Large-N_{c} relationships among two-derivative pionless effective field theory couplings

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Large-N_c Relationships Among Two-Derivative Pionless Effective Field Theory Couplings

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We analyze two-derivative two-nucleon interactions in a combined pionless effective field theory and large-N_c expansion. At leading order in the large-N_c expansion, relationships among low-energy constants emerge. We find these to be consistent with experiment. However, it is critical to correctly address the subtraction-point dependence of the low-energy constants. These results provide additional confidence that the dual-expansion procedure is useful for analyzing low-energy few-body observables.

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I. INTRODUCTION

In the effective field theory (EFT) approach to nucleon-nucleon interactions, the underlying short-distance details of quantum chromodynamics (QCD) are encoded in the values of the low-energy couplings (LECs) that accompany each operator. At energies well below the pion mass, the nucleon-nucleon interactions are given in terms of two-nucleon contact operators with an increasing number of derivatives as well as any needed external fields. This approach is commonly referred to as pionless EFT (EFT$_{\not\pi}$), see e.g., Refs. [1–5] and references therein. While in the future it may be possible to predict the LECs from QCD directly, currently their values are determined from fits to experimental observables or phase shift analyses of data. As we attempt to reconcile increasingly accurate low-energy measurements with increasingly accurate calculations, the number of LECs needed in EFT$_{\not\pi}$ increases. Additional theoretical constraints can provide relationships between the LECs and reduce the number of independent couplings.

Here we analyze the two-derivative nucleon-nucleon contact operators contributing to low-energy scattering by considering the large-N_c expansion of QCD, where N_c is the number of colors [6, 7]. In the large-N_c limit additional symmetries emerge. Fortunately, in many instances these symmetries survive as approximate symmetries in actual ($N_c = 3$) QCD. Combining the EFT$_{\not\pi}$ and large-N_c expansions decreases the number of independent LECs and increases the predictive power of the combined expansions. For example, when calculating an observable for which insufficient data is available to fit the higher order (in the EFT$_{\not\pi}$ expansion) LECs, the large-N_c relationships may make predictions possible.

Large-N_c methods were applied to nucleon interactions in reference [8], who showed that in the large-N_c limit the non-derivative S-wave interactions should be identical. The general two-nucleon potential was analyzed in Ref. [9], predicting the relative strength of central, tensor, and spin-orbit forces. These results compared favorably with the couplings as predicted by the Nijmegen potential model [10] derived from an analysis of a large set of scattering data over a wide range of energies. Additional phenomenological models were considered in Ref. [11]. Reference [12] investigated Wigner-SU(4) and Serber symmetries, and explored their connection to large-N_c in NN potential models. The large-N_c approach has also been applied to three-nucleon interactions [13, 14].

EFTs have emerged as powerful tools for describing nucleon interactions (see, e.g., Refs. [2, 4, 5, 15, 16] and references therein). In Ref. [17] some of us applied the dual EFT$_{\not\pi}$ and large-N_c expansions to low-energy two-nucleon parity-violating interactions. Since there is not enough data to confirm (although current data is not in contradiction with) those results, we want to test the formalism on a system where more data is available. Here, in addition to the large-N_c expansion, we focus on low energies and parity-conserving interactions in partial waves with orbital angular momentum $L \leq 2$ as described by EFT$_{\not\pi}$. In this formalism, these interactions are described by nucleon contact operators with up to two derivatives. The two-derivative two-nucleon operators are interesting for several reasons. First, the lowest-order (no-derivative) EFT$_{\not\pi}$ operators (in the $^1S_0$ and $^3S_1$ channels) can obscure the large-N_c relationships if not analyzed carefully. The large-N_c prediction that the LECs in these two channels are the same at leading order in N_c (LO-in-N_c) [8] suggests that the S-wave scattering lengths in the large-N_c limit should be identical. In reality, the scattering lengths are not close to one another and even differ in sign, which seems to contradict the large-N_c prediction. However, the S-wave scattering lengths are known to be fine-tuned and anomalously large. (See also Ref. [18]). Further, the LECs themselves, to which the large-N_c predictions apply, are not observables and are renormalization scheme dependent. The values of the LECs agree within the expected accuracy once an appropriate renormalization point is chosen.

A more robust test of the large-N_c relationships may be found by considering two-nucleon scattering at higher orders in EFT$_{\not\pi}$ because the effective range expansion parameters at these orders (the effective

range, shape parameter, etc.) are expected to be of natural size. The two-body P-wave and S-D-mixing operators, which first occur at two derivatives, are important for the role they play in currently unexplained asymmetries in three body scattering at very low energies ($E_{\text{nucleon}} < 12\,\text{MeV}$). The suite of polarization observables in nucleon-deuteron scattering depends upon both two-nucleon S-D mixing and two-body scattering in the ^3P_J, $J = 0, 1, 2$ channels. In particular, the so-called A_y (nucleon-deuteron) analyzing power is very sensitive to two-body scattering in the ^3P_J channels. When the ^3P_J phase shifts are modified by only 15% it can yield changes in A_y by 50%, which easily accounts for the current A_y discrepancies between theory and experiment. This issue is reviewed in depth in Ref. [19] and references therein and considered in EFT$_{\not\pi}$ in Ref. [20].

In the next section we will present the needed terminology for defining and using EFT$_{\not\pi}$ and large-N_c techniques.

II.   PIONLESS EFT AND LARGE-N_c CONSTRAINTS

EFT$_{\not\pi}$ describes nucleon-nucleon interactions at energies of less than 10's of MeV in terms of contact terms that only involve nucleon fields and external currents. These operators are organized according to a power counting based on a small expansion parameter $Q/\Lambda_{\not\pi}$. All short-distance/high-energy details are encoded in the corresponding LECs. Because of the number of LECs and the sparse data available, some with potentially large error bars at the lowest scattering energies, we will explore the enhanced symmetry that QCD acquires in the limit $N_c \to \infty$. This symmetry is approximate for real ($N_c = 3$) QCD, but corrections are expected to be perturbative in $1/N_c$. Combining the EFT$_{\not\pi}$ and large-N_c expansions allows us to reduce the number of LECs occurring at any given order compared to the standard EFT$_{\not\pi}$ expansion.

The EFT$_{\not\pi}$ expansion parameter is $Q/\Lambda_{\not\pi}$, where Q is the momentum and/or energy transfer and $\Lambda_{\not\pi} \sim m_\pi$ is the breakdown scale of the theory. However, a naive counting of derivatives, as is possible in for example chiral perturbation theory, is not sufficient to establish Q scaling in EFT$_{\not\pi}$. In particular, the large scattering lengths in the S-wave channels for two-nucleon scattering require a modification of this naive scaling [21–24]. The power divergent subtraction (PDS) scheme [23] introduces a scale (subtraction point) $\mu \sim Q$ in the problem to establish a consistent power counting in the anomalous S-wave channels. For example, the leading (no-derivative) S-wave interactions are given by

$$\mathcal{L}_0^{(S)} = -C_0^{(^3S_1)}(N^T P_i N)^\dagger (N^T P_i N) - C_0^{(^1S_0)}(N^T P_a N)^\dagger (N^T P_a N) \,, \tag{1}$$

where $P_i = \frac{1}{\sqrt{8}}\sigma_2\sigma_i\tau_2$ and $P_a = \frac{1}{\sqrt{8}}\sigma_2\tau_2\tau_a$, with σ_i ($i = 1, 2, 3$) and τ_a ($a = 1, 2, 3$) the Pauli matrices in spin and isospin space. The operators are dimension six, so their coefficients have units of MeV^{-2}. However, these two energy dimensions are not both from large scales in the problem, as is typical in many EFTs. Instead, in the PDS scheme, the coefficients scale as

$$C_0^{(S)} = \frac{4\pi}{M} \frac{1}{\frac{1}{a^{(S)}} - \mu} \,, \tag{2}$$

where M is the nucleon mass, μ is the subtraction scale in the PDS scheme, and $a^{(S)}$ is the scattering length in the S channel, either $^3S_1$ or $^1S_0$. For $\mu \sim Q$, this yields an S-wave scattering amplitude of the order $1/(MQ)$. As we will see below, the modification of naive power counting in the S-wave channels will impact the EFT$_{\not\pi}$ Q scaling of higher order S-wave coefficients as well. But see Sec. IV for a discussion on restrictions on μ when analyzing μ-dependent (non-physical) quantities.

The EFT$_{\not\pi}$ operators of Eq. (1) are given in the so-called "partial-wave" basis, where the two incoming and the two outgoing nucleons are identified by their $^{2S+1}L_J$ quantum numbers, where

$S = 0, 1$ is the total intrinsic spin, L is the relative orbital angular momentum of the two nucleons, and J is the total angular momentum of the two-nucleon system. The J quantum numbers are conserved but others may mix. This basis is particularly convenient because it maps most directly onto experimental results.

