; instead, it tracks the *larger* of n_{eq} and $1/d^3$. Effectively, $1/d^3$ gives a floor abundance on n_χ .

To determine how this alters the freeze-out abundance, we make the same change of variables as in Eq. (6), so that

$$\frac{dY}{dx} = -\lambda x^{-n-2}[Y^2\theta(Y - Y_d) - Y_{eq}^2], \quad (10)$$

where we have defined the new quantity Y_d to be given by

$$Y_d = (1/d^3)/s. \quad (11)$$

The quantity Y_d has a simple physical interpretation; it is the inverse of the total entropy in a diffusion volume d^3 .

To keep our argument as general as possible, we let d be an arbitrary power-law in T , namely

$$d(T) = d_0(T/T_0)^{-\alpha}, \quad (12)$$

where d_0 and T_0 are arbitrary fiducial values of the diffusion length and the temperature, and we are implicitly assuming that the diffusion length increases as the temperature decreases ($\alpha > 0$). Because s scales as $a^{-3} \sim T^3$, we can then write Y_d as

$$Y_d = Y_1 x^{3-3\alpha}, \quad (13)$$

where the fiducial quantity Y_1 is defined by Eq. (13): Y_1 is the value of Y_d at $x = 1$. Hence, Y_1 is the inverse of the entropy in a diffusion volume d^3 at a temperature $T = m_\chi$. Larger Y_1 corresponds to smaller diffusion length and vice-versa.

A rough approximation to the evolution of Y is sketched out in Fig. 1 for the case of s -wave annihilation. For illustrative purposes, we take $\alpha = 11/4$ here and in Fig. 2, as this corresponds to a diffusion length limited by scattering off of thermal background particles with a constant scattering cross section (see the next section). However, our qualitative results do not depend on the value of α . When χ is in thermal equilibrium, with $Y \approx Y_{eq}$, a finite diffusion length has no effect on the evolution of Y as long as $Y_{eq} > Y_d$. However, when $Y_{eq} < Y_d$, the value of Y tracks Y_d instead of Y_{eq} . Thus, Y tracks the *larger* of Y_{eq} or Y_d , until $n_\chi\langle\sigma_A v\rangle$ drops below H and the abundance freezes out at the value shown in Fig. 1 by the horizontal black lines. (Note that if $Y_d > Y_{eq}$ when χ is highly relativistic, we will have $Y > Y_{eq}$ and $n_\chi > n_{eq}$. This is a rare case in which it is possible for the density of a particle to exceed its relativistic equilibrium density. Physically, this occurs because the finite diffusion length prevents particle annihilations, while production from the thermal background continues to produce χ particles. This would require an exceedingly small diffusion length).

This approximation assumes a sudden sharp freeze out, but in reality residual annihilations continue to occur even after $n_\chi\langle\sigma_A v\rangle$ drops below H . This is illustrated in Fig. 2, in which we display the exact evolution, derived from a numerical integration of Eq. (10), for s -wave annihilation. The solid blue curve in Fig. 2 corresponds to the exact evolution where $Y_{eq} < Y_d$ at $x = x_f$, where x_f is the freeze-out value of x in the standard scenario without diffusion effects. The dashed blue curve shows the equivalent abundance in the sudden freeze-out approximation. As in the case of standard freeze out, residual annihilations reduce the final abundance below that obtained by using the sudden freeze out approximation.

We now calculate an expression for the final abundance that takes into account these residual annihilations. For the diffusion-limited case, we have $Y \approx Y_d$ instead of $Y \approx Y_{eq}$ at $x = x_f$. The reactions that create χ particles effectively shut off at x_f , and Eq. (10) becomes

$$\frac{dY}{dx} = -\lambda x^{-n-2}Y^2\theta(Y - Y_d), \quad (14)$$

At this point, Y continues to track Y_d , which is a decreasing function of x given Eq. (13). The rate at which Y_d decreases with x is then

