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Defining the Proton Radius: a Unified Treatment

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Background: There is significant current interest in knowing the value of the proton radius and also its proper definition.

Purpose: Combine the disparate literatures of hydrogen spectroscopy and diverse modern parton distributions to understand the meaning of the proton radius in a manner consistent with the separate bodies of work.

Methods: Use perturbation theory, light-front dynamics and elementary techniques to find relativistically correct definitions of the proton radius and charge density.

Results: It is found that the very same proton radius is accessed by measurements of hydrogen spectroscopy and elastic lepton scattering. The derivation of the mean-square radius as a moment of a spherically symmetric three-dimensional density is shown to be incorrect. A relativistically-correct, two-dimensional charge density is related to the diverse modern literature of various parton distributions. Relativistically invariant moments thereof are derived in a new relativistic moment expansion, the RME.

Conclusions: The equation $r_p^2 \equiv -6G'_E(0)$ is the definition of the proton radius.

I. INTRODUCTION

What is the value of the radius of the proton? This question has generated much interest since the publication of the results of the muon-hydrogen spectroscopy experiment in 2010 [1] and its confirmation [2]. The proton radius was measured to be $r_p = 0.84184(67)$ fm, which contrasted with the value obtained from electron-hydrogen spectroscopy $r_p = 0.8768(69)$ fm. At that time the large value was consistent with that obtained (with much larger uncertainties [3]) from electron scattering. This difference of about 4% has become known as the proton radius puzzle [4]. The proton radius puzzle was reviewed in 2013 [5] and 2015 [6]. New experimental results for hydrogen have appeared since that time [7, 8] without resolving the puzzle. More results are planned. The PRAD experiment [9] seeks to make previous electron scattering determinations of r_p more precise by making measurements at very small values of momentum transfer. A new measurement of the $2S_{1/2} - 2P_{1/2}$ transition in hydrogen is expected to appear (E. A. Hessels, 2016 talk at ECT*). All possible explanations of the proton radius puzzle will be addressed in the Muon-Scattering Experiment (MUSE) by measuring $e^{\pm} - p$ and $\mu^{\pm} - p$ scattering [10] .

One might wonder whether or not a 4% difference really matters. After all, 4% is pretty small and (at the present time) the value of r_p cannot be calculated to that accuracy. Perhaps the most interesting issue is whether or not the fundamental electron-proton interaction is the same as the muon-proton interaction. To find that this is not the case is to discover a violation of the principle of lepton-universality, a cornerstone of the standard model.

But there is another basic question that must be addressed: what **is** the radius of the proton? How does one define the radius of a quantum-field theoretic system made of nearly massless quarks and gluons? This quantity can be measured in the hydrogen atom and also in electron-proton scattering. There is a separate, but clear, literature in the fields of atomic and nuclear physics. Both fields obtain the same answer that

$$-6G'_E(0) \equiv r_p^2, \tag{1}$$

where $G_E(Q^2)$ is the Sachs electric form factor. This form factor is a specifically defined probability amplitude that an interaction between a photon of four-momentum q^μ ($Q^2 = -q^2$) and a charged constituent of the proton can absorb such a momentum with the proton remaining in its ground state. The meaning of Eq. (1) is that the quantity $-6G'_E(0)$ appears in both hydrogen spectroscopy and lepton-proton elastic scattering measurements. The quantity r_p^2 is merely an abbreviation. The expression Eq. (1) is uniquely used in spectroscopy and scattering experiments to determine the proton radius from the slope of G_E.

The aims of the present paper are:

- unite the hydrogen spectroscopy literature with that of lepton-proton scattering and show how Eq. (1) emerges from the separate bodies of work
- show that a three-dimensional charge density cannot be defined, so that r_p^2 is **not** a second moment of a density distribution.
- remind readers how it is that a two-dimensional charge density which is a matrix element of a density operator between identical initial and final states, can be defined and determined by the Dirac form factor, F_1
- place the two-dimensional charge density in the modern context of generalized parton distributions and Wigner functions
- derive a relativistically correct moment expansion of F_1.

The second item may be considered controversial by some. This is because the text-book interpretation [11–18] of G_E is that its Fourier transform is a three-dimensional charge density. This interpretation is deeply embedded in the thinking of nuclear and particle physicists and therefore continues to guide intuition, as it has since the days of the Nobel prize-winning work of Hofstadter [19–21]. Nevertheless, the relativistic motion

of the nearly massless fermionic constituents of the proton causes the text-book interpretation to be incorrect because relativistic invariance is ignored in defining the three-dimensional density.

The modern day literature regarding the measurable aspects of the proton, which is consistent with relativity, is much deeper than the understanding from 1956. The increasing availability of high energies and high luminosities at fixed target and collider experiments [22, 23] allows for unprecedented access to the internal transverse spatial and momentum distributions of charge distributions inside nucleons and in nuclei. The standard framework [24] is that of Wigner distributions [25] that allow simultaneous knowledge of both spatial and momentum aspects of the nucleon wave function. Knowledge of the Wigner distributions allows the construction of generalized parton distributions (GPDs) [26–38] and transverse momentum distributions (TMDs) [40–45] that are generalizations of the usual collinear parton distributions. The variables of the widely used relativistic formalism involve three dimensions– one is the longitudinal momentum of a parton and the other two involve either the transverse positions (GPD) or momenta (TMD). The longitudinal and transverse degrees of freedom are treated separately. This is necessary to maintain symmetries and sum rules provided by relativistic invariance. Electromagnetic form factors may be obtained by doing integrals over the longitudinal momentum coordinate of GPDs. These form factors must be described using the same variables as the other observables. Thus, only a two-dimensional charge density may be defined.

Let's outline the remainder of this paper. The appearance of the proton radius, r_p, in hydrogen spectroscopy is discussed in Sect.II. It explains how the key points related to extracting the value of the proton radius were already clearly explained in Refs [46–48]. It is nevertheless worthwhile to repeat, publicize this earlier discussion, try to re-emphasize the key points and strengthen the connection with treatments of lepton-proton scattering. Sect. III shows that the only existing derivation of a three-dimensional, spherically symmetric charge density is faulty. A properly defined relativistic three-dimensional charge density with modern formulations is discussed in Sect. IV. This quantity is intimately connected with modern formulations of the diverse set of possible parton distributions. The ensuing phenomenology is discussed in Sect. V in which a correctly defined moment expansion RME is derived. Some details are placed in Appendices.