However, the partial-wave basis is not the basis in which large-N_c counting is clearly manifest. The large-N_c counting of two-nucleon operators is determined by considering the two-nucleon matrix element of the Hamiltonian H [9],

$$V(\mathbf{p}_-, \mathbf{p}_+) = \langle (\mathbf{p}'_1, C), (\mathbf{p}'_2, D) | H | (\mathbf{p}_1, A), (\mathbf{p}_2, B) \rangle \ , \tag{3}$$

where the A, B, C, D denote combined spin and isospin quantum numbers of the nucleons and

$$\mathbf{p}_\pm \equiv \mathbf{p}' \pm \mathbf{p} \ , \tag{4}$$

with the outgoing and incoming relative momenta given by

$$\mathbf{p}' = \mathbf{p}'_1 - \mathbf{p}'_2 \ , \quad \mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2 \ , \tag{5}$$

in terms of the momenta in Eq. (3). H is the Hartree Hamiltonian [7, 9],

$$H = N_c \sum_n \sum_{s,t} v_{stn} \left(\frac{S}{N_c} \right)^s \left(\frac{I}{N_c} \right)^t \left(\frac{G}{N_c} \right)^{n-s-t} \ , \tag{6}$$

expressed in terms of the operators

$$S_i = q^\dagger \frac{\sigma_i}{2} q \ , \quad I_a = q^\dagger \frac{\tau_a}{2} q \ , \quad G_{ia} = q^\dagger \frac{\sigma_i \tau_a}{2} q \ , \tag{7}$$

where q are bosonic (colorless) SU(2) doublet operators. The coefficients v_{stn} account for momentum dependence and the desired symmetry properties.

The large-N_c counting of matrix elements was explored in Refs. [8, 9, 25, 26], who found scaling of the form

$$\langle N(\mathbf{p}') | \frac{O^{(n)}_{I,S}}{N_c^n} | N(\mathbf{p}) \rangle \lesssim \frac{1}{N_c^{|I-S|}} \ , \tag{8}$$

where $O^{(n)}_{I,S}$ is an n-body quark operator of isospin I and spin S (e.g., $q^\dagger q$ is an $n{=}1$ one-body operator). Using Eq. (8) on the quark-model operators of Eq. (7), the large-N_c scaling of these operators is

$$\langle N(\mathbf{p}') | \frac{S}{N_c} | N(\mathbf{p}) \rangle \sim \langle N(\mathbf{p}') | \frac{I}{N_c} | N(\mathbf{p}) \rangle \lesssim \frac{1}{N_c} \ , \quad \langle N(\mathbf{p}') | \frac{G}{N_c} | N(\mathbf{p}) \rangle \lesssim 1 \ . \tag{9}$$

In addition, the identity operator scales as $\langle N(\mathbf{p}') | \mathbb{1} | N(\mathbf{p}) \rangle \sim N_c$.

In principle, the momenta $\mathbf{p}_-$ and $\mathbf{p}_+$ are both considered to be independent of N_c. However, as discussed in Ref. [9], there is a possible additional $1/N_c$ suppression related to these momenta. In the large-N_c limit, it is consistent to interpret the potential in the meson-exchange picture [27, 28]. In the t-channel, contributions proportional to $\mathbf{p}_+$ only arise as relativistic corrections and are always accompanied by a factor of $1/M \sim 1/N_c$. Analogously, in the u-channel $\mathbf{p}_-$ only appears as a relativistic correction with a factor of $1/M$. Instead of considering all possible contractions of the operators between two-nucleon states, it is convenient to only consider t-channel contributions, for

which the large-N_c suppression of relativistic corrections can be taken into account by counting the momenta as

$$\mathbf{p}_- \sim 1 \ , \quad \mathbf{p}_+ \sim 1/N_c \ . \tag{10}$$

Analysis of the u-channel contributions would lead to equivalent results, see also the discussion in Ref. [13].

In the large-N_c limit the matrix element of Eq. (3) factorizes into matrix elements of single-nucleon operators; for our two-body-scattering purposes we perform the large-N_c analysis on operators of the form $(N^\dagger O_1 N)(N^\dagger O_2 N)$, where the O_i are spin-isospin operators. The large-N_c scaling found from these matrix elements can be mapped onto the EFT$_{\not\pi}$ LECs. By using Fierz transformations, the large-N_c behavior of the LECs in the EFT$_{\not\pi}$ partial-wave basis can then be extracted.

One issue that remains an open question in this procedure is the role of the Δ resonance in the large-N_c limit of QCD. From an EFT viewpoint we are free to integrate out the Δ as well as the pion to form a low-energy theory. But the large-N_c limit does not necessarily respect that point of view. In particular, in the large-N_c limit the nucleon-Δ mass splitting goes to zero and it is reasonable to be concerned about whether one can legitimately talk about the large-N_c limit in EFT$_{\not\pi}$ without the Δ degree of freedom. In the following we will restrict the discussion to the matrix element defined in Eq. (3) without considering the effects of virtual Δs in intermediate states. A more detailed discussion about the impact of integrating out Δ degrees of freedom can be found in [29].

In the next section we consider the two-derivative EFT$_{\not\pi}$ operators needed to describe S-wave, P-wave, and S-D-mixed two-nucleon scattering.

III. TWO-DERIVATIVE OPERATORS

In EFT$_{\not\pi}$, there are seven independent two-nucleon operators with two derivatives. In the partial-wave basis, these correspond to the two S-wave, the S-D-mixing, and the four P-wave terms. In what we will call the "large-N_c-counting basis," there appears to be a great deal of freedom in choosing a set of seven independent terms to take as the basis. In the absence of large-N_c considerations, Fierz relationships show the equivalence of these choices. However, as discussed in Ref. [17], the large-N_c counting is most transparent when performed on an overcomplete set of the large-N_c-basis operators, because the Fierz transformations that are used to reduce the Lagrangian to a minimal set can relate terms that are of the same order in the EFT$_{\not\pi}$ counting, but have different large-N_c scalings.

The overcomplete basis required for our analysis contains a total of 14 terms [30–32]. Since we are interested in isospin-singlet terms, each operator of a given spin-momentum structure appears in two possible forms: either with the isospin identity operators, or with $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$.

The two-derivative operators of the form $(N^\dagger O_1 N)(N^\dagger O_2 N)$ that are LO-in-N_c are $\mathcal{O}(N_c)$ and can be chosen to be:

$$\begin{aligned}
\mathcal{L}_{\text{LO-in-}N_c} = {} & C_{1\cdot 1} \nabla_i (N^\dagger N) \nabla_i (N^\dagger N) \\
& + C_{G\cdot G} \nabla_i (N^\dagger \sigma_j \tau_a N) \nabla_i (N^\dagger \sigma_j \tau_a N) \\
& + C'_{G\cdot G} \nabla_i (N^\dagger \sigma_i \tau_a N) \nabla_j (N^\dagger \sigma_j \tau_a N) \ .
\end{aligned} \tag{11}$$

The subscripts on the coefficients are designed to echo the operators they multiply. When more than one contraction is possible a prime superscript is used.

Given that the expansion parameter is $1/N_c = 1/3$ for real QCD, corrections to these leading (in N_c) terms might be expected at the 30% level. However, the next terms in the large-N_c expansion for

these processes occurs two orders higher, $\mathcal{O}(1/N_c)$. Those operators may be chosen to be:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{N^2LO\text{-}in\text{-}}N_c} = & \, C_{\tau\cdot\tau}\nabla_i(N^\dagger\tau_a N)\nabla_i(N^\dagger\tau_a N) \\
& + \overset{\leftrightarrow}{C}_{1\cdot 1}(N^\dagger\overset{\leftrightarrow}{\nabla}_i N)(N^\dagger\overset{\leftrightarrow}{\nabla}_i N) \\
& + C_{\sigma\cdot\sigma}\nabla_i(N^\dagger\sigma_j N)\nabla_i(N^\dagger\sigma_j N) \\
& + \overset{\leftrightarrow}{C}_{G\cdot G}(N^\dagger\overset{\leftrightarrow}{\nabla}_i\sigma_j\tau_a N)(N^\dagger\overset{\leftrightarrow}{\nabla}_i\sigma_j\tau_a N) \\
& - \frac{i}{2}\overset{\leftrightarrow}{C}_{1\cdot\sigma}\,\epsilon_{ijk}\left[\nabla_j(N^\dagger\sigma_i N)(N^\dagger\overset{\leftrightarrow}{\nabla}_k N) + \nabla_j(N^\dagger N)(N^\dagger\overset{\leftrightarrow}{\nabla}_k\sigma_i N)\right] \\
& - \frac{i}{2}\overset{\leftrightarrow}{C}_{G\cdot\tau}\,\epsilon_{ijk}\left[\nabla_j(N^\dagger\sigma_i\tau_a N)(N^\dagger\overset{\leftrightarrow}{\nabla}_k\tau_a N) + \nabla_j(N^\dagger\tau_a N)(N^\dagger\overset{\leftrightarrow}{\nabla}_k\sigma_i\tau_a N)\right] \\
& + C'_{\sigma\cdot\sigma}\nabla_i(N^\dagger\sigma_i N)\nabla_j(N^\dagger\sigma_j N) \\
& + \overset{\leftrightarrow}{C}'_{G\cdot G}(N^\dagger\overset{\leftrightarrow}{\nabla}_i\sigma_i\tau_a N)(N^\dagger\overset{\leftrightarrow}{\nabla}_j\sigma_j\tau_a N)\ .
\end{aligned}
\tag{12}
$$

The remaining three operators from the initial overcomplete set of 14 are suppressed by additional powers of $1/N_c$ and will not be considered here.

As discussed above, only 7 of the 14 operator structures are independent. There is some freedom in choosing which set of operators to retain. In EFT calculations in this basis, Fierz transformations can be applied to remove the contributions containing $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ in favor of the simpler identity operators. However, this approach masks the large-N_c counting, since eliminating the terms proportional to $C_{G\cdot G}$ and $C'_{G\cdot G}$ induces LO-in-N_c contributions in terms that are naively of higher order in N_c. It is possible but tedious to keep track of these "hidden" LO-in-N_c contributions. Instead, it is more convenient to keep all terms in Eq. (11) and to apply Fierz transformations to the N^{2}LO-in-N_c terms in Eq. (12). Four of these terms are redundant and can be removed. We choose to keep the terms proportional to $C_{\tau\cdot\tau}$, $C_{\sigma\cdot\sigma}$, $\overset{\leftrightarrow}{C}_{1\cdot\sigma}$, and $C'_{\sigma\cdot\sigma}$.