$$\frac{dY_d}{dx} = Y_1(3 - 3\alpha)x^{2-3\alpha}. \quad (15)$$

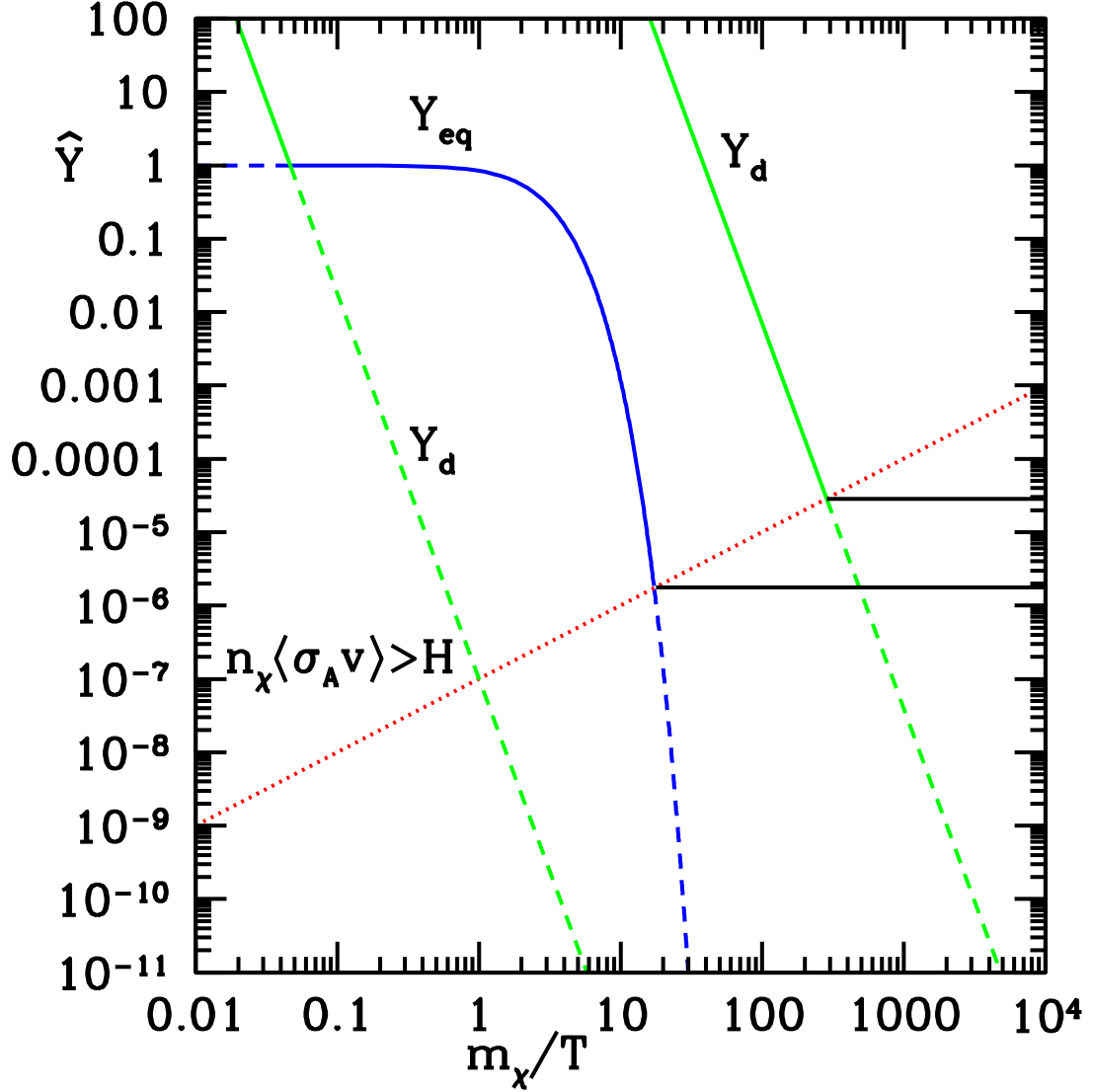


FIG. 1: A rough sketch of the evolution of \hat{Y} , the value of Y normalized to its relativistic value, n_{eq}/s , as a function of m_χ/T , for the case of s -wave annihilations. (Here $\hat{Y} \equiv Y/(n_{eq}/s)$, with $n_{eq}/s = 0.208g_\chi/g_*$ for a relativistic fermion). Blue curve is the thermal equilibrium abundance Y_{eq} . Area above the dotted red line is the region in which $n_\chi \langle \sigma_A v \rangle > H$, so annihilations can take place. (We have fixed $\langle \sigma_A v \rangle$ to a single illustrative value). Green lines give Y_d as defined in Eq. (13) for $\alpha = 11/4$ and two different values of Y_1 . For each value of Y_1 , the value of \hat{Y} tracks the larger of Y_{eq} and Y_d until \hat{Y} drops below the threshold for annihilations given by the dotted red line, at which point the abundances freeze out at the values shown by the horizontal black lines. In both cases, the value of \hat{Y} traces out the solid portion of the displayed curves.

However, Y cannot decrease faster than the annihilation rate given by Eq. (14) in the absence of diffusion effects, namely

$$\frac{dY}{dx} = -\lambda x^{-n-2} Y^2. \quad (16)$$

Hence, Y tracks Y_d only until $|dY_d/dx|$ from Eq. (15) becomes larger than $|dY/dx|$ from Eq. (16). At this point, diffusion effects become irrelevant and the further evolution of Y is determined only by Eq. (16), as in standard freeze out.

We can now calculate the final abundance of Y in the presence of diffusion effects, which we will denote \tilde{Y}_∞ . Equating dY_d/dx from Eq. (15) with dY/dx from Eq. (16) and taking $Y = Y_d$ given by Eq. (13) gives the value of x