II. HYDROGEN ATOM

This section is concerned with understanding the role of the proton radius in hydrogen spectroscopy. The starting point is to understand the leading relativistic corrections to the basic Dirac energy levels. The standard procedure is well-documented in Refs. [46, 47], and their discussion is used here. In the center of mass system the non-relativistic Hamiltonian for a system of a proton and a lepton (of mass m) with a Coulomb interaction is given by

$$H_0 = \frac{\vec{p}^2}{2m} + \frac{\vec{p}^2}{2M} - \frac{\alpha}{r}. \tag{2}$$

In a non-relativistic loosely bound system an expansion in powers of α^2 corresponds to an expansion in powers of v^2/c^2. To proceed one needs an effective Hamiltonian including terms of order v^2/c^2. Breit [49, 50] considered such a Hamiltonian, realizing that all corrections to the non-relativistic Hamiltonian of order v^2/c^2 may be obtained from the sum of the free relativistic Hamiltonians of each of the particles along with relativistic one-photon exchange between the fermions. An explicit expression for the resulting Breit potential was derived [51] from the one-photon exchange amplitude using the Foldy-Wouthhuysen transformation. If hyperfine effects are ignored, the result to order v^2/c^2 is given by

$$V_{\rm Breit} = \frac{\pi\alpha}{2}\left(\frac{1}{m^2}+\frac{1}{M^2}\right)\delta(\vec{r}) - \frac{\alpha}{2mMr}\left(\vec{p}^2 + \frac{\vec{r}(\vec{r}\cdot\vec{p})\cdot\vec{p}}{r^2}\right) + \frac{\alpha}{r^3}\left(\frac{1}{4m^2}+\frac{1}{2mM}\right)[\vec{r}\times\vec{p}]\cdot\vec{\sigma}. \tag{3}$$

All contributions to the energy levels up to order α^4 may be calculated from the total Hamiltonian $H_0+V_{\rm Breit}$. The corrections of order α^4 are the first-order matrix elements of the Breit interaction between the Coulomb-Schroedinger eigenfunctions of H_0. The result is

$$E_{nj} = m + M - \frac{m_r\alpha^2}{2n^2} - \frac{m_r\alpha^4}{2n^3}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n} + \frac{m_r}{4n(m+M)}\right) + \frac{\alpha^4 m_r^3}{2n^3M^2}\left(\frac{1}{j+\frac{1}{2}} - \frac{1}{l+\frac{1}{2}}\right)(1-\delta_{l0}), \tag{4}$$

where the reduced mass $m_r = mM/(m+M)$ and j in the total angular momentum quantum number of the lepton. Note the presence of the last term of Eq. (4) which removes the degeneracy of the Dirac spectrum between levels with the same j and $l = j \pm \frac{1}{2}$.

The expressions Eq. (3) and Eq. (4) are obtained assuming the proton is a point-like proton. Electromagnetic form factors are introduced to include the effects of its non-zero spatial extent. The photon-proton vertex operator Γ^μ is given by

$$\Gamma^\mu = \gamma^\mu F_1(Q^2) + i\frac{\sigma^{\mu\nu}}{2M}\kappa F_2(Q^2), \tag{5}$$

where $Q^2 > 0$ is the negative of the square of the virtual space-like photon momentum, M is the proton mass, F_1 is the Dirac form factor, F_2 is the Pauli form factor and κ is the proton anomalous magnetic moment. It is useful to define the Sachs form factors:

$$G_E(Q^2) = F_1(Q^2) - \tau\kappa F_2,\; G_M(Q^2) = F_1(Q^2) + \kappa F_2(Q^2), \tag{6}$$

where $\tau \equiv \frac{Q^2}{4M^2}$. With this notation $F_{1,2}(0) = 1$.

The photon-electron vertex function in a hydrogen-like atom is given [46, 47] as the matrix element: $\bar{u}(\vec{p}', s')\Gamma^\mu u(\vec{p}, s)$ with spinors normalized as $u^\dagger u = 1$. Calculation, see *eg.* [46, 47, 52], to order $1/M^2$ (and ignoring a spin-orbit term) reveals that

$$\bar{u}(\vec{p}', s')\Gamma^0 u(\vec{p}, s) = (1 - \frac{\vec{q}^2}{8M^2})G_E(\vec{q}^2) \tag{7}$$

This key equation is derived in Appendix A. The left-hand side of Eq. (7) is the time component of a four-vector. The right-hand-side does not depend on $\vec{p}$; it is frame-independent, and may be used to identify the leading proton-radius effect.

One uses the lowest order Taylor expansion, keeping only the spin-independent term, to extract the proton radius from the measured energy levels. Thus one writes:

$$G_E(\vec{q}^2) = 1 + \vec{q}^2 G'_E(0). \tag{8}$$

The difference between Q^2 and $\vec{q}^2$, $-q_0^2$, is of order $\alpha^2 m^2/M^2$ and is a higher order correction, and any correction arising from a $G''_E(0)$ term is completely negligible [53]. One then finds (to order $\vec{q}^2$) that

$$\bar{u}(\vec{p}', s')\Gamma^0 u(\vec{p}, s) \to 1 - \frac{\vec{q}^2}{8M^2} + \vec{q}^2 G'_E(0). \tag{9}$$

The term $\frac{\vec{q}^2}{8M^2}$ leads to the Darwin term in the lepton-proton interaction [51] that provides the term proportional to δ_{l0} in Eq. (4), so it is already included. The net result is that the effect of the proton size is given simply by including the term $\vec{q}^2 G'_E(0)$ in the lepton-proton vertex function. This procedure avoids defining a radius as a moment of a spherically-symmetric, three-dimensional charge density.

Keeping the non-zero size of the proton leads to the momentum space version of the Coulomb potential:

$$V_C(\vec{q}^2) = -\frac{4\pi\alpha}{\vec{q}^2}(1 + \vec{q}^2 G'_E(0)) = -4\pi\alpha(\frac{1}{\vec{q}^2} + G'_E(0)) \tag{10}$$

The coordinate space potential $V_C(r)$ is given by the three-dimensional Fourier transform:

$$V_C(r) = -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} 4\pi\alpha(\frac{1}{\vec{q}^2} + G'_E(0)) = -\frac{\alpha}{r} - 4\pi\alpha G'_E(0)\delta(\vec{r}). \tag{11}$$

Since G_E falls with increasing $\vec{q}^2$ one finds a repulsive correction to the Coulomb potential, ΔV_C, given by

$$\Delta V_C(\vec{r}) = -4\pi\alpha G'_E(0)\delta(\vec{r}). \tag{12}$$

The typical value of $\vec{q}^2$ is of the order of the square of the inverse of the Bohr radius of the atom. The muonic hydrogen atom Bohr radius is about 200 times smaller than that for the electronic one. This huge difference does not influence the potential ΔV_C because of the cancellation of the factor $\vec{q}^2$ by its inverse that arises from the photon propagator. The net result is the delta function appearing in Eq. (12). The difference between Bohr radii would enter if one included the $\vec{q}^4$ term in the Taylor expansion of G_E, but such terms are smaller by the ratio of the proton size to the Bohr radius [53].

The shift in the energy, ΔE, is given by the matrix element:

$$\Delta E = \langle \psi_{nl} | \Delta V_C | \psi_{nl} \rangle = -4\pi\alpha G'_E(0) |\psi_{n0}(0)|^2 \delta_{l0}. \tag{13}$$

The net result is that the energy shift in the hydrogen atom is determined by the slope of G_E at its origin. The effect, Eq. (13), is of order α^4 because $|\psi_{n0}(0)|^2$ is of order α^3, so that this term should be included with the others of order α^4 displayed in Eq. (4). Effects of higher order in α are not considered here.

For historical reasons, to be discussed in the next Section, the slope is redefined as

$$G'_E(0) = -\frac{r_p^2}{6}, \tag{14}$$

with

$$G_E(Q^2) = 1 - \frac{r_p^2}{6}\vec{q}^2 \tag{15}$$

for sufficiently small values of $\vec{q}^2$. This means that one may also write

$$\Delta E = 4\pi\alpha \frac{r_p^2}{6} |\psi_{n0}(0)|^2 \delta_{l0}, \tag{16}$$

as is often done.