We wish to use the large-N_c scaling of the above operators to indicate how the partial-wave operators scale with large-N_c. The two-derivative EFT$_{\not\pi}$ operators in the partial-wave basis are

$$
\mathcal{L}_2^{(^3S_1)} = \frac{1}{8}C_2^{(^3S_1)}\left[(N^T P_i N)^\dagger(N^T P_i\overset{\leftrightarrow}{\nabla}^2 N) + \mathrm{h.c.}\right]\ ,
\tag{13}
$$

$$
\mathcal{L}_2^{(^1S_0)} = \frac{1}{8}C_2^{(^1S_0)}\left[(N^T P_a N)^\dagger(N^T P_a\overset{\leftrightarrow}{\nabla}^2 N) + \mathrm{h.c.}\right]\ ,
\tag{14}
$$

$$
\mathcal{L}_2^{(SD)} = \frac{1}{4}C^{(SD)}\left[(N^T P_i N)^\dagger(N^T P_j\overset{\leftrightarrow}{\nabla}_x\overset{\leftrightarrow}{\nabla}_y N)(\delta_{ix}\delta_{jy} - \frac{1}{3}\delta_{ij}\delta_{xy}) + \mathrm{h.c.}\right]\ ,
\tag{15}
$$

$$
\mathcal{L}_2^{(^1P_1)} = \frac{1}{4}C^{(^1P_1)}(N^T P_0\overset{\leftrightarrow}{\nabla}_i N)^\dagger(N^T P_0\overset{\leftrightarrow}{\nabla}_i N)\ ,
\tag{16}
$$

$$
\begin{aligned}
\mathcal{L}_2^{(^3P_J)} = \frac{1}{4}\Big[& C^{(^3P_0)}\delta_{xy}\delta_{wz} + C^{(^3P_1)}(\delta_{xw}\delta_{yz} - \delta_{xz}\delta_{yw}) \\
& + C^{(^3P_2)}(2\delta_{xw}\delta_{yz} + 2\delta_{xz}\delta_{yw} - \frac{4}{3}\delta_{xy}\delta_{wz})\Big](N^T P_{y,a}\overset{\leftrightarrow}{\nabla}_x N)^\dagger(N^T P_{z,a}\overset{\leftrightarrow}{\nabla}_w N)\ ,
\end{aligned}
\tag{17}
$$

where $N^T\mathcal{O}\overset{\leftrightarrow}{\nabla}_i N \equiv N^T\mathcal{O}\nabla_i N - (\nabla_i N^T)\mathcal{O}N$ with $\mathcal{O}$ some spin-isospin operator, and the projection

operators are defined by

$$P_i = \frac{1}{\sqrt{8}}\sigma_2\sigma_i\tau_2 , \qquad P_a = \frac{1}{\sqrt{8}}\sigma_2\tau_2\tau_a , \qquad P_0 = \frac{1}{\sqrt{8}}\sigma_2\tau_2 , \qquad P_{i,a} = \frac{1}{\sqrt{8}}\sigma_2\sigma_i\tau_2\tau_a . \tag{18}$$

The Fierz identities in Appendix A can be used to relate the large-N_c-counting basis to the partial-wave basis. The EFT$_{\not{\pi}}$ partial-wave coefficients have large-N_c scaling provided by

$$C^{(^{2S+1}L_J)} = C_{\text{LO-in-}N_c}^{(^{2S+1}L_J)} + C_{\text{N}^2\text{LO-in-}N_c}^{(^{2S+1}L_J)} , \tag{19}$$

where each $C_{\text{LO-in-}N_c}^{(^{2S+1}L_J)}$ is a linear combination of $C_{1\cdot1}$, $C_{G\cdot G}$, and $C'_{G\cdot G}$ and each $C_{\text{N}^2\text{LO-in-}N_c}^{(^{2S+1}L_J)}$ is a linear combination of $C_{\tau\cdot\tau}$, $C_{\sigma\cdot\sigma}$, $\overset{\leftrightarrow}{C}_{1\cdot\sigma}$, and $C'_{\sigma\cdot\sigma}$:

$$
\begin{aligned}
C_2^{(^1S_0)} = & -4(C_{1\cdot1} - 3C_{G\cdot G} - C'_{G\cdot G}) & \text{LO-in-}N_c \\
& -4(C_{\tau\cdot\tau} - 3C_{\sigma\cdot\sigma} - C'_{\sigma\cdot\sigma}) & \text{N}^2\text{LO-in-}N_c \\
C_2^{(^3S_1)} = & -4(C_{1\cdot1} - 3C_{G\cdot G} - C'_{G\cdot G}) & \text{LO-in-}N_c \\
& -4(-3C_{\tau\cdot\tau} + C_{\sigma\cdot\sigma} + \frac{1}{3}C'_{\sigma\cdot\sigma}) & \text{N}^2\text{LO-in-}N_c \\
C^{(SD)} = & -4(3C'_{G\cdot G}) & \text{LO-in-}N_c \\
& -4(-C'_{\sigma\cdot\sigma}) & \text{N}^2\text{LO-in-}N_c \\
C^{(^1P_1)} = & -4(C_{1\cdot1} + 9C_{G\cdot G} + 3C'_{G\cdot G}) & \text{LO-in-}N_c \\
& -4(-3C_{\tau\cdot\tau} - 3C_{\sigma\cdot\sigma} - C'_{\sigma\cdot\sigma}) & \text{N}^2\text{LO-in-}N_c \\
C^{(^3P_0)} = & -\frac{4}{3}(C_{1\cdot1} + C_{G\cdot G} - 3C'_{G\cdot G}) & \text{LO-in-}N_c \\
& -\frac{4}{3}(C_{\tau\cdot\tau} + C_{\sigma\cdot\sigma} - 2\overset{\leftrightarrow}{C}_{1\cdot\sigma} - 3C'_{\sigma\cdot\sigma}) & \text{N}^2\text{LO-in-}N_c \\
C^{(^3P_1)} = & -2(C_{1\cdot1} + C_{G\cdot G} + 2C'_{G\cdot G}) & \text{LO-in-}N_c \\
& -2(C_{\tau\cdot\tau} + C_{\sigma\cdot\sigma} - \overset{\leftrightarrow}{C}_{1\cdot\sigma} + 2C'_{\sigma\cdot\sigma}) & \text{N}^2\text{LO-in-}N_c \\
C^{(^3P_2)} = & -(C_{1\cdot1} + C_{G\cdot G}) & \text{LO-in-}N_c \\
& -(C_{\tau\cdot\tau} + C_{\sigma\cdot\sigma} + \overset{\leftrightarrow}{C}_{1\cdot\sigma}) . & \text{N}^2\text{LO-in-}N_c
\end{aligned}
\tag{20}
$$

While the large-N_c basis of Eqs. (11) and (12) is the one where large-N_c scaling is manifest, and the partial-wave basis has obvious physical significance, there is another basis that is physically illuminating; the one that is driven by whether the operator is of central, tensor, or spin-orbit type. From inspection we can see that the leading order EFT$_{\not{\pi}}$ terms up to N^{2}LO-in-N_c counting can be categorized as central (derivatives contract with themselves): $C_{1\cdot1}$, $C_{G\cdot G}$, $C_{\tau\cdot\tau}$, and $C_{\sigma\cdot\sigma}$; tensor (derivative contracts with $\vec{\sigma}$): $C'_{G\cdot G}$ and $C'_{\sigma\cdot\sigma}$; and spin-orbit (cross product): $\overset{\leftrightarrow}{C}_{1\cdot\sigma}$. The contribution of central vs. tensor vs. spin-orbit to partial-wave channels is known, as is their large-N_c counting from Ref. [9], see, for example, Ref [12]; here we delineate them in terms of EFT$_{\not{\pi}}$ counting as well. The tensor interaction also provides a clear example of the necessity to consider the overcomplete Lagrangian for the large-N_c analysis. As shown in Ref. [9] and in agreement with our calculation, the tensor interaction is of LO in N_c. However, it is the operator $C'_{G\cdot G}$ proportional to $\tau_1 \cdot \tau_2$ that gives the LO

contribution. Had we eliminated this term using Fierz identities before applying the large-N_c analysis, the tensor interaction would have erroneously been considered to be N^{2}LO in N_c.

Before we explore how well the relationships of Eq. (20) are satisfied experimentally in Sec. V we will address some subtleties related to the subtraction point dependence of our results.

