Model-independent form-factor constraints for electromagnetic spin-1 currents

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Model-independent form-factor constraints for electromagnetic spin-1 currents

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Using local gauge invariance in the form of the Ward-Takahashi identity and the fact that properly constructed current operators must be free of kinematic singularities, it is shown that the magnetic moment μ and the quadrupole moment Q of an elementary spin-1 particle with mass m and charge e are related by $2m\mu + m^2 Q = e$, thus constraining the normalizations of the Sachs form factors. This relation holds true as a matter of course at the tree level in the standard model, but we prove it remains true in general for dressed spin-1 states derived from elementary fields. General expressions for spin-1 propagators and currents with arbitrary hadronic dressing are given showing the result to be independent of any dressing effect or model approach.

I. INTRODUCTION

The electromagnetic structure of a massive spin-1 particle has been discussed for some time (see Refs. [1–7] and references therein). The early work of Lee and Yang [1] shows that at the tree level, the particle's magnetic moment μ and the quadrupole moment Q are given by ($\hbar = c = 1$) $\mu = e(1+\kappa)/2m$ and $Q = -e\kappa/m^2$ in terms of one common constant κ. Although usually not written in this manner, this correlation may also be expressed independent of κ as

$$2m\mu + m^2 Q = e\,, \tag{1}$$

where m is the mass and e the charge. This relation is also true for the canonical moments of the $W^\pm$ gauge boson in electroweak gauge theory at the tree level where $\mu = e/m$ and $Q = -e/m^2$ [4], which corresponds to putting $\kappa = 1$ in the Lee-Yang result. The same expressions have also been obtained by Brodsky and Hiller [6] in the strong binding limit based on a generalization of the Drell-Hearn-Gerasimov sum rule [8, 9]. The experimental value of μ, in particular, of the DELPHI Collaboration [10], quoted as the most recent one by PDG [11], is also compatible with this standard-model result.

A more general electromagnetic structure allowing for the quadrupole moment to be independent of charge and magnetic moment was considered in Refs. [3–7, 12–14] (see also references therein), thus exploiting the full multipole degrees of freedom of a spin-1 object. With the usual parametrization $\mu = e(1+\kappa+\lambda)/2m$ and $Q = -e(\kappa-\lambda)/m^2$ [4], resulting in

$$2m\mu + m^2 Q = e\,(1+2\lambda)\;, \tag{2}$$

the value of λ indicates the degree of independence of μ and Q. The results tabulated in Ref. [7] obtained for the ρ meson by various authors providing independent model determinations of μ and Q correspond to values of λ ranging from 0.1 to about 0.5, at variance with the simple correlation (1).

In Sec. II, we consider here the ramifications of imposing local gauge invariance on the structure of the electromagnetic current operator of a spin-1 particle based on an elementary field, and we will show in a model-independent manner that Eq. (1) is strictly valid (i.e., $\lambda = 0$) simply based on demanding a nonsingular current operator that must satisfy the Ward-Takahashi identity [15–17] as a necessary and sufficient condition for *local* gauge invariance. Concluding remarks are provided in Sec. III.

II. SPIN-1 CURRENT

To be locally gauge invariant, the spin-1 current $J^{\lambda\mu\nu}$ must reproduce the Ward-Takahashi identity (WTI) of the form

$$k_\mu J^{\lambda\mu\nu}(q',q) \stackrel{!}{=} e\left[P^{-1}(q') - P^{-1}(q)\right]^{\lambda\nu}\;, \tag{3}$$

where $P^{\lambda\nu}(q)$ is the propagator of the elementary spin-1 particle with four-momentum q and $k = q' - q$ is the (incoming) photon four-momentum (see Fig. 1). We emphasize here that except for the charge parameter e, the right-hand side of the WTI contains no additional information about the particle's electromagnetic structure. Moreover, the WTI is an *off-shell* relation at the operator level that necessarily requires a commensurate off-shell structure for the associated current. The WTI must be true irrespective of whether the spin-1 particle is a stable particle or a resonance with nonzero width. It also must be true independent of the hadronic gauge one chooses for, in general, the spin-1 propagator will be gauge dependent [17]. This gauge dependence will drop out when considering physical matrix elements, however, to be consistent, it must be carried through at all intermediate steps.