The reader may wonder why the three-dimensional Fourier transform is allowed in proceeding from Eq. (10) to Eq. (11), but not in proceeding from G_E to a spherically-symmetric, three-dimensional charge density. Eq. (10) and Eq. (11), are used when the degrees of freedom are leptons and protons. For that case, the situation is basically non-relativistic and there is a well-defined scheme [46, 47] to compute the relativistic corrections. The non-zero proton size is a correction on the order of the terms in the Breit interaction, Eq. (4). On the other hand G_E involves photons that hit the light up and down quarks. These always move relativistically. For systems made of such particles, the internal wave functions of the initial and final states are different, as discussed in detail in the next Section. No matter how small the non-zero momentum transfer is, the initial and final proton states differ in a way that involves an exponential of the full QCD Hamiltonian.

III. LEPTON-PROTON SCATTERING

The electron-proton elastic scattering cross section, obtained under the assumption that the lepton-proton interaction is mediated by a single photon (and neglecting the electron mass) is most simply expressed in terms of G_E, G_M [55]:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_M \times [G_E^2 + \frac{\tau}{\epsilon} G_M^2] \frac{1}{1+\tau}, \tag{17}$$

where $\left(\frac{d\sigma}{d\Omega}\right)_M$ is the Mott cross-section in which the proton is treated as point-like ($F_{1,2}(Q^2) = 1$), and ϵ is a kinematic factor. Including the non-zero value of the muon mass leads to a slightly more complicated expression [54].

Elastic electron-proton scattering experiments were first performed at Stanford by Hofstadter and collaborators, and summarized in Ref. [20]. This early work, which famously [21] discovered that the proton was not a point-particle, assumed that $F_1 = F_2 \equiv F$. Their analysis used the equations:

$$\lim_{Q_L^2 \to 0} F(Q_L^2) = 1 - \frac{Q_L^2 a^2}{6} + \cdots , \tag{18}$$

in which a was associated with the physical extent of the proton and Q_L is the *laboratory value* of the magnitude of the three-momentum transfer. It is further asserted that

$$F(Q_L^2) = \int_0^\infty \rho(\mathbf{r}) e^{i\vec{Q}_L \cdot \mathbf{r}} d^3r, \tag{19}$$

where The authors are very careful about using this expression. They state that Eq. (19) *"applies in the non-relativistic limit in which $\rho(\mathbf{r})$ is the static density distribution"*. This definition is frame-dependent and so violates the principle of relativity.

Expansion of the exponential appearing in Eq. (19) leads to the well-known moment expansion:

$$F(Q_L^2) = 1 - \frac{Q_L^2}{6}\langle r^2 \rangle + \frac{Q_L^2}{120}\langle r^4 \rangle + \cdots \tag{20}$$

$$\langle r^n \rangle = \int r^n \rho(\mathbf{r}) d^3r. \tag{21}$$

The validity of this expansion depends upon the validity of the non-relativistic limit. But there is **no** reason to believe that any non-relativistic treatment is valid for treating the proton form factor because elastic electron-proton scattering proceeds mainly via the absorption of a virtual photon by a nearly massless up or down quark.

Sachs *et. al.* [56, 57] introduced the so-called Breit frame in which $q^0 = 0$ so that here (and in all following equations) Q^2 is the Lorentz scalar quantity, $Q^2 = -q^2$, and q^μ is the four-momentum of the single-photon mediator. Sachs [57] argued that in this frame the charge density is given by the Fourier transform of G_E. The resulting non-relativistic (NR) density as $\rho_{\rm NR}(r)$ is defined by the equation:

$$\rho_{\rm NR}(r) \equiv \int \frac{d^3Q}{(2\pi)^3} e^{-i\mathbf{Q}\cdot\mathbf{r}} G_E(Q^2). \tag{22}$$

The next step is to show that the definition Eq. (22) (despite its wide use) has no connection with well-defined matrix elements of quantum field theory. Equation (22) can be used to obtain the results that

$$\langle r^2 \rangle_{\rm NR} \equiv \int d^3r r^2 \rho_{\rm NR}(r) = -6\, G_E'(0) \tag{23}$$

$$\langle r^4 \rangle_{\rm NR} \equiv \int d^3r r^4 \rho_{\rm NR}(r) = 60\, G_E''(0). \tag{24}$$

Eq. (23) has often been used to analyze the charge distribution of the neutron n. The result is that the mean square charge radius, $\langle r_n^2 \rangle_{\rm NR}$, is almost completely accounted for numerically by the anomalous magnetic term, $3\kappa_n/(2M^2)$, arising from the F_2 contribution to G_E [17]. This seems very strange and looks like a puzzle. The puzzle dissipears if one realizes that the slope of G_E is not related to the expectation value of r^2 in a spherically symmetric charge density. The pion form factor is another example of oddity. The 3-dimensional Fourier transform of the monopole form factor (that approximates the data) is singular at the origin [58].

There is a more technical way of explaining why the non-relativistic approach fails. The difficulty comes because proton wave functions of differing momentum are different. These functions are related by a boost,

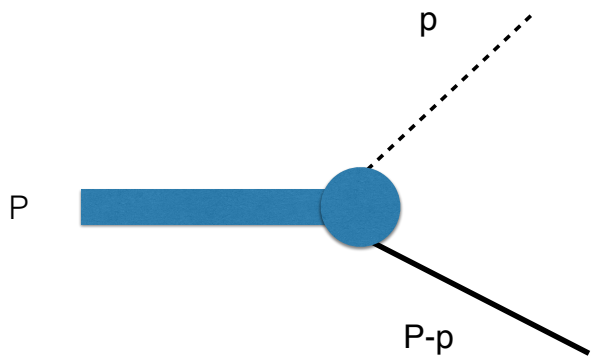

FIG. 1. [Color online] Bethe-Salpeter wave function. A hadron of momentum P fluctuates into a constituent of four-momentum p and another of momentum $P-p$

which in general contains an exponential of an operator as complicated as the strong interaction Hamiltonian [59, 60]. To see this, consider a very simple example in which a scalar proton of mass M is made of a scalar quark (of mass m_q and a scalar di-quark (of mass M_S). In this case the Bethe-Salpeter wave function is given by

$$\psi_{\rm B.S.}(P,p) = \frac{1}{p^2-m_q^2+i\epsilon}\frac{1}{(P-p)^2-M_S^2+i\epsilon} = \frac{1}{p^2-m_q^2+i\epsilon}\frac{1}{M^2-2P\cdot p+p^2-M_S^2+i\epsilon}, \tag{25}$$

in which various constants have not been displayed. This wave function is shown in Fig. 1.

Boosting the wave function in this simple model is achieved merely by changing the value of the four-momentum P. In lepton-proton elastic scattering the four momentum of the initial proton is P and that of the final one is $P'=P+q$. The initial-state wave function depends upon $P\cdot p$, which in the laboratory frame is given by Mp^0. The final-state wave function depends on $P'\cdot p=(P+q)\cdot p=(M+q^0)p^0-\mathbf{q}\cdot\mathbf{p}$. Thus electron scattering involves two different wave functions. A density requires the appearance of the square of a wave function, which does not appear here.