IV. SUBTRACTION POINT DEPENDENCE

The large-N_c analysis provides estimates of the relative sizes of *coefficients* in EFT$_{\slashed{\pi}}$. In general only observables, which are linear combinations of coefficients multiplied by matrix elements, are subtraction point (μ) independent. Individual EFT coefficients, however, can be μ dependent, such as $C_2^{(^3S_1)}$, $C_2^{(^1S_0)}$, and $C^{(SD)}$ in our case. To compare the large-N_c predictions, which are relationships among EFT coefficients, to values extracted from experiment, it is therefore necessary to choose a value for μ. As shown in Ref. [8] for the LO EFT couplings, some choices can completely hide the additional symmetry emerging in the large-N_c limit. The form of the LO-in-EFT$_{\slashed{\pi}}$ terms can be chosen as (see Refs. [33, 34])

$$\mathcal{L}_{\text{LO-in-EFT}_{\slashed{\pi}}} = -\frac{1}{2}C_S(N^\dagger N)(N^\dagger N) - \frac{1}{2}C_T(N^\dagger \sigma_i N)(N^\dagger \sigma_i N) \ . \tag{21}$$

As discussed in previous sections, this also turns out to be the basis convenient for determining large-N_c scaling. In the large-N_c limit $C_S \sim N_c$, while $C_T \sim 1/N_c$ [8]. The LECs $C_{S/T}$ are related to the LECs in the partial-wave basis of Eq. (1) by

$$C_0^{(^1S_0)} = (C_S - 3C_T) \ , \quad C_0^{(^3S_1)} = (C_S + C_T) \ , \tag{22}$$

which suggests that in the large-N_c limit

$$\frac{C_0^{(^3S_1)}}{C_0^{(^1S_0)}} = \left.\frac{\frac{1}{a^{(^1S_0)}} - \mu}{\frac{1}{a^{(^3S_1)}} - \mu}\right|_{\text{LO-in-}N_c} = 1 \ , \tag{23}$$

where we have used the PDS expression of Eq. (2), with corrections suppressed by a factor of $1/N_c^2$. If $\mu = 0$ is chosen as the subtraction point, this expression yields a ratio that is not only far from unity but negative. This is an example of a choice for μ that violates the large-N_c prediction. As discussed in Ref. [8], this problem is related to the fine-tuning of the scattering lengths and can be avoided by an appropriate choice of the matching scale. In the PDS formalism, this corresponds to choosing a μ that is larger than the scattering lengths. With increasing μ, the ratio approaches 1 from below, reaching a value of 0.9 (i.e., within $10\% \approx 1/N_c^2$ of the LO prediction) for $\mu \approx 440\,\text{MeV}$. However, the 10% prediction comes from expected corrections to the coefficients in the large-N_c-counting basis, which in this case are the LECs C_T and C_S rather than the partial-wave coefficients. The factor of 3 in Eq. (22) modifies the error estimate on the ratio of Eq. (23) to $\approx 4/N_c^2 \sim 1/N_c$. Given this estimate, we might expect the ratio in Eq. (23) to be no closer than 0.7 (from $1/N_c \sim 30\%$ corrections), which is obtained at a value of $\mu = 140\,\text{MeV}$. As we will see below, this is indeed in the range of μ where the large-N_c predictions are satisfied for the two-derivative partial-wave operators. On the other hand, at this value of $\mu = 140\,\text{MeV}$, the ratio $C_T/C_S \approx 0.08$, consistent with the expected $1/N_c^2$ suppression.

When attempting to relate quantities that have μ dependence, the aim is to avoid obscuring large-N_c relationships, as well as to minimize the μ dependence and also maintain other features (such as power counting) of the theory. In EFT$_{\slashed{\pi}}$ power counting, the large scattering lengths are counted as

$1/a \sim Q$, with Q the generic small scale in the theory. Using the PDS scheme, choosing μ such that $\mu \sim Q$ and simultaneously $(1/a - \mu) \sim Q$ results in LO (in EFT$_{\not\pi}$ counting) coefficients that justify the resummation of an infinite series of diagrams. While the second condition is simple to satisfy in the $^1S_0$ channel with a negative scattering length, the positive and smaller scattering length in the $^3S_1$ channel requires special care. As can be seen from Eq. (2), choosing $\mu \approx 1/a^{(^3S_1)}$ results in a large denominator. The same large denominator problem occurs in the NLO-in-EFT$_{\not\pi}$-counting coefficient $C_2^{(^3S_1)}$:

$$C_2^{(^3S_1)} = \frac{2\pi}{M} \frac{r^{(^3S_1)}}{(1/a^{(^3S_1)} - \mu)^2} \; , \tag{24}$$

where $r^{(^3S_1)}$ is the effective range in the spin-triplet channel. The EFT$_{\not\pi}$ power counting requires $p^2 C_2^{(^3S_1)}$ to be suppressed relative to the LO coefficient $C_0^{(^3S_1)}$. For the large EFT$_{\not\pi}$ momentum $p \sim 70\,\mathrm{MeV}$, the ratio $p^2 C_2^{(^3S_1)}/C_0^{(^3S_1)}$ is less than $1/3$ for $\mu > 100\,\mathrm{MeV}$. Alternatively, one can consider the expansion of $p \cot \delta^{(^3S_1)}$ about the deuteron pole [35, 36] instead of about $p = 0$ as done above. In that case, the LEC $C_2^{(^3S_1)}$ will have an expansion in powers of the EFT$_{\not\pi}$ parameter $Q/\Lambda_{\mathrm{EFT}_{\not\pi}}$,

$$C_2^{(^3S_1)} = C_{2,-2}^{(^3S_1)} + C_{2,-1}^{(^3S_1)} + \cdots \; , \tag{25}$$

(see Eqn. (2.19) of Ref. [1]), where the subscript on the left-hand side and the first subscripts on the right-hand side indicate the number of derivatives of the corresponding operator and the second subscripts on the right-hand side are the scaling of the coefficient with Q. The second term on the right-hand side is suppressed by a factor of $1/3$ relative to the first only for $\mu \gtrsim 100\,\mathrm{MeV}$. We therefore do not expect to be able to take $\mu < 100\,\mathrm{MeV}$ without encountering unnaturally large values of the LECs. On the other hand, Eq. (24) shows that taking μ to be very large compared to $1/a$ results in unnaturally small values of the LECs.

The possibility of choosing a renormalization scale of the order of the pion mass is discussed in Ref. [37], whose authors argue that such a choice results in an EFT with LECs that are of natural size, but in which the relative ordering of the perturbative expansion is modified. Ref. [12] also emphasizes the importance of scale dependence when investigating large-N_c and related symmetries in NN potential models.

V. LO-IN-LARGE-N_c RELATIONSHIPS AND COMPARISONS

How well do these relationships work in the real world? There are seven partial-wave terms, but only three that are leading-in-N_c; the large-N_c analysis predicts relationships between the different partial-wave couplings. This is just as anticipated; the number of independent LECs dictated by EFT$_{\not\pi}$ is reduced when the approximate symmetry of large-N_c is imposed.

The LO-in-N_c relationships between the LECs in the large-N_c counting and partial-wave bases can

be written in matrix form as

$$
\begin{pmatrix}
C_2^{(^1S_0)} \\
C_2^{(^3S_1)} \\
C^{(SD)} \\
C^{(^1P_1)} \\
C^{(^3P_0)} \\
C^{(^3P_1)} \\
C^{(^3P_2)}
\end{pmatrix}
=
\begin{pmatrix}
-4 & 12 & 4 \\
-4 & 12 & 4 \\
0 & 0 & -12 \\
-4 & -36 & -12 \\
-\frac{4}{3} & -\frac{4}{3} & 4 \\
-2 & -2 & -4 \\
-1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
C_{1\cdot1} \\
C_{G\cdot G} \\
C'_{G\cdot G}
\end{pmatrix} .
\tag{26}
$$

The linear combinations

$$
C_1^D \equiv C_{1\cdot1} + 9C_{G\cdot G} + 3C'_{G\cdot G} , \quad C_2^D \equiv C_{1\cdot1} + C_{G\cdot G} , \quad C_3^D \equiv C'_{G\cdot G}
\tag{27}
$$

'block-diagonalize' the P-waves,

$$
\begin{pmatrix}
C_2^{(^1S_0)} \\
C_2^{(^3S_1)} \\
C^{(SD)} \\
C^{(^1P_1)} \\
C^{(^3P_0)} \\
C^{(^3P_1)} \\
C^{(^3P_2)}
\end{pmatrix}
=
\begin{pmatrix}
2 & -6 & -2 \\
2 & -6 & -2 \\
0 & 0 & -12 \\
-4 & 0 & 0 \\
0 & -\frac{4}{3} & 4 \\
0 & -2 & -4 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
C_1^D \\
C_2^D \\
C_3^D
\end{pmatrix} .
\tag{28}
$$

The dependence of the partial-wave-basis coefficients on the leading-order coefficients in the large-N_c-counting basis can vary by large (compared to $N_c = 3$) factors. For example, there is a relative factor of 9 between the $C_{1\cdot1}$ and $C_{G\cdot G}$ contributions to the $^1P_1$ coupling in Eq. (26). These large numerical factors highlight that while the large-N_c expansion corrections are $1/N_c^2 = 1/9 \sim 10\%$ at this order, other physics can obscure this counting. This is why a dual-expansion (large-N_c along with $\mathrm{EFT}_{\not\pi}$) treatment is important, and why large-N_c predictions alone need to be treated as trends with potentially substantial errors rather than strict rules for ordering the relative size of observables.