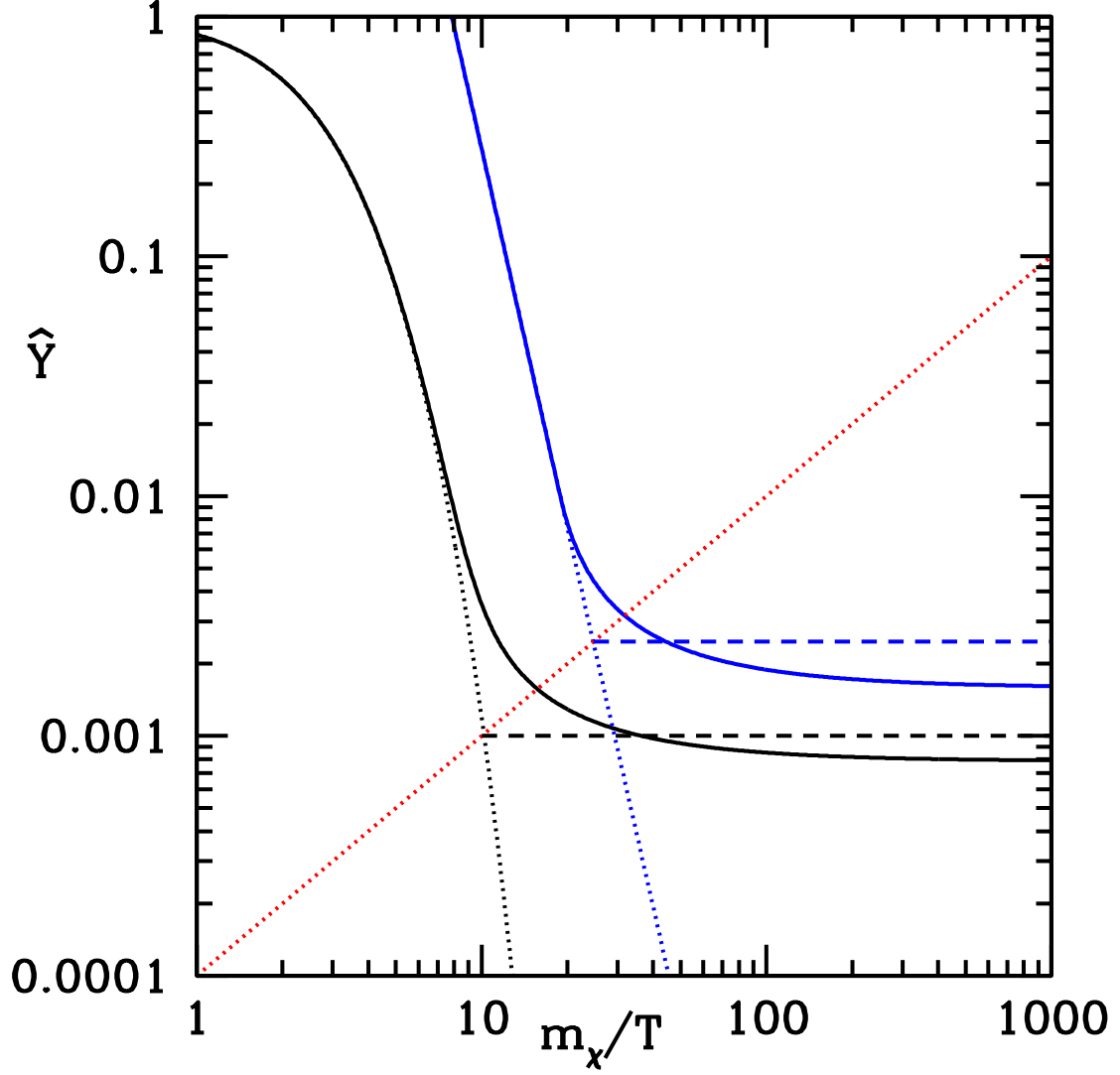


FIG. 2: The evolution of \hat{Y} , defined as in Fig. 1, as a function of m_χ/T , from a numerical integration of Eq. (10), for s -wave annihilations with $\lambda(n_{eq}/s) = 10^4$ and $\alpha = 11/4$. Solid black curve gives the evolution in the standard model with no diffusion effects. Dotted black curve is $Y_{eq}/(n_{eq}/s)$. Solid blue curve gives evolution for $Y_1/(n_{eq}/s) = 50000$; dotted blue line gives the corresponding value of $Y_d/(n_{eq}/s)$. Dotted red line corresponds to $n_\chi \langle \sigma_A v \rangle = H$. The points at which this line intersects the dotted blue line and the dotted black line correspond to the sudden freeze-out approximation in the diffusion-limited case and in the standard model, respectively, with the resulting abundances given by the dashed blue and dashed black lines.

at which annihilations can proceed without diffusion effects, which we will denote x_d :

$$x_d = \left(\frac{\lambda Y_1}{3\alpha - 3} \right)^{1/(3\alpha - 2 + n)}. \quad (17)$$