A. Breit Frame Falibility

The Breit frame (introduced in Ref. [56]) is the one in which the three-momentum of the initial proton, $\vec{P}$, is -1/2 that of the incident virtual photon, $\vec{q}$, $\vec{P}=-\vec{q}/2$. The final proton momentum $\vec{P'}=\vec{P}+\vec{q}=\vec{q}/2$, and the initial and final protons have the same energy. However, the initial and final wave functions are different because the quantities $P\cdot p$ and $P'\cdot p$ must appear in any relativistic wave function and

$$P\cdot p=\sqrt{M^2+\mathbf{q}^2/4}p^0+\mathbf{q}\cdot\mathbf{p}/2,\ P'\cdot p=\sqrt{M^2+\mathbf{q}^2/4}p^0-\mathbf{q}\cdot\mathbf{p}/2 \tag{26}$$

These differ, so that again one is not dealing with the square of a given wave function.

Ref. [56] shows that the matrix element of the time component of Γ^μ, Γ^0 is proportional to G_E if the helicity is changed (spin direction is not changed). This argument is also given in *e.g.* [15]. In that reference a quantity ρ is *defined* as the matrix element of Γ^0:

$$\rho\equiv\bar{u}(\vec{q}/2,s_z)\Gamma^0u(-\vec{q}/2,s_z)=G_E(Q^2), \tag{27}$$

with the normalization again being $u^\dagger u=1$, and where the direction of $\vec{q}$ defines the z-axis. The equality follows from evaluating the spinor matrix element. However, *no spatial dependence is implied by this definition*

of ρ.

To the best of my knowledge the derivation of a relationship between G_E and a three-dimensional charge density is attempted only in Appendix II of Ref. [57]. In that formalism the initial and final states are "*brought to rest*" through the use of narrow wave packets. This is an attempt to avoid the previously mentioned problem associated with the boosts.

Although Ref. [57] made a strong attempt to derive the charge density as a three-dimensional Fourier transform of G_E, the derivation is simply wrong. To show this, I redo the calculation of Appendix II of that paper, avoiding an incorrect assumption used there.

Sachs writes the proton wave packet state as

$$|\Psi\rangle = \int d^3P g(\vec{P})|P,s\rangle, \tag{28}$$

with $g(\vec{P})$ representing a narrow wave packet. The function $g(\vec{P})$ is sufficiently narrow so that

$$|g(\vec{P})|^2 \to \delta(\vec{P}). \tag{29}$$

It is worthwhile to use an explicit representation. Using a Gaussian

$$g(\vec{P}) = \frac{R^{3/2}}{\pi^{3/4}} \exp\left[-\vec{P}^2R^2/2\right], \tag{30}$$

where R is assumed to be infinite, as a first choice is convenient. The meaning of the expression Eq. (30) is the standard one of a distribution in which the first step is to do all of the relevant integrals, keeping R finite. Then, with the values of the integrals in hand take R to infinity [61, 62].

Sachs proceeded by computing moments of the putative charge distribution. Only the time-component, and a quadratic moment is relevant to display the so-called charge density. In this case:

$$M^{(2)} \equiv \int d^3q \int d^3p\, g^*(\vec{p}+\vec{q}/2)g(\vec{p}-\vec{q}/2) \int d^3x\, x^2 \langle P',\lambda|j^0(x)|\vec{P},\lambda\rangle, \tag{31}$$

with the integration variables chosen as $\vec{p}$ and $\vec{q}$ with $\vec{P'} = \vec{p}+\vec{q}/2, \vec{P} = \vec{p}-\vec{q}/2$, and $x^2 \equiv \sum_i x_i^2$ with x_i as the Cartesian component of the three vector $\vec{x}$.

The matrix element of the time-component of the current is given by

$$\langle P'|j^0(x)|P\rangle = (2\pi)^{-3} e^{iq\cdot x}\bar{u}(\vec{P'},\lambda)\Gamma^0 u(\vec{P},\lambda), \tag{32}$$

which (after replacing x^2 by $-\nabla_q^2$, integration over $\vec{x}$, and integration by parts) leads to the expression:

$$M^{(2)} = -\int d^3q\delta(\vec{q}){\nabla_q}^2 \int d^3p\, g^*(\vec{p}+\vec{q}/2)g(\vec{p}-\vec{q}/2)\bar{u}(\vec{p}+\vec{q}/2,\lambda)\Gamma^0 u(\vec{p}-\vec{q}/2,\lambda)e^{iq^0t}. \tag{33}$$

At this stage Sachs made the replacement $g^*(\vec{p}+\vec{q}/2)g(\vec{p}-\vec{q}/2) \to |g(p)|^2$. The justification is that "*terms resulting from the shape of the wave packet are not of interest here and are therefore dropped.*". However, this is not a correct justification for the replacement because (as a δ function) the quantity $|g(P)|^2$ must vary rapidly. One needs to be careful about the derivatives. To see this, use the specific form of Eq. (30) in Eq. (33). Then

$$M^{(2)} = -\lim_{R\to\infty} \int d^3q\delta(\vec{q})\nabla_q^2 \int d^3p \tfrac{R^3}{\pi^{3/2}} \exp\left(-\vec{p}^2R^2 - \vec{q}^2R^2/4\right)\bar{u}(\vec{p}+\vec{q}/2,\lambda)\Gamma^0 u(\vec{p}-\vec{q}/2,\lambda)e^{iq^0t}. \tag{34}$$

The term $\frac{R^3}{\pi^{3/2}} \exp\left(-\vec{p}^2R^2\right)$ leads to a delta function setting $\vec{p}=0$, so that $q^0=0$ and the Breit-frame result, Eq. (27), for the matrix element of Γ^0 may be used. Then:

$$M_i^{(2)} = -\lim_{R\to\infty} \int d^3q\delta(\vec{q})\nabla_q^2\, e^{-\vec{q}^2R^2/4}G_E(\vec{q}^2) \tag{35}$$

$$= -\lim_{R\to\infty} \int d^3q\delta(\vec{q})\left(-R^2/2G_E(\vec{q}^2)) + \nabla_q^2 G_E(\vec{q}^2)\right) \tag{36}$$

$$= \lim_{R\to\infty}(R^2/2 - {\nabla_q}^2 G_E(\vec{q}^2))|_{\vec{q}^2=0} = \infty. \tag{37}$$

This means that the quadratic moment is actually infinite! This moment expansion fails. The underlying reason is that ∇_q^2 must involve the square of some distance and the infinite parameter R^2 must appear in addition to any length scales in the proton.

If one asserts that $G_E(\vec{q}^2) = \int d^3x e^{-i\vec{q}\cdot\vec{x}}\rho_{\rm NR}|(|\vec{x}|)$, then one finds

$$-\nabla_q{}^2 G_E(\vec{q}^2))|_{\vec{q}^2=0} = \int d^3x\, x^2 \rho_{\rm NR}(|\vec{x}|), \tag{38}$$

which looks like the expressions of the usual literature. However, this term (which does appear in Eq. (37)) is overwhelmed by the infinite term that also appears. Note that the infinite result does not rely on using the specific Gaussian form of Eq. (30). It would occur with any specific representation of a delta function, as shown in Appendix B. Thus the derivation of Sachs is fatally flawed.[1]

One may understand the failure of the Sachs procedure in simple terms. The wave function $|\Psi\rangle$ is meant to represent a proton of 0 three-momentum, so that (via the uncertainty principle) its position is totally undetermined. This is the origin of the infinite result. Another procedure would be to use Eq. (30) in the opposite limit that R is very, very small. This would lead to a wave packet that is concentrated in a narrow region of space, taken as the origin. However, the use of such a wave packet in Eq. (31) would not allow the use of the Breit frame result because the integral over $\vec{p}$ would go over all of its values.