A. S-waves

As $N_c \to \infty$ the prediction from Eq. (26) is

$$
\left. \frac{C_2^{(^3S_1)}}{C_2^{(^1S_0)}} \right|_{\text{LO-in-}N_c} = 1 .
\tag{29}
$$

As discussed above, the LECs are not observables and are renormalization point dependent. In the PDS renormalization scheme, the ratio of the S-wave couplings is given by

$$
\frac{C_2^{(^3S_1)}}{C_2^{(^1S_0)}} = \frac{r^{(^3S_1)}}{r^{(^1S_0)}} \frac{(\mu - 1/a^{(^1S_0)})^2}{(\mu - 1/a^{(^3S_1)})^2} ,
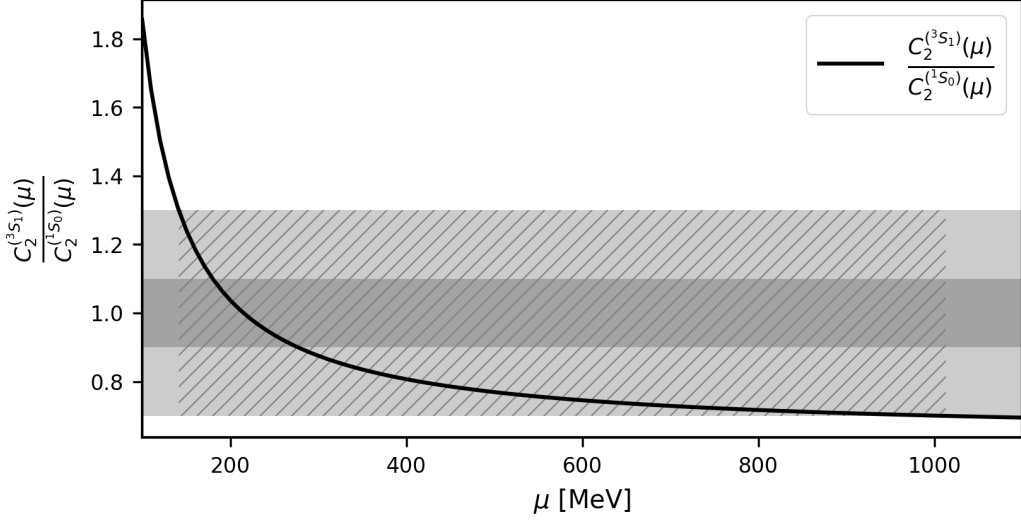\tag{30}
$$

FIG. 1. The ratio $C_2^{(^3S_1)}/C_2^{(^1S_0)}$ of Eq. (29) as a function of the renormalization scale μ (black curve). The gray band shows the large-N_c prediction with 10% (dark gray) and 30% (light gray) variation. The hatched region indicates the range of μ over which the large-N_c prediction is satisfied to 30%.

where the experimentally extracted values [38]

$$a^{(^1S_0)} = -23.7148 \pm 0.0043 \,\text{fm} , \qquad r^{(^1S_0)} = 2.750 \pm 0.018 \,\text{fm} , \qquad (31)$$

$$a^{(^3S_1)} = 5.4112 \pm 0.0015 \,\text{fm} , \qquad r^{(^3S_1)} = 1.7436 \pm 0.0019 \,\text{fm} , \qquad (32)$$

are the scattering lengths and effective ranges in the $^1S_0$ and $^3S_1$ channels. Fig. 1 shows the μ dependence of this ratio. The LO-in-N_c prediction is satisfied within a 30% error over a wide range of μ. Placing a 10% or 30% error bar on the large-N_c prediction is not meant to be a rigorous analysis of how a 10% error on the large-N_c basis coefficients translates to the expected error on the ratio, but is simply an estimate to see if expected trends are satisfied. Although corrections to individual LECs are expected to be $O(1/N_c^2) \approx 10\%$, the inclusion of the 30% error on the ratio is motivated by the analogous analysis of the LO-in-EFT$_{\not\pi}$ results discussed in Sec. IV.

This range of μ agrees with the one found for the LO-in-EFT$_{\not\pi}$ LECs discussed in Sec. IV, which is consistent with the fact that the μ dependence of the two-derivative LEC in a given S-wave channel is related to that of the corresponding zero-derivative LEC through the renormalization group.

B. P-waves

The P-wave LECs are renormalization scale independent to this order in EFT$_{\not\pi}$. Their values can be determined from fits to P-wave phase shifts (see App. B for details). For the following discussion

we will use the central values

$$C^{(^3P_0)} = 6.6\,\mathrm{fm}^4 \;, \quad C^{(^3P_1)} = -6.0\,\mathrm{fm}^4 \;,$$
$$C^{(^3P_2)} = 0.57\,\mathrm{fm}^4 \;, \quad C^{(^1P_1)} = -22\,\mathrm{fm}^4 \;. \tag{33}$$

These values are in agreement with those of Ref. [20], which were extracted from the Nijmegen potential model at 1 MeV. Taking into account different energy ranges used in the fits and different models, the potential model extraction of scattering data yields uncertainties on the values of Eq. (33) of less than 5%. Below we will extract values for the large-N_c basis coefficients to compare with our predictions. The truncation of the large-N_c expansion at a given order introduces theoretical uncertainties that should be taken into account in the fits. A rigorous treatment of these errors is beyond the scope of this work. Instead, as very rough estimates of the expected theoretical uncertainty of $O(1/N_c^2)$, we include 10% errors on the "experimental" coefficients of Eq. (33) and apply naive error propagation to estimate errors on the large-N_c LECs extracted from the fit. But this should be interpreted with caution and the quoted errors should not be considered rigorous. As usual with large-N_c predictions it is more appropriate to speak of trends than of predictions with rigorous error estimates.

1. ^3P_J-waves

At LO-in-N_c, the ^3P_J-wave LECs depend upon only two independent parameters; a relationship exists amongst the ^3P_J-wave LECs. There are several ways to express this. First, consider

$$\left.\frac{C^{(^3P_0)} + C^{(^3P_1)}}{C^{(^3P_2)}}\right|_{\text{LO-in-}N_c} = \frac{10}{3} \approx 3.3 \;. \tag{34}$$

As can be seen from the numerical values in Eq. (33), this is a ratio of small numbers that depends on large cancellations and is therefore very sensitive to the numerical inputs. Using the central values of Eq. (33) gives a ratio of about 1, apparently in serious disagreement with the large-N_c prediction. However, if the procedure described in Sec. V B is used for roughly estimating errors the result is 0.95 $\pm$ 1.6. An equivalent alternative is

$$\left.\frac{C^{(^3P_0)} - \frac{4}{3}C^{(^3P_2)}}{-C^{(^3P_1)} + 2C^{(^3P_2)}}\right|_{\text{LO-in-}N_c} = 1 \;. \tag{35}$$

Using the central values of Eq. (33) gives 0.82. Normally distributed uncorrelated 10% errors about the central values of the partial-wave couplings yield 0.82 $\pm$ 0.12, which is consistent with an LO-in-N_c prediction.

2. $^1P_1$- and ^3P_J-waves

At LO-in-N_c, the LEC for the $^1P_1$-wave depends on an independent set of couplings compared to the ^3P_J-waves, as is most easily seen from the lowest four rows of Eq. (28). In particular, the ^3P_J-waves in isolation only depend on two independent combinations of LO LECs in the large-N_c basis. Including the $^1P_1$ requires all three LO-in-N_c LECs and it is interesting to discover which values for the large-N_c-basis LECs at LO are preferred by only the μ-independent partial waves. Using the last

four rows of Eq. (26) we find that a simultaneous fit of the three LO-in-N_c LECs to the central values of the $^1P_1$ and ^3P_J couplings yields

$$C_{1\cdot1} = -0.36 \pm 0.26 \, \text{fm}^4 \, , \qquad C_{G\cdot G} = 0.12 \pm 0.08 \, \text{fm}^4 \, , \qquad C'_{G\cdot G} = 1.60 \pm 0.12 \, \text{fm}^4 \, , \qquad (36)$$

where the errors are obtained by propagating 10% errors assigned to Eq. (33). But such a result is not to be taken too seriously because we have not attempted to fit to all seven partial waves. In the next two sections we consider large-N_c relationships that involve the μ-dependent LECs.

C. S-D-mixing term and ^3P_J-waves

The EFT$_{\not\pi}$ LEC for the S-D-mixing term is given at its leading order by [1, 39]

$$C^{(SD)} = E_1^{(2)} \frac{3}{\sqrt{2}} C_0^{(^3S_1)} \, , \qquad (37)$$

where $E_1^{(2)}$ is the (μ-independent) LO coefficient in a momentum expansion of the S-D-mixing parameter $\bar{\epsilon}_1$.[1] On dimensional grounds $E_1^{(2)}$ scales as $(1/\Lambda_{\not\pi})^2 \sim 2 \, \text{fm}^2$. But the value extracted from partial-wave data is about five times smaller: $E_1^{(2)} \sim 0.4 \, \text{fm}^2$ [10, 39]. This has led to the conclusion that the S-D-mixing coefficient $C^{(SD)}$ is unnaturally small.[2]

A LO-in-N_c relationship involving the S-D-mixing LEC and ^3P_J-wave LECs is

$$\frac{1}{3} \frac{C^{(SD)}}{C^{(^3P_1)} - 2C^{(^3P_2)}} \bigg|_{\text{LO-in-}N_c} = 1 \, . \qquad (38)$$

Fig. 2 shows the μ dependence of this ratio. Agreement with the LO-in-N_c prediction is found at lower values of μ compared to those found in Fig. 1. But taking into account the suppression of $C^{(SD)}$ brings the preferred μ value of the large-N_c prediction of Eq. (38) back into alignment with the μ range preferred by the large-N_c prediction of Eq. (29). For example, with the physical value of $E_1^{(2)}$, matching the large-N_c expectation for the ratio of Eq. (38) requires $\mu \sim 70 \, \text{MeV}$. Allowing for some of the suppression to be due to variations within natural ranges and taking $C^{(SD)}$ to be a factor of 3 (instead of 5) larger than its physical value, Fig. 3 shows agreement with the LO-in-N_c prediction in the range $105 \, \text{MeV} \lesssim \mu \lesssim 150 \, \text{MeV}$.