As usual, we assume here the spin-1 particle to be stable, described by a propagator $P^{\lambda\nu}(q)$ that has a physical pole with unit residue at a real squared four-momentum $q^2 = m^2$. [More general expressions will be discussed at the end of this note, in Eqs. (13) and (16).] For a stable particle, the on-shell matrix element of the inverse propagator vanishes, which will make

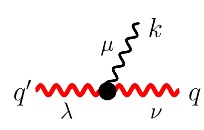

FIG. 1. Depiction of electromagnetic current vertex for the ρ meson, $\gamma(k) + \rho(q) \to \rho(q')$, with associated four-momenta and Lorentz indices. (Time runs from right to left.)

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the right-hand side of the WTI (3) vanish for $q'^2 = q^2 = m^2$, thus indicating a gauge-invariant conserved current.

The electromagnetic spin-1 current operator with form factors is usually written as (see, e.g., Refs. [6, 14])

$$\begin{aligned} J_0^{\lambda\mu\nu}(q',q) &= -eG_1(k^2)(q'+q)^\mu g^{\lambda\nu} \\ &\quad - eG_2(k^2)\left(k^\lambda g^{\mu\nu} - g^{\lambda\mu}k^\nu\right) \\ &\quad + eG_3(k^2)(q'+q)^\mu \frac{k^\lambda k^\nu}{2m^2}\,, \end{aligned} \tag{4}$$

where the form factors G_1, G_2, and G_3 are related to the charge, magnetic, and quadrupole form factors. This current ansatz comprises the most general Lorentz structure available for on-shell matrix elements imposing time-reversal invariance and current conservation. The four-momenta and Lorentz indices appearing here are defined in Fig. 1 where the (charged) ρ meson is used as a generic template for an elementary spin-1 particle.

Introducing Sachs form factors $G_C(k^2)$, $G_M(k^2)$, and $G_Q(k^2)$ describing charge, magnetic moment, and quadrupole moment, respectively, by [6, 14]

$$\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{2}{3}\eta \\ 0 & 1 & 0 \\ -\frac{1}{1+\eta} & \frac{1}{1+\eta} & \frac{3+2\eta}{3+3\eta} \end{pmatrix} \begin{pmatrix} G_C \\ G_M \\ G_Q \end{pmatrix}, \tag{5}$$

where $\eta = -k^2/4m^2$, their normalizations are given by

$$eG_C(0) = e \qquad \text{(charge } e\text{)}\,, \tag{6a}$$

$$eG_M(0) = 2m\mu \qquad \text{(magnetic moment } \mu\text{)}\,, \tag{6b}$$

$$eG_Q(0) = m^2 Q \qquad \text{(quadrupole moment } Q\text{)}\,, \tag{6c}$$

which introduce the three electromagnetic multipole moments of the spin-1 particle. The corresponding normalizations of the form factors G_i $(i = 1, 2, 3)$ then are found as

$$G_1(0) = G_C(0) = 1\,, \tag{7a}$$

$$G_2(0) = G_M(0) = \frac{2m}{e}\mu\,, \tag{7b}$$

$$\begin{aligned} G_3(0) &= -G_C(0) + G_M(0) + G_Q(0) \\ &= -1 + \frac{2m}{e}\mu + \frac{m^2}{e}Q\,. \end{aligned} \tag{7c}$$

It is evident here in the last equation that $G_3(0) = 0$ is equivalent to the validity of Eq. (1) and, indeed, we will show here that the vanishing of $G_3(0)$ is a necessary condition for a well-defined current that satisfies the WTI (3).

The four-divergence of the current (4),

$$k_\mu J_0^{\lambda\mu\nu} = e(q'^2 - q^2)\left[-G_1(k^2)g^{\lambda\nu} + G_3(k^2)\frac{k^\lambda k^\nu}{2m^2}\right], \tag{8}$$

vanishes for $q'^2 = q^2 = m^2$ and thus indeed provides a conserved current. However, this is not the correct form of the WTI for an elementary particle. Clearly, to reproduce the WTI of the generic form (3), one must be able to separate the four-divergence expression into a difference of two terms, individually depending on q' and q, respectively, without any k^2 dependence. This is simply not possible with form factors depending on k^2.