At $x = x_d$, the value of Y is given by the value of Y_d in Eq. (13), namely

$$Y(x = x_d) = Y_1 \left(\frac{3\alpha - 3}{\lambda Y_1} \right)^{(3\alpha - 3)/(3\alpha - 2 + n)} \quad (18)$$

In order to calculate the new asymptotic value of Y , denoted by \tilde{Y}_∞ , we integrate Eq. (16) from $x = x_d$ to $x = \infty$,

using the value of $Y(x = x_d)$ that we have just derived. The final result is

$$\tilde{Y}_\infty = \left(\frac{(3\alpha - 3)(1 + n)}{3\alpha - 2 + n} \right) \frac{1}{\lambda} \left[\frac{\lambda Y_1}{3\alpha - 3} \right]^{(1+n)/(3\alpha-2+n)}. \quad (19)$$

Alternately, we can express the value of \tilde{Y}_∞ relative to Y_∞ , the asymptotic value of Y for the case where diffusion effects are negligible, which is given by Eq. (8). This gives

$$\frac{\tilde{Y}_\infty}{Y_\infty} = \left(\frac{3\alpha - 3}{3\alpha - 2 + n} \right) \frac{1}{x_f^{n+1}} \left[\frac{\lambda Y_1}{3\alpha - 3} \right]^{(1+n)/(3\alpha-2+n)}. \quad (20)$$