The net result of all of this is that the relation between $G_E'(0)$ and r_p^2, Eq. (1), is merely a definition.

IV. TRUE CHARGE DENSITY

A proper determination of a charge density requires the measurement of a matrix element of a density operator taken between initial and final states that are the same. The aim here is to show that the proton form factor, F_1 is a specific integral of the three-dimensional charge density of partons in the infinite momentum frame, $\hat{\rho}_\infty(x^-, \mathbf{b})$.

It is necessary to provide a brief introduction to light- front coordinates. Instead of the usual $x^0 = ct, x^3 = z$, the light front approach uses $x^\pm = (x^- \pm x^3)/\sqrt{2}$. By convention the term x^+ corresponds to the time and x^- corresponds to the longitudinal distance coordinate.

In the infinite momentum frame, IMF, the time coordinate $ct = x^0/\sqrt{2}$ is expressed in a frame moving along the negative z direction with a velocity nearly that of light using the Lorentz transformation as the variable $x^+ = (x^0 + x^3)/\sqrt{2}$, with the usual γ factor absorbed by a Jacobean of an integral over volume [63]. The x^+ variable is canonically conjugate to the minus-component of the momentum operator $p^- \equiv (p^0 - p^3)/\sqrt{2}$. The longitudinal spatial variable is $x^- = (x^0 - x^3)/\sqrt{2}$ and its conjugate momentum is $p^+ = (p^0 + p^3)/\sqrt{2}$. It is this plus-component of momentum that is associated with the usual Bjkoren variable. The transverse coordinates x, y are written as $\mathbf{b}$ with the conjugate momentum denoted $\mathbf{p}$. Boldface is used here to denote the two-dimensional transverse components of position and momentum vectors to distinguish these from the three-dimensional vectors (*e.g.* $\vec{q}$) of previous sections.

Light-front time-quantization, which sets x^+ and the plus-component of all spatial variables to zero, is used. This means that x^- can be thought of as the longitudinal variable $-\sqrt{2}x^3$. One extremely useful aspect of using these variables is that Lorentz transformations to frames moving with different transverse velocities do not depend on interactions. These transformations form the kinematic subgroup of the Poincaré group, so that boosts in the transverse direction are accomplished as in the non-relativistic theory; the dependence on

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[1] The evaluation of Eq. (34) proceeded by first obtaining $\delta(\vec{p})$ and then handling the dependence of $\vec{q}$. The same result, Eq. (37) is obtained if one first differentiates with respect to $\vec{q}$.

the total transverse momentum of any system appears only as an overall phase factor.

This language may seem a bit abstract. All it means the wave function of a proton with a given $(p^+, \mathbf{p})$ is related to the one of momentum $(p^+, \mathbf{0})$ by a factor that is independent of the relative momenta of the partons that make up the wave function. The necessary integrations to compute form factors (in a frame in which $Q^2 = \mathbf{q}^2$) only involve the relative variables that appear in light-front wave functions. Examples can be found in [64–66].

The density that is relevant here has been known for a long time [67] and often been exploited [58, 64, 68]. In the IMF, the electromagnetic charge density J^0 operator becomes J^+ and

$$\hat{\rho}_\infty(x^-, \mathbf{b}) = J^+(x^-, \mathbf{b}) = \sum_q e_q \bar{q}(x^-, \mathbf{b})\gamma^+ q(x^-, \mathbf{b}) = \sum_q e_q \sqrt{2} q_+^\dagger(x^-, \mathbf{b}) q_+(x^-, \mathbf{b}), \tag{39}$$

where $q_+(x^\mu) = \gamma^0\gamma^+/\sqrt{2}q(x^\mu)$, the independent part of the quark-field operator $q(x^\mu)$. The time variable, x^+ is set to zero. Note the appearance of the absolute square of quark field-operators, which is the signature of a true density operator. An analogous expression is widely used to describe color charge densities [69, 70].

The purpose of this section is to show how matrix elements of $\hat{\rho}_\infty(x^-, \mathbf{b})$ (which are true densities) emerge from modern quantum field theory treatments of nucleon structure. The vast literature concerning the diverse set of functions that are used to describe nucleon structure includes generalized parton distributions GPDs [26, 27, 30–37, 71, 72] transverse momentum distributions (TMDs) [40, 41, 73–77] and, more recently generalized transverse momentum distributions (GTMDs) [78, 79].

Generalized parton distributions are of high current interest because they can be related to the total angular momentum carried by quarks in the nucleon and can be determined using deeply virtual Compton scattering experiments [27]. The opportunity of determining all of these is greatly enhanced by the possible creation of an electron-ion collider [80].

These distributions are specific matrix elements of quark-field operators, between nucleon states, which in contrast to the usual quark distribution functions, do not necessarily have the same momenta. The specific case in which the longitudinal momentum transfer vanishes, and the initial and final states have the same helicity $\lambda' = \lambda$ is relevant in the present context. Then, in the light-cone gauge, $A^+ = 0$, the matrix element defining the GPD, H_q for a quark of flavor q is

$$H_q(x,t) = \int \frac{dx^-}{4\pi} \langle p^+, \mathbf{p}', \lambda | \bar{q}(-\frac{x^-}{2}, \mathbf{0})\gamma^+ q(\frac{x^-}{2}, \mathbf{0}) | p^+, \mathbf{p}, \lambda \rangle e^{ixp^+x^-}. \tag{40}$$

where the normalization is $\langle p'^+, \mathbf{p}', \lambda | p^+, \mathbf{p}, \lambda \rangle = 2p^+(2\pi)^3\delta(p'^+ - p^+)\delta^{(2)}(\mathbf{p}' - \mathbf{p})$. The variable λ denotes the helicity, and only the helicity non-flip term needed to compute F_1 appear here. The four-momentum transfer $q_\alpha = p'_\alpha - p_\alpha$ is space-like, with the square of the space-like four-momentum transfer $q^2 = -Q^2$ and use the Drell-Yan (DY) frame with $(q^+ = 0, Q^2 = \mathbf{q}^2)$. No longitudinal momentum is transferred, so that initial and final states are related only by kinematic transformations. Moreover, the current operator links Fock-state components with the same number of constituents. The abbreviation $-t = -(p'-p)^2 = (\mathbf{p}'-\mathbf{p})^2 = -q^2 = Q^2$ is used. The presence of the operator γ^+ insures that independent field operators appear in the matrix element.