To resolve the discrepancy, one must move the electromagnetic form factors to manifestly transverse terms, without changing the on-shell limit, similar to the treatment of currents for spin-0 and spin-1/2 in Ref. [18]. To this end, we may add an *off-shell* term to the current (4) according to

$$J_1^{\lambda\mu\nu} = J_0^{\lambda\mu\nu} + ek^\mu(q'^2 - q^2)\left(\frac{G_1 - 1}{k^2}g^{\lambda\nu} - \frac{G_3}{k^2}\frac{k^\lambda k^\nu}{2m^2}\right) \tag{9}$$

that clearly is irrelevant for any physical matrix element and thus will not change the electromagnetic form-factor content of the current as defined by Eq. (4). However, this modification is absolutely essential for considerations of local gauge invariance in view of the fact that the Ward-Takahashi identity itself is an off-shell relation. For the modified current,

$$\begin{aligned} J_1^{\lambda\mu\nu}(q',q) &= -e(q'+q)^\mu g^{\lambda\nu} - eG_2(k^\lambda g^{\nu\mu} - k^\nu g^{\mu\lambda}) \\ &\quad - e\left(\frac{G_1 - 1}{k^2}g^{\lambda\nu} - \frac{G_3}{k^2}\frac{k^\lambda k^\nu}{2m^2}\right) \\ &\quad \times \left[(q'+q)^\mu k^2 - k^\mu(q'^2 - q^2)\right]\,, \end{aligned} \tag{10}$$

the form-factor dependence does not appear in the four-divergence,

$$k_\mu J_1^{\lambda\mu\nu}(q',q) = -g^{\lambda\nu}e\left[(q'^2 - m^2) - (q^2 - m^2)\right]\,, \tag{11}$$

which has the correct structure of the WTI (3) and vanishes for on-shell hadrons.

It should be emphasized here that the additional off-shell term in Eq. (9) is unique because the resulting current expression (10) is comprised of the only three linearly-independent, time-reversal-invariant *transverse* operators available for spin 1 that survive on shell.[1] In other words, given $J_0^{\lambda\mu\nu}$ of Eq. (4), one cannot construct an alternative subtraction curent that reproduces the WTI.

While the form (11) of the WTI is only true for stable particles, without any explicit hadronic dressing effects, it is sufficient for the present purpose for it illustrates the basic mechanism how the dependence on electromagnetic form factors is eliminated from the WTI.

The assertion that Eq. (1) is true in general now simply follows from noting that the operator structure of an electromagnetic current must be free of kinematic singularities. We may demand, therefore, that the additional current in Eq. (9) and thus the transverse term in the modified current (10) be well defined for all values of q' and q. In particular, it may not have singularities at the photon point, $k^2 = 0$, which immediately provides the necessary conditions

$$G_1(0) = 1 \qquad \text{and} \qquad G_3(0) = 0 \tag{12}$$

[1] There exists a fourth independent transverse operator, but since its on-shell matrix element vanishes, its coefficient function cannot be associated with a physical form factor and it can be put to zero without lack of generality.

to make $(G_1 - 1)/k^2$ and G_3/k^2 well behaved. The first condition is trivially true because of the normalization (7a). The second condition then makes the right-hand side of Eq. (7c) vanish, which is equivalent to (1), and thus proves the point that the validity of Eq. (1) is not limited to the assumptions of the original Lee-Yang approach [1], but remains true in general.

A. Fully dressed spin-1 current

We complete the presentation here by showing that even allowing for arbitrary dressing effects will not alter this conclusion.

Without going into details, the most general fully dressed spin-1 propagator may be written in terms of two (in general, complex) scalar dressing functions as

$$P^{\lambda\nu}(q) = \frac{-g^{\lambda\nu} + \frac{q^\lambda q^\nu}{m^2} N(q^2)}{q^2 - m^2 - \Sigma(q^2)} \ . \tag{13}$$

Here, $N(q^2)$ is a gauge-dependent function that is irrelevant for physical matrix elements. The selfenergy function $\Sigma(q^2)$, on the other hand, determines all physically relevant dressing effects. To make m the physical mass, it is assumed here that the selfenergy vanishes at $q^2 = m^2$, but this can be arranged easily. The inverse of the propagator, as it appears in the generic WTI (3), reads

$$\left(P^{-1}(q)\right)^{\lambda\nu} = -g^{\lambda\nu} D(q^2) + q^\lambda q^\nu C(q^2) \tag{14}$$

where

$$D(q^2) = q^2 - m^2 - \Sigma(q^2) \tag{15}$$

is a short-hand notation for the denominator of the propagator (13). The function $C(q^2)$ contains $N(q^2)$ and thus is gauge dependent; its details can easily be worked out by explicitly constructing the inverse (14), but since they are not relevant, they will be omitted here.