Eqs. (19)-(20) are the main results of our paper. They take a particularly simple form for s -wave annihilations ($n = 0$), namely

$$\tilde{Y}_\infty = \left(\frac{3\alpha - 3}{3\alpha - 2} \right) \frac{1}{\lambda} \left[\frac{\lambda Y_1}{3\alpha - 3} \right]^{1/(3\alpha-2)}, \quad (21)$$

and

$$\frac{\tilde{Y}_\infty}{Y_\infty} = \left(\frac{3\alpha - 3}{3\alpha - 2} \right) \frac{1}{x_f} \left[\frac{\lambda Y_1}{3\alpha - 3} \right]^{1/(3\alpha-2)}. \quad (22)$$

Comparing to a numerical integration of Eq. (10), we find that these expressions for \tilde{Y}_∞ are accurate to better than 2% for both the s -wave and p -wave cases.

III. FINITE DIFFUSION LENGTH FROM SCATTERING

Our results in the previous section apply to the general case in which the diffusion length of the particles is small enough to affect the freeze-out process, without reference to any specific model of diffusion. Here we will consider the specific case of scattering of the χ particles off of standard-model thermal background particles. This process has been explored in detail in connection with the process of kinetic decoupling, which generally occurs later than chemical decoupling for most particle species [16–23]. Here we will be interested in the extreme case for which the scattering rate is large enough to affect the freeze-out process itself, as discussed in the previous section.

Assume that the χ particles can annihilate into thermal Standard Model (SM) particles, and also scatter off of those same particles:

$$\chi + \chi \rightarrow SM + SM, \quad (23)$$

and

$$\chi + SM \rightarrow \chi + SM. \quad (24)$$

We will assume s -wave annihilations, so that $\langle \sigma_A v \rangle = \sigma_0$, while the cross section for scattering off of standard model particles will be σ_S .

As the χ particles scatter off of the thermal background particles, they undergo a random walk with step size l , which gives a diffusion length d :

$$d = \sqrt{lv t}. \quad (25)$$

When $T > m_\chi$, the value of l is just the mean free path, given by $l = (n_S \sigma_S)^{-1}$. However, for $T < m_\chi$, multiple scatterings are required to significantly change the momentum of the χ particles. Following Ref. [19], we will assume that the number of such collisions required to alter the trajectory of a given χ is $\sim m_\chi/T$. Thus, the step size l in this case is $l \sim (n_S \sigma_S)^{-1} (m_\chi/T)$. Since freeze out occurs when $m_\chi/T > 1$, we will use the latter expression to derive the diffusion length. Then we obtain

$$d = \sqrt{(n_S \sigma_S)^{-1} (m_\chi/T) v t}. \quad (26)$$

We take $v \approx \sqrt{3T/m}$ and $t = 0.3g_*^{-1/2}m_{Pl}/T^2$, which assumes a radiation-dominated universe. The values of n_S and σ_S are necessarily model dependent. To make an estimate of the circumstances under which scattering can alter the freeze-out process, we will assume that χ can scatter off of all of the degrees of freedom in the relativistic background, so that n_S is given by $n_S = \zeta(3)g_n T^3/\pi^2$, where $g_n = 3/4$ ($= 1$) per fermionic (bosonic) degree of freedom. Because the scattering cross section itself is completely model dependent, we will, for simplicity, take σ_S to be a constant. (For a discussion of other possibilities, see, e.g., Ref. [16]). It is straightforward to generalize our results to other functional forms for σ_S .