GPDs allow for a unified description of a number of hadronic properties [28]. Notice that if $t = 0$ they reduce to conventional PDFs $H_q(x, 0) = q(x)$, and, of most relevance here, that the integration of H_q over x yields the nucleon electromagnetic form factor:

$$F_1(t) = \sum_q e_q \int dx H_q(x,t), \tag{41}$$

with the defining equation

$$F_1(Q^2) = \frac{\langle p'^+, \mathbf{p}', \lambda | J^+(0) | p^+, \mathbf{p}, \lambda \rangle}{2p^+}. \tag{42}$$

The spatial structure of a nucleon can be examined if one uses the fact that transverse boosts are independent of interactions in the infinite momentum frame [81, 82] to define [38, 39, 67] nucleonic states that are transversely localized. The state with transverse center of mass $\mathbf{R}$ set to 0 is formed by taking a linear superposition of states of transverse momentum. In particular,

$$\left|p^+, \mathbf{R}=\mathbf{0}, \lambda\right\rangle \equiv \mathcal{N} \int \frac{d^2\mathbf{p}}{(2\pi)^2\sqrt{2p^+}} \left|p^+, \mathbf{p}, \lambda\right\rangle, \tag{43}$$

where $|p^+, \mathbf{p}, \lambda\rangle$ are light-cone helicity eigenstates [67] and $\mathcal{N}$ is a normalization factor satisfying $|\mathcal{N}|^2 \int \frac{d^2\mathbf{p}_\perp}{(2\pi)^2} = 1$. References [83, 84] use wave packet treatments that avoid states normalized to δ functions, but this leads to the same results as using Eq. (43). Note however, the relevant range of integration in Eq. (43) must be restricted to $|\mathbf{p}| \ll p^+$ to maintain the interpretation of a nucleon moving with well-defined longitudinal momentum [83]. Thus the infinite momentum frame, with p^+ as the large momentum, is used. This is a frame in which the interpretation of a nucleon as a set of a large number of partons is valid.

Using Eq. (43) sets the transverse center of momentum of a state of total very large momentum p^+ to zero, so that a transverse distance $\mathbf{b}$ relative to $\mathbf{R}$ can be defined. To use this feature generalize the quark-field operator appearing in Eq. (40) by making a translation:

$$\hat{O}_q(x, \mathbf{b}) \equiv \int \frac{dx^-}{4\pi} q_+^\dagger \left(-\frac{x^-}{2}, \mathbf{b}\right) q_+ \left(\frac{x^-}{2}, \mathbf{b}\right) e^{ixp^+x^-}. \tag{44}$$

The impact parameter dependent PDF is defined [83] as the matrix element of this operator in the state of Eq. (43):

$$q(x, \mathbf{b}) \equiv \left\langle p^+, \mathbf{R}=\mathbf{0}, \lambda\right| \hat{O}_q(x, \mathbf{b}) \left|p^+, \mathbf{R}=\mathbf{0}, \lambda\right\rangle. \tag{45}$$

The use of Eq. (43) in Eq. (45) allows one to show that $q(x, \mathbf{b})$ is the two-dimensional Fourier transform of the GPD H_q:

$$q(x, \mathbf{b}) = \int \frac{d^2q}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} H_q(x, t=-\mathbf{q}^2), \tag{46}$$

with H_q appearing because the initial and final helicities are each λ. A complete determination of $H_q(x,t)$ (with $t \leq 0$) would determine $q(x, \mathbf{b})$.

One finds a probability interpretation [67] by integrating $q(x, \mathbf{b})$ over all values of x. This sets the differences in longitudinal distances, appearing in Eq. (44), to 0. Then the use of translational invariance leads to the result

$$\int dx\, q(x, \mathbf{b}) = \left\langle p^+, \mathbf{R}=\mathbf{0}, \lambda\right| q_+^\dagger(x^-, \mathbf{b}) q_+(x^-, \mathbf{b}) \left|p^+, \mathbf{R}=\mathbf{0}, \lambda\right\rangle. \tag{47}$$

This equation shows that the matrix element of a true density operator (square of a quark-field operator) taken between identical initial and final states is experimentally accessible.

Furthermore, multiplying Eq. (47) by the quark charge e_q (in units of e), sums over quark flavors, uses Eq. (43) with $\hat{O}_q(x, \mathbf{b}) = e^{-i\hat{\mathbf{p}}\cdot\mathbf{b}} \hat{O}_q(x, \mathbf{0}) e^{i\hat{\mathbf{p}}\cdot\mathbf{b}}$ along with Eq. (41), the resulting infinite-momentum-frame IMF parton charge density in transverse space is

$$\rho(b) \equiv \sum_q e_q \int dx\, q(x, \mathbf{b}) = \int \frac{d^2q}{(2\pi)^2} F_1(Q^2=\mathbf{q}^2) e^{-i\,\mathbf{q}\cdot\mathbf{b}}. \tag{48}$$

This relation shows that a properly-defined charge density, the transverse charge density, is obtained using the same formulations that is used to define generalized densities.

A. Wigner distributions

There is now a broader perspective involving a diverse set of distributions that can be used to characterize nucleon structure [79]. This subsection is intended to place the transverse density in the context of Wigner distributions. Wigner distributions in QCD were first explored in Refs. [24, 85]. Neglecting relativistic effects, those authors used the standard three-dimensional Fourier transform in the Breit frame and introduced six-dimensional Wigner distributions (three position and three momentum coordinates). The modern perspective involves instead five-dimensional Wigner distributions (two position and three momentum coordinates) as seen from the infinite momentum frame (IMF). These three momentum variables of a quark are $k^+, \mathbf{k}$, so there is no spherically-symmetric charge density. These light front variables were exploited [79] to arrive at a definition of Wigner distributions that is completely consistent with relativity.

The first step is to use Wigner operators for quarks of flavor q at a fixed light-cone time $y^+ = 0$:

$$\widehat{W}_q(\mathbf{b}, \mathbf{k}, x) \equiv \frac{1}{2} \int \frac{\mathrm{d}z^- \, \mathrm{d}^2 z}{(2\pi)^3} \, e^{i(xp^+ z^- - \mathbf{k}\cdot\mathbf{z})} \, \overline{q}(y - \tfrac{z}{2}) \gamma^+ \mathcal{W} \, q(y + \tfrac{z}{2})\big|_{z^+ = 0}, \tag{49}$$

with $y^\mu = [0, 0, \mathbf{b}]$, p^+ is the average of the initial and final nucleon longitudinal momentum and $x = k^+/p^+$ is the average fraction of nucleon longitudinal momentum carried by the struck quark. The above equation is a specific Wigner operator that involves γ^+ that is relevant here. More generally one could use any twist-two Dirac operator $\Gamma = \gamma^+, \gamma^+\gamma_5, i\sigma^{j+}\gamma_5$ with $j = 1, 2$. A Wilson line, $\mathcal{W}$, ensures the color gauge invariance of the Wigner operator by connecting the points $(y - \frac{z}{2})$ and $(y + \frac{z}{2})$ see *e.g.* [86].