D. Relationship involving all $L \leq 2$

The block-diagonal form of Eq. (28) shows that at LO in the large-N_c expansion

$$\frac{C_2^{(^3S_1)} - \frac{1}{6} C^{(SD)}}{-\frac{1}{2} C^{(^1P_1)} + 6C^{(^3P_2)}} \bigg|_{\text{LO-in-}N_c} = 1 \, . \qquad (39)$$

----

[1] Because of the slow convergence of the expansion in $E_1^{(n)}$ (where n is the order in the expansion), $C^{(SD)}$ is sometimes expressed in terms of η_{SD}, the asymptotic D/S ratio of the deuteron [39].

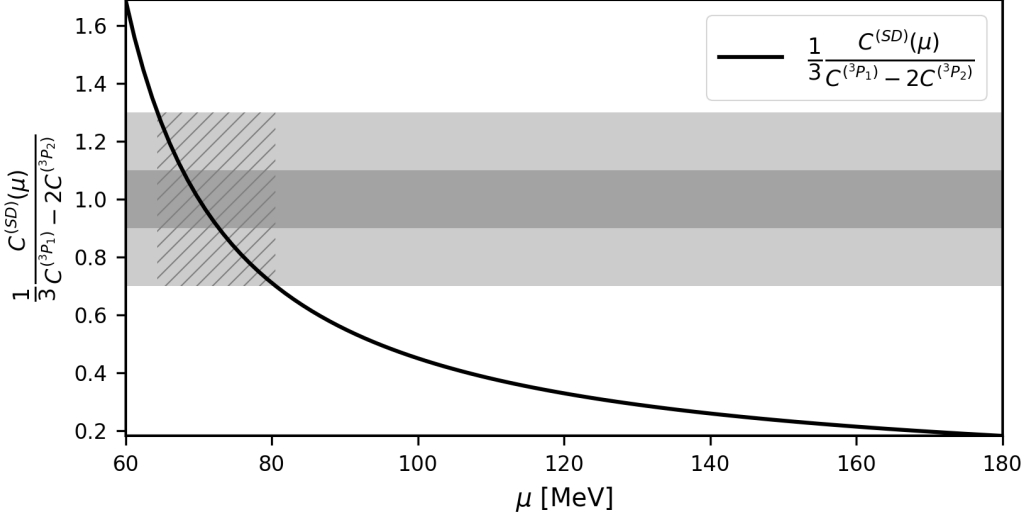[2] We thank G. Rupak for a discussion of this point.

FIG. 2. The ratio of Eq. (38) as a function of the renormalization scale μ (black curve). The gray band shows the large-N_c prediction with 10% (dark gray) and 30% (light gray) variation. The hatched region indicates the range of μ over which the large-N_c prediction is satisfied to 30%.
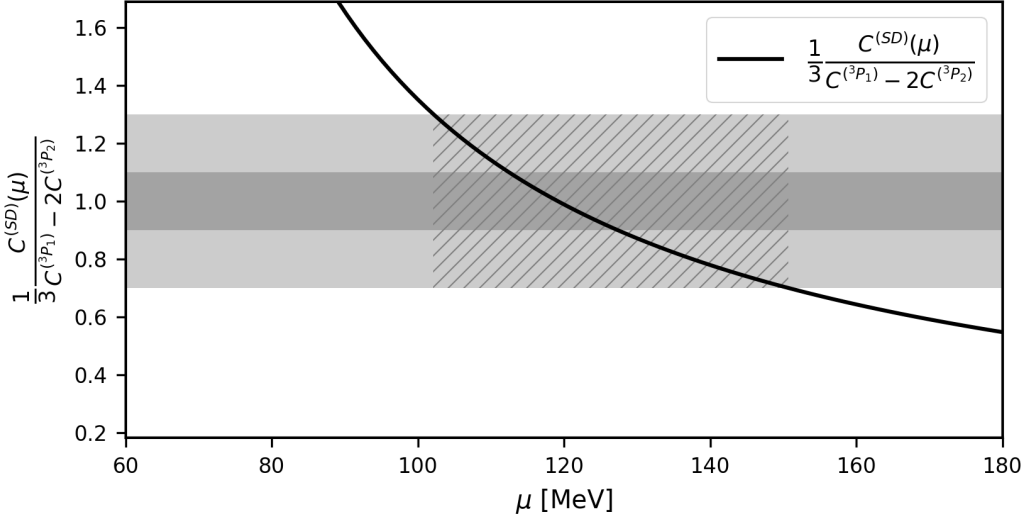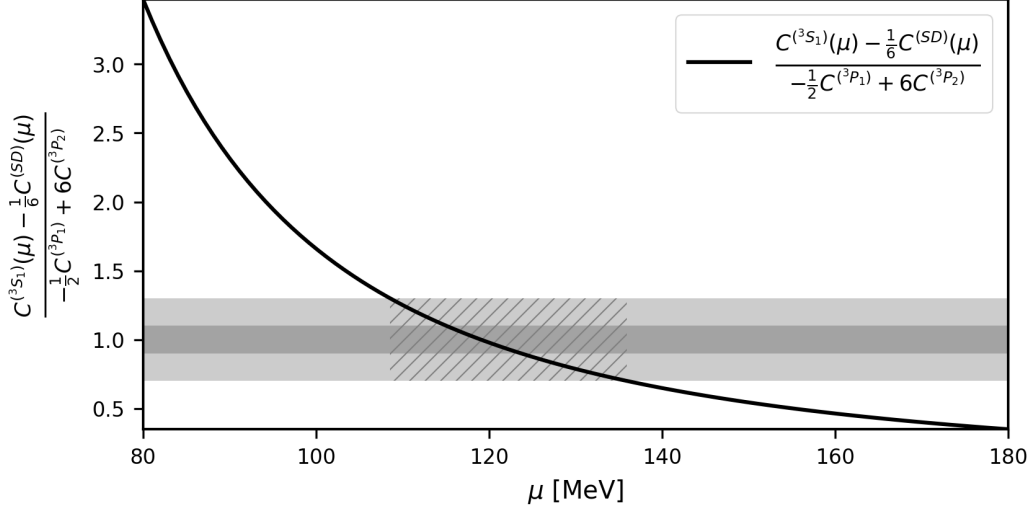

FIG. 3. The ratio of Eq. (38) as a function of the renormalization scale μ (black curve) with the value of $C^{(SD)}$ a factor of 3 larger than its physical value. The gray band shows the large-N_c prediction with 10% (dark gray) and 30% (light gray) variation. The hatched region indicates the range of μ over which the large-N_c prediction is satisfied to 30%.

FIG. 4. The ratio of Eq. (39) as a function of the renormalization scale μ (black curve). The gray band shows the large-N_c prediction with 10% (dark gray) and 30% (light gray) variation. The hatched region indicates the range of μ over which the large-N_c prediction is satisfied to 30%.

Fig. 4 shows this ratio as a function of μ. Agreement within expected 30% corrections is found for the range $110\,\text{MeV} \lesssim \mu \lesssim 135\,\text{MeV}$. A fit of the three LO-in-N_c coefficients to all seven partial-wave couplings at $\mu = 120\,\text{MeV}$ yields

$$C_{1\cdot 1} = -0.58 \pm 0.17\,\text{fm}^4\,, \qquad C_{G\cdot G} = 0.42 \pm 0.05\,\text{fm}^4\,, \qquad C'_{G\cdot G} = 0.76 \pm 0.05\,\text{fm}^4\,, \qquad (40)$$

where the errors are estimated by assuming normally distributed uncorrelated 10% errors about the central values of the partial-wave couplings extracted from experimental inputs for a choice of $\mu = 120\,\text{MeV}$ (see the discussion of error estimates in Sec. V B). While the central values of Eq. (40) do not appear to be very close to those obtained in Sec. V B 2, these are the more appropriate LECs because the fit involves all partial waves. Following the discussion of Sec. V C we now consider how these coefficients change if we allow the anomalously small $C^{(SD)}$ coefficient to be three times its experimental value. In that case, we obtain (still at $\mu = 120\,\text{MeV}$) the following large-N_c coefficients:

$$C_{1\cdot 1} = -0.59 \pm 0.17\,\text{fm}^4, \qquad C_{G\cdot G} = 0.10 \pm 0.07\,\text{fm}^4, \qquad C'_{G\cdot G} = 1.72 \pm 0.15\,\text{fm}^4. \qquad (41)$$

Interestingly, these values are more consistent with those found in Sec. V B 2, indicating that a larger value for $C^{(SD)}$ is more compatible with physics in the P-wave sector. The plot for the large-N_c ratio of Eq. (39) with $C^{(SD)}$ three times its physical value is shown in Fig. 5. Agreement with the large-N_c prediction is found for $115\,\text{MeV} \lesssim \mu \lesssim 145\,\text{MeV}$.