Combining these expressions, and taking $g_n \approx g_*$, we obtain

$$d = 2.1g_*^{-3/4}\sigma_S^{-1/2}m_{Pl}^{1/2}m_\chi^{1/4}T^{-11/4}. \quad (27)$$

Using the standard expression for entropy density, we can substitute this value for the diffusion length into Eq. (11) to derive an expression for Y_d :

$$Y_d = 0.26g_*^{5/4}\sigma_S^{3/2}m_{Pl}^{-3/2}m_\chi^{9/2}\left(\frac{m_\chi}{T}\right)^{-21/4}. \quad (28)$$

In terms of the parameters of Eqs. (13) and (21), we then have:

$$Y_1 = 0.26g_*^{5/4}\sigma_S^{3/2}m_{Pl}^{-3/2}m_\chi^{9/2}, \quad (29)$$

and

$$\alpha = 11/4. \quad (30)$$

Using the values for Y_1 and α from Eqs. (29) and (30), along with the definition of λ from Eq. (7), Eq. (22) becomes

$$\frac{\tilde{Y}_\infty}{Y_\infty} = \frac{21}{25} \frac{1}{x_f} \left[0.013g_*^{7/4}m_{Pl}^{-1/2}m_\chi^{11/2}\sigma_0\sigma_S^{3/2} \right]^{4/25}. \quad (31)$$

Clearly, the change in the relic χ abundance relative to its standard abundance is an increasing function of the annihilation cross section, the scattering cross section, and the particle mass. Diffusion effects become important when $\tilde{Y}_\infty/Y_\infty > 1$, which corresponds to

$$m_\chi > 2.7 x_f^{25/22} g_*^{-7/22} \sigma_0^{-2/11} \sigma_s^{-3/11} m_{Pl}^{1/11}. \quad (32)$$

Taking $g_* \sim 100$ and $x_f \sim 10$ (both good order-of-magnitude approximations), we graph the region for which scattering affects the freeze-out abundance in Fig. 3. In this figure, the area above the solid line corresponding to each indicated mass represents the region in parameter space for which a finite diffusion length from scattering will increase the relic freeze-out abundance relative to its value in the absence of diffusion effects. For reference, we also include the value of σ_0 that corresponds to the observed dark matter abundance for a thermal relic, $\sigma_0 \sim 1.0 \times 10^{-36} \text{ cm}^2 = 2.6 \times 10^{-9} \text{ GeV}^{-2}$.

Note also that m_χ is bounded from above by unitarity, which requires that [7]

$$\sigma_0 m_\chi^2 < 4\pi. \quad (33)$$

This bound requires particles with masses above roughly 100 TeV to have an annihilation cross section below the standard weak-scale σ_0 cited above, which results in a freeze-out abundance larger than the observed dark matter abundance. (See, however, Ref. [24] for mechanisms to evade this bound). For particle masses below the unitarity bound and $\sigma_0 = 2.6 \times 10^{-9} \text{ GeV}^{-2}$, we see from Fig. 3 that the scattering cross section must be much larger than the annihilation cross section in order for a finite diffusion length to alter the freeze-out abundance.

IV. DIFFUSION-LIMITED FREEZE IN

Another mechanism to produce relic particles, called freeze in, was first proposed by Hall et al. [25]. In this scenario, a feebly interacting massive particle (FIMP) is coupled too weakly to standard model particles to ever be in equilibrium with the thermal background. However the FIMP can be produced by annihilations of other particles, albeit at a very low rate. Ref. [25] gives a number of variations on this theme involving a combination of freeze in,