Wigner distributions are defined as matrix elements of the Wigner operators sandwiched between nucleon states with polarization $\vec{S}$:

$$\rho_q(\mathbf{b}, \mathbf{k}, x, \lambda) \equiv \int \frac{\mathrm{d}^2\Delta}{(2\pi)^2} \, \langle p^+, \tfrac{\mathbf{\Delta}}{2}, \vec{S} | \widehat{W}_q(\mathbf{b}, \mathbf{k}, x) | p^+, -\tfrac{\mathbf{\Delta}}{2}, \vec{S} \rangle. \tag{50}$$

Ref. [79] shows how four different three-dimensional densities can be defined. The task here is to connect the transverse density, $\rho(b)$, of Eq. (48) with the Wigner distribution defined above. This is done by integrating the quantity $\rho_q(\mathbf{b}, \mathbf{k}, x, \lambda)$ over all values of x and $\mathbf{k}$. This sets z^- and $\mathbf{z}$ to 0, so that the Wilson line becomes unity, and the result is

$$\int dx \, d^2k \langle p^+, \tfrac{\mathbf{\Delta}}{2}, \lambda | \widehat{W}_q(\mathbf{b}, \mathbf{k}, x) | p^+, -\tfrac{\mathbf{\Delta}}{2}, \lambda \rangle = \frac{1}{2p^+} \langle p^+, \tfrac{\mathbf{\Delta}}{2}, \lambda | \overline{q}(x^- = 0, \mathbf{b}) \, \gamma^+ \, q(x^- = 0, \mathbf{b}) | p^+, -\tfrac{\mathbf{\Delta}}{2}, \lambda \rangle, \tag{51}$$

where the polarization vector $\vec{S}$ has been set to the light front helicity λ. Using translational invariance in the transverse direction and Eq. (50) shows that

$$\int dx \, d^2k \rho_q(\mathbf{b}, \mathbf{k}, x, \lambda) = \int \frac{d^2\Delta}{(2\pi)^2} \frac{e^{-i\mathbf{\Delta}\cdot\mathbf{b}}}{2p^+} \langle p^+, \tfrac{\mathbf{\Delta}}{2}, \lambda | \overline{q}(0) \, \gamma^+ \, q(0) | p^+, -\tfrac{\mathbf{\Delta}}{2}, \lambda \rangle, \tag{52}$$

so that the charge density operator appears. Multiplying this expression by e_q, summing over quark flavors q and using Eq. (42) and Eq. (48) shows that

$$\sum_q e_q \int dx \, d^2k \rho_q(\mathbf{b}, \mathbf{k}, x, \lambda) = \rho(b). \tag{53}$$

This means that the transverse charge density exhibits a specific aspect of quark Wigner distributions. Observe that proton electromagnetic form factors occupy a small, but important, corner of a vast field.

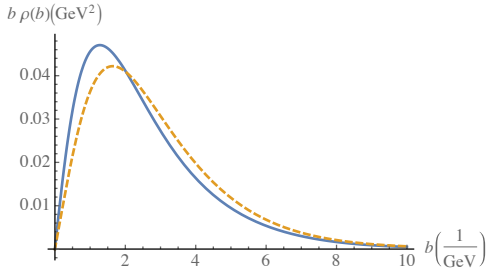

FIG. 2. True (solid) vs non-relativistic (dashed) density

V. TRUE *VS.* NON-RELATIVISTIC DENSITY

The quantity $\rho(b)$ of Eq. (48) is a true density. It is properly defined as the matrix element of a density operator between identical initial and final states. It depends only on the two-dimensional transverse variable $\mathbf{b}$ because boosts in the longitudinal momentum depend on interactions. Furthermore, it depends only on the magnitude b because of its independence of λ.

The purpose of this Section is to understand the differences between $\rho(b)$ and $\rho_{\rm NR}(r)$ even though these are inherently different. The former is well-defined in quantum field theory, and the latter is defined as a three-dimensional Fourier transform of G_E. However, it is useful to compare $\rho(b)$ with a two-dimensional version of $\rho_{\rm NR}(r)$. A transverse non-relativistic density can be defined in analogy with the transverse densities that are used in relativistic heavy ion physics, and which generally appear in Glauber theory (eikonal approximation) calculations of scattering processes. To this end, one writes $\vec{r} = z\,\widehat{\mathbf{k}} + \mathbf{b}$ and then integrates $\rho_{\rm NR}(r)$ over all values of z. Then one obtains a non-relativistic transverse density

$$\rho_{\rm NR,T}(b) = \int_{-\infty}^{\infty} dz\, \rho_{\rm NR}(\sqrt{b^2+z^2}) \tag{54}$$

The use of Eq. (22) in this equation followed by integration over angles yields the result

$$\rho_{\rm NR,T}(b) = \frac{1}{2\pi}\int_0^{\infty} dQQJ_0(bQ)G_E(Q^2), \tag{55}$$

where J_0 is a cylindrical Bessel function. The non-relativistic transverse density $\rho_{\rm NR,T}(b)$ may be compared with a more detailed version of the true transverse density of Eq. (48):

$$\rho(b) = \frac{1}{2\pi}\int_0^{\infty} dQQJ_0(bQ)F_1(Q^2). \tag{56}$$

The difference is simple: to obtain a relativistic transverse charge density one need only replace G_E by F_1.

Next specific values of F_1 and G_E are used to obtain an explicit comparison. The form factors $G_{E,M}$ have been measured [87, 88]. Thus the quantity F_1 is readily available from the relation

$$F_1(Q^2) = \frac{G_E(Q^2) + \frac{Q^2}{4M^2}G_M(Q^2)}{1+\frac{Q^2}{4M^2}}. \tag{57}$$

Then the functions G_E and F_1, obtained from a recent parameterization [89], are used to obtain the densities shown in Fig. 2. The non-relativistic transverse density is seen to be very different from the correct transverse density. In particular, the non-relativistic version has a larger spatial extent.

The spatial extent can be understood by computing the average value of b^2:

$$\langle b^2\rangle_{\rm NR} = \int d^2b\, b^2 \rho_{\rm NR}(b) = -4G'_E(0) = \frac{2}{3}r_p^2 = \frac{2}{3}\langle r^2\rangle_{\rm NR} \tag{58}$$

$$\langle b^2\rangle = \int d^2b\, b^2 \rho(b) = -4F'_1(0) \tag{59}$$

The use of the relation Eq. (6) leads to the result:

$$\langle b^2\rangle_{\rm NR} = \langle b^2\rangle + \frac{\kappa}{4M^2} = \langle b^2\rangle + 0.02\,{\rm fm}^2 \tag{60}$$

The difference between the true value and the non-relativistic one is very significant on the scale of distances relevant to the proton radius puzzle. But to be clear: hydrogen spectroscopy measures the slope of G_E at its origin and relates that quantity to r_p^2.

A. Relativistic Moment Expansion- RME

A moment expansion analogous to Eq. (20) can be derived from the relation between $F_1(Q^2)$ and the true density $\rho(b)$. Invert Eq. (48) to obtain

$$F_1(Q^2) = \int d^2b\rho(b)e^{i\mathbf{Q}\cdot\mathbf{b}}. \tag{61}$$

Then expand the exponential in a power series in $i\mathbf{Q}\cdot\mathbf{b}$ and also expand $F_1(Q^2)$ in powers of Q^2. Equating the two expansions gives the result

$$F_1(Q^2) = \sum_{n=0}^{\infty}(-1)^n \frac{I_n}{(2n)!}\langle b^{2n}\rangle Q^{2n} F_1^{(n)}(0) \tag{62}$$

$$I_n \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi \cos^{2n}\phi = \frac{(2n-1)!!}{2^n\, n!} \tag{63}$$

$$\langle b^{2n}\rangle \equiv \int d^2b\rho(b) b^{2n}, \tag{64}$$

where the notation $F_1^{(n)}(0)$ denotes taking the n'th derivative of F_1 with respect to Q^2 at $Q^2 = 0$ The first terms are given by

$$F_1(Q^2) \approx 1 - \frac{Q^2}{4}\langle b^2\rangle F^{(1)}(0) + \frac{Q^4}{64}\langle b^4\rangle F_1^{(2)}(0) + \cdots. \tag{65}$$

Each of the moments $\langle b^{2n}\rangle$ is invariant under Lorentz transformations.