VI.   RESULTS TO N^{2}LO IN N_c

As seen in Sec. III, four additional independent operators contribute at N^{2}LO in the large-N_c expansion. While both central- $(C_{1\cdot 1}, C_{G\cdot G})$ and tensor-type $(C'_{G\cdot G})$ interactions are present at LO-in-N_c
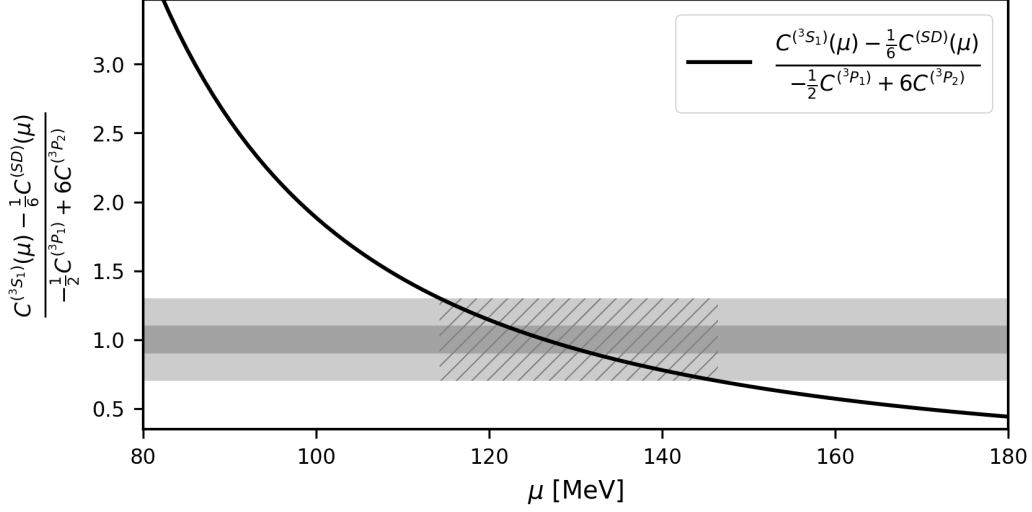
FIG. 5. The ratio of Eq. (39) as a function of the renormalization scale μ (black curve) with the value of $C^{(SD)}$ a factor of 3 larger than its physical value. The gray band shows the large-N_c prediction with 10% (dark gray) and 30% (light gray) variation. The hatched region indicates the range of μ over which the large-N_c prediction is satisfied to 30%.

and receive additional N²LO-in-N_c contributions ($C_{\tau \cdot \tau}, C_{\sigma \cdot \sigma}$, and $C'_{\sigma \cdot \sigma}$, respectively), the spin-orbit contribution ($\overset{\leftrightarrow}{C}_{1 \cdot \sigma}$) to the ^3P_J-waves is a new feature at N²LO-in-N_c. As a result, with seven independent couplings and seven partial waves, the LO relations discussed above no longer hold. For example, as can be seen from Eq. (20), the two S-wave LECs are no longer predicted to be the same. Instead their ratio takes the form

$$\left. \frac{C_2^{(^3S_1)}}{C_2^{(^1S_0)}} \right|_{\text{N}^2\text{LO-in-}N_c} = \frac{C_{1\cdot 1} - 3C_{G\cdot G} - C'_{G\cdot G} - 3C_{\tau \cdot \tau} + C_{\sigma \cdot \sigma} + \frac{1}{3}C'_{\sigma \cdot \sigma}}{C_{1\cdot 1} - 3C_{G\cdot G} - C'_{G\cdot G} + C_{\tau \cdot \tau} - 3C_{\sigma \cdot \sigma} - C'_{\sigma \cdot \sigma}} \ . \tag{42}$$

The modification of the other ratios considered at LO-in-N_c is provided in Appendix C.

When all seven N²LO-in-EFT$_{\slashed{\pi}}$ LECs are fit to the seven partial-wave LECs at $\mu = 120\,\text{MeV}$ we obtain

$$\left. \begin{array}{l} C_{1\cdot 1} = -0.58 \pm 0.11\,\text{fm}^4 \ , \ C_{G\cdot G} = 0.40 \pm 0.05\,\text{fm}^4 \ , \\ C'_{G\cdot G} = 0.84 \pm 0.05\,\text{fm}^4 \end{array} \right\} \text{LO} \tag{43}$$

$$\left. \begin{array}{l} C_{\tau \cdot \tau} = 0.15 \pm 0.07\,\text{fm}^4 \ , \ C_{\sigma \cdot \sigma} = -0.39 \pm 0.07\,\text{fm}^4 \ , \\ C'_{\sigma \cdot \sigma} = 0.78 \pm 0.10\,\text{fm}^4 \ , \ \overset{\leftrightarrow}{C}_{1 \cdot \sigma} = -0.17 \pm 0.12\,\text{fm}^4 \ , \end{array} \right\} \text{N}^2\text{LO} \tag{44}$$

where the errors are again obtained by propagating 10% errors on the partial-wave LECs. From this set of values it appears that, at least for this choice of μ, there is no clear evidence that the "N²LO-in-N_c" coefficients are smaller than the "LO-in-N_c" coefficients. However, all that the large-N_c analysis tells us is that the LO-in-N_c coefficients *can* start at order N_c, not that they do. There

could be cancellations that cause them to be smaller than expected. It is interesting to see that the spin-orbit term is indeed suppressed compared to the LO-in-N_c terms as predicted, although the suppression is less pronounced than the $1/N_c^2$ expectation.

As discussed in Sec. V C, obtaining consistent values of μ to satisfy the large-N_c ratios tends to confirm the claim of Ref. [40] that the S-D-mixing LEC is unnaturally small. Following the approach at LO and taking the S-D-mixing LEC to be a factor of three larger than its experimental value yields

$$\left.\begin{array}{l} C_{1\cdot 1} = -0.59 \pm 0.10\,\mathrm{fm}^4 \ , \quad C_{G\cdot G} = 0.11 \pm 0.06\,\mathrm{fm}^4 \ , \\ C'_{G\cdot G} = 1.72 \pm 0.13\,\mathrm{fm}^4 \end{array}\right\}\mathrm{LO} \tag{45}$$

$$\left.\begin{array}{l} C_{\tau\cdot\tau} = 0.16 \pm 0.07\,\mathrm{fm}^4 \ , \quad C_{\sigma\cdot\sigma} = -0.10 \pm 0.08\,\mathrm{fm}^4 \ , \\ C'_{\sigma\cdot\sigma} = -0.10 \pm 0.16\,\mathrm{fm}^4 \ , \quad \overset{\leftrightarrow}{C}_{1\cdot\sigma} = -0.17 \pm 0.12\,\mathrm{fm}^4 \ , \end{array}\right\}\mathrm{N}^2\mathrm{LO} \tag{46}$$

It is notable that performing the fit of the N^{2}LO-in-N_c LECs while excluding the S-D term gives very similar results. These values are closer to the expected pattern that the three LO-in-N_c LECs are larger than the N^{2}LO-in-N_c LECs, with the $C_{G\cdot G}$ term the exception. This result emphasizes again that the large-N_c analysis only provides trends and that other physics can have a significant impact on the size of the couplings.

VII. CONCLUSIONS

We analyzed the two-derivative nucleon-nucleon contact interactions in a combined EFT$_{\not\pi}$ and large-N_c approach. The symmetries of QCD as encoded in EFT$_{\not\pi}$ restrict the form of the operators, while the corresponding EFT$_{\not\pi}$ LECs encode short-distance details of the underlying interactions. Because QCD has not been solved, these LECs are treated as free parameters in EFT$_{\not\pi}$. The feature of the large-N_c analysis is that QCD attains additional symmetries in the large-N_c limit that constrain the relative sizes of the EFT$_{\not\pi}$ LECs. We showed that in the large-N_c limit only three of the seven EFT$_{\not\pi}$ LECs are independent; we derived four independent relationships between the LECs that should hold in this limit. Critically, the LECs that multiply operators involving S-waves are not observables but contain a subtraction point (μ) dependence. Since large-N_c relationships are expressed in terms of these coefficients it is important to choose values of μ so that the large-N_c relationships are not obscured. By using empirical values extracted from partial-wave analyses, we showed that the LO-in-N_c relationships are reasonably well satisfied even in the real world with $N_c = 3$ for appropriate values of μ. Consistency among the relations is improved by implementing the observation that the S-D-mixing coefficient $C^{(SD)}$ is unnaturally small and adjusting it upward.

At LO-in-N_c the large-N_c basis coefficients favored by experiment are all of the same order, which suggests that the large-N_c counting is compatible with nature. At N^{2}LO-in-N_c the values of the three LO-in-N_c coefficients do not change dramatically compared to the LO extraction. But contrary to expectation the additional four LECs are not 10% smaller. However, the large-N_c counting only establishes an upper limit on the size of the "LO-in-N_c" coefficients. It is possible that other effects cause cancellations that make them smaller than the naive large-N_c estimate. Interestingly, if we take $C^{(SD)}$ to be three times its actual value, in acknowledgment that it is experimentally unnaturally small, the three LO-in-N_c coefficients again do not change dramatically in the fit involving all seven LECs at N^{2}LO-in-N_c, but for this enhanced value of $C^{(SD)}$ there is some evidence that the N^{2}LO-in-N_c coefficients tend to be smaller compared to the LO-in-N_c coefficients.

Our analysis shows that the large-N_c approach can provide useful guidance in constraining the LECs in EFT$_{\not\pi}$. In particular, when little data are available to constrain LECs it may be useful to employ large-N_c relationships to restrict the number of unknown LECs. However, as illustrated by

the S-D-mixing term, other physics that is not captured in the large-N_c expansion can influence the relative size of couplings. In addition, in the case of subtraction-point-dependent LECs, special care has to be taken to not obscure large-N_c relations. The results of the large-N_c analysis should therefore not be considered as exact predictions, but instead should be interpreted as providing overall trends.