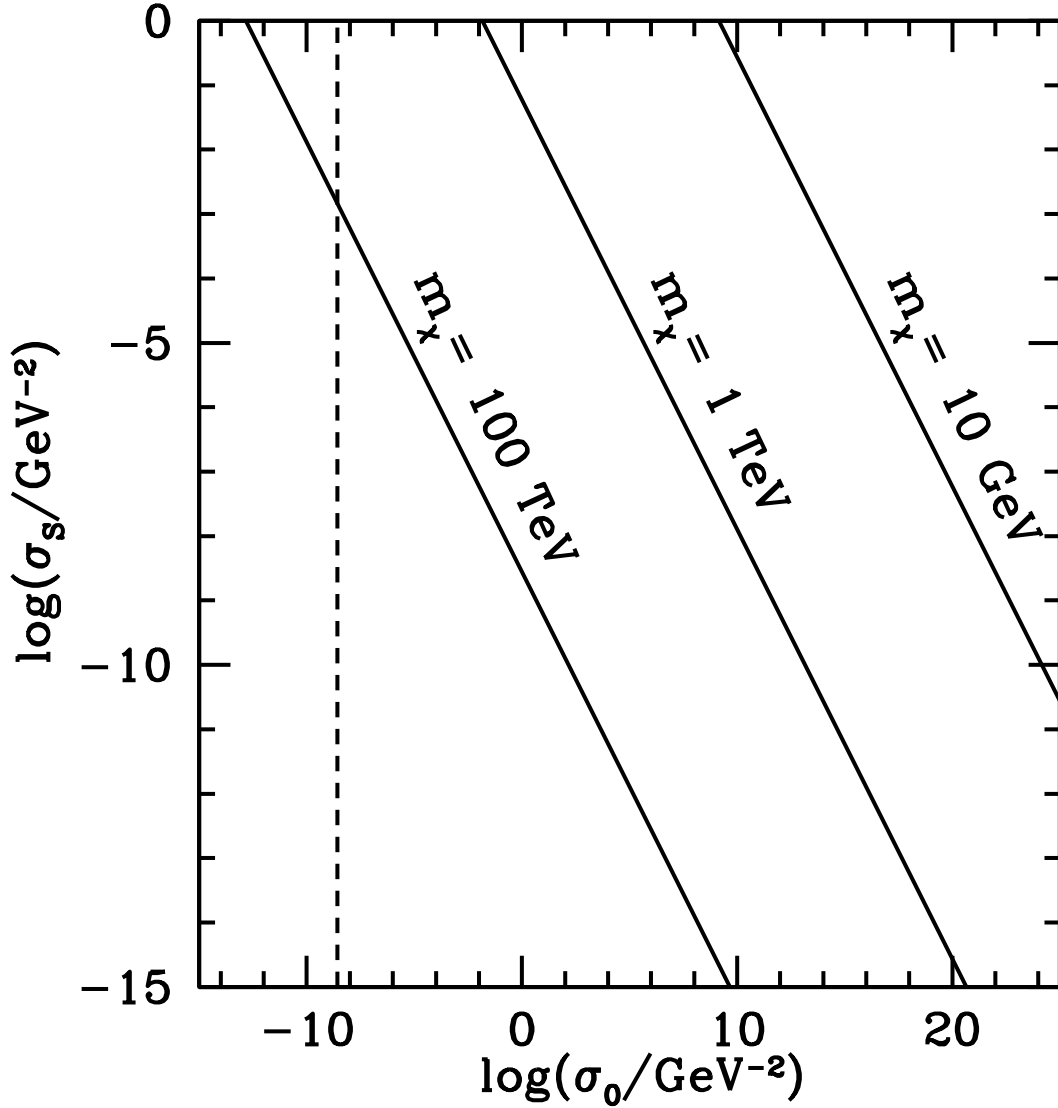


FIG. 3: Region in parameter space defined by the particle mass m_χ , s -wave annihilation cross section $\sigma_0 \equiv \langle \sigma_A v \rangle$, and scattering cross section σ_S , for which a finite diffusion length from scattering off of the thermal background will alter the standard freeze-out abundance of χ : this region lies above the solid lines corresponding to each indicated mass. Vertical dashed line is the annihilation cross section corresponding to the observed dark matter abundance today in the absence of diffusion effects ($\sigma_0 = 2.6 \times 10^{-9} \text{ GeV}^{-2}$).

freeze out, and particle decays, but we will confine our attention here to the simplest scenario, in which there is only freeze in of a relic particle. Because of the variety of possible scenarios, we will keep our discussion as general as possible. Nonetheless, we will be able to derive some interesting results for freeze in.