The fully dressed current compatible with the propagator (13) then is obtained by applying the gauge derivative [18, 19] to the inverse propagator (14) resulting in

$$J^{\lambda\mu\nu}(q', q) = J_1^{\lambda\mu\nu}(q', q) \frac{D(q'^2) - D(q^2)}{q'^2 - q^2} + J_{\text{gauge}}^{\lambda\mu\nu}(q', q) \ , \tag{16}$$

with a gauge-dependent current piece that reads

$$J_{\text{gauge}}^{\lambda\mu\nu}(q', q) = e q'^\lambda g^{\mu\nu} C(q'^2) + e g^{\lambda\mu} q^\nu C(q^2) + e q'^\lambda (q' + q)^\mu q^\nu \frac{C(q'^2) - C(q^2)}{q'^2 - q^2} \ , \tag{17}$$

whose on-shell matrix elements vanish. The $0/0$ situations arising here at $q'^2 = q^2$ from the finite-difference derivatives of the denominator function D in (16) and of the function C in (17) are well behaved and nonsingular. For a stable particle, in particular, the on-shell value of the finite-difference derivative of D is directly related to the unit residue of the propagator and thus unity as well. Hence, the on-shell matrix elements of the current $J^{\lambda\mu\nu}$ with full hadronic dressing, of the modified undressed current $J_1^{\lambda\mu\nu}$, and of the usual current expression $J_0^{\lambda\mu\nu}$ of Eq. (4) are identical. The normalizations in Eqs. (6) and (7), therefore, are not affected by hadronic dressing.

Evaluating now the four-divergences of the gauge-dependent current contribution,

$$k_\mu J_{\text{gauge}}^{\lambda\mu\nu}(q', q) = e \left[q'^\lambda q'^\nu C(q'^2) - q^\lambda q^\nu C(q^2)\right] \ , \tag{18}$$

and of the entire dressed current,

$$k_\mu J^{\lambda\mu\nu}(q', q) = -g^{\lambda\nu} e \left[D(q'^2) - D(q^2)\right] + k_\mu J_{\text{gauge}}^{\lambda\mu\nu}(q', q) \ , \tag{19}$$

we indeed obtain the WTI (3) in terms of the fully dressed inverse propagator (14). The dressed current (16), therefore, is locally gauge invariant. Moreover, for a stable spin-1 particle, the physical on-shell matrix element of the four-divergence (19) vanishes, thus providing a conserved current.

All electromagnetic form factors appear here only in $J_1^{\lambda\mu\nu}$ in Eq. (16) in manifestly transverse contribution, as detailed in Eq. (10). Hence, the demand that these contributions should be well behaved and free of kinematic singularities carries over directly to the present case with full hadronic dressing. The conditions (12), therefore, are valid here as well, independent of the details of dressing effects.

III. CONCLUSION AND DISCUSSION

We may thus conclude that the relationship (1) linking the three multipole moments of an elementary spin-1 particle holds true in general and that it is model independent. While this correlation is trivially satisfied by the canonical moment values (i.e., $\mu = e/m$, $Q = -e/m^2$) discussed in the first paragraph of the Introduction, the relationship as such does not make any demand on individual values other than that they must be linked to satisfy (1). This correlation is important not just on general theoretical grounds because it reduces the multipole degrees of freedom, but since it imposes restrictions on approximations in model treatments of the moments, it will also allow for more realistic assessments of the reliability of various approaches. Other than possessing spin 1, the derivation makes no special demands on the nature of the particle as long as its description is based on an elementary field. It therefore applies to the $W^\pm$ gauge bosons of electroweak theory as well as to strongly interacting spin-1 particles like the ρ meson, etc.

However, the present considerations do *not* apply to hadronic spin-1 bound states like the deuteron because the requirement of the WTI (3) as a necessary and sufficient statement of local gauge invariance only applies to elementary particles. For hadronic bound states, in principle, their electromagnetic structures can be described microscopically in terms of how their (observable) hadronic constituents couple to the electromagnetic field. Comprehensive gauge-invariance considerations for hadronic spin-1 bound states like the deuteron, therefore, would need to consider also the possibility of asymptotically free constituent particles, including their final-state interactions. A somewhat simplified (incomplete) description along such lines can be found in Ref. [12]. Hence, utilizing the current (4) for the deuteron [12–14] provides an *effective* description of its electromagnetic spin-1 properties, with a conserved current because the four-divergence (8) vanishes on shell, however, there is no associated "deuteron propagator" to satisfy the WTI (3) independent of its effective electromagnetic properties.

Finally, we mention without further discussion that in the elementary-particle case, the respective expressions for the dressed propagator, Eq. (13), and the dressed current, Eq. (16), remain valid even if the spin-1 particle is a resonance, with nonzero width described by the imaginary part of the dressing function Σ. The mass m and the moments μ and Q then are parameters tied together by the normalizations (7), but they will not necessarily retain their usual physical meanings if the width is too large.

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