Appendix A: Fierz identities

The Fierz identies needed to relate the two bases are:

$$\delta_{CB}\delta_{DA} = -\frac{1}{2}(\sigma_2)_{DC}(\sigma_2)_{AB} + \frac{1}{2}(\sigma_i\sigma_2)_{DC}(\sigma_2\sigma_i)_{AB} \ , \tag{A1}$$

$$\delta_{CA}\delta_{DB} = \frac{1}{2}(\sigma_2)_{DC}(\sigma_2)_{AB} + \frac{1}{2}(\sigma_i\sigma_2)_{DC}(\sigma_2\sigma_i)_{AB} \ , \tag{A2}$$

$$(\sigma_i)_{CB}(\sigma_j)_{DA} = \frac{1}{2}\delta_{ij}(\sigma_2)_{DC}(\sigma_2)_{AB} - \frac{1}{2}i\epsilon_{ijk}\left[(\sigma_k\sigma_2)_{DC}(\sigma_2)_{AB} + (\sigma_2)_{DC}(\sigma_2\sigma_k)_{AB}\right]$$
$$- \frac{1}{2}\left(\delta_{ik}\delta_{jn} + \delta_{jk}\delta_{in} - \delta_{ij}\delta_{kn}\right)(\sigma_k\sigma_2)_{DC}(\sigma_2\sigma_n)_{AB} \ , \tag{A3}$$

$$(\sigma_i)_{CA}(\sigma_j)_{DB} = -\frac{1}{2}\delta_{ij}(\sigma_2)_{DC}(\sigma_2)_{AB} + \frac{1}{2}i\epsilon_{ijk}\left[(\sigma_k\sigma_2)_{DC}(\sigma_2)_{AB} - (\sigma_2)_{DC}(\sigma_2\sigma_k)_{AB}\right]$$
$$- \frac{1}{2}\left(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk} - \delta_{ij}\delta_{kn}\right)(\sigma_k\sigma_2)_{DC}(\sigma_2\sigma_n)_{AB} \ , \tag{A4}$$

where lowercase indices run from 1 through 3 with summation over repeated indices, and uppercase letters run from 1 to 2 as the spin indices. Analogous relations hold for isospin matrices with the substitution $\boldsymbol{\sigma} \to \boldsymbol{\tau}$.

Appendix B: Extracting LECs from data

To determine how well the P-wave large-N_c predictions are satisfied we need to extract the values of the P-wave LECs from phase shifts. They are related using

$$S^{(c)} = e^{2i\delta^{(c)}} = 1 + i\frac{pM}{2\pi}A^{(c)} \ , \tag{B1}$$

where p is the center-of-mass momentum, M is the nucleon mass, $S^{(c)}$ is the scattering S-matrix, and $A^{(c)}$ is the scattering amplitude in the c channel. The P-wave phase shifts $\delta^{(c)}$ can be expanded as [41]

$$p^3\cot \delta^{(c)} = -\frac{1}{a_P^{(c)}} + \frac{1}{2}r_P^{(c)}p^2 + \cdots \ , \tag{B2}$$

where $a_P^{(c)}$ is a P-wave scattering volume and $r_P^{(c)}$ is a P-wave effective range in channel c. For small energies and small phase shifts Eqs. (B1) and (B2) reduce to

$$A^{(c)} = \frac{4\pi}{pM}\delta^{(c)} + \cdots = \frac{4\pi a_P^{(c)}}{M}p^2 + \mathcal{O}(p^4) , \tag{B3}$$

where to the order we are considering the amplitudes are related to the LECs by

$$A^{(^1P_1)} = \frac{1}{3}p^2 C^{(^1P_1)}, \quad A^{(^3P_0)} = p^2 C^{(^3P_0)}, \quad A^{(^3P_1)} = \frac{2}{3}p^2 C^{(^3P_1)}, \quad A^{(^3P_2)} = \frac{4}{3}p^2 C^{(^3P_2)} . \tag{B4}$$

At very low energies, the contribution of partial waves with $L > 1$ to a differential cross section can be neglected, but at very low energies the P-wave contribution itself is a small percentage of the differential cross section. Also, low-energy data points are scarce, may have large uncertainties, and are mostly available from proton-proton scattering, where Coulomb corrections might be important. Hence there is tension in deciding the most appropriate energy range to use to fit the partial-wave LECs in the $L = 1$ sector. Databases (e.g., NNOnline [42] and references therein) helpfully compile available data as encoded via phenomenological potential models. The fit here is performed for lab energies in the range $[0, T]$ MeV. The central values of the fit for $T = 10\,\text{MeV}$ are given in Eq. (33). For a given potential model the error from using different energy ranges is small. The variations between different models are about 1%, with the exception of the $^3P_2$-wave, for which they are still less than 5%. For a detailed study of systematic errors from the extraction of partial wave parameters from potential models see Ref. [43].

It may also be useful to collect here the relationships relevant for the other two-derivative (but non-P-wave) operators. The LECs in the S-wave channels at two derivatives involve the scattering lengths $a^{(^1S_0)}$ and $a^{(^3S_1)}$ and the effective ranges $r^{(^1S_0)}$ and $r^{(^3S_1)}$. The relationships with the LECs for channel $S = {}^1S_0$ or $^3S_1$ in the PDS scheme are

$$C_2^{(S)}(\mu) = \frac{2\pi}{M}\left(\frac{1}{-\mu + 1/a^{(S)}}\right)^2 r^{(S)} , \tag{B5}$$

and the connection to the phase shifts is given by the effective range expansion [35, 44, 45]

$$p\cot\delta^{(S)} = -\frac{1}{a^{(S)}} + \frac{1}{2}r^{(S)}p^2 + \cdots .$$

The LEC $C^{(SD)}$ is related to the SD-mixing parameter $\bar{\epsilon}_1$. Performing an expansion of the mixing parameter in powers of Q, $\bar{\epsilon}_1 = \epsilon_1^{(2)} + \cdots$, the leading-order contribution appears at order Q^2 and is given by [39]

$$\epsilon_1^{(2)} = E_1^{(2)}\frac{p^3}{\sqrt{p^2 + \gamma^2}} , \tag{B6}$$

where $E_1^{(2)} = 0.386\,\text{fm}^2$ and γ is the deuteron binding momentum. The μ-dependent LEC $C^{(SD)}$ is given by

$$C^{(SD)} = \frac{3}{\sqrt{2}}\frac{4\pi}{M}\frac{1}{(\gamma - \mu)}E_1^{(2)} . \tag{B7}$$

Appendix C: N²LO modification of LO large-N_c ratios

Analogous to the S-wave ratio of Eq. (42), the other ratios discussed in Sec. V are also modified by N²LO large-N_c terms. The ratio of Eq. (34) takes the form

$$\frac{C^{(^3P_0)} + C^{(^3P_1)}}{C^{(^3P_2)}}\Bigg|_{\text{N}^2\text{LO-in-}N_c} = \frac{\frac{10}{3}(C_{1\cdot1} + C_{G\cdot G} + C_{\tau\cdot\tau} + C_{\sigma\cdot\sigma}) - \frac{14}{3}\overset{\leftrightarrow}{C}_{1\cdot\sigma}}{(C_{1\cdot1} + C_{G\cdot G} + C_{\tau\cdot\tau} + C_{\sigma\cdot\sigma}) + \overset{\leftrightarrow}{C}_{1\cdot\sigma}} \; . \tag{C1}$$

The result is written with suggestive parentheses to indicate that for this large-N_c expression only the spin-orbit term $(\overset{\leftrightarrow}{C}_{1\cdot\sigma})$ modifies the LO-in-N_c result. If $\overset{\leftrightarrow}{C}_{1\cdot\sigma}$ is smaller than the other $1/N_c$ terms, this relationship may be considered more robust than those that receive corrections from the central and tensor $1/N_c$ terms. Equation (35) is modified to

$$\frac{C^{(^3P_0)} - \frac{4}{3}C^{(^3P_2)}}{-C^{(^3P_1)} + 2C^{(^3P_2)}}\Bigg|_{\text{N}^2\text{LO-in-}N_c} = \frac{(C'_{G\cdot G} + C'_{\sigma\cdot\sigma}) + \overset{\leftrightarrow}{C}_{1\cdot\sigma}}{(C'_{G\cdot G} + C'_{\sigma\cdot\sigma}) - \overset{\leftrightarrow}{C}_{1\cdot\sigma}} \; . \tag{C2}$$

Again, since this is an equivalent relationship to the one above, we see that the change in the ratio is only due to the spin-orbit term. If that term is small for some reason then this ratio is less sensitive to N²LO-in-N_c corrections. From Eq. (38) we find

$$\frac{1}{3}\frac{C^{(SD)}}{C^{(^3P_1)} - 2C^{(^3P_2)}}\Bigg|_{\text{N}^2\text{LO-in-}N_c} = \frac{1}{3}\frac{3C'_{G\cdot G} - C'_{\sigma\cdot\sigma}}{C'_{G\cdot G} - \overset{\leftrightarrow}{C}_{1\cdot\sigma} + C'_{\sigma\cdot\sigma}} \; , \tag{C3}$$

while the relation of Eq. (39) takes the form

$$\frac{C_2^{(^3S_1)} - \frac{1}{6}C^{(SD)}}{-\frac{1}{2}C^{(^1P_1)} + 6C^{(^3P_2)}}\Bigg|_{\text{N}^2\text{LO-in-}N_c} =$$
$$\frac{-2C_{1\cdot1} + 6C_{G\cdot G} + 3C'_{G\cdot G} + 6C_{\tau\cdot\tau} - 2C_{\sigma\cdot\sigma} - C'_{\sigma\cdot\sigma}}{-2C_{1\cdot1} + 6C_{G\cdot G} + 3C'_{G\cdot G} - 6C_{\tau\cdot\tau} - 6C_{\sigma\cdot\sigma} - C'_{\sigma\cdot\sigma} - 3\overset{\leftrightarrow}{C}_{1\cdot\sigma}} \; . \tag{C4}$$

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