In this simple scenario, we assume that the particle χ is produced through annihilation of one or more particles A , so that the production rate is

$$\frac{dn_\chi}{dt} + 3Hn_\chi = \langle \sigma_A v \rangle n_A^2. \quad (34)$$

Again, we define Y_χ and Y_A to be the ratio of the particle number densities to the entropy density, so Eq. (34) becomes

$$\frac{dY_\chi}{dt} = s \langle \sigma_A v \rangle Y_A^2. \quad (35)$$

In the treatment of Ref. [25], it is assumed that A is a standard-model particle in equilibrium with the thermal background, so that when T drops below m_χ , the interactions producing the χ are Boltzmann suppressed and freeze in terminates. To keep our results as general as possible, we will not make these assumptions, but we will take A to be relativistic in our epoch of interest, as this simplifies the calculation.

Now consider the effect of a finite diffusion length on the freeze-in process. In this case, the diffusion length of χ is irrelevant, since it is the annihilation of A that produces the final χ abundance, so it is the finite diffusion length of A that has the potential to alter this abundance. As before, let d be the diffusion length of A . Then the freeze in process will be altered whenever $n_A d^3 < 1$.

Now consider the case where the finite diffusion length is produced by the scattering of A off of particles in the thermal background. Since we are taking A to be relativistic at $T \sim m_\chi$, Eq. (26) becomes

$$d = \sqrt{(n_S \sigma_S)^{-1} t}, \quad (36)$$

and the condition for the finite diffusion length to alter the relic abundance, $n_A d^3 < 1$, becomes

$$n_A n_S^{-3/2} \sigma_S^{-3/2} t^{3/2} < 1. \quad (37)$$

We now take $n_S \sim T^3$, $n_A \sim Y_A T^3$, and $t \sim m_{Pl}/T^2$, to obtain

$$Y_A T^{-9/2} m_{Pl}^{3/2} \sigma_S^{-3/2} < 1. \quad (38)$$

Equation (38) is our condition for which a finite diffusion length for A alters the freeze-in process at temperature T , due to scattering of A off of the thermal background. In the case of freeze in, finite diffusion effects suppress the annihilation of A into χ particles, reducing the final freeze-in abundance of χ . The size of this effect, and whether it occurs at all, depend upon the detailed scenario for freeze in.

V. CONCLUSIONS

It is remarkable that after 50 years, we are still finding new aspects of the thermal evolution of relic particle abundances. It is important to note that the effects we have outlined here are not “optional.” Any particle undergoing annihilation into standard model particles will also scatter off of the thermal background, and the only question is the magnitude of this effect. For particles satisfying the unitarity bound, scattering effects will affect the freeze-out abundance only if the scattering cross section is much larger than the annihilation cross section. However, the scattering effects outlined in the previous section are unlikely to exhaust the possibilities of diffusion-limited freeze out; one can also consider scattering from relic magnetic fields, domain walls, or other exotic early-universe phenomena.

The main approximation we made in our derivation of the effect of a finite diffusion length is the sharp cut-off in the annihilation rate when the mean particle separation is larger than the diffusion length. A more detailed calculation would show a more gradual effect. However, we expect the derivation in Sec. 2 to be qualitatively accurate and, furthermore, independent of the particular mechanism limiting the particle diffusion. In this derivation we have assumed nothing about the actual mechanism producing a finite diffusion length; our result depends only the value of the diffusion length as a function of temperature.

Our calculation in Sec. 3 is more model-dependent; the effect of scattering will depend on the total scattering cross section, the particles off of which scattering occurs, and the scaling of scattering with temperature. However, the formalism we have developed in that section is easily extended to other scattering scenarios such as those we have mentioned above.

A finite diffusion length from scattering can also affect the freeze-in process for relic particle production. In this case, the effects depend not on the scattering of the relic particle itself, but on the scattering of the particles that annihilate into the relic particle. For freeze in, the effect is the opposite of the effect on freeze out: a finite diffusion length decreases the relic particle abundance.

While we have not found an especially compelling example for which a finite diffusion length would significantly alter previous results for freeze out or freeze in, the current heightened interest in new dark matter candidates suggests the possibility that diffusion could be an important consideration in computing relic abundances in the future. We can say more generally that the effects discussed here are likely to be most important when the relic particle mass or the scattering cross section is very large.

Acknowledgments

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