Hydrogen spectroscopy depends on the moments $\langle r^{2,4}\rangle_{\rm NR}$. The RME of Eq. (62) can be used to determine the true moments $\langle b^{2n}\rangle$ in terms of the non-relativistic ones. The use of Eq. (60) leads to the result:

$$\langle b^2\rangle = \frac{2}{3}\langle r^2\rangle_{\rm NR} - \frac{\kappa}{4M^2}, \tag{66}$$

and

$$\langle b^4\rangle = \frac{8}{15}\langle r^4\rangle_{\rm NR} + \frac{8}{3M^2}\langle r^2\rangle_{\rm NR} - \frac{4\mu}{3M^2}\langle r_M^2\rangle_{\rm NR} - \frac{4\kappa}{M^4}, \tag{67}$$

where $\mu = 1+\kappa$, $\langle r_M^2\rangle_{\rm NR} \equiv -6G'_M(0)$.

The moments of b are closely tied to GPDs, which are accessible experimentally and through lattice calculations.

VI. SUMMARY

This paper unites the hydrogen spectroscopy literature with that of lepton-proton scattering to show how Eq. (1) emerges from the separate literatures.

The appearance of the proton radius r_p in hydrogen spectroscopy is discussed in Sect.II, which shows that the energy shift caused by the non-zero extent of the proton (Eq. (13)) is proportional to the slope of $G_E(Q^2)$ at its origin. An explicit three-dimensional charge density does not appear. There is no need to define r_p^2 as a moment of such a density.

Sect. III begins with a brief historical review of how a non-relativistic, frame-dependent spherically-symmetric, three-dimensional charge density was postulated in the early work of Hofstadter and co-workers. There is only one attempted derivation of this density in the literature [57]. This derivation is shown to be faulty because it used states of completely uncertain position, which leads to an infinite contribution, Eq. (37).

A properly defined relativistic three-dimensional charge density is discussed in Sect. IV. This quantity is intimately connected with modern formulations of the diverse set of possible parton distributions. It depends on longitudinal and transverse momentum or longitudinal and transverse position. The two-dimensional transverse density ($\rho(b)$), obtained as an integral over the longitudinal coordinate, is a two-dimensional Fourier transform of the Dirac form factor F_1, Eq. (48). The transverse density is shown to be related to specific integrals of a Wigner distribution, Eq. (53).

The phenomenology of $\rho(b)$ is discussed in Sect. V. The non-relativistic version is shown to be significantly different from the correct density, and a correctly defined moment expansion RME is derived, Eq. (62). These moments are related to the non-relativistic ones.

APPENDIX A-DERIVATION OF EQ. (7)

The expression $\bar{u}(p',s')\Gamma^0 u(p,s)$ (Eq. (5) with $\vec{p}' = \vec{p} + \vec{q}$) is evaluated here. The spinor is given to order $1/M^2$ by

$$u^\dagger(p,s) = [1 \quad \frac{\vec{\sigma}\cdot\vec{p}}{2m}]/(1+\vec{p}^2/(8M^2)), \tag{68}$$

in which the spin-dependent term is neglected. Then first evaluate

$$\bar{u}(p',s')\gamma^0 u(p,s)F_1 = (1+\tfrac{\vec{\sigma}\cdot\vec{p}'}{2M}\tfrac{\vec{\sigma}\cdot\vec{p}}{2M})(1-\tfrac{\vec{p}'^2+\vec{p}^2}{8M^2})F_1 \tag{69}$$

$$= (1-\tfrac{\vec{q}^2}{8M^2})F_1. \tag{70}$$

Next evaluate the term proportional to F_2. Use $i\sigma^{0\nu}q_n = ii\frac{1}{2}[\gamma^0, -\vec{\sigma}\cdot\vec{q}] = \gamma^0\vec{\gamma}\cdot\vec{q}$, so that

$$\bar{u}(p',s')\frac{i\sigma^{0\nu}q_\nu}{2M}u(p,s)F_2 = [\vec{\sigma}\cdot\vec{q}\vec{\sigma}\cdot\vec{p} - \vec{\sigma}\cdot(\vec{p}+\vec{q})\vec{\sigma}\cdot\vec{q}]F_2/(4M^2) = -\frac{\vec{q}^2}{4M^2}F_2, \tag{71}$$

in which a spin-dependent term is omitted. Combining the results Eq. (70) and Eq. (71) and recalling the definition Eq. (5) leads immediately to Eq. (7).

APPENDIX B-GENERAL WAVE PACKET

Sect. III showed that the derivation of the relation between G_E and a three-dimensional charge density was incorrect because it ignores an infinite term. A specific representation, Eq. (30), of the delta function was used to demonstrate this flaw. The aim here is to show that an infinite term appears for any representation of

the delta function. The evaluation of Eq. (33) involves the combination $\nabla_q^2 \int d^3p\, g(\vec{p}+\vec{q}/2)g^*(\vec{p}-\vec{q}/2)$. Other terms in which $\vec{\nabla}_q$ acts on the product of the g_R factors times the matrix element of Γ^0 or only on Γ^0 are not infinite. But this term is actually infinite. To see this introduce the Fourier transform:

$$g(\vec{p}) = \int d^3x \tilde{g}(\vec{x}) e^{i\vec{p}\cdot\vec{x}}. \tag{72}$$

The function $g(\vec{p})$ is very narrow in momentum space, so $\tilde{g}(\vec{x})$ must be very broad in coordinate space. Using Fourier transforms one finds

$$\nabla_q^2 \int d^3p\, g(\vec{p}+\vec{q}/2)g^*(\vec{p}-\vec{q}/2) = -(2\pi)^3 \int d^3r\, r^2 |\tilde{g}(\vec{r})|^2. \tag{73}$$

Thus it is immediately plausible that this quantity is infinite for any representation of the delta function.

The infinite result can proved. Let the general form, dictated by dimensional analysis, of $g(\vec{p})$ be given by

$$g_R(\vec{p}) = R^{3/2} F(pR), \tag{74}$$

where F is a dimensionless, real-valued function and it is well-understand that the limit $R \to \infty$ is to be taken after doing the relevant integrals. Then

$$\tilde{g}_R(\vec{r}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} g_R(\vec{p}) = \frac{1}{2\pi^2 R^{3/2}} \frac{R}{r} \int_0^\infty du\, u\, \sin(ur/R)\, F(u) \equiv \frac{1}{R^{3/2}} G(r/R), \tag{75}$$

and

$$\int d^3r\, r^2 |\tilde{g}_R(\vec{r})|^2 = \frac{1}{R^3} \int d^3r r^2 |G(r/R)|^2 = R^2 \int d^3z z^2 |G(z)|^2, \tag{76}$$

where z is a dimensionless variable. The function G is normalizable, and its z^2-weighted integral is finite, because it is a Fourier transform of a normalizable function. However, the entire expression is proportional to R^2 which is infinite.

The net result is that the wave packet treatment of Sachs is not feasible, no matter how a delta function is represented